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SUPERSTABILITY OF GENERALIZED DERIVATIONS ON NON-ARCHIMEDEAN RANDOM BANACH ALGEBRAS VIA FIXED POINT METHOD

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Abstract Using the fixed point method, we prove the superstability of generalized derivations on non-Archimedean random Banach algebras associated with the Cauchy functional equation.

Keywords p-Adic field, Non-Archimedean random normed space, non-Archimedean random Banach algebra, Generalized derivation, Hyers-Ulam stability, Fixed point alternative

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1. Introduction and preliminaries

Ulam [55] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated. We shall call such an $f : G \to G'$ an *approximate homomorphism*.

Hyers [21] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

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exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

Rassias [43] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1. *[43] Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

*for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (**??**) for $x \neq 0$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.*

Rassias [44] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [18] following the same approach as in Rassias [43], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [18], as well as by Rassias and Šemrl [47] that one cannot prove a Rassias' type theorem when $p = 1$. The counterexamples of Gajda [18], as well as of Rassias and Šemrl [47] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. Găvruta [19], Jung [26], who among others studied the Hyers-Ulam stability of functional equations.

Beginning around the year 1980 the topic of approximate homomorphisms and their stability theory in the field of functional equations and inequalities was taken up by several mathematicians (cf. Hyers and Rassias [23], Rassias [45] and the references therein). Several mathematician have contributed works on these subjects (see [1–3, 5–10, 12, 13, 17, 22, 25, 27–32, 34, 35, 38, 40, 41, 46, 48, 49, 52–54, 57]).

A functional equation $\Im$ is superstable if every approximately solution of $\Im$ is an exact solution of it. For more information on superstability of functional equations and applications, see [15, 16].

A non-Archimedean field is a field $\mathcal{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathcal{K}$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathcal{K}$ (see [4]).

In 1897, Hensel [20] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. During the last three decades p-adic numbers have gained the interest of physicists for their research, in particular in problems coming from quantum physics, p-adic strings and superstrings (cf. [33, 56]).

Let $\mathcal{X}$ be a vector space over a scalar field $\mathcal{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\|$ for all $r \in \mathcal{K}$ and $x \in \mathcal{X}$;

(iii) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in \mathcal{X}$ (the strong triangle inequality).

A sequence $\{x_m\}$ in a non-Archimedean space is Cauchy if and only if $\{x_{m+1} - x_m\}$ converges to zero. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

We recall some definitions and results which will be used later in the article.

A triangular norm (shorter t-norm) is a binary operation on the unit interval $[0, 1]$, i.e., a function $T : [0, 1] \times [0, 1] \to [0, 1]$ such that for all $a, b, c \in [0, 1]$ the following four axioms hold:

(i) $T(a, b) = T(b, a)$ (commutativity);

(ii) $T(a, (T(b, c))) = T(T(a, b), c)$ (associativity);

(iii) $T(a, 1) = a$ (boundary condition);

(iv) $T(a, b) \leq T(a, c)$ whenever $b \leq c$ (monotonicity).

In the sequel, we adopt the usual terminology, notations, and conventions of the theory of random normed spaces as in [11, 36, 50, 51]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. $\mathcal{D}^+$ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^-F(+\infty) = 1$, where $l^-f(x)$ denotes the left limit of the function f at the point x, that is, $l^-f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$.

Definition 1.1. (cf. [11,36,50,51]) A non-Archimedean RN-space is a triple $(\mathcal{X}, \mu, T)$, where $\mathcal{X}$ is a linear space over a non-Archimedean field $\mathcal{K}$, T is a continuous t-norm, and μ is a mapping from $\mathcal{X}$ into D^+ such that the following conditions hold:

$(NA - RN1)$ $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

$(NA - RN2)$ $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in \mathcal{X}$, $t > 0$, and $\alpha \neq 0$;

$(NA - RN3)$ $\mu_{x+y}(\max(t, s)) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y, z \in \mathcal{X}$ and $t, s \geq 0$.

It is easy to show that if $(NA - RN3)$ holds, then

$(RN3)$ $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$.

Definition 1.2. Let $(\mathcal{X}, \mu, T)$ be a non-Archimedean RN-space and $\{x_n\}$ be a sequence in $\mathcal{X}$. Then $\{x_n\}$ is said to be convergent if there exists $x \in \mathcal{X}$ such that

$$\lim_{n \to \infty} \mu_{x_n - x}(t) = 1$$

for all $t > 0$. Then x is called the limit of the sequence $\{x_n\}$.

A sequence $\{x_n\}$ in $\mathcal{X}$ is called Cauchy if for each $\epsilon > 0$ and $t > 0$, there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have

$$\mu_{x_{n+p}-x_n}(t) > 1 - \epsilon.$$

If each Cauchy sequence is convergent, then the random norm is said to be complete, and the non-Archimedean RN-space is called a non-Archimedean random Banach space.

Definition 1.3. [37] A non-Archimedean random normed algebra (X, μ, T, T') is a non-Archimedean random normed space (X, μ, T) with an algebraic structure such that

$$\mu_{xy}(t) \geq T'(\mu_x(t), \mu_y(t))$$

for all $x, y \in X$ and all $t > 0$, in which T' is a continuous t-norm.

Note that a complete non-Archimedean random normed algebra is called a non-Archimedean random Banach algebra.

We recall a fundamental result in fixed point theory. Let $\mathcal{E}$ be a set. A function $d : \mathcal{E} \times \mathcal{E} \to [0, \infty)$ is called a generalized metric on $\mathcal{E}$ if d satisfies:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{E}$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{E}$.

Definition 1.4. Let $\mathcal{A}$ be a Banach algebra. An additive mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ is said to be a ring derivation if $\mathcal{D}(xy) = \mathcal{D}(x)y + x\mathcal{D}(y)$ for all $x, y \in \mathcal{A}$. An additive mapping $\mathcal{H} : \mathcal{A} \to \mathcal{A}$ is said to be a generalized ring derivation if there exists a ring derivation $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ such that

$$\mathcal{H}(xy) = x\mathcal{H}(y) + \mathcal{D}(x)y$$

for all $x, y \in \mathcal{A}$.

Theorem 1.2. *[14, 42] Suppose that a complete generalized metric space $(\mathcal{E}, d)$ (i.e. one for which d may assume infinite values) and a strictly contractive mapping $J : \mathcal{E} \to \mathcal{E}$ with the Lipschitz constant $0 < L < 1$ is given. Then for each given $x \in \mathcal{E}$, exactly one of the following assertions is true: either*

(i) $d(J^m x, J^{m+1}x) = \infty$ for all $m \geq 0$ or

(ii) there exists k such that $d(J^m x, J^{m+1}x) < \infty$ for all $m \geq k$.

Actually, if (ii) holds, then the sequence $\{J^m x\}$ is convergent to a fixed point w of J and

(iii) w is the unique fixed point of J in $\mathcal{Y} = \{y \in \mathcal{E} : d(J^k x, y) < \infty\}$;

(iv) $d(y, w) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \mathcal{Y}$.

In 1996, Isac and Rassias [24] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [15,39,42]).

2. Non-Archimedean random superstability of generalized derivations

Hereafter, we will assume that (A, μ, T) is a non-Archimedean random Banach algebra with unit e over a non- Archimedean field $\mathcal{K}$ and assume that ψ, φ are two distribution functions on $A \times A \times [0, \infty)$ such that $\psi(a, b, \cdot)$ and $\varphi(a, b, \cdot)$ are nondecreasing, $\psi(ax, ax, t) \geq \psi(x, x, \frac{t}{|a|})$ and $\varphi(ax, ax, t) \geq \varphi(x, x, \frac{t}{|a|})$ for all $x \in A$ and $a \neq 0$.

Theorem 2.1. *Suppose that $f : A \to A$ and $g : A \to A$ are mappings such that*

$$\mu_{f(x+y)-f(x)-f(y)}(t) \geq \varphi(x, y, t), \tag{2.1}$$

$$\mu_{f(xy)-xf(y)-g(x)y}(t) \geq \psi(x, y, t) \tag{2.2}$$

for all $x, y \in A$ and all $t > 0$. If there exist a natural number $k \in \mathcal{K}$, constant $0 < L < 1$ and $0 < \beta < 1$ such that

$$\varphi(kx, ky, t) \geq \varphi(x, y, \frac{1}{L|k|}t), \tag{2.3}$$

$$\psi(kx, y, t) \geq \psi(x, y, \frac{1}{\beta|k|}t)$$

for all $x, y \in A$ and all $t > 0$, then f is a generalized derivation.

Proof. By induction on $\jmath$, we shall show that for each $x \in A$, $t > 0$ and $\jmath \geq 2$,

$$\mu_{f(\jmath x)-\jmath f(x)}(t) \geq M_\jmath(x, t) := T\Big(\varphi(x, x, t), \varphi(2x, x, t), \cdots, \varphi((\jmath - 1)x, x, t)\Big). \tag{2.4}$$

Putting $x = y$ in (2.1), we obtain

$$\mu_{f(2x)-2f(x)}(t) \geq \varphi(x, x, t) \qquad (x \in A, t > 0).$$

This proves (2.4) for $\jmath = 2$. Let (2.4) hold for some $\jmath > 2$. Replacing x by $\jmath x$ and y by x in (2.1), we get

$$\mu_{f((\jmath+1)x)-f(x)-f(\jmath x)}(t) \geq \varphi(\jmath x, x, t) \qquad (x \in A, t > 0).$$

Hence

$$\begin{aligned}
\mu_{f((\jmath+1)x)-(\jmath+1)f(x)}(t) &= \mu_{\Big(f((\jmath+1)x)-f(x)-f(\jmath x)+f(\jmath x)-\jmath f(x)\Big)}(t) \\
&\geq T\Big(\mu_{f((\jmath+1)x)-f(x)-f(\jmath x)}(t), \mu_{f(\jmath x)-\jmath f(x)}(t)\Big) \\
&\geq T\Big(\varphi(\jmath x, x, t), M_\jmath(x, t)\Big) \\
&= M_{\jmath+1}(x, t)
\end{aligned}$$

for all $x \in A$. Thus (2.4) holds for all $\jmath \geq 2$. In particular,

$$\mu_{f(kx)-kf(x)}(t) \geq M(x, t) \tag{2.5}$$

for all $x \in A$ and all $t > 0$, where

$$M(x, t) = T\Big(\varphi(x, x, t), \varphi(2x, x, t), \cdots, \varphi((k - 1)x, x, t)\Big) \qquad (x \in A, t > 0).$$

Let $\mathcal{E}$ be the set of all functions $r : A \to A$. We define $d : \mathcal{E} \times \mathcal{E} \to [0, \infty)$ as follows:

$$d(r, s) = inf\{\alpha > 0 : \mu_{r(x)-s(x)}(\alpha t) \geq M(x, t), \forall x \in A, \forall t > 0\}.$$

It is easy to show that d is a generalized complete metric on $\mathcal{E}$. Define $\Lambda : \mathcal{E} \to \mathcal{E}$ by $(\Lambda r)(x) = \frac{r(kx)}{k}$. Then Λ is strictly contractive on $\mathcal{E}$, in fact, if $d(r, s) = \epsilon$, then, by (2.3),

$$\mu_{(\Lambda r)(x)-(\Lambda s)(x)}(L\epsilon t) = \mu_{\frac{r(kx)}{k}-\frac{s(kx)}{k}}(L\epsilon t) = \mu_{r(kx)-s(kx)}(|k|L\epsilon t) \\ \geq M(kx, |k|Lt) \geq M(x, t)$$

for all $x \in A$ and all $t > 0$. So $d(r, s) = \epsilon$ implies that $d(\Lambda r, \Lambda s) \leq L\epsilon$. From this it is easy to show that $d(\Lambda r, \Lambda s) \leq Ld(r, s)$ for all $r, s \in \mathcal{E}$.
Hence Λ is a strictly contractive mapping with Lipschitz constant L. By (2.5),

$$\mu_{\frac{f(kx)}{k}-f(x)}(\frac{t}{|k|}) \geq M(x, t)$$

for all $x \in A$ and all $t > 0$. So $d(\Lambda f, f) \leq \frac{1}{|k|} < \infty$. By the fixed point alternative, Λ has a unique fixed point $h : A \to A$ in the set

$$\mathcal{U} = \{r \in \mathcal{E} : d(r, \Lambda r) < \infty\}$$

and for each $x \in A$,

$$\lim_{n\to\infty} \mu_{\frac{f(k^n x)}{k^n}-h(x)}(t) = 1 \tag{2.6}$$

for all $x \in A$ and all $t > 0$, since $\lim_{n\to\infty} d(\Lambda^n f, h) = 0$. Using the fixed point alternative, we have

$$d(f, h) \leq \frac{1}{1-L} d(f, \Lambda f) \leq \frac{1}{|k|(1-L)}.$$

This implies that

$$\mu_{f(x)-h(x)}(t) \geq M(x, |k|(1-L)t) \tag{2.7}$$

for all $x \in A$ and all $t > 0$. It follows from (2.1) and (2.3) that

$$\mu_{\left(\frac{f(k^n(x+y))}{k^n}-\frac{f(k^n x)}{k^n}-\frac{f(k^n y)}{k^n}\right)}(\frac{t}{|k|^n}) \geq \varphi(k^n x, k^n y, t)$$

for all $x, y \in A$, all $t > 0$ and all $n \in \mathbb{N}$. Hence

$$\mu_{\left(\frac{f(k^n(x+y))}{k^n}-\frac{f(k^n x)}{k^n}-\frac{f(k^n y)}{k^n}\right)}(t) \geq \varphi(k^n x, k^n y, |k|^n t) \geq \varphi(x, y, \frac{|k|^n}{L^n|k|^n}t) \\ = \varphi(x, y, \frac{1}{L^n}t)$$

for all $x, y \in A$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \varphi(x, y, (\frac{1}{L})^n t) = 1$ for all $x, y \in A$ and all $t > 0$,

$$\mu_{h(x+y)-h(x)-h(y)}(t) = 1$$

for all $x, y \in A$ and all $t > 0$. Thus h is additive.

Assume that $\Delta : A^2 \to A$ is a mapping defined by

$$\Delta(x,y) = f(xy) - xf(y) - g(x)y$$

for all $x, y \in A$. Letting $x := k^n x$ and $t := |k|^n s$ in (2.2), we get

$$\mu_{\left(\frac{f(k^n xy)}{k^n} - xf(y) - \frac{g(k^n x)}{k^n} y\right)}(s) \geq \psi(x, y, \frac{1}{\beta^n} s)$$

for all $x, y \in A$ and all $s > 0$, which implies that

$$\lim_{n\to\infty} \mu_{\frac{\Delta(k^n x, y)}{k^n}}(s) = 1 \tag{2.8}$$

for all $x, y \in A$ and all $s > 0$. Putting $x := k^n x$, $y := e$ and $t := |k|^n s$ in (2.2), we have

$$\mu_{\left(\frac{g(k^n x)}{k^n} - \frac{f(k^n x)}{k^n} + xf(e)\right)}(s) \geq \psi(x, e, \frac{1}{\beta^n} s)$$

for all $x \in A$ and all $s > 0$. So we deduce that

$$\lim_{n\to\infty} \mu_{\left(\frac{g(k^n x)}{k^n} - \frac{f(k^n x)}{k^n} + xf(e)\right)}(s) = 1 \tag{2.9}$$

for all $x \in A$ and all $s > 0$. Note that

$$\mu_{\left(\frac{g(k^n x)}{k^n} - (h(x) - xf(e))\right)}(s) \geq T\Big(\mu_{\left(\frac{g(k^n x)}{k^n} - \frac{f(k^n x)}{k^n} + xf(e)\right)}(\frac{s}{2}), \mu_{\left(\frac{f(k^n x)}{k^n} - h(x)\right)}(\frac{s}{2})\Big)$$

for all $x \in A$ and all $s > 0$. By (2.6) and (2.9), we get

$$\lim_{n\to\infty} \mu_{\left(\frac{g(k^n x)}{k^n} - (h(x) - xf(e))\right)}(s) = 1$$

for all $x \in A$. If we define a mapping $\delta : A \to A$ by $\delta(x) = h(x) - xf(e)$ for all $x \in A$, then, by the additivity of h, δ is additive.

Letting $x := k^n x$ and $y := e$ in (2.2), we have

$$\begin{aligned}\mu_{\left(g(k^n x) - f(k^n x) + k^n x f(e)\right)}(s) &\geq \psi(k^n x, e, s) \geq \psi(x, e, \frac{1}{\beta^n |k|^n} s) \\ &\geq \psi(x, e, \frac{1}{\beta^n} s)\end{aligned} \tag{2.10}$$

for all $x \in A$ and all $s > 0$, since $|k| \leq 1$. Setting $x := k^n x$ in (2.7), we obtain

$$\begin{aligned}&\mu_{f(k^n x) - h(k^n x)}(t) \geq M(k^n x, |k|(1-L)t) \\ &= T\Big(\varphi(k^n x, k^n x, |k|(1-L)t), \varphi(2k^n x, k^n x, |k|(1-L)t), \cdots, \\ &\varphi((k-1)k^n x, k^n x, |k|(1-L)t)\Big) \\ &\geq T\Big(\varphi(x, x, \frac{1-L}{L^n |k|^{n-1}} t), \varphi(2x, x, \frac{1-L}{L^n |k|^{n-1}} t), \cdots, \\ &\varphi((k-1)x, x, \frac{1-L}{L^n |k|^{n-1}} t)\Big) \\ &\geq T\Big(\varphi(x, x, \frac{1-L}{L^n} t), \varphi(2x, x, \frac{1-L}{L^n} t), \cdots, \varphi((k-1)x, x, \frac{1-L}{L^n} t)\Big) \\ &= M(x, \frac{1-L}{L^n} t)\end{aligned} \tag{2.11}$$

for all $x \in A$ and all $t > 0$. The conditions (2.10) and (2.11) imply that

$$\lim_{n\to\infty} \mu_{\left(g(k^n x)-f(k^n x)+k^n x f(e)\right)}(s) = 1 \tag{2.12}$$

and

$$\lim_{n\to\infty} \mu_{f(k^n x)-h(k^n x)}(t) = 1 \tag{2.13}$$

for all $x \in A$ and all $s, t > 0$. In particular, we show that

$$\begin{aligned}&\mu_{\left(g(k^n x)-f(k^n x)+k^n x f(e)+f(k^n x)-h(k^n x)\right)}(s)\\ &\geq T\Big(\mu_{\left(g(k^n x)-f(k^n x)+k^n x f(e)\right)}(\frac{s}{2}), \mu_{\left(f(k^n x)-h(k^n x)\right)}(\frac{s}{2})\Big)\end{aligned}$$

for all $x \in A$ and all $s > 0$. Thus, by virtue of (2.12) and (2.13), we have

$$\lim_{n\to\infty} \mu_{\left(g(k^n x)-f(k^n x)+k^n x f(e)+f(k^n x)-h(k^n x)\right)}(s) = 1 \tag{2.14}$$

for all $x \in A$ and all $s > 0$. It follows from the additivity of h that

$$\begin{aligned}\mu_{\left(\frac{g(k^n x)}{k^n}-\delta(x)\right)y}(s) &= \mu_{\left(\frac{g(k^n x)}{k^n}-(h(x)-xf(e))\right)y}(s)\\ &= \mu_{\left(g(k^n x)-f(k^n x)+k^n x f(e)+f(k^n x)-h(k^n x)\right)y}(|k|^n s)\\ &\geq \mu_{\left(g(k^n x)-f(k^n x)+k^n x f(e)+f(k^n x)-h(k^n x)\right)}(s).\mu_y(|k|^n)\end{aligned}$$

for all $x, y \in A$ and all $s > 0$. By (2.14), we have

$$\lim_{n\to\infty} \mu_{\left(\frac{g(k^n x)}{k^n}-\delta(x)\right)y}(s) = 1 \tag{2.15}$$

for all $x, y \in A$ and all $s > 0$. We now note that

$$\mu_{\left(xf(y)+\frac{g(k^n x)}{k^n}y+\frac{\Delta(k^n x, y)}{k^n}-xf(y)-\delta(x)y\right)}(s) \geq T\Big(\mu_{\left(\frac{g(k^n x)}{k^n}-\delta(x)\right)y}(\frac{s}{2}), \mu_{\left(\frac{\Delta(k^n x,y)}{k^n}\right)}(\frac{s}{2})\Big)$$

for all $x, y \in A$ and all $s > 0$. In view of (2.8) and (2.15), we show that

$$\lim_{n\to\infty} \mu_{\left(xf(y)+\frac{g(k^n x)}{k^n}y+\frac{\Delta(k^n x, y)}{k^n}-xf(y)-\delta(x)y\right)}(s) = 1 \tag{2.16}$$

for all $x, y \in A$ and all $s > 0$. Now, using (2.6) and (2.16), we get

$$\begin{aligned}\mu_{h(xy)-xf(y)-\delta(x)y}(t) &= \lim_{n\to\infty} \mu_{\left(\frac{f(k^n xy)}{k^n}-xf(y)-\delta(x)y\right)}(t)\\ &= \lim_{n\to\infty} \mu_{\left(xf(y)+\frac{g(k^n x)}{k^n}y+\frac{\Delta(k^n x,y)}{k^n}-xf(y)-\delta(x)y\right)}(t) = 1\end{aligned} \tag{2.17}$$

for all $x, y \in A$ and all $t > 0$. Hence we get

$$h(xy) = xf(y) + \delta(x)y \tag{2.18}$$

for all $x, y \in A$. Applying (2.18) and the additivity of δ, we obtain

$$xf(k^n y) + \delta(x) \cdot k^n y = h(x \cdot k^n y) = h(k^n x \cdot y) = k^n x f(y) + \delta(x) \cdot k^n y,$$

which means that

$$x\frac{f(k^n y)}{k^n} = xf(y)$$

for all $x, y \in A$. Hence we get

$$\lim_{n\to\infty} \mu_{\left(x\frac{f(k^n y)}{k^n} - xf(y)\right)}(s) = 1 \tag{2.19}$$

for all $x, y \in A$ and all $s > 0$. In particular, we have by the additivity of h,

$$\begin{aligned}\mu_{\left(x\frac{f(k^n y)}{k^n} - xh(y)\right)}(s) &= \mu_{x\cdot\left(f(k^n y) - h(k^n y)\right)}(|k|^n s)\\ &\geq T\Big(\mu_{f(k^n y)-h(k^n y)}(s), \mu_x(|k|^n)\Big)\end{aligned}$$

for all $x, y \in A$ and all $s > 0$. This inequality and (2.13) guarantee the following

$$\lim_{n\to\infty} \mu_{\left(x\frac{f(k^n y)}{k^n} - xh(y)\right)}(s) = 1 \tag{2.20}$$

for all $x, y \in A$ and all $s > 0$.

On the other hand,

$$\mu_{xf(y)-xh(y)}(s) = \mu_{\left(xf(y) - x\frac{f(k^n y)}{k^n} + x\frac{f(k^n y)}{k^n} - xh(y)\right)}(s)$$

for all $x, y \in A$, all $s > 0$ and $n \in \mathbb{N}$.

By taking $n \to \infty$ in the last inequality and using (2.19) and (2.20), we have

$$xf(y) = xh(y) \tag{2.21}$$

for all $x, y \in A$. Consequently, (2.18) becomes

$$h(xy) = xh(y) + \delta(x)y$$

for all $x, y \in A$.

On the other hand,

$$\delta(xy) = h(xy) - xyf(e) = x(h(y) - yf(e)) + \delta(x)y = x\delta(y) + \delta(x)y,$$

for all $x, y \in A$. That is, δ is a derivation.

Letting $x = e$ in (2.21), we have $f = h$. Therefore, we conclude that f is a generalized derivation, which completes the proof. □

Corollary 2.1. *Suppose that $f : A \to A$ is a mapping such that*

$$\mu_{f(x+y)-f(x)-f(y)}(t) \geq \varphi(x, y, t),$$

$$\mu_{f(xy)-xf(y)-f(x)y}(t) \geq \psi(x, y, t)$$

for all $x, y \in A$ *and all* $t > 0$*. If there exist a natural number* $k \in \mathcal{K}$*, constant* $0 < L < 1$ *and* $0 < \beta < 1$ *such that*

$$\varphi(kx, ky, t) \geq \varphi(x, y, \frac{1}{L|k|}t),$$

$$\psi(kx, y, t) \geq \psi(x, y, \frac{1}{\beta|k|}t)$$

for all $x, y \in A$ *and all* $t > 0$*, then* f *is a derivation.*

Declarations

Availablity of data and materials
Not applicable.

Human and animal rights
We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest
The authors declare that they have no competing interests.

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Authors' contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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