The study of nonlinear fractional partial differential equations via the Khalouta-Atangana-Baleanu operator

Ali KHALOUTA

Laboratory of Fundamental and Numerical Mathematics
Department of Mathematics, Faculty of Sciences
Ferhat Abbas Sétif University 1, 19000 Sétif, Algeria.
E-mail: nadjibkh@yahoo.fr; ali.khalouta@univ-setif.dz

Abstract

This paper studies nonlinear fractional partial differential equations via the Khalouta-Atangana-Baleanu operator. Using Banach's fixed point theorem we obtain new results on the existence and uniqueness of solutions to the proposed problem. Furthermore, two new semi-analytical methods called Khalouta homotopy perturbation method (KHHPM) and Khalouta variational iteration method (KHVIM) are presented to find new approximate analytical solutions to our nonlinear fractional problem. The first of the two new proposed methods, KHHPM, is a hybrid method that combines homotopy perturbation method and Khalouta transform in the sense of Atangana-Baleanu-Caputo derivative. The other method, KHVIM is also a hybrid method that combines variational iteration method and Khalouta transform in the sense of Atangana-Baleanu-Caputo derivative. Convergence and absolute error analysis of KHHPM and KHVIM were also performed. A numerical example is provided to support our results. The results obtained showed that the proposed methods are very impressive, effective, reliable, and easy methods for dealing with complex problems in various fields of applied sciences and engineering.

Keywords: Fractional partial differential equations, Atangana-Baleanu operator, Banach space, Khalouta transform method, Homotopy perturbation method, Variational iteration method.

2020 Mathematics Subject Classification: 35R11, 26A33, 35A22, 47H10.

1 Introduction

Fractional calculus and its applications have attracted the attention of many scientistes and researchers in recent years, not only in mathematics but also in a variety of scientific disciplines such as physics [17], chemical kinetics [21], fluid dynamics [18], viscoelastic [5], electrochemistry [16], elasticity [2], engineering [23], economics [22], financial systems [19], biology [9], medicine [20], statistics [1], computing image [25], nonlinear heat conduction [6], optimal control [4], etc. Moreover, many cosmic events that classical differential equations cannot describe can be described by fractional differential equations.
One of the most exciting and challenging studies today is the search for the exact or approximate solutions of nonlinear fractional differential equations in mathematical and physical sciences. Recently, various methods have been presented for solving nonlinear fractional differential equations. For example, in [11], the author proposed the modified fractional Taylor series method (MFTSM) to achieve an approximate solution for nonlinear fractional Lienard's equations with Caputo fractional derivative. In [12], presented a novel iterative method to approximate the solution of nonlinear wave-like equations of fractional order with variable coefficients. Homotopy perturbation transform method (HPTM) was presented in [13] for the approximate solution of Caputo time-fractional nonlinear system of equations describing the unsteady flow of a polytropic gas. It was proven that the method converges to the exact solution. In [14] it was introduced Elzaki differential transform method (EDTM) to get to numerical solution of the the fractional SIS epidemic model.
The aim of this paper is to determine sufficient conditions for the existence and uniqueness of the solution of nonlinear fractional partial differential equations involving Atangana-Baleanu-Caputo fractional derivative of arbitrary order $\alpha \in (0,1)$ of the form

$$^{ABC}D_{\varsigma}^{\alpha}\Theta(\varkappa,\varsigma)=R\Theta(\varkappa,\varsigma)+N\Theta(\varkappa,\varsigma)+F(\varkappa,\varsigma), \tag{1.1}$$

with the initial condition

$$\Theta(\varkappa,0)=\Theta_0(\varkappa), \tag{1.2}$$

where $\varkappa\in[a,b], b>a, \varsigma\geq 0$, $^{ABC}D_{\varsigma}^{\alpha}$ is the Atangana-Baleanu-Caputo fractional derivative operator of order $\alpha\in(0,1)$, R and N are a linear and nonlinear operators, respectively, and F is the source term. In addition, , we propose two semi analytical methods called Khalouta homotopy perturbation method (KHHPM) and Khalouta variational iteration method (KHVIM) to find new approximate analytical solution to the proposed problem. The KHHPM and KHVIM are a combination of the homotopy perturbation method and the variational iteration method which were first proposed by Ji-Haun-He [7, 8] and the Khalouta transform which is a generalization of the several well-known integral transforms [10]. The most important features of the proposed methods can be summarized in the following points.
1- The KHHPM and KHVIM can be applied to analyze the solution of linear or nonlinear fractional problems without any type of discretization, linearization, perturbation, or restrictive assumptions.
2- The KHHPM and KHVIM gives a series solutions which converge rapidly within few number of iterations.
3- The KHHPM and KHVIM are accurate and effective with minimal effort to achieve results.
4- The KHHPM and KHVIM are used to investigate the analytical and numerical solutions of fractional partial differential equations which naturally arises in applied sciences and engineering.
The remainder of this paper is organized as follows. In Section 2, we start with some basic definitions and theorems of fractional calculus and Khalouta transform, respectively. In Section 3 and 4, we study the existence and uniqueness theorem and propose two algorithms for KHHPM and KHVIM to solve the nonlinear fractional partial differential equations via the Atangana-Baleanu-Caputo fractional derivative which are considered as the main contributions in this paper. In Section 5, we present an example illustrating the main result. In Section 6, we provide results and discussions on the proposed methods. Finally, concluding remarks are given in Section 7.

2 Preliminaries and results

In this section, we demonstrate some important ideas and consequences of fractional calculus that have recently been developed by [3]. In addition, the Khalouta transform and some of its useful theorems used in this paper are also presented.

Definition 2.1. Let a function $\Theta(.,\varsigma)\in H^1(0,T), T>0$ for each fixed $\varkappa\in I=[a,b]\subset\mathbb{R}$ and $0<\alpha<1$, then the Atangana-Baleanu fractional derivative in Riemann-Liouville sense is defined as

$$^{ABR}D_{\varsigma}^{\alpha}\Theta(\varkappa,\varsigma)=\frac{\mathcal{AB}(\alpha)}{1-\alpha}\frac{d}{d\varsigma}\int_0^{\varsigma}\Theta(\varkappa,\tau)E_{\alpha}\left(-\frac{\alpha\left(\varsigma-\tau\right)^{\alpha}}{1-\alpha}\right)d\tau, \tag{2.1}$$

and the Atangana-Baleanu fractional derivative in Caputo sense is defined as

$$^{ABC}D_{\varsigma}^{\alpha}\Theta(\varkappa,\varsigma)=\frac{\mathcal{AB}(\alpha)}{1-\alpha}\int_0^{\varsigma}\Theta'(\varkappa,\tau)E_{\alpha}\left(-\frac{\alpha\left(\varsigma-\tau\right)^{\alpha}}{1-\alpha}\right)d\tau, \tag{2.2}$$

where $\mathcal{AB}(\alpha)$ represents the normalization function that satisfies the conditions $\mathcal{AB}(0)=\mathcal{AB}(1)=1$ and $E_{\alpha}(.)$ represents the special function known as Mittag-Leffler function for one parameter is defined as [15]

$$E_{\alpha}\left(z\right)=\sum_{i=0}^{\infty}\frac{z^i}{\Gamma(i\alpha+1)}, \alpha>0, z\in\mathbb{C}. \tag{2.3}$$

Definition 2.2. The fractional integral associate to the Atangana-Baleanu fractional derivative of order $0 < \alpha < 1$ is defined as

$$^{AB}I_{\varsigma}^{\alpha}\Theta(\varkappa,\varsigma) = \frac{1-\alpha}{\mathcal{AB}(\alpha)}\Theta(\varkappa,\varsigma) + \frac{\alpha}{\mathcal{AB}(\alpha)\Gamma(\alpha)}\int_0^{\varsigma}\Theta(\varkappa,\tau)\left(\varsigma-\tau\right)^{\alpha-1}d\tau. \tag{2.4}$$

Now, we present a new result related to the Khalouta transform of the Atangana-Baleanu fractional derivative. The Khalouta transform is a new integral transform that is applied to solve ordinary and partial differential equations, defined and developed by [10].

Definition 2.3. The Khalouta transform of the piecewise continuous function $\Theta(\varkappa,\varsigma)$ on $I \times [0,T]$ and of exponential order is given by the following integral

$$\mathbb{KH}\left[\Theta(\varkappa,\varsigma)\right] = \mathcal{K}(\varkappa,s,\gamma,\eta) = \frac{s}{\gamma\eta}\int_0^{\infty}\exp\left(-\frac{s\varsigma}{\gamma\eta}\right)\Theta(\varkappa,\varsigma)d\varsigma, s,\gamma,\eta > 0. \tag{2.5}$$

The basic properties of the Khalouta transform are given in the following theorem [10].

Theorem 2.1. *1) If $\Theta(\varkappa,\varsigma)$ and $\Psi(\varkappa,\varsigma)$ be piecewise continuous and of exponential order, then for all constants λ and μ, we have*

$$\mathbb{KH}\left[\lambda\Theta(\varkappa,\varsigma)+\mu\Psi(\varkappa,\varsigma)\right] = \lambda\mathbb{KH}\left[\Theta(\varkappa,\varsigma)\right] + \mu\mathbb{KH}\left[\Psi(\varkappa,\varsigma)\right]. \tag{2.6}$$

2) If the n^{th} derivative of $\Theta(\varkappa,\varsigma)$ withrespect to t is $\Theta^{(n)}(\varkappa,\varsigma)$, then its Khalouta transform is given as

$$\mathbb{KH}\left[\Theta^{(n)}(\varkappa,\varsigma)\right] = \frac{s^n}{\gamma^n\eta^n}\mathcal{K}(\varkappa,s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}\Theta^{(k)}(\varkappa,0), n \geq 1. \tag{2.7}$$

3) If the Khalouta transform of $\Theta(\varkappa,\varsigma)$ and $\Psi(\varkappa,\varsigma)$ are $\mathcal{K}(\varkappa,s,\gamma,\eta)$ and $\mathcal{V}(\varkappa,s,\gamma,\eta)$ respectively, then

$$\mathbb{KH}\left[\left(\Theta*\Psi\right)(\varkappa,\varsigma)\right] = \int_0^{\infty}\Theta(\varkappa,\varsigma)\Psi(\varkappa,\varsigma-\tau)d\tau = \frac{\gamma\eta}{s}\mathcal{K}(\varkappa,s,\gamma,\eta)\mathcal{V}(\varkappa,s,\gamma,\eta), \tag{2.8}$$

where $\mathbb{KH}\left[\left(\Theta*\Psi\right)(\varkappa,\varsigma)\right]$ is the Khalouta convolution of the functions $\Theta(\varkappa,\varsigma)$ and $\Psi(\varkappa,\varsigma)$.
4) The Khalouta transforms of some special functions are as follows

$$\begin{aligned}
\mathbb{KH}[1] &= 1,\\
\mathbb{KH}[\varsigma] &= \frac{\gamma\eta}{s},\\
\mathbb{KH}\left[\frac{\varsigma^n}{n!}\right] &= \frac{\gamma^n\eta^n}{s^n}, n = 0,1,2,...\\
\mathbb{KH}\left[\frac{\varsigma^{\alpha}}{\Gamma(\alpha+1)}\right] &= \frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}, \alpha > -1,\\
\mathbb{KH}\left[E_{\alpha}(-a\varsigma^{\alpha})\right] &= \frac{s^{\alpha}}{s^{\alpha}+a\gamma^{\alpha}\eta^{\alpha}}, a \in \mathbb{R}.
\end{aligned} \tag{2.9}$$

Theorem 2.2. *Let $\mathcal{K}(\varkappa,s,\gamma,\eta)$ be the Khalouta transform of the function $\Theta(\varkappa,\varsigma)$. Then the Khalouta transform of the Atangana-Baleanu fractional derivative in Riemann-Liouville sense is expressed as*

$$\mathbb{KH}\left[^{ABR}D_{\varsigma}^{\alpha}\Theta(\varkappa,\varsigma)\right] = \left(\frac{s^{\alpha}\mathcal{AB}(\alpha)}{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}\right)\mathcal{K}(\varkappa,s,\gamma,\eta). \tag{2.10}$$

Proof Using the definition of Khaloua transform (2.5) and the Atangana-Baleanu fractional derivative in Riemann-Liouville sense (2.1), we get

$$\mathbb{KH}\left[^{ABR}D_{\varsigma}^{\alpha}\Theta(\varkappa,\varsigma)\right] = \mathbb{KH}\left[\frac{\mathcal{AB}(\alpha)}{1-\alpha}\frac{d}{d\varsigma}\int_0^{\varsigma}\Theta(\varkappa,\tau)E_{\alpha}\left(-\frac{\alpha\left(\varsigma-\tau\right)^{\alpha}}{1-\alpha}\right)d\tau\right]. \tag{2.11}$$

Applying the properties of the Khalouta transform given in equations (2.7) and (2.8), we get

$$\begin{aligned}
\mathbb{KH}\left[{}^{ABR}D_{\varsigma}^{\alpha}\Theta(\varkappa,\varsigma)\right] &= \frac{\mathcal{AB}(\alpha)}{1-\alpha}\mathbb{KH}\left[\frac{d}{d\varsigma}\left(\Theta(\varkappa,\varsigma)*E_{\alpha}\left(-\frac{\alpha\varsigma^{\alpha}}{1-\alpha}\right)\right)\right] \\
&= \frac{\mathcal{AB}(\alpha)}{1-\alpha}\left(\frac{s}{\gamma\eta}\mathbb{KH}\left[\Theta(\varkappa,\varsigma)*E_{\alpha}\left(-\frac{\alpha\varsigma^{\alpha}}{1-\alpha}\right)\right]-\frac{s}{\gamma\eta}\mathbb{KH}\left[\Theta(\varkappa,0)*E_{\alpha}(0)\right]\right) \\
&= \frac{\mathcal{AB}(\alpha)}{1-\alpha}\left(\frac{s^{\alpha}}{s^{\alpha}+\frac{\alpha}{1-\alpha}\gamma^{\alpha}\eta^{\alpha}}\mathcal{K}(\varkappa,s,\gamma,\eta)\right) \\
&= \left(\frac{s^{\alpha}\mathcal{AB}(\alpha)}{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}\right)\mathcal{K}(\varkappa,s,\gamma,\eta). \qquad (2.12)
\end{aligned}$$

This completes the proof. ■

Theorem 2.3. *Let $\mathcal{K}(\varkappa,s,\gamma,\eta)$ be the Khalouta transform of the function $\Theta(\varkappa,\varsigma)$. Then the Khalouta transform of the Atangana-Baleanu fractional derivative in Caputo sense is expressed as*

$$\mathbb{KH}\left[{}^{ABC}D_{\varsigma}^{\alpha}\Theta(\varkappa,\varsigma)\right] = \left(\frac{s^{\alpha}\mathcal{AB}(\alpha)}{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}\right)\left(\mathcal{K}(\varkappa,s,\gamma,\eta)-\Theta(\varkappa,0)\right). \qquad (2.13)$$

Proof Using the definition of Khaloua transform (2.5) and the Atangana-Baleanu fractional derivative in Caputo sense (2.2), we get

$$\mathbb{KH}\left[{}^{ABC}D_{\varsigma}^{\alpha}\Theta(\varkappa,\varsigma)\right] = \mathbb{KH}\left[\frac{\mathcal{AB}(\alpha)}{1-\alpha}\int_0^{\varsigma}\Theta'(\varkappa,\tau)E_{\alpha}\left(-\frac{\alpha\left(\varsigma-\tau\right)^{\alpha}}{1-\alpha}\right)d\tau\right]. \qquad (2.14)$$

Applying the properties of the Khalouta transform given in equations (2.7) and (2.8), we get

$$\begin{aligned}
\mathbb{KH}\left[{}^{ABC}D_{\varsigma}^{\alpha}\Theta(\varkappa,\varsigma)\right] &= \frac{\mathcal{AB}(\alpha)}{1-\alpha}\mathbb{KH}\left[\left(\Theta'(\varkappa,\varsigma)*E_{\alpha}\left(-\frac{\alpha\varsigma^{\alpha}}{1-\alpha}\right)\right)\right] \\
&= \frac{\mathcal{AB}(\alpha)}{1-\alpha}\left(\frac{\gamma\eta}{s}\mathbb{KH}\left[\Theta'(\varkappa,\varsigma)\right]\mathbb{KH}\left[E_{\alpha}\left(-\frac{\alpha\varsigma^{\alpha}}{1-\alpha}\right)\right]\right) \\
&= \frac{\mathcal{AB}(\alpha)}{1-\alpha}\frac{s^{\alpha}}{s^{\alpha}+\frac{\alpha}{1-\alpha}\gamma^{\alpha}\eta^{\alpha}}\left(\mathcal{K}(\varkappa,s,\gamma,\eta)-\Theta(\varkappa,0)\right) \\
&= \left(\frac{s^{\alpha}\mathcal{AB}(\alpha)}{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}\right)\left(\mathcal{K}(\varkappa,s,\gamma,\eta)-\Theta(\varkappa,0)\right). \qquad (2.15)
\end{aligned}$$

This completes the proof. ■

3 Existence and uniqueness results

In this section, our main objective is to use Banach fixed point theorem to establish existence and uniqueness results to the nonlinear Atangana-Baleanu-Caputo fractional partial differential equation

$${}^{ABC}D_{\varsigma}^{\alpha}\Theta(\varkappa,\varsigma) = R\Theta(\varkappa,\varsigma)+N\Theta(\varkappa,\varsigma)+F(\varkappa,\varsigma), \qquad (3.1)$$

with the initial condition

$$\Theta(\varkappa,0) = \Theta_0(\varkappa), \qquad (3.2)$$

where $\varkappa\in[a,b]$, $b>a$, $\varsigma\geq 0$, ${}^{ABC}D_{\varsigma}^{\alpha}$ is the Atangana-Baleanu-Caputo fractional derivative operator of order $0<\alpha\leq 1$, R and N are a linear and nonlinear operators, respectively, and F is the source term.

Theorem 3.1. *Let $\Theta(.,\varsigma) \in H^1(0,T), T>0$ for each fixed $\varkappa \in I=[a,b] \subset \mathbb{R}$, then $\Theta(\varkappa,\varsigma)$ is a solution of equations (3.1)-(3.2), if and only if it is a solution of the integral equation*

$$\begin{aligned}\Theta(\varkappa,\varsigma) &= \Theta_0(\varkappa)+\frac{1-\alpha}{\mathcal{AB}(\alpha)}(R\Theta(\varkappa,\varsigma)+N\Theta(\varkappa,\varsigma)+F(\varkappa,\varsigma)) \\ &+\frac{\alpha}{\mathcal{AB}(\alpha)\Gamma(\alpha)}\int_0^{\varsigma}(R\Theta(\varkappa,\eta)+N\Theta(\varkappa,\eta)+F(\varkappa,\eta))(\varsigma-\tau)^{\alpha-1}\,d\tau. \end{aligned} \tag{3.3}$$

Proof The proof is clear and direct. Applying the integral operator ${}^{AB}I_{\varsigma}^{\alpha}$ to equation (3.1), we have

$$\Theta(\varkappa,\varsigma)-\Theta_0(\varkappa) = {}^{AB}I_{\varsigma}^{\alpha}(R\Theta(\varkappa,\varsigma)+N\Theta(\varkappa,\varsigma)+F(\varkappa,\varsigma)). \tag{3.4}$$

Using Definition 2.2, we can write

$$\begin{aligned}\Theta(\varkappa,\varsigma) &= \Theta_0(\varkappa)+\frac{1-\alpha}{\mathcal{AB}(\alpha)}(R\Theta(\varkappa,\varsigma)+N\Theta(\varkappa,\varsigma)+F(\varkappa,\varsigma)) \\ &+\frac{\alpha}{\mathcal{AB}(\alpha)\Gamma(\alpha)}\int_0^{\varsigma}(R\Theta(\varkappa,\eta)+N\Theta(\varkappa,\eta)+F(\varkappa,\eta))(\varsigma-\tau)^{\alpha-1}\,d\tau. \end{aligned} \tag{3.5}$$

This completes the proof. ■

Theorem 3.2. *Consider the nonlinear Atangana-Baleanu-Caputo fractional partial differential equation (3.1) with initial conditions (3.2). If $R\Theta(\varkappa,\varsigma)$ and $N\Theta(\varkappa,\varsigma)$ are Lipschitz functions with $\|R\Theta_1-R\Theta_2\|+\|N\Theta_1-N\Theta_2\| \le L\|\Theta_1-\Theta_2\|$, where Θ_1 and Θ_2 are different functions and L is a Lipschitz constant which verifies the following condition*

$$L<\frac{\mathcal{AB}(\alpha)\Gamma(\alpha)}{(1-\alpha)\Gamma(\alpha)+T^{\alpha}}, \tag{3.6}$$

then equations (3.1)-(3.2) has a unique solution in $H^1(0,T)$.

Proof To prove this result, let $\mathcal{B}$ is the Banach space with the norm on $\Omega = I\times\,[0,T]$ defined by

$$\|\Theta(\varkappa,\varsigma)\| = \max_{(\varkappa,\varsigma)\in\Omega}|\Theta(\varkappa,\varsigma)| \text{ for all } \Theta(.,\varsigma)\in H^1(0,T), \varkappa\in I=[a,b]\subset\mathbb{R}, \tag{3.7}$$

and consider the operator $\mathcal{T}: H^1(0,T)\to H^1(0,T)$, defined by

$$\begin{aligned}(\mathcal{T}\Theta)(\varkappa,\varsigma) &= \Theta_0(\varkappa)+\frac{1-\alpha}{\mathcal{AB}(\alpha)}(R\Theta(\varkappa,\varsigma)+N\Theta(\varkappa,\varsigma)+F(\varkappa,\varsigma)) \\ &+\frac{\alpha}{\mathcal{AB}(\alpha)\Gamma(\alpha)}\int_0^{\varsigma}(R\Theta(\varkappa,\eta)+N\Theta(\varkappa,\eta)+F(\varkappa,\eta))(\varsigma-\tau)^{\alpha-1}\,d\tau. \end{aligned} \tag{3.8}$$

Finding a solution to equations (3.1)-(3.2) is equivalent to finding a fixed point of $\mathcal{T}$.

Now, for all $\Theta_1(\varkappa,\varsigma), \Theta_2(\varkappa,\varsigma) \in H^1(0,T)$ and $\varkappa \in I = [a,b] \subset \mathbb{R}$, we have

$$\begin{aligned}
\|(\mathcal{T}\Theta_1)(\varkappa,\varsigma) - (\mathcal{T}\Theta_2)(\varkappa,\varsigma)\| &= \left\| \frac{1-\alpha}{\mathcal{AB}(\alpha)} (R\Theta_1(\varkappa,\varsigma) - R\Theta_2(\varkappa,\varsigma) + N\Theta_2(\varkappa,\varsigma) - N\Theta_2(\varkappa,\varsigma)) \right. \\
&\quad \left. + \frac{\alpha}{\mathcal{AB}(\alpha)\Gamma(\alpha)} \int_0^\varsigma \begin{pmatrix} R\Theta_1(\varkappa,\varsigma) - R\Theta_2(\varkappa,\varsigma) \\ +N\Theta_2(\varkappa,\varsigma) - N\Theta_2(\varkappa,\varsigma) \end{pmatrix} (\varsigma-\tau)^{\alpha-1} d\tau \right\| \\
&\leq \frac{1-\alpha}{\mathcal{AB}(\alpha)} L \|\Theta_1(\varkappa,\varsigma) - \Theta_2(\varkappa,\varsigma)\| \\
&\quad + \frac{\alpha}{\mathcal{AB}(\alpha)\Gamma(\alpha)} L \|\Theta_1(\varkappa,\varsigma) - \Theta_2(\varkappa,\varsigma)\| \left| \int_0^\varsigma (\varsigma-\tau)^{\alpha-1} d\tau \right| \\
&= \frac{1-\alpha}{\mathcal{AB}(\alpha)} L \|\Theta_1(\varkappa,\varsigma) - \Theta_2(\varkappa,\varsigma)\| \\
&\quad + \frac{\alpha}{\mathcal{AB}(\alpha)\Gamma(\alpha)} L \|\Theta_1(\varkappa,\varsigma) - \Theta_2(\varkappa,\varsigma)\| \frac{\varsigma^\alpha}{\alpha} \\
&\leq \left(\frac{(1-\alpha)\Gamma(\alpha) + T^\alpha}{\mathcal{AB}(\alpha)\Gamma(\alpha)} \right) L \|\Theta_1(\varkappa,\varsigma) - \Theta_2(\varkappa,\varsigma)\|. \qquad (3.9)
\end{aligned}$$

Since $\left(\frac{(1-\alpha)\Gamma(\alpha) + T^\alpha}{\mathcal{AB}(\alpha)\Gamma(\alpha)} \right) L < 1$, then $\mathcal{T}$ is contraction and by Banach fixed point theorem, $\mathcal{T}$ has a unique fixed point $\Theta(\varkappa,\varsigma) \in H^1(0,T)$. Thanks to Theorem 3.1, then $\Theta(\varkappa,\varsigma) \in H^1(0,T)$ is a unique solution for equations (3.1)-(3.2).
The proof is complete. ■

4 Semi-analytical methods

In this section, we discuss two different methods that are used to solve the nonlinear Atangana-Baleanu-Caputo fractional partial differential equation (3.1) with initial conditions (3.2). Furthermore, we illustrate the convergence and absolute error analysis of these methods.

4.1 Khalouta homotopy perturbation method (KHHPM)

Theorem 4.1. *The solution of the nonlinear Atangana-Baleanu-Caputo fractional partial differential equation described by equation (3.1) is given by*

$$\Theta(\varkappa,\varsigma) = \lim_{m \longrightarrow \infty} \sum_{k=0}^{m} \Theta_k(\varkappa,\varsigma) = \sum_{k=0}^{\infty} \Theta_k(\varkappa,\varsigma), \qquad (4.1)$$

with

$$\Theta_k(\varkappa,\varsigma) = \mathbb{KH}^{-1} \left(\frac{s^\alpha + \alpha\gamma^\alpha\eta^\alpha}{s^\alpha \mathcal{AB}(\alpha)} \mathbb{KH}\left[R\Theta_{k-1}(\varkappa,\varsigma) + H_{k-1}(\Theta)\right] \right), \qquad (4.2)$$

where $\mathbb{KH}[.]$ *is the Khalouta transform and* $H_k(\Theta)$ *are He's polynomials.*

Proof Operating the Khalouta transform on both sides of equation (3.1), we get

$$\mathbb{KH}\left[{}^{ABC}D_\varsigma^\alpha \Theta(\varkappa,\varsigma)\right] = \mathbb{KH}\left[R\Theta(\varkappa,\varsigma) + N\Theta(\varkappa,\varsigma) + F(\varkappa,\varsigma)\right]. \qquad (4.3)$$

Using Theorem 2.3 and the initial condition (3.2), we have

$$\begin{aligned}
\mathbb{KH}[\Theta(\varkappa,\varsigma)] &= \Theta(\varkappa,0) + \left(\frac{s^\alpha - \alpha(s^\alpha - \gamma^\alpha\eta^\alpha)}{s^\alpha \mathcal{AB}(\alpha)} \right) \mathbb{KH}[F(\varkappa,\varsigma)] \\
&\quad + \left(\frac{s^\alpha - \alpha(s^\alpha - \gamma^\alpha\eta^\alpha)}{s^\alpha \mathcal{AB}(\alpha)} \right) \mathbb{KH}[R\Theta(\varkappa,\varsigma) + N\Theta(\varkappa,\varsigma)]. \qquad (4.4)
\end{aligned}$$

Taking the inverse Khalouta transform to both sides of equation (4.4) to get

$$\Theta(\varkappa,\varsigma)=G(\varkappa,\varsigma)+\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[R\Theta(\varkappa,\varsigma)+N\Theta(\varkappa,\varsigma)\right]\right), \tag{4.5}$$

where $G(\varkappa,\varsigma)$ represents the term arising from the source term and the prescribed initial conditions. According to the homotopy perturbation method, we can write

$$\Theta(\varkappa,\varsigma)=\sum_{m=0}^{\infty}p^{m}\Theta_{m}(\varkappa,\varsigma), \tag{4.6}$$

where $p\in[0,1]$ is the homotopy parameter.
The nonlinear term can be decomposed as

$$N\Theta(\varkappa,\varsigma)=\sum_{m=0}^{\infty}p^{m}H_{m}(\Theta), \tag{4.7}$$

where $H_m(\Theta)$ are He's polynomials [7], that are given by

$$H_{m}(\Theta_0,\Theta_1,...,\Theta_m)=\frac{1}{m!}\frac{\partial^{m}}{\partial p^{m}}\left[N\left(\sum_{i=0}^{m}p^{i}\Theta_i\right)\right]_{p=0}, m=0,1,2,... \tag{4.8}$$

Substituting equaions (4.6) and (4.7) into equation (4.5), we get

$$\sum_{m=0}^{\infty}p^{m}\Theta_{m}(\varkappa,\varsigma)=G(\varkappa,\varsigma)+p\left[\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[\begin{array}{c}R\sum\limits_{m=0}^{\infty}p^{m}\Theta_{m}(\varkappa,\varsigma)\\+\sum\limits_{m=0}^{\infty}p^{m}H_{m}(\Theta)\end{array}\right]\right)\right]. \tag{4.9}$$

Comparing the coefficients of like powers of p, the following approximations are obtained

$$\begin{aligned}
p^{0} &: \Theta_0(\varkappa,\varsigma)=G(\varkappa,\varsigma),\\
p^{1} &: \Theta_1(\varkappa,\varsigma)=\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[R\Theta_0(x,t)+H_0(\Theta)\right]\right),\\
p^{2} &: \Theta_2(\varkappa,\varsigma)=\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[R\Theta_1(x,t)+H_1(\Theta)\right]\right),\\
p^{3} &: \Theta_3(\varkappa,\varsigma)=\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[R\Theta_2(x,t)+H_2(\Theta)\right]\right),\\
&\ \ \vdots
\end{aligned} \tag{4.10}$$

In general, the recursive relation is given by

$$p^{n}:\Theta_m(\varkappa,\varsigma)=\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[R\Theta_{m-1}(\varkappa,\varsigma)+H_{m-1}(\Theta)\right]\right). \tag{4.11}$$

Then, the solution according to homotopy $(p\rightarrow 1)$ is given by

$$\Theta(\varkappa,\varsigma)=\Theta_0(\varkappa,\varsigma)+\Theta_1(\varkappa,\varsigma)+\Theta_2(\varkappa,\varsigma)+...+\Theta_m(\varkappa,\varsigma). \tag{4.12}$$

The above series solution converges to the closed form of equation (3.1) as $m\rightarrow\infty$, that is

$$\Theta(\varkappa,\varsigma)=\lim_{m\longrightarrow\infty}\sum_{k=0}^{m}\Theta_k(\varkappa,\varsigma)=\sum_{k=0}^{\infty}\Theta_k(\varkappa,\varsigma). \tag{4.13}$$

This completes the proof. ■
The following theorems study the condition for the convergence and absolute error of the solution using KHHPM.

Theorem 4.2. *For $0<\omega<1$, where $\omega = L\left(\frac{(1-\alpha)\Gamma(\alpha+1)+\alpha\varsigma^{\alpha}}{\mathcal{AB}(\alpha)\Gamma(\alpha+1)}\right)$, then the KHHPM series solution defined by equation (4.1) is convergent.*

Proof Suppose $\mathcal{S}_m$ be the m^{th} partial sum, i.e., $\mathcal{S}_m = \sum\limits_{k=0}^{m}\Theta_k(\varkappa,\varsigma)$. Firstly, we prove that $\{\mathcal{S}_m\}_{m\geq 0}$ is a Cauchy sequence in Banach space $\mathcal{B}$. Taking into account a new form of He's polynomial described in equation (4.8), we obtain

$$N(S_m) = \widetilde{H}_m + \sum_{k=0}^{m-1}\widetilde{H}_k. \tag{4.14}$$

Now,

$$\begin{aligned}
\|\mathcal{S}_m-\mathcal{S}_n\| &= \max_{(\varkappa,\varsigma)\in\Omega}|\mathcal{S}_m-\mathcal{S}_n| = \max_{(\varkappa,\varsigma)\in\Omega}\left|\sum_{k=n+1}^{m}\Theta_k(\varkappa,\varsigma)\right| \\
&\leq \max_{(\varkappa,\varsigma)\in\Omega}\left|\begin{array}{c}\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha})}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[\sum_{k=n+1}^{m}R\Theta_{k-1}(\varkappa,\varsigma)\right]\right)\\ +\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha})}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[\sum_{k=n+1}^{m}\widetilde{H}_{k-1}(\varkappa,\varsigma)\right]\right)\end{array}\right| \\
&= \max_{(\varkappa,\varsigma)\in\Omega}\left|\begin{array}{c}\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha})}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[\sum_{k=n}^{m-1}R\Theta_{k}(\varkappa,\varsigma)\right]\right)\\ +\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha})}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[\sum_{k=n}^{m-1}\widetilde{H}_{k}(\varkappa,\varsigma)\right]\right)\end{array}\right| \\
&\leq \max_{(\varkappa,\varsigma)\in\Omega}\left|\begin{array}{c}\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha})}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[R\left(\mathcal{S}_{m-1}\right)-R\left(\mathcal{S}_{n-1}\right)\right]\right)\\ +\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha})}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[N\left(\mathcal{S}_{m-1}\right)-N\left(\mathcal{S}_{n-1}\right)\right]\right)\end{array}\right| \\
&\leq \max_{(\varkappa,\varsigma)\in\Omega}\left[\begin{array}{c}\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha})}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left|R\left(\mathcal{S}_{m-1}\right)-R\left(\mathcal{S}_{n-1}\right)\right|\right)\\ +\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha})}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left|N\left(\mathcal{S}_{m-1}\right)-N\left(\mathcal{S}_{n-1}\right)\right|\right)\end{array}\right].
\end{aligned} \tag{4.15}$$

Since R and N are Lipschitz functions with a Lipschitz constant L, then we have

$$\begin{aligned}
\|\mathcal{S}_m-\mathcal{S}_n\|_{\mathcal{B}} &\leq L\max_{(\varkappa,\varsigma)\in\Omega}\left[\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left|\mathcal{S}_{m-1}-\mathcal{S}_{n-1}\right|\right)\right] \\
&= L\max_{(\varkappa,\varsigma)\in\Omega}\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left|\mathcal{S}_{m-1}-\mathcal{S}_{n-1}\right|\right) \\
&= L\max_{(\varkappa,\varsigma)\in\Omega}\mathbb{KH}^{-1}\left(\left(\frac{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\left|\mathcal{S}_{m-1}-\mathcal{S}_{n-1}\right|\right) \\
&= L\max_{(\varkappa,\varsigma)\in\Omega}\left(\frac{1-\alpha}{\mathcal{AB}(\alpha)}+\frac{\alpha}{\mathcal{AB}(\alpha)}\frac{\varsigma^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)\left|\mathcal{S}_{m-1}-\mathcal{S}_{n-1}\right| \\
&= L\left(\frac{(1-\alpha)\Gamma\left(\alpha+1\right)+\alpha\varsigma^{\alpha}}{\mathcal{AB}(\alpha)\Gamma\left(\alpha+1\right)}\right)\left\|\mathcal{S}_{m-1}-\mathcal{S}_{n-1}\right\|.
\end{aligned} \tag{4.16}$$

Consider $m=n+1$, then we have

$$\begin{aligned}
\|\mathcal{S}_{n+1}-\mathcal{S}_n\| &\leq \omega\left\|\mathcal{S}_n-\mathcal{S}_{n-1}\right\| \leq \omega^2\left\|\mathcal{S}_{n-1}-\mathcal{S}_{n-2}\right\| \\
&\leq \ldots \leq \omega^n\left\|\mathcal{S}_1-\mathcal{S}_0\right\|,
\end{aligned} \tag{4.17}$$

where

$$\omega = L\left(\frac{(1-\alpha)\Gamma\left(\alpha+1\right)+\alpha\varsigma^{\alpha}}{\mathcal{AB}(\alpha)\Gamma\left(\alpha+1\right)}\right). \tag{4.18}$$

Using the triangle inequality, we have

$$\begin{aligned}
\|\mathcal{S}_m-\mathcal{S}_n\| &= \|\mathcal{S}_{n+1}-\mathcal{S}_n+\mathcal{S}_{n+2}-\mathcal{S}_{n+1}+...+\mathcal{S}_m-\mathcal{S}_{m-1}\| \\
&\leq \|\mathcal{S}_{n+1}-\mathcal{S}_n\|+\|\mathcal{S}_{n+2}-\mathcal{S}_{n+1}\|+...+\|\mathcal{S}_m-\mathcal{S}_{m-1}\| \\
&\leq \omega^n\|\mathcal{S}_1-\mathcal{S}_0\|+\omega^{n+1}\|\mathcal{S}_1-\mathcal{S}_0\|+...+\omega^{m-1}\|\mathcal{S}_1-\mathcal{S}_0\| \\
&= \omega^n\left(1+\omega+...+\omega^{m-n-1}\right)\|\mathcal{S}_1-\mathcal{S}_0\| \\
&\leq \omega^n\left(\frac{1-\omega^{m-n}}{1-\omega}\right)\|\Theta_1\|. \qquad (4.19)
\end{aligned}$$

Since $0<\omega<1$, we have $1-\omega^{m-n}<1$, then

$$\|\mathcal{S}_m-\mathcal{S}_n\| \leq \frac{\omega^n}{1-\omega}\|\Theta_1\|. \qquad (4.20)$$

$\|\Theta_1\|$ is finite, thus as $n\to+\infty$, then $\|\mathcal{S}_m-\mathcal{S}_n\|=0$. Hence $\{\mathcal{S}_m\}_{m\geq 0}$ is a Cauchy sequence in the Banach space $\mathcal{B}$. Consequently, the series solution $\sum\limits_{k=0}^{\infty}\Theta_k(\varkappa,\varsigma)$ is convergent.

This completes the proof. ■

Corollary 4.1. *If the series solution* $\sum\limits_{k=0}^{\infty}\Theta_k(\varkappa,\varsigma)$ *converges then it is an exact solution of the nonlinear Atangana-Baleanu-Caputo fractional partial differential equation described by equation (3.1) with the initial condition (3.2).*

Theorem 4.3. *Let* $\Theta(\varkappa,\varsigma)$ *be the approximate solution of the truncated finite series* $\sum\limits_{k=0}^{l}\Theta_k(\varkappa,\varsigma)$. *Suppose that it is possible to obtain a real number* $\omega\in(0,1)$ *such that* $\|\Theta_{k+1}(\varkappa,\varsigma)\|\leq\omega\|\Theta_k(\varkappa,\varsigma)\|\forall k\in\mathbb{N}$. *Then the maximum absolute error is*

$$\left\|\Theta(\varkappa,\varsigma)-\sum_{k=0}^{l}\Theta_k(\varkappa,\varsigma)\right\| \leq \frac{\omega^{l+1}}{1-\omega}\|\Theta_0(\varkappa,\varsigma)\|. \qquad (4.21)$$

Proof Let the series $\sum\limits_{k=0}^{l}\Theta_k(\varkappa,\varsigma)$ be finite. Then

$$\begin{aligned}
\left\|\Theta(\varkappa,\varsigma)-\sum_{k=0}^{l}\Theta_k(\varkappa,\varsigma)\right\| &\leq \left\|\sum_{k=l+1}^{\infty}\Theta_k(\varkappa,\varsigma)\right\| \\
&\leq \sum_{k=l+1}^{\infty}\|\Theta_k(\varkappa,\varsigma)\| \\
&\leq \sum_{k=l+1}^{\infty}\omega^k\|\Theta_0(\varkappa,\varsigma)\| \\
&\leq \omega^{l+1}\left(1+\omega+\omega^2+\omega^3+...\right)\|\Theta_0(\varkappa,\varsigma)\| \\
&\leq \frac{\omega^{l+1}}{1-\omega}\|\Theta_0(\varkappa,\varsigma)\|. \qquad (4.22)
\end{aligned}$$

This completes the proof. ■

4.2 Khalouta variational iteration method (KHVIM)

Theorem 4.4. *The exact solution of the nonlinear Atangana-Baleanu-Caputo fractional partial differential equation described by equation (3.1) using the KHVIM, is given as a limit of the successive approximations* $\Theta_m(\varkappa,\varsigma), m=0,1,2,...$, *in other words*

$$\Theta(\varkappa,\varsigma)=\lim_{m\to\infty}\Theta_m(\varkappa,\varsigma). \qquad (4.23)$$

Proof According to the variational iteration transform [24], the correction functional of equation (3.1), is given as

$$\Theta_{m+1}(\varkappa,\varsigma)=\Theta_m(\varkappa,\varsigma)+\int_0^{\varsigma}\lambda(\varkappa,\varsigma-\tau)\left({}^{ABC}D_{\varsigma}^{\alpha}\Theta_m(\varkappa,\tau)-R\Theta_m(\varkappa,\tau)-N\Theta_m(\varkappa,\tau)-F(\varkappa,\tau)\right)d\tau, \tag{4.24}$$

where $\lambda(\varkappa,\varsigma-\tau)$ is a general lagrange multiplier, the subscript $n\geq 0$ denotes the n^{th} approximation. Taking the Khalouta transform on both sides of equation (4.24) and using part (3) of Theorem 2.1, we have

$$\begin{aligned}
\mathbb{KH}\left[\Theta_{m+1}(\varkappa,\varsigma)\right] &= \mathbb{KH}\left[\Theta_m(\varkappa,\varsigma)\right]\\
&+\mathbb{KH}\left[\int_0^{\varsigma}\lambda(\varkappa,\varsigma-\tau)\left({}^{ABC}D_{\varsigma}^{\alpha}\Theta_m(\varkappa,\tau)-R\Theta_m(\varkappa,\tau)-N\Theta_m(\varkappa,\tau)F(\varkappa,\tau)\right)d\tau\right]\\
&= \mathbb{KH}\left[\Theta_m(\varkappa,\varsigma)\right]\\
&+\frac{\gamma\eta}{s}\mathbb{KH}\left[\lambda(\varkappa,\varsigma)\right]\mathbb{KH}\left[{}^{ABC}D_{\varsigma}^{\alpha}\Theta_m(\varkappa,\varsigma)-R\Theta_m(\varkappa,\varsigma)-N\Theta_m(\varkappa,\varsigma)-F(\varkappa,\varsigma)\right]\\
&= \mathbb{KH}\left[\Theta_m(\varkappa,\varsigma)\right]+\frac{\gamma\eta}{s}\mathbb{KH}\left[\lambda(\varkappa,\varsigma)\right]\left(\left(\frac{s^{\alpha}\mathcal{AB}(\alpha)}{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}\right)\left(\mathcal{K}(\varkappa,s,\gamma,\eta)-\Theta(\varkappa,0)\right)\right.\\
&\left.-\mathbb{KH}\left[R\Theta_m(\varkappa,\varsigma)+N\Theta_m(\varkappa,\varsigma)+F(\varkappa,\varsigma)\right]\right).
\end{aligned} \tag{4.25}$$

The optimal value of $\lambda(\varkappa,\varsigma)$ can be identified by making the equation (4.25) stationary with respect to $\Theta_m(\varkappa,\varsigma)$

$$\begin{aligned}
\delta\left(\mathbb{KH}\left[\Theta_{m+1}(\varkappa,\varsigma)\right]\right) &= \delta\left(\mathbb{KH}\left[\Theta_m(\varkappa,\varsigma)\right]\right)\\
&+\frac{\gamma\eta}{s}\delta\left(\mathbb{KH}\left[\lambda(\varkappa,\varsigma)\right]\left(\left(\frac{s^{\alpha}\mathcal{AB}(\alpha)}{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}\right)\left(\mathcal{K}(\varkappa,s,\gamma,\eta)-\Theta(\varkappa,0)\right)\right.\right.\\
&\left.\left.+\mathbb{KH}\left[R\Theta_m(\varkappa,\varsigma)+N\Theta_m(\varkappa,\varsigma)+F(\varkappa,\varsigma)\right]\right)\right).
\end{aligned} \tag{4.26}$$

Considering $R\Theta_m(\varkappa,\varsigma)+N\Theta_m(\varkappa,\varsigma)$ as restricted variation, i.e., $\delta\left(\mathbb{KH}\left[R\Theta_m(\varkappa,\varsigma)+N\Theta_m(\varkappa,\varsigma)\right]\right)=0$, we have

$$1+\mathbb{KH}\left[\lambda(\varkappa,\varsigma)\right]\left(\frac{s^{\alpha}\gamma\eta\mathcal{AB}(\alpha)}{\left(s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)\right)s}\right)=0, \tag{4.27}$$

which implies that

$$\mathbb{KH}\left[\lambda(\varkappa,\varsigma)\right]=-\frac{\left(s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)\right)s}{s^{\alpha}\gamma\eta\mathcal{AB}(\alpha)}. \tag{4.28}$$

Using (4.28) in equation (4.25) and taking the inverse Khalouta transform, we attain a new correction functional

$$\Theta_{m+1}(\varkappa,\varsigma)=\Theta(\varkappa,0)+\mathbb{KH}^{-1}\left[\left(\frac{s^{\alpha}-\alpha\left(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha}\right)}{s^{\alpha}\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[R\Theta_m(\varkappa,\varsigma)+N\Theta_m(\varkappa,\varsigma)+F(\varkappa,\varsigma)\right]\right]. \tag{4.29}$$

The initial value $\Theta_0(\varkappa,\varsigma)$ can be find as

$$\Theta_0(\varkappa,\varsigma)=\Theta(\varkappa,0). \tag{4.30}$$

Consequently, the exact solution of equation (3.1) can be obtained by using

$$\Theta(\varkappa,\varsigma)=\lim_{m\to\infty}\Theta_m(\varkappa,\varsigma). \tag{4.31}$$

This completes the proof. ■

Theorem 4.5. *Let $\Theta_m(\varkappa,\varsigma)$ and $\Theta(\varkappa,\varsigma)$ be in Banach space $\mathcal{B}$. If there exists a positive constant $\varrho=L\left(\frac{(1-\alpha)\Gamma(\alpha+1)+\alpha\varsigma^{\alpha}}{\mathcal{AB}(\alpha)\Gamma(\alpha+1)}\right)\in(0,1)$ such that $\|\Theta_{m+1}(\varkappa,\varsigma)\|\leq\varrho\|\Theta_m(\varkappa,\varsigma)\|$ for all $(\varkappa,\varsigma)\in\Omega=I\times[0,T]$ with $\|\Theta_1(\varkappa,\varsigma)-\Theta_0(\varkappa,\varsigma)\|<\infty$, then the sequence defined by equation (4.29) with $\Theta_0(\varkappa,\varsigma)=\Theta(\varkappa,0)$ converges to $\Theta(\varkappa,\varsigma)$, i.e. the exact solution of equation (3.1).*

Proof To achieve this result, we must show that $\{\Theta_m(\varkappa,\varsigma)\}$ is a Cauchy sequence in Banach space $\mathcal{B}$.

$$\begin{aligned}
\|\Theta_m(\varkappa,\varsigma)-\Theta_n(\varkappa,\varsigma)\| &= \max_{(\varkappa,\varsigma)\in\Omega}|\Theta_m(\varkappa,\varsigma)-\Theta_n(\varkappa,\varsigma)| \quad (4.32)\\
&\le \max_{(\varkappa,\varsigma)\in\Omega}\left|\begin{array}{c}\mathbb{KH}^{-1}\left(\left(\frac{s^\alpha-\alpha(s^\alpha-\gamma^\alpha\eta^\alpha)}{s^\alpha\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[R\Theta_m(\varkappa,\varsigma)-R\Theta_n(\varkappa,\varsigma)\right]\right)\\ +\mathbb{KH}^{-1}\left(\left(\frac{s^\alpha-\alpha(s^\alpha-\gamma^\alpha\eta^\alpha)}{s^\alpha\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[N\Theta_m(\varkappa,\varsigma)-N\Theta_m(\varkappa,\varsigma)\right]\right)\end{array}\right|\\
&\le \max_{(\varkappa,\varsigma)\in\Omega}\left[\begin{array}{c}\mathbb{KH}^{-1}\left(\left(\frac{s^\alpha-\alpha(s^\alpha-\gamma^\alpha\eta^\alpha)}{s^\alpha\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left|R\Theta_m(\varkappa,\varsigma)-R\Theta_n(\varkappa,\varsigma)\right|\right)\\ +\mathbb{KH}^{-1}\left(\left(\frac{s^\alpha-\alpha(s^\alpha-\gamma^\alpha\eta^\alpha)}{s^\alpha\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left|N\Theta_m(\varkappa,\varsigma)-N\Theta_m(\varkappa,\varsigma)\right|\right)\end{array}\right].
\end{aligned}$$

Since R and N are Lipschitz functions with a Lipschitz constant L, then we have

$$\begin{aligned}
\|\Theta_m(\varkappa,\varsigma)-\Theta_n(\varkappa,\varsigma)\| &\le L\max_{(\varkappa,\varsigma)\in\Omega}\left[\mathbb{KH}^{-1}\left(\left(\frac{s^\alpha-\alpha\left(s^\alpha-\gamma^\alpha\eta^\alpha\right)}{s^\alpha\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left|\Theta_m(\varkappa,\varsigma)-\Theta_n(\varkappa,\varsigma)\right|\right)\right]\\
&= L\max_{(\varkappa,\varsigma)\in\Omega}\mathbb{KH}^{-1}\left(\left(\frac{s^\alpha-\alpha\left(s^\alpha-\gamma^\alpha\eta^\alpha\right)}{s^\alpha\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left|\Theta_m(\varkappa,\varsigma)-\Theta_n(\varkappa,\varsigma)\right|\right)\\
&= L\max_{(\varkappa,\varsigma)\in\Omega}\mathbb{KH}^{-1}\left(\left(\frac{s^\alpha-\alpha\left(s^\alpha-\gamma^\alpha\eta^\alpha\right)}{s^\alpha\mathcal{AB}(\alpha)}\right)\left|\Theta_m(\varkappa,\varsigma)-\Theta_n(\varkappa,\varsigma)\right|\right)\\
&= L\max_{(\varkappa,\varsigma)\in\Omega}\left(\frac{1-\alpha}{\mathcal{AB}(\alpha)}+\frac{\alpha}{\mathcal{AB}(\alpha)}\frac{\varsigma^\alpha}{\Gamma(\alpha+1)}\right)\left|\Theta_m(\varkappa,\varsigma)-\Theta_n(\varkappa,\varsigma)\right|\\
&= L\left(\frac{(1-\alpha)\Gamma(\alpha+1)+\alpha\varsigma^\alpha}{\mathcal{AB}(\alpha)\Gamma(\alpha+1)}\right)\|\Theta_m(\varkappa,\varsigma)-\Theta_n(\varkappa,\varsigma)\|. \quad (4.33)
\end{aligned}$$

Let $m=n+1$, then

$$\begin{aligned}
\|\Theta_{n+1}(\varkappa,\varsigma)-\Theta_n(\varkappa,\varsigma)\| &\le \varrho\|\Theta_n(\varkappa,\varsigma)-\Theta_{n-1}(\varkappa,\varsigma)\|\le\varrho^2\|\Theta_{n-1}(\varkappa,\varsigma)-\Theta_{n-2}(\varkappa,\varsigma)\|\\
&\le ...\le\varrho^n\|\Theta_1(\varkappa,\varsigma)-\Theta_0(\varkappa,\varsigma)\|, \quad (4.34)
\end{aligned}$$

where

$$\varrho=L\left(\frac{(1-\alpha)\Gamma(\alpha+1)+\alpha\varsigma^\alpha}{\mathcal{AB}(\alpha)\Gamma(\alpha+1)}\right). \quad (4.35)$$

From the triangle inequality, we have

$$\begin{aligned}
\|\Theta_m(\varkappa,\varsigma)-\Theta_n(\varkappa,\varsigma)\| &= \|\Theta_{n+1}(\varkappa,\varsigma)-\Theta_n(\varkappa,\varsigma)+\Theta_{n+2}(\varkappa,\varsigma)-\Theta_{n+1}(\varkappa,\varsigma)\\
&\quad +...+\Theta_m(\varkappa,\varsigma)-\Theta_{m-1}(\varkappa,\varsigma)\|\\
&\le \|\Theta_{n+1}(\varkappa,\varsigma)-\Theta_n(\varkappa,\varsigma)\|+\|\Theta_{n+2}(\varkappa,\varsigma)-\Theta_{n+1}(\varkappa,\varsigma)\|\\
&\quad +...+\|\Theta_m(\varkappa,\varsigma)-\Theta_{m-1}(\varkappa,\varsigma)\|\\
&\le \varrho^n\|\Theta_1(\varkappa,\varsigma)-\Theta_0(\varkappa,\varsigma)\|+\varrho^{n+1}\|\Theta_1(\varkappa,\varsigma)-\Theta_0(\varkappa,\varsigma)\|\\
&\quad +...+\varrho^{m-1}\|\Theta_1(\varkappa,\varsigma)-\Theta_0(\varkappa,\varsigma)\|\\
&= \varrho^n\left(1+\varrho+...+\varrho^{m-n-1}\right)\|\Theta_1(\varkappa,\varsigma)-\Theta_0(\varkappa,\varsigma)\|\\
&\le \varrho^n\left(\frac{1-\varrho^{m-n}}{1-\varrho}\right)\|\Theta_1(\varkappa,\varsigma)-\Theta_0(\varkappa,\varsigma)\|. \quad (4.36)
\end{aligned}$$

Since $0<\varrho<1$, so $1-\varrho^{m-n}<1$, then

$$\|\Theta_m(\varkappa,\varsigma)-\Theta_n(\varkappa,\varsigma)\|\le\frac{\varrho^n}{1-\varrho}\|\Theta_1(\varkappa,\varsigma)-\Theta_0(\varkappa,\varsigma)\|. \quad (4.37)$$

But $\|\Theta_1(\varkappa,\varsigma)-\Theta_0(\varkappa,\varsigma)\|<\infty$, then $\|\Theta_m(\varkappa,\varsigma)-\Theta_n(\varkappa,\varsigma)\|\to 0$ as $m\to\infty$. We conclude that $\{\Theta_m(\varkappa,\varsigma)\}$ is a Cauchy sequence in the Banach space $\mathcal{B}$. Consequently, the sequence converges.
This completes the proof. ■

5 Numerical application

Let us consider the following nonlinear Atangana-Baleanu-Caputo fractional partial differential equation

$$ {}^{ABC}D_{\varsigma}^{\alpha}\Theta(\varkappa,\varsigma) = -\frac{\partial}{\partial\varkappa}\left(\frac{12}{\varkappa}\Theta(\varkappa,\varsigma)-\varkappa\right)\Theta(\varkappa,\varsigma)+\frac{\partial^2}{\partial\varkappa^2}\Theta^2(\varkappa,\varsigma), 0<\alpha\leq 1 \tag{5.1} $$

with the initial condition

$$ \Theta(\varkappa,0)=\varkappa^2, \tag{5.2} $$

The exact solution of equations (5.1)-(5.2) for ordinary motion, i.e. $\alpha=1$, is given by

$$ \Theta(\varkappa,\varsigma)=\varkappa^2\exp(\varsigma). \tag{5.3} $$

In this section, we apply KHHPM and KHVIM to demonstrate the effectiveness and accuracy of these methods for solving the nonlinear Atangana-Baleanu-Caputo fractional partial differential equation (5.1) with the initial condition (5.2).

5.1 KHHPM-Solution

Applying the KHHPM described in the subsection 4.1 to equations (5.1)-(5.2), we obtain

$$ \sum_{m=0}^{\infty}p^m\Theta_m(\varkappa,\varsigma)=\varkappa^2+p\left[\mathbb{KH}^{-1}\left(\left(\frac{s^\alpha-\alpha\left(s^\alpha-\gamma^\alpha\eta^\alpha\right)}{s^\alpha\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[\begin{array}{c}\sum\limits_{m=0}^{\infty}p^m\Theta_m(\varkappa,\varsigma)+\sum\limits_{n=0}^{\infty}p^mH_m(\Theta)\\ -\sum\limits_{n=0}^{\infty}p^mG_m(\Theta)+\sum\limits_{n=0}^{\infty}p^mJ_m(\Theta)\end{array}\right]\right)\right] \tag{5.4} $$

where $H_m(\Theta), G_m(\Theta)$ and $J_m(\Theta)$ are He's polynomials that represents the nonlinear terms, $\frac{12}{\varkappa^2}\Theta^2(\varkappa,\varsigma)$, $\frac{12}{\varkappa}\Theta(\varkappa,\varsigma)\frac{\partial}{\partial\varkappa}\Theta(\varkappa,\varsigma)$ and $\frac{\partial^2}{\partial\varkappa^2}\Theta^2(\varkappa,\varsigma)$ respectively.
From relation (4.8), the first few components of He's polynomials are given as follows

$$ \begin{aligned} H_0(\Theta) &= \frac{12}{\varkappa^2}\Theta_0^2(\varkappa,\varsigma),\\ H_1(\Theta) &= \frac{12}{\varkappa^2}\left(2\Theta_0(\varkappa,\varsigma)\Theta_1(\varkappa,\varsigma)\right),\\ H_2(\Theta) &= \frac{12}{\varkappa^2}\left(2\Theta_0(\varkappa,\varsigma)\Theta_2(\varkappa,\varsigma)+\Theta_1^2(\varkappa,\varsigma)\right),\\ &\vdots \end{aligned} \tag{5.5} $$

and

$$ \begin{aligned} G_0(\Theta) &= \frac{12}{\varkappa}\Theta_0(\varkappa,\varsigma)\frac{\partial}{\partial\varkappa}\Theta_0(\varkappa,\varsigma),\\ G_1(\Theta) &= \frac{12}{\varkappa}\left(\Theta_0(\varkappa,\varsigma)\frac{\partial}{\partial\varkappa}\Theta_1(\varkappa,\varsigma)+\Theta_1(\varkappa,\varsigma)\frac{\partial}{\partial\varkappa}\Theta_0(\varkappa,\varsigma)\right),\\ G_2(\Theta) &= \frac{12}{\varkappa}\left(\Theta_0(\varkappa,\varsigma)\frac{\partial}{\partial\varkappa}\Theta_2(\varkappa,\varsigma)+\Theta_1(\varkappa,\varsigma)\frac{\partial}{\partial\varkappa}\Theta_1(\varkappa,\varsigma)+\Theta_2(\varkappa,\varsigma)\frac{\partial}{\partial\varkappa}\Theta_0(\varkappa,\varsigma)\right),\\ &\vdots \end{aligned} \tag{5.6} $$

and

$$\begin{aligned} J_0(\Theta) &= \frac{\partial^2}{\partial\varkappa^2}\Theta_0^2(\varkappa,\varsigma), \\ J_1(\Theta) &= \frac{\partial^2}{\partial\varkappa^2}\left(2\Theta_0(\varkappa,\varsigma)\Theta_1(\varkappa,\varsigma)\right), \\ J_2(\Theta) &= \frac{\partial^2}{\partial\varkappa^2}\left(2\Theta_0(\varkappa,\varsigma)\Theta_2(\varkappa,\varsigma)+\Theta_1^2(\varkappa,\varsigma)\right), \\ &\vdots \end{aligned} \tag{5.7}$$

Comparing the coefficient of like powers of p in equation (5.4), the following result is obtained

$$\begin{aligned} p^0 &: \Theta_0(\varkappa,\varsigma)=\varkappa^2, \\ p^1 &: \Theta_1(\varkappa,\varsigma)=\mathbb{KH}^{-1}\left(\left(\frac{s^\alpha-\alpha\left(s^\alpha-\gamma^\alpha\eta^\alpha\right)}{s^\alpha\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[\Theta_0(\varkappa,\varsigma)+H_0(\Theta)-G_0(\Theta)+J_0(\Theta)\right]\right), \\ p^2 &: \Theta_2(\varkappa,\varsigma)=\mathbb{KH}^{-1}\left(\left(\frac{s^\alpha-\alpha\left(s^\alpha-\gamma^\alpha\eta^\alpha\right)}{s^\alpha\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[\Theta_1(\varkappa,\varsigma)+H_1(\Theta)-G_1(\Theta)+J_1(\Theta)\right]\right), \\ p^3 &: \Theta_3(\varkappa,\varsigma)=\mathbb{KH}^{-1}\left(\left(\frac{s^\alpha-\alpha\left(s^\alpha-\gamma^\alpha\eta^\alpha\right)}{s^\alpha\mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[\Theta_2(\varkappa,\varsigma)+H_2(\Theta)-G_2(\Theta)+J_2(\Theta)\right]\right), \\ &\vdots \end{aligned} \tag{5.8}$$

By the above algorithm we get

$$\begin{aligned} \Theta_0(\varkappa,\varsigma) &= \varkappa^2, \\ \Theta_1(\varkappa,\varsigma) &= \frac{\varkappa^2}{\mathcal{AB}(\alpha)}\left((1-\alpha)+\alpha\frac{\varsigma^\alpha}{\Gamma(\alpha+1)}\right), \\ \Theta_2(\varkappa,\varsigma) &= \frac{\varkappa^2}{\mathcal{AB}^2(\alpha)}\left((1-\alpha)^2+2\alpha(1-\alpha)\frac{\varsigma^\alpha}{\Gamma(\alpha+1)}+\alpha^2\frac{\varsigma^{2\alpha}}{\Gamma(2\alpha+1)}\right) \\ \Theta_3(\varkappa,\varsigma) &= \frac{\varkappa^2}{\mathcal{AB}^3(\alpha)}\left((1-\alpha)^3+3\alpha(1-\alpha)^2\frac{\varsigma^\alpha}{\Gamma(\alpha+1)}+3\alpha^2(1-\alpha)\frac{\varsigma^{2\alpha}}{\Gamma(2\alpha+1)}+\alpha^3\frac{\varsigma^{3\alpha}}{\Gamma(3\alpha+1)}\right), \\ &\vdots \end{aligned} \tag{5.9}$$

and so on.
Finally, our KHHPM-solution $\Theta(\varkappa,\varsigma)$ in series form is given by

$$\begin{aligned} \Theta(\varkappa,\varsigma) &= \varkappa^2\left(1+\frac{(1-\alpha)\mathcal{AB}^2(\alpha)+(1-\alpha)^2\mathcal{AB}(\alpha)+(1-\alpha)^3}{\mathcal{AB}^3(\alpha)}\right. \\ &+\left(\frac{\alpha\mathcal{AB}^2(\alpha)+2\alpha(1-\alpha)\mathcal{AB}(\alpha)+3\alpha(1-\alpha)^2}{\mathcal{AB}(\alpha)}\right)\frac{\varsigma^\alpha}{\Gamma(\alpha+1)} \\ &\left.+\left(\frac{\alpha^2\mathcal{AB}(\alpha)+3\alpha^2(1-\alpha)}{\mathcal{AB}^2(\alpha)}\right)\frac{\varsigma^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3}{\mathcal{AB}^3(\alpha)}\frac{\varsigma^{3\alpha}}{\Gamma(3\alpha+1)}+...\right) \end{aligned} \tag{5.10}$$

Assuming $\mathcal{AB}(\alpha)=1$ and taking $\alpha=1$ in equation (5.10), then the KHHPM-solution reduced as

$$\Theta(\varkappa,\varsigma)=\varkappa^2\left(1+\varsigma+\frac{\varsigma^2}{2!}+\frac{\varsigma^3}{3!}+...\right). \tag{5.11}$$

This result converges to the exact solution in a closed form

$$\Theta(\varkappa,\varsigma)=\varkappa^2\exp(\varsigma). \tag{5.12}$$

5.2 KHVIM-solution

Applying the KHVIM presented in the subsection 4.2 to equations (5.1)-(5.2), we get

$$\Theta_{m+1}(\varkappa,\varsigma) = \varkappa^2 + \mathbb{KH}^{-1}\left[\left(\frac{s^\alpha - \alpha\left(s^\alpha - \gamma^\alpha\eta^\alpha\right)}{s^\alpha \mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[\begin{array}{c}\Theta_m(\varkappa,\varsigma) + \frac{12}{\varkappa^2}\Theta_m^2(\varkappa,\varsigma) \\ -\frac{12}{\varkappa}\Theta_m(\varkappa,\varsigma)\frac{\partial}{\partial\varkappa}\Theta_m(\varkappa,\varsigma) + \frac{\partial^2}{\partial\varkappa^2}\Theta_m^2(\varkappa,\varsigma)\end{array}\right]\right]. \tag{5.13}$$

Now we find the successive approximate solutions as follows

$$\begin{aligned}
\Theta_0(\varkappa,\varsigma) &= \varkappa^2 \\
\Theta_1(\varkappa,\varsigma) &= \varkappa^2 + \mathbb{KH}^{-1}\left[\left(\frac{s^\alpha - \alpha\left(s^\alpha - \gamma^\alpha\eta^\alpha\right)}{s^\alpha \mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[\begin{array}{c}\Theta_0(\varkappa,\varsigma) + \frac{12}{\varkappa^2}\Theta_0^2(\varkappa,\varsigma) \\ -\frac{12}{\varkappa}\Theta_0(\varkappa,\varsigma)\frac{\partial}{\partial\varkappa}\Theta_0(\varkappa,\varsigma) + \frac{\partial^2}{\partial\varkappa^2}\Theta_0^2(\varkappa,\varsigma)\end{array}\right]\right] \\
&= \varkappa^2\left(1 + \frac{(1-\alpha)}{\mathcal{AB}(\alpha)} + \frac{\alpha}{\mathcal{AB}(\alpha)}\frac{\varsigma^\alpha}{\Gamma(\alpha+1)}\right), \\
\Theta_2(\varkappa,\varsigma) &= \varkappa^2 + \mathbb{KH}^{-1}\left[\left(\frac{s^\alpha - \alpha\left(s^\alpha - \gamma^\alpha\eta^\alpha\right)}{s^\alpha \mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[\begin{array}{c}\Theta_1(\varkappa,\varsigma) + \frac{12}{\varkappa^2}\Theta_1^2(\varkappa,\varsigma) \\ -\frac{12}{\varkappa}\Theta_1(\varkappa,\varsigma)\frac{\partial}{\partial\varkappa}\Theta_1(\varkappa,\varsigma) + \frac{\partial^2}{\partial\varkappa^2}\Theta_1^2(\varkappa,\varsigma)\end{array}\right]\right], \\
&= \varkappa^2\left(1 + \frac{(1-\alpha)\mathcal{AB}(\alpha) + (1-\alpha)^2}{\mathcal{AB}^2(\alpha)} + \left(\frac{\alpha\mathcal{AB}(\alpha) + 2\alpha(1-\alpha)}{\mathcal{AB}^2(\alpha)}\right)\frac{\varsigma^\alpha}{\Gamma(\alpha+1)} + \frac{\alpha^2}{\mathcal{AB}^2(\alpha)}\frac{\varsigma^{2\alpha}}{\Gamma(2\alpha+1)}\right), \\
\Theta_3(\varkappa,\varsigma) &= \varkappa^2 + \mathbb{KH}^{-1}\left(\frac{s^\alpha - \alpha\left(s^\alpha - \gamma^\alpha\eta^\alpha\right)}{s^\alpha \mathcal{AB}(\alpha)}\right)\mathbb{KH}\left[\begin{array}{c}\Theta_2(\varkappa,\varsigma) + \frac{12}{\varkappa^2}\Theta_2^2(\varkappa,\varsigma) \\ -\frac{12}{\varkappa}\Theta_2(\varkappa,\varsigma)\frac{\partial}{\partial\varkappa}\Theta_2(\varkappa,\varsigma) + \frac{\partial^2}{\partial\varkappa^2}\Theta_2^2(\varkappa,\varsigma)\end{array}\right] \\
&= \varkappa^2\left(1 + \frac{(1-\alpha)\mathcal{AB}^2(\alpha) + (1-\alpha)^2\mathcal{AB}(\alpha) + (1-\alpha)^3}{\mathcal{AB}^3(\alpha)}\right. \\
&\quad + \left(\frac{\alpha\mathcal{AB}^2(\alpha) + 2\alpha(1-\alpha)\mathcal{AB}(\alpha) + 3\alpha(1-\alpha)^2}{\mathcal{AB}^3(\alpha)}\right)\frac{\varsigma^\alpha}{\Gamma(\alpha+1)} \\
&\quad \left. + \left(\frac{\alpha^2\mathcal{AB}(\alpha) + 3\alpha^2(1-\alpha)}{\mathcal{AB}^3(\alpha)}\right)\frac{\varsigma^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\alpha^3}{\mathcal{AB}^3(\alpha)}\frac{\varsigma^{3\alpha}}{\Gamma(3\alpha+1)}\right), \\
&\quad \vdots
\end{aligned} \tag{5.14}$$

and so on.
Finally, our KHVIM-solution $\Theta(\varkappa,\varsigma)$ is given by

$$\begin{aligned}
\Theta(\varkappa,\varsigma) &= \varkappa^2\left(1 + \frac{(1-\alpha)\mathcal{AB}^2(\alpha) + (1-\alpha)^2\mathcal{AB}(\alpha) + (1-\alpha)^3}{\mathcal{AB}^3(\alpha)}\right. \\
&\quad + \left(\frac{\alpha\mathcal{AB}^2(\alpha) + 2\alpha(1-\alpha)\mathcal{AB}(\alpha) + 3\alpha(1-\alpha)^2}{\mathcal{AB}^3(\alpha)}\right)\frac{\varsigma^\alpha}{\Gamma(\alpha+1)} \\
&\quad \left. + \left(\frac{\alpha^2\mathcal{AB}(\alpha) + 3\alpha^2(1-\alpha)}{\mathcal{AB}^3(\alpha)}\right)\frac{\varsigma^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\alpha^3}{\mathcal{AB}^3(\alpha)}\frac{\varsigma^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right).
\end{aligned} \tag{5.15}$$

Assuming $\mathcal{AB}(\alpha) = 1$ and taking $\alpha = 1$ in equation (5.15), then the KHVIM-solution reduced as

$$\Theta(\varkappa,\varsigma) = \varkappa^2\left(1 + \varsigma + \frac{\varsigma^2}{2!} + \frac{\varsigma^3}{3!} + ...\right). \tag{5.16}$$

This result converges to the exact solution in a closed form

$$\Theta(\varkappa,\varsigma) = \varkappa^2 \exp(\varsigma). \tag{5.17}$$

6 Results and discussion

In this section, we study the behavior of the nonlinear Atangana-Baleanu-Caputo fractional partial differential equation using KHHPM and KHVIM. Additionally, MATLAB software was used to produce 2D and 3D graphs representing the solutions of equations (5.1)-(5.2) for different values of α.
The 3D graphs of the KHHPM-solution and exact solution are shown in Figure 1. Figure 2 shows the 3D graphs of the KHVIM-solution and exact solution. Figure 3 shows a comparison of the KHHPM, KHVIM and exact solutions in 2D graphs for different values of α. Tables 1 and 2 evaluate the values of the approximate KHHPM-solution, approximate KHVIM-solution and exact solutions of $\Theta(\varkappa, \varsigma)$ at different values of $\varkappa, \varsigma$ and α, and compares the absolute error between KHHPM, KHVIM and the exact solution with $\alpha = 1$.
It should be noted that we obtained a good approximation with the exact solution of the our problem and that we used four order approximate solutions during the calculations. If we had increased the order of approximation, which would have increased the number of terms in the solution, there would have been better approximation solutions. Additionally, the graphs and tables demonstrate that the approximate solution to the nonlinear Atangana-Baleanu-Caputo fractional partial differential equation described by (5.1) which is obtained by KHHPM and KHVIM, converges to the precise solution when the value of α approaches the classical value 1 of the problem, this indicates a good agreement between the exact solution and the proposed methods. It is confirmed that the KHHPM and KHVIM are the best tool for solving nonlinear Atangana-Baleanu-Caputo fractional partial differential equations.

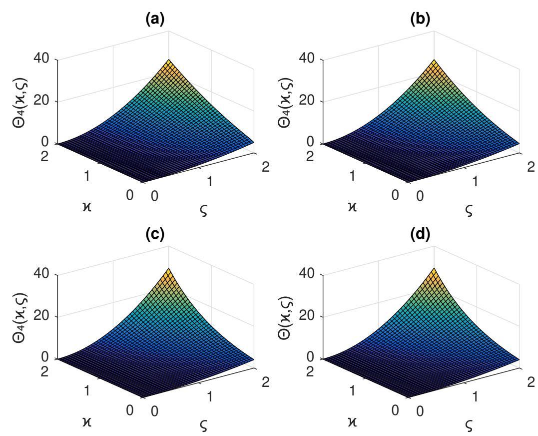

Figure 1: 3D plots of the approximate KHHPM-solutions for different values of α : (a) $\alpha = 0.8$, (b) $\alpha = 0.9$, (c) $\alpha = 1$, (d) Exact solution

7 Conclusions

The present paper was devoted to studying nonlinear fractional partial differential equations via the Khalouta-Atangana-Baleanu operator. Banach's fixed point theorem was used to determine sufficient conditions of the existence and uniqueness results of this problem. In addition, KHHPM and KHVIM are used in this paper to solve nonlinear fractional partial differential equations using the Atangana-Baleanu-Caputo sense. We investigated the convergence and absolute error of the methods . The given example shows a high degree of agreement between the KHHPM and KHVIM results and the remarkable results show how simple and effective these approaches are and how they can be applied to nonlinear fractional problems. Finally, in the future, we plan to apply the Khalouta transform to explore solutions of other fractional partial differential equations with variable-order fractional derivatives.

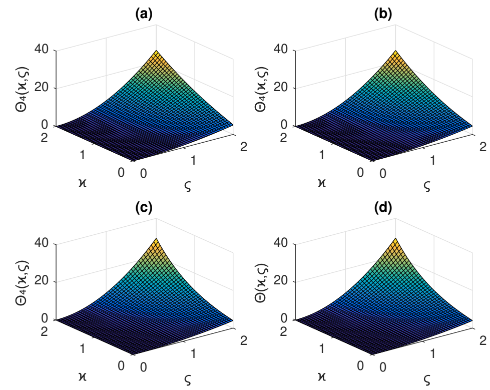

Figure 2: 3D plots of the approximate KHVIM-solutions for different values of α : (a) $\alpha = 0.8$, (b) $\alpha = 0.9$, (c) $\alpha = 1$, (d) Exact solution

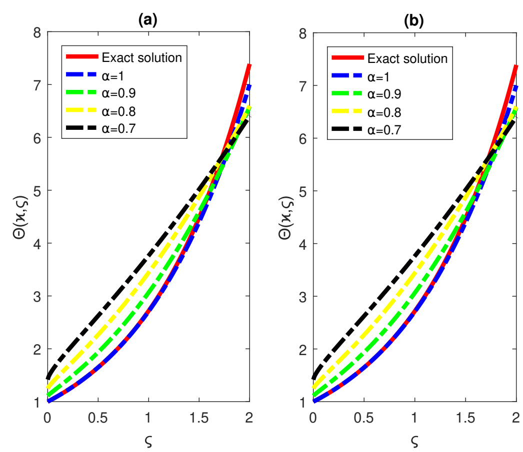

Figure 3: 2D plots of the approximate KHHPM-solutions, KHVIM-solutions and exact solution for different values of α and $\varkappa = 0.5$

$\varkappa$	ς	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	Θ_{exact}	$\lvert\Theta_{exact} - \Theta_{KHHPM}\rvert$
0.1	0.1	0.017355	0.014736	0.012662	0.011052	0.011052	4.2514×10^{-8}
0.2	0.2	0.07872	0.066657	0.056733	0.048856	0.048856	1.1033×10^{-7}
0.3	0.3	0.19697	0.16742	0.14213	0.12149	0.12149	1.9177×10^{-6}
0.4	0.4	0.38498	0.32954	0.28016	0.23868	0.23869	1.4618×10^{-5}
0.5	0.5	0.65605	0.56667	0.48377	0.41211	0.41218	7.0943×10^{-5}
0.6	0.6	1.0240	0.89364	0.76761	0.65570	0.65596	2.5877×10^{-4}
0.7	0.7	1.5033	1.3265	1.1482	0.98596	0.98674	7.7512×10^{-4}
0.8	0.8	2.1090	1.8828	1.6439	1.4223	1.4243	2.0102×10^{-3}
0.9	0.9	2.8569	2.5811	2.2753	1.9876	1.9923	4.6701×10^{-3}
1	1	3.7632	3.4416	3.0649	2.7083	2.7183	9.9485×10^{-3}

Table 1: The values of of the approximate KHHPM-solutions and exact solution and at different values of $\varkappa, \varsigma$ and α

$\varkappa$	ς	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	Θ_{exact}	$\lvert\Theta_{exact} - \Theta_{KHVIM}\rvert$
0.1	0.1	0.017355	0.014736	0.012662	0.011052	0.011052	4.2514×10^{-8}
0.2	0.2	0.07872	0.066657	0.056733	0.048856	0.048856	1.1033×10^{-7}
0.3	0.3	0.19697	0.16742	0.14213	0.12149	0.12149	1.9177×10^{-6}
0.4	0.4	0.38498	0.32954	0.28016	0.23868	0.23869	1.4618×10^{-5}
0.5	0.5	0.65605	0.56667	0.48377	0.41211	0.41218	7.0943×10^{-5}
0.6	0.6	1.0240	0.89364	0.76761	0.65570	0.65596	2.5877×10^{-4}
0.7	0.7	1.5033	1.3265	1.1482	0.98596	0.98674	7.7512×10^{-4}
0.8	0.8	2.1090	1.8828	1.6439	1.4223	1.4243	2.0102×10^{-3}
0.9	0.9	2.8569	2.5811	2.2753	1.9876	1.9923	4.6701×10^{-3}
1	1	3.7632	3.4416	3.0649	2.7083	2.7183	9.9485×10^{-3}

Table 2: The values of of the approximate KHVIM-solutions and exact solution at different values of $\varkappa, \varsigma$ and α

Declarations

Funding: No funds, grants, or other support was received.

Conflict of interest: Authors declare that they have no conflict of interest.

Author contributions: All authors contributed equally to each part of this work.

Availability of data and materials: All data generated or analyzed during this study are included in this published article.

Acknowledgement: We are very grateful to the anonymous reviewers for their careful reading and suggestions which have greatly improved this article.

References

[1] I. Alhribat and M. H. Samuh , *Generating Statistical Distributions Using Fractional Differential Equations*, Jordan Journal of Mathematics and Statistics, 16(2) (2023), 379-396.

[2] G. Alotta, M. Di Paola, and F. P. Pinnola, *An unified formulation of strong non-local elasticity with fractional order calculus*, Meccanica, 57 (2022), 793-805.

[3] A. Atangana and D. Baleanu, *New fractional derivatives with non-local and non-singular kernel Theory and Application to Heat Transfer Model*, Therm. Sci. 20(2) (2016), 763-769.

[4] R. Dhayal, M. Malik, S. Abbas, *Solvability and optimal controls of non-instantaneous impulsive stochastic fractional differential equation of order* $q \in (1, 2)$, Stochastics, 93(5) (2021), 780-802.

[5] A. Giusti, *On infinite order differential operators in fractional viscoelasticity*, Fractional Calculus and Applied Analysis, 20(4) (2017), 854–867, http://dx.doi.org/10.1515/fca-2017-0045.

[6] P.A. Harris and R. Garra, *Nonlinear heat conduction equations with memory: Physical meaning and analytical results*, Journal of Mathematical Physics, 58(6) (2017) 063501, http://dx.doi.org/10.1063/1.4984583.

[7] J. H. He. *Homotopy perturbation technique*. Comput. Methods Appl. Mech. Engrg., 178(3) (1999), 25-262.

[8] J. H. He. *Variational iteration method a kind of non linear analytical technique: some exemples*. Int. J.Nonlinear Mech., 34 (1999), 699-708.

[9] C. Ionescu, A. Lopes, D. Copot, J.A.T. Machado, and J.H.T. Bates, *The role of fractional calculus in modeling biological phenomena: A review*, Communications in Nonlinear Science and Numerical Simulation, 51 (2017), 141-159.

[10] A. Khalouta, *A New Exponential Type Kernel Integral Transform: Khalouta Transform and its Applications*, Mathematica Montisnigri, 57 (2023), 5-23, DOI: 10.20948/mathmontis-2023-57-1

[11] A. Khalouta, *A New Analytical Series Solution with Convergence for Nonlinear Fractional Lienard's Equations with Caputo Fractional Derivative*, Kyungpook Mathematical Journal , 62(2022), 583-593.

[12] A. Khalouta, *A novel iterative method to solve nonlinear wave-like equations of fractional order with variable coefficients*, Revista Colombiana de Matematicas, 56(1) (2022), 13-34.

[13] A. Khalouta, *New approaches for solving Caputo time-fractional nonlinear system of equations describing the unsteady flow of a polytropic gas*, International Journal of Nonlinear Analysis and Applications, 14(3) (2023), 33-46.

[14] A. Khalouta, *A novel computational method for solving the fractional SIS epidemic model of two different fractional operators*, Annals of the University of Craiova, Mathematics and Computer Science Series, 50(1) (2023), 136-151.

[15] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Application of Fractional Differential equations*, Elsevier, Amsterdam, 2006.

[16] K.B. Oldham, *Fractional differential equations in electrochemistry*, Advances in Engineering Software, 41(1) (2010), 9-12.

[17] R. Ozarslan, E. Bas, D. Baleanu, and B. Acay, *Fractional physical problems including wind-influenced projectile motion with Mittag-Leffler kernel*, AIMS Mathematics, 5(1) (219) 467-481.

[18] J. C. Prajapati, A. D. Patel, K. N. Pathak and A. K. Shukla, *Fractional calculus approach in the study of instability phenomenon in fluid dynamics*, Palestine Journal of Mathematics, 1(2) (2012), 95-103.

[19] Z.U. Rehman, S. Boulaaras, R. Jan, I Ahmad, and S. Bahramand, *Computational analysis of financial system through non-integer derivative*, Journal of Computational Science, 75 (2024), 102204.

[20] B. Shiria and D. Baleanu, *Numerical solution of some fractional dynamical systems in medicine involving non-singular kernel with vector order*, Results in Nonlinear Analysis 2(4) (2019), 160-168

[21] J. Singh, D. Kumar, and D. Baleanu, *On the analysis of chemical kinetics system pertaining to a fractional derivative with Mittag-Leffler type kernel*, Chaos, 27 (2017), 103113.

[22] A. Traore and N. Sene, *Model of economic growth in the context of fractional derivative,* Alexandria Engineering Journal, 59(6) (2020), 4843-4850.

[23] V. V. Uchaikin, *Fractional derivatives for physicists and engineers*, Berlin, Springer, 2013.

[24] G. C. Wu, *Challenge in the variational iteration method–A new approach to identification of the Lagrange multipliers*, Journal of King Saud University-Science, 25(2) (2013), 175-178.

[25] X. Yang, J. Zeng, C. Xu, L. Peng, and J. Alsultan, *Modeling of fractional differential equation in cloud computing image fusion algorithm*, Applied Mathematics and Nonlinear Sciences, 8(1) (2023), 1125-1134