Directed search process driven by Lévy motion with stochastic resetting

Yan Xu * Hexin Zhu

College of Mathematics and Information Science, Hebei University, Baoding, 071002, CHINA

Abstract. In this paper, we demonstrate how certain active transport processes in living cells can be modeled based on a directed search process driven by Lévy motion with stochastic resetting. We focus on the motor-driven intracellular transport of vesicles to synaptic targets in the axons and dendrites of neurons, where the restart duration of the search process after reset is finite, and comprises a finite return time and a refractory period. We employ a probabilistic renewal method to calculate the splitting probabilities and conditional mean first passage times (MFPTs) for capture by a finite array of contiguous targets. We consider two different search scenarios: bounded search on the interval $[0, L]$, where L denotes the length of the array, with a refractory boundary at $x = 0$ and a reflecting boundary at $x = L$ (Model A), and partially bounded search on the half-line (Model B). In the latter case, the probability that the particle cannot find a target in the absence of resetting is nonzero. We show that both models have the same splitting probability, and that increasing the resetting rate r increases the splitting probability. Furthermore, the MFPTs of Model A are monotonically increasing with respect to r, whereas the MFPTs of Model B are nonmonotone with respect to r, with a minimum at an optimal resetting rate.

Keywords. stochastic resetting; search processes; renewal theory; Lévy motion.

2010 Mathematics Subject Classification. 34F05; 37H10; 35R60.

*Corresponding author. Email:xuyanbd@hbu.edu.cn

1 Introduction

Due to its prevalence in diverse domains of nature, the search problem is prompting increasing interest in identifying optimal strategies [1–4]. Assuming there is a specific target in the population that needs to be searched if the target cannot be found for a long time during the search process, the most effective way to address this issue is to restart the process. Evans and Majumdar [5] first studied diffusion under stochastic resetting. They defined and studied a simple stochastic resetting diffusion model in which a Brownian particle is randomly reset to its initial position at a constant rate r, which can be considered the basic model of a random intermittent search process. The act of finding a specific target in a crowd can also be considered a restart behavior. Manrubia and Zanette [6] first studied discrete-time stochastic multiplicative processes with reset events. Gelenbe [2] investigated the impact of restart behavior using different methods. Stochastic resetting has become a central part of our daily life. Common resetting methods involve resetting the router when the network signal is weak and resetting the computer when it becomes stuck. In our lifestyle, taking a day off from work and going home to rest is also a form of resetting. Stochastic resetting has optimized our lives in a way that allows us to restart jobs with high efficiency and saves us time. The emergence and optimization of resetting in daily life are major reasons for widespread analysis and research in various fields.

Several attempts have been made to study stochastic resetting in different fields, including biochemistry [7–9], biology [10], and computer science [11]. The reset position can be arbitrary, except for a random reset to the origin. Majumdar et al. [8] considered a model where the position is reset to the farthest distance the searcher has previously reached. According to [12, 13], this model is akin to the random walk model that characterizes animal foraging behavior. During the foraging process, foraging animals tend to start each search from the farthest point they have previously reached, which increases their probability of finding food. The specific processes of stochastic resetting have also been analyzed and studied. Examples include Lévy flights [14,15], continuous time random walks with or without drift [16–18], and scaled Brownian motion [19, 20]. Furthermore, diffusion processes with stochastic resetting have been comprehensively developed.

Currently, most search processes involving stochastic resetting are related to Brownian motion, while few involve Lévy motion. Therefore, it is essential to apply Lévy motion to directional search processes with stochastic resetting.

The remainder of this paper is organized as follows. Section 2 introduces the search-and-capture model driven by Lévy motion. To develop the analysis of stochastic resetting with delays, we first consider the simpler problem of directed search on the half-line with a single target in Section 3. In Section 4, we analyze the splitting probabilities and mean first passage times(MFPTs) for the particle to be captured by one of the multiple targets. We show that Models A and B have the same splitting probabilities but different MFPTs. In Section 5, we explore the dependence of the splitting probabilities and conditional MFPTs on various model parameters, including the search phase speed (v_+), return phase speed (v_-), resetting rate r and other parameters. Our findings show that the MFPTs of Model A are monotonically increasing with respect to r, whereas those of Model B are non-monotonic with a minimum at an optimal resetting rate. We also demonstrate how increasing the search speed leads to a more uniform distribution of statistical quantities across the target array. Finally, Section 6 presents the conclusions and proposes some unresolved issues.

2 Directed search-and-capture model driven by Lévy motion

Consider a particle moving on a finite interval $[0, L]$ driven by Lévy motion, which is a standard Brownian motion combined with a Poisson process with parameter λ. The particle can either have a right-moving (anterograde) state with speed v_+ or a left-moving (retrograde) state with speed v_-. It is worth noting that this process satisfies the corresponding Fokker-Planck (FP) equation [21]:

$$\frac{\partial p_n(x,t)}{\partial t} = -\frac{\partial}{\partial x}[v_n p_n(x,t)] + \frac{1}{2}\frac{\partial^2 p_n(x,t)}{\partial x^2} + \lambda \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\frac{\partial^k p_n(x,t)}{\partial x^k}.$$

The particle can undergo the state transition $v_+ \to v_-$ at a resetting rate r, after which it returns to the origin. At the origin, the particle enters a refractory state characterized by an exponentially distributed waiting time with rate η. After this waiting period, the particle reenters the domain in the anterograde state. Finally, we impose a reflecting boundary condition at $x = L$ such that if the particle reaches the end $x = L$, it switches to the retrograde state and returns to the origin. Figure 1 shows the schematic representation of different particle states. In this paper, we focus on the first passage time (FPT) problem of the particle to find one of N contiguous targets of size l with $Nl = L$. Therefore, we introduce an additional assumption that the particle can be absorbed anywhere in the domain $[0, L]$ at a rate κ.

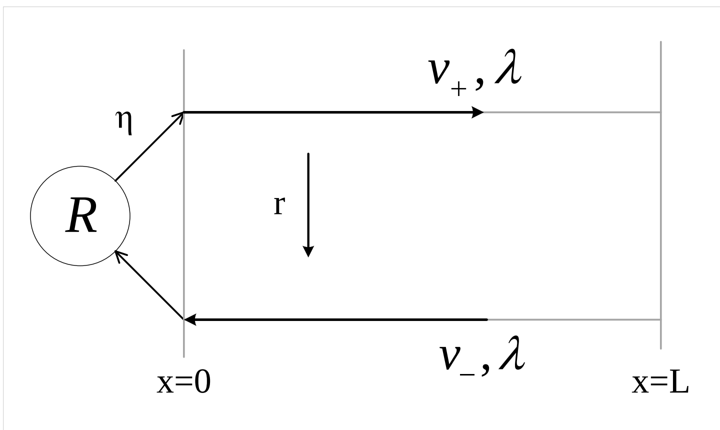

Figure 1: Schematic representation of particle states: anterograde state with speed v_+, retrograde state with speed v_-, and refractory state R. The refractory period τ is generated by an exponential waiting time density $\psi(\tau) = \eta e^{-\eta\tau}$, where r denotes the resetting rate.

Let $p_n(x,t)$ be the probability density that at time t the particle is at $X(t) = x$ and in either the anterograde state ($n = +$) or the retrograde state ($n = -$). Similarly, let $P_0(t)$ denote the probability that the particle is in the refractory state at time t. The corresponding Chapman-Kolmogorov (CK) equation is given by

Model A

$$\frac{\partial p_+}{\partial t} = -v_+ \frac{\partial p_+}{\partial x} + \frac{1}{2}\frac{\partial^2 p_+}{\partial x^2} + \lambda \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k p_+}{\partial x^k} - r p_+ - \kappa p_+, x \in (0, L), \tag{2.1a}$$

$$\frac{\partial p_-}{\partial t} = v_- \frac{\partial p_-}{\partial x} - \frac{1}{2}\frac{\partial^2 p_-}{\partial x^2} - \lambda \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k p_-}{\partial x^k} + r p_+, \tag{2.1b}$$

$$\frac{dP_0}{dt} = v_- p_-(0,t) - \frac{1}{2}\frac{\partial p_-}{\partial x}\bigg|_{x=0} - \lambda \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\frac{\partial^{k-1} p_-}{\partial x^{k-1}}\bigg|_{x=0} - \eta P_0(t). \tag{2.1c}$$

with the boundary conditions

$$\begin{aligned} v_+ p_+(0,t) - \frac{1}{2}\frac{\partial p_+}{\partial x}\bigg|_{x=0} - \lambda \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\frac{\partial^{k-1} p_+}{\partial x^{k-1}}\bigg|_{x=0} &= \eta P_0(t), \\ v_- p_-(L,t) - \frac{1}{2}\frac{\partial p_-}{\partial x}\bigg|_{x=L} - \lambda \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\frac{\partial^{k-1} p_-}{\partial x^{k-1}}\bigg|_{x=L} &= \\ v_+ p_+(L,t) - \frac{1}{2}\frac{\partial p_+}{\partial x}\bigg|_{x=L} - \lambda \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\frac{\partial^{k-1} p_+}{\partial x^{k-1}}\bigg|_{x=L}&. \end{aligned} \tag{2.1}$$

The boundary conditions indicate that the instantaneous state of the particle reaches two boundaries. The first equation indicates that the particle enters the anterograde state from the refractory period at $x = 0$. The second equation indicates that the particle switches from the anterograde state to the retrograde state at $x = L$. We assume that the particle starts in the anterograde state at $x = 0$. The probability $P_k(t)$ of the particle being captured by the kth target at time t is given by

$$\frac{dP_k}{dt} = \kappa \int_{(k-1)l}^{kl} p_+(x,t)dx. \tag{2.2}$$

By adding equations (2.1a) and (2.1b) and then integrating with respect to x over the interval $[0, L]$,and applying the boundary conditions (2.1), we obtain

$$\frac{d}{dt}\int_0^L p(x,t)dx + \sum_{k=0}^{N} \frac{dP_k}{dt} = 0,$$

where $p = p_+ + p_-$. This ensures the conservation of total probability over all events, i.e.,

$$\int_0^L p(x,t)dx + \sum_{k=0}^{N} P_k(t) = 1. \tag{2.3}$$

We also consider another model in which there is no reflecting boundary at $x = L$. Specifically, we consider the case where the particle can continue beyond the array of targets until it resets and switches to the return phase. Then, equations (2.1a)-(2.1) become

Model B

$$\frac{\partial p_+}{\partial t} = -v_+\frac{\partial p_+}{\partial x} + \frac{1}{2}\frac{\partial^2 p_+}{\partial x^2} + \lambda \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\frac{\partial^k p_+}{\partial x^k} - r p_+ - \kappa H(L-x) p_+, \tag{2.4a}$$

$$\frac{\partial p_-}{\partial t} = v_- \frac{\partial p_-}{\partial x} - \frac{1}{2}\frac{\partial^2 p_-}{\partial x^2} - \lambda \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\frac{\partial^k p_-}{\partial x^k} + rp_+, \tag{2.4b}$$

$$\frac{dP_0}{dt} = v_- p_-(0,t) - \frac{1}{2}\frac{\partial p_-}{\partial x}\bigg|_{x=0} - \lambda \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\frac{\partial^{k-1} p_-}{\partial x^{k-1}}\bigg|_{x=0} - \eta P_0(t),$$

$$v_+ p_+(0,t) - \frac{1}{2}\frac{\partial p_+}{\partial x}\bigg|_{x=0} - \lambda \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\frac{\partial^{k-1} p_+}{\partial x^{k-1}}\bigg|_{x=0} = \eta P_0(t), \tag{2.4c}$$

where $H(x)$ denotes the Heaviside function, and $x \in (0, \infty)$ in equations (2.4a) and (2.4b). The first equality in equation (2.4c) means that the loss of probability particle from the retrograde state into the refractory state is equal to the probability of instantaneous entry into the refractory state.

There is a major difference between the two models in the absence of resetting. In Model A, the probability of the particle being captured by one of the targets is 1 when $r = 0$, which is a result of the reflecting boundary condition at $x = L$. In contrast, in Model B, the particle may pass beyond the target array without being captured by any targets, leading to differences in how the conditional MFPTs vary with the resetting rate.

3 For a single target: MFPT

In this section, we first consider a simple problem of a particle moving rightward on the half-line at a constant speed v_+ governed by a Poisson process with parameter λ(as in Model B), a single target at a fixed location $X^* > 0$ and resetting to the origin. If the particle is within a distance l of the target, $l \leqslant X^*$, then the particle can be absorbed by the target at a rate κ.

3.1 MFPT with instantaneous resetting

In the case of instantaneous resetting, we have

$$\frac{\partial p_+}{\partial t} = -v_+ \frac{\partial p_+}{\partial x} + \frac{1}{2}\frac{\partial^2 p_+}{\partial x^2} + \lambda \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\frac{\partial^k p_+}{\partial x^k} - \kappa\chi(x - X^*)p_+ - rp_+ + r\delta(x). \tag{3.1}$$

Here $\chi(x)$ denotes an indicator function: $\chi(x) = 1$ if $|x| < l$ and is zero otherwise. The fifth term on the right-hand side represents the negative probability flux rp_+ at each point x. The sixth term represents the corresponding positive probability flux into $x = 0$ which sums to r. Let $Q_r(x_0,t)$ denote the survival probability that the particle has not been absorbed by the target up to time t, having started at x_0. Then, we have

$$Q_r(x_0,t) = \int_0^\infty p_+(x,t|x_0,0)dx. \tag{3.2}$$

In particular, we set $Q_r(t) = Q_r(0,t)$. The MFPT T_r to be absorbed by the target can be expressed in terms of the survival probability as follows:

$$T_r = -\int_0^\infty t\frac{dQ_r(t)}{dr} = \int_0^\infty Q_r(t)dt. \tag{3.3}$$

Note that Q_r is related to the survival probability without resetting Q_0 using the last renewal equation [22]:

$$Q_r(x_0,t) = e^{-rt}Q_0(x_0,t) + r\int_0^t Q_0(t')Q_r(x_0,t-t')e^{-rt'}dt'.$$

The first term on the right-hand side represents trajectories without resetting. The integrand in the second term is the contribution from trajectories that last reset at time $t-t'$; it consists of the product of the survival probability starting from $x=x_0$ with resetting up to time $t-t'$ and the survival probability starting from $x=x_0$ without any resetting for the time interval of duration t'. Since we have a convolution, it is natural to introduce the Laplace transform as follows:

$$\widetilde{Q}_r(x_0,s) = \int_0^\infty Q_r(x_0,t)e^{-st}dt.$$

By applying the Laplace transform to the last renewal equation, we have

$$\begin{aligned}\int_0^\infty Q_r(x_0,t)e^{-st}dt &= \int_0^\infty e^{-rt}Q_0(x_0,t)e^{-st}dt + r\int_0^\infty\int_0^t Q_0(t')Q_r(x_0,t-t')e^{-rt'}dt'e^{-st}dt \\ &= \widetilde{Q}_0(x_0,r+s) + r\int_0^\infty Q_r(x_0,t-t')e^{-s(t-t')}dt\int_0^t Q_0(t')e^{-rt'}e^{-st'}dt' \\ &= \widetilde{Q}_0(x_0,r+s) + r\widetilde{Q}_r(x_0,s)\widetilde{Q}_0(r+s),\end{aligned}$$

and by rearranging, we obtain

$$\widetilde{Q}_r(x_0,s) = \frac{\widetilde{Q}_0(x_0,r+s)}{1-r\widetilde{Q}_0(r+s)}. \tag{3.4}$$

Substituting into equation (3.3) with $x_0 = 0$, we obtain that the MFPT to reach the target is

$$T_r = \widetilde{Q}_0(r) = \frac{\widetilde{Q}_0(r)}{1 - r\widetilde{Q}_0(r)}. \tag{3.5}$$

3.2 MFPT in the presence of refractory periods

Let $\psi(t)$ denote the waiting time density of the refractory period following each return to the origin, with a finite mean $\overline{\tau}$. The generalized renewal equation is given by

$$\begin{aligned} Q_r(t) = e^{-rt}Q_0(t) + r\int_0^t \Big(1 - \Psi(\sigma)\Big)e^{-r(t-\sigma)}Q_0(t-\sigma)d\sigma \\ +r\int_0^t Q_0(t')e^{-rt'}\left[\int_0^{t-t'} \psi(\tau)Q_r(t-t'-\tau)d\tau\right]dt'. \end{aligned} \tag{3.6}$$

The first term on the right-hand side represents trajectories without resetting. The second term sums over all trajectories that first reset at some time $t-\sigma$, $0 \leqslant \sigma \leqslant t$ and are still in the refractory state at time t. The probability of the remaining refractory for a period t is $1 - \Psi(t)$ with

$$\Psi(t) = \int_0^t \psi(\sigma)d\sigma,$$

The third term is the contribution from trajectories that last reset at time $t-t'-\sigma$, spend a time σ in the refractory state, and then exit the refractory state at time $t-t'$ without further resetting. By applying the Laplace transform to equation (3.6), we obtain

$$\begin{aligned} \widetilde{Q}_r(s) &= \widetilde{Q}_0(r+s) + r\int_0^\infty \int_0^t \Big(1 - \Psi(\sigma)\Big)e^{-r(t-\sigma)}Q_0(t-\sigma)d\sigma e^{-st}dt \\ &\quad + r\int_0^\infty \int_0^t Q_0(t')e^{-rt'}\left[\int_0^{t-t'} \psi(\tau)Q_r(t-t'-\tau)d\tau\right]dt'e^{-st}dt \\ &= \widetilde{Q}_0(r+s) + r\int_0^\infty \int_0^t e^{-r(t-\sigma)}e^{-s(t-\sigma)}Q_0(t-\sigma)d(t-\sigma)e^{-s\sigma}dt \\ &\quad - r\int_0^\infty \int_0^t \left(\int_0^\sigma \psi(y)dy\right)e^{-r(t-\sigma)}Q_0(t-\sigma)d\sigma e^{-st}dt \\ &\quad + r\int_0^\infty \int_0^t Q_0(t')e^{-rt'}e^{-st'}\left[\int_0^{t-t'} \psi(\tau)e^{-s\tau}Q_r(t-t'-\tau)e^{-s(t-t'-\tau)}d\tau\right]dt'dt, \end{aligned}$$

we have

$$\widetilde{Q}_r(s) = \widetilde{Q}_0(r+s) + r\frac{1-\widetilde{\psi}(s)}{s}\widetilde{Q}_0(r+s) + r\widetilde{Q}_0(r+s)\widetilde{\psi}(s)\widetilde{Q}_r(s), \tag{3.7}$$

which can be rearranged to obtain

$$\widetilde{Q}_r(s) = \frac{\widetilde{Q}_0(r+s)\left[1 + \frac{r(1-\widetilde{\psi}(s))}{s}\right]}{1 - r\widetilde{Q}_0(r+s)\widetilde{\psi}(s)}. \tag{3.8}$$

Taking the limit $s \to 0$ in equation (3.8), using $\widetilde{\psi}(0) = 1$ with

$$\lim_{s\to 0} \frac{1-\widetilde{\psi}(s)}{s} = -\psi'(0) = \int_0^\infty \tau\psi(\tau)d\tau \equiv \overline{\tau},$$

we obtain that the MFPT with a refractory period is

$$T_r = \frac{\widetilde{Q}_0(r)[1 + r\overline{\tau}]}{1 - r\widetilde{Q}_0(r)}. \tag{3.9}$$

Moreover, without a refractory period, this equation reduces to equation (3.5).

3.3 MFPT in the presence of finite return times

Rather than a refractory period, suppose that the particle returns to the origin at a speed v_- following each resetting event. Equation (3.1) becomes

$$\frac{\partial p_+}{\partial t} = -v_+\frac{\partial p_+}{\partial x} + \frac{1}{2}\frac{\partial^2 p_+}{\partial x^2} + \lambda\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\frac{\partial^k p_+}{\partial x^k} - \kappa\chi(x - X^*)p_+ - rp_+, \tag{3.10a}$$

$$\frac{\partial p_-}{\partial t} = v_-\frac{\partial p_-}{\partial x} - \frac{1}{2}\frac{\partial^2 p_-}{\partial x^2} - \lambda\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\frac{\partial^k p_-}{\partial x^k} + rp_+,\ \ x \in (0,\infty). \tag{3.10b}$$

$$v_+p_+(0,t) - \frac{1}{2}\frac{\partial p_+}{\partial x}\bigg|_{x=0} - \lambda\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\frac{\partial^{k-1}p_+}{\partial x^{k-1}}\bigg|_{x=0}$$
$$= v_-p_-(0,t) - \frac{1}{2}\frac{\partial p_-}{\partial x}\bigg|_{x=0} - \lambda\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\frac{\partial^{k-1}p_-}{\partial x^{k-1}}\bigg|_{x=0}. \tag{3.10c}$$

Turning to the calculation of the MFPT, we have to write down the appropriate generalization of the renewal equation (3.4):

$$Q_r(t) = e^{-rt}Q_0(t) + r\int_0^{t/\xi_+} e^{-r(t-\sigma)}Q_0(t-\sigma)d\sigma$$
$$+ \int_0^t Q_0(t')re^{-rt'}\left[\int_0^{(t-t')/\xi_-} Q_r(t - t' - \sigma\xi_-)re^{-r\sigma}d\sigma\right]dt', \tag{3.11}$$

where

$$\xi_+ = \frac{v_+ + v_-}{v_+} \quad , \quad \xi_- = \frac{v_-}{v_+}.$$

Similar to equation (3.4), the first term on the right-hand side represents trajectories without resetting. The second term sums over all trajectories that first reset at some time $t-\sigma$, $0 \leqslant \sigma \leqslant t/\xi_+$ and are still in the process of returning to the origin. Since the particle has been in the ballistic state for time $t-\sigma$, it has traveled a distance $v_+(t-\sigma)$, which means that

$$\sigma \leqslant \frac{v_+}{v_-}(t-\sigma).$$

Rearranging this equation yields the constraint that $\sigma \leqslant t/\xi_+$. The third term is the contribution from trajectories whose last reset occurred at time $t-t'-v_+\sigma/v_-$, where σ is the time spent in the ballistic state before resetting, after which the particle takes a time $v_+\sigma/v_-$ to return to the origin, and then an additional time t' in the ballistic state without further resetting. We also require

$$\sigma\frac{v_+}{v_-} \leqslant (t-t'),$$

which yields the constraint $\sigma \leqslant (t-t')\xi_-$.

Next, we apply Laplace transformation to all the terms in equation (3.11). First, we set

$$A(t) := \int_0^{t/\xi_+} e^{-r(t-\sigma)}Q_0(t-\sigma)d\sigma = \int_{t/\xi}^{t} e^{-r\sigma}Q_0(\sigma)d\sigma.$$

with $\xi^{-1} = 1-\xi_+^{-1}$. It follows that

$$\frac{dA(t)}{dt} = Q_0(t)e^{-rt} - \xi^{-1}Q_0(t/\xi)e^{-rt/\xi}.$$

Applying Laplace transformation to this equation with $A(0)=0$ gives

$$\begin{aligned}
\int_0^\infty \frac{dA(t)}{dt}e^{-st}dt &= \int_0^\infty Q_0(t)e^{-rt}e^{-st}dt - \int_0^\infty \xi^{-1}Q_0(t/\xi)e^{-rt/\xi}e^{-st}dt, \\
s\int_0^\infty A(t)e^{-st}dt &= \widetilde{Q}_0(r+s) - \int_0^\infty Q_0(t/\xi))e^{-rt/\xi}e^{-st}d(t/\xi), \\
s\widetilde{A}(s) &= \widetilde{Q}_0(r+s) - \widetilde{Q}_0(r+s\xi),
\end{aligned}$$

which yields

$$\widetilde{A}(s) = \frac{\widetilde{Q}_0(r+s) - \widetilde{Q}_0(r+\xi s)}{s}. \tag{3.12}$$

Next, let

$$B(t) := \int_0^{t/\xi_-} Q_r(t - \sigma\xi_-) r e^{-r\sigma} d\sigma = \xi_-^{-1} \int_0^t Q_r(t-\tau) r e^{-r\tau/\xi_-} d\tau.$$

Similarly, we have

$$\widetilde{B}(s) = \frac{\widetilde{Q}_r(s)}{s\xi_- + r}. \tag{3.13}$$

Finally, by applying Laplace transformation to equation (3.11) and using equations (3.12) and (3.13), we obtain

$$\begin{aligned}
\widetilde{Q}_r(s) &= \widetilde{Q}_0(r+s) + r\left[\frac{\widetilde{Q}_0(r+s) - \widetilde{Q}_0(r+\xi s)}{s}\right] + r\widetilde{Q}_0(r+s) r \frac{1}{s\xi_+ r}\widetilde{Q}_r(s) \\
&= \widetilde{Q}_0(r+s) + r\left[\frac{\widetilde{Q}_0(r+s) - \widetilde{Q}_0(r+\xi s)}{s}\right] + \frac{r^2}{s\xi_- + r}\widetilde{Q}_0(r+s)\widetilde{Q}_r(s),
\end{aligned}$$

which can be rearranged to obtain

$$\widetilde{Q}_r(s) = \frac{\widetilde{Q}_0(r+s) + r\left[\widetilde{Q}_0(r+s) - \widetilde{Q}_0(r+\xi s)\right]/s}{1 - r^2\widetilde{Q}_0(r+s)/(r+s\xi_-)}. \tag{3.14}$$

Taking the limit $s \to 0$ in equation (3.14) and using

$$\lim_{s\to 0} \frac{\widetilde{Q}_0(r+s) - \widetilde{Q}_0(r+\xi s)}{s} = (1-\xi)\widetilde{Q}_0'(r) = -\frac{v_+}{v_-}\widetilde{Q}_0'(r),$$

we obtain that the MFPT in the presence of finite return times is

$$T_r = \frac{\widetilde{Q}_0(r) - r\frac{v_+}{v_-}\widetilde{Q}_0'(r)}{1 - r\widetilde{Q}_0(r)}. \tag{3.15}$$

Taking the limit $v_- \to \infty$ (instantaneous resetting), this reduces to equation (3.5). Finally, equations (3.9) and (3.15) demonstrate that the effect of refractory periods and finite return times gives MFPT of the form

$$T_r = \frac{\widetilde{Q}_0(r) + r\overline{\tau}\widetilde{Q}_0(r) - r\frac{v_+}{v_-}\widetilde{Q}_0'(r)}{1 - r\widetilde{Q}_0(r)}. \tag{3.16}$$

3.4 Alternative renewal method

We now describe an alternative renewal method for calculating MFPTs in the presence of delays, which employs some classical concepts from probability theory: stopping times and strong Markov properties. This approach was previously applied to the search-and-capture model of cytoneme-based morphogen transport [23] and has recently been used to derive a general expression for a search process with stochastic resetting and delays. We will extend this method to multiple targets in Section 4.

The basic idea of this approach is to exploit the fact that resetting eliminates any memory of previous search stages. We introduce the discrete random variable $K(t) \in \{0, 1\}$, which determines whether the particle has been captured by the target ($K(t) = 1$) or is still free ($K(t) = 0$) in the time interval $[0, t]$. Consider the following set of FPT:

$$\mathcal{T} = \inf\{t > 0; X^* - l \leqslant X(t) \leqslant X^* + l, K(t) = 1\},$$
$$\mathcal{S} = \inf\{t > 0; X(t) = 0, K(t) = 0\},$$
$$\mathcal{R} = \inf\{t > 0; X^* - l \leqslant X(t + \mathcal{S} + \mathcal{N}) \leqslant X^* + l, K(t + \mathcal{S} + \mathcal{N}) = 1\}.$$

Here, $\mathcal{T}$ denotes the FPT for finding the target irrespective of the number of resettings; $\mathcal{S}$ denotes the FPT for the first resetting and return to the origin given that the particle is still free; $\mathcal{N}$ denotes the first refractory period; $\mathcal{R}$ denotes the FPT for finding the target given that at least one resetting has occurred. Next, we introduce the sets

$$\Omega = \{\mathcal{T} < \infty\}, \quad \Gamma = \{\mathcal{S} < \mathcal{T} < \infty\}.$$

That is, Ω is the set of all events for which the particle is eventually absorbed by the target, and Γ is the subset of Ω for which the particle resets at least once. It then follows that

$$\Omega \backslash \Gamma = \{\mathcal{T} < \mathcal{S} = \infty\}.$$

In other words, $\Omega \backslash \Gamma$ is the set of all events for which the particle is captured by the target without any resetting. We now use a probabilistic argument to calculate the MFPT $T_r = \mathbb{E}[\mathcal{T}]$ in the presence of resetting ($r > 0$).

Consider the decomposition

$$\mathbb{E}[\mathcal{T}] = \mathbb{E}[\mathcal{T} 1_{\Omega \backslash \Gamma}] + \mathbb{E}[\mathcal{T} 1_{\Gamma}]. \tag{3.17}$$

The first expectation on the right-hand side can be evaluated by noting that it is the MFPT captured by the target without any resetting, and the probability density function of such an event is $-e^{-r\tau}\partial_t Q_0(t)$. Hence,

$$\mathbb{E}[\mathcal{T}1_{\Omega\backslash\Gamma}] = -\int_0^\infty \tau e^{-r\tau}\frac{\partial Q_0(\tau)}{\partial\tau}d\tau = \left(1 + r\frac{d}{dr}\right)\widetilde{Q}_0(r). \tag{3.18}$$

The second expectation can be further decomposed as

$$\begin{aligned}\mathbb{E}[\mathcal{T}1_\Gamma] = \mathbb{E}[(\mathcal{S}+\mathcal{N}+\mathcal{R})1_\Gamma] &= \mathbb{E}[\mathcal{S}1_\Gamma] + \overline{\tau}\mathbb{P}[\Gamma] + \mathbb{E}[\mathcal{R}1_\Gamma] \\ &= \mathbb{E}[\mathcal{S}1_\Gamma] + (\overline{\tau} + T_r)\mathbb{P}[\Gamma]. \end{aligned} \tag{3.19}$$

Here, $\mathbb{E}[\mathcal{N}] = \overline{\tau}$ denotes the mean refractory period; we used the result $\mathbb{E}[\mathcal{R}1_\Gamma] = T_r\mathbb{P}[\Gamma]$. The latter follows from the fact that return to the origin restarts the stochastic process without any memory.

To calculate $\mathbb{E}[\mathcal{S}1_\Gamma]$, it is necessary to incorporate the time to return to the origin following the first return event. The first resetting occurs with probability $re^{-r\tau}Q_0(\tau)d\tau$ on the interval $[\tau, \tau + d\tau]$. At time τ, the particle is at position $v_+\tau$ and thus takes an additional time $v_+\tau/v_-$ to return to $x = 0$. Therefore, we have

$$\mathbb{E}[\mathcal{S}1_\Gamma] = \int_0^\infty re^{-r\tau}\tau\left(1 + \frac{v_+}{v_-}\right)Q_0(\tau)d\tau = -r\left(1 + \frac{v_+}{v_-}\right)\frac{d}{dr}\widetilde{Q}_0(r). \tag{3.20}$$

Moreover, from the definitions of the FPTs and the effect of resetting, we have

$$\mathbb{P}[\Gamma] = \mathbb{P}[\mathcal{S} < \infty]\mathbb{P}[\mathcal{R} < \infty], \tag{3.21}$$

with $\mathbb{P}[\mathcal{R} < \infty] = 1$ and

$$\mathbb{P}[\mathcal{S} < \infty] = \int_0^\infty re^{-r\tau}Q_0(\tau)d\tau = r\widetilde{Q}_0(r). \tag{3.22}$$

Combining equations (3.18) and (3.22), we obtain the implicit equation

$$T_r = \left(1 + r\frac{d}{dr}\right)\widetilde{Q}_0(r) + r\overline{\tau}\widetilde{Q}_0(r) - r\left(1 + \frac{v_+}{v_-}\right)\frac{d}{dr}\widetilde{Q}_0(r) + r\widetilde{Q}_0(r)T_r\ . \tag{3.23}$$

Then, rearranging this equation recovers the general result (3.16).

4 For multiple targets: splitting probabilities and conditional MFPTs

Next, we extend the above probabilistic renewal method to calculate the splitting probability $\pi_k^{(r)}$ that a particle evolving according to Models A and B in Section 2 is eventually captured by the kth target,

$$\pi_k^{(r)} = \lim_{t\to\infty} P_k(t), \quad \sum_{k=1}^{N} \pi_k^{(r)} = 1, \tag{4.1}$$

and the corresponding conditional MFPT $T_k^{(r)}$. This generates expressions for $\pi_k^{(r)}$ and $T_k^{(r)}$ in terms of statistical quantities for the search process without a return phase(no resetting nor reflection at $x = L$). To achieve this, we first analyze target capture in the absence of a return phase. This means that if the particle reaches the end $x = L$, it cannot be captured by any target.

4.1 Target capture without a return phase

Consider the splitting probability π_k and conditional MFPT T_k for the particle to be captured by the kth target when $p_-(x,t) \equiv 0$ (no return phase), having started in the search phase at position $x = 0$ and time $t = 0$. The first step is to solve equation (2.1a) with $r = 0$, which becomes

$$\frac{\partial p_+}{\partial t} = -v_+\frac{\partial p_+}{\partial x} + \frac{1}{2}\frac{\partial^2 p_+}{\partial x^2} + \lambda\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\frac{\partial^k p_+}{\partial x^k} - \kappa p_+ \ . \tag{4.2}$$

Generally, the contribution of higher-order terms of the generalized FPK equation to the entire equation is small [24]; therefore, we keep the equation until the second-order term. Consequently, we obtain

$$\frac{\partial p_+}{\partial t} = -(v_+ + \lambda)\frac{\partial p_+}{\partial x} + \frac{1+\lambda}{2}\frac{\partial^2 p_+}{\partial x^2} - \kappa p_+ \ . \tag{4.3}$$

Then we solve equation(4.3). Let $p_+(x,t) = f(x)g(t)$. Substituting this expression into equation(4.3), we obtain a probability density for variable separation. Hence, equation(4.3) has the solution of the form

$$p_+(x,t|0,0) = \delta(x - v_+t)e^{\frac{2(v_++\lambda)}{1+\lambda}x-\kappa t}, \quad 0 < t < \frac{L}{v_+}.$$

The probability flux into the kth target is

$$\begin{aligned} J_k(t) &= \kappa\int_{(k-1)l}^{kl} p_+(x,t|0,0)dx = \kappa\int_{(k-1)l}^{kl} \delta(x - v_+t)e^{\frac{2(v_++\lambda)}{1+\lambda}x-\kappa t}dx \\ &= \frac{1+\lambda}{2(v_++\lambda)}\kappa\chi_k(t)e^{\frac{2(v_++\lambda)}{1+\lambda}x-\kappa t}, \end{aligned} \tag{4.4}$$

where $\chi_k(t) = 1$ if $\tau_{(k-1)} < t < \tau_k$ and zero otherwise; we have set $\tau_k = kl/v_+$. Let $\mathcal{T}_k$ denote the FPT that the particle is captured by the kth target, with $\mathcal{T}_k = \infty$ indicating that it is not captured. Then, the splitting probability that the particle is captured by the kth target is given by

$$\begin{aligned}\pi_k := \mathbb{P}[0 < \mathcal{T}_k < \infty] = \int_0^\infty J_k(x,t)dt = \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x} \kappa \int_0^\infty \chi_k(t) e^{-\kappa t} dt \\ = \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x} [e^{-\kappa \tau_{k-1}} - e^{-\kappa \tau_k}]. \end{aligned} \tag{4.5}$$

Here $\frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x} e^{-\kappa \tau_{k-1}}$ is the probability of reaching the kth target without being captured by any upstream targets. Thus, π_k is the probability that the particle is captured by the kth target before passing to the $(k+1)$th target. It follows that

$$\sum_{k=1}^N \pi_k = 1 - \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x} e^{-\kappa L/v_+}, \tag{4.6}$$

where $\frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x} e^{-\kappa L/v_+}$ is the probability of the particle reaching the end without being captured by any target in the array.

Given the splitting probability π_k, we define the corresponding conditional MFPT by

$$T_k = \mathbb{E}[\mathcal{T}_k | \mathcal{T}_k < \infty]. \tag{4.7}$$

To determine T_k, it is convenient to consider the probability $\Pi_k(t)$ that the particle is captured by the kth target after time t:

$$\Pi_k(t) = \mathbb{P}[t < \mathcal{T}_k < \infty] = \int_t^\infty J_k(t')dt'. \tag{4.8}$$

Substituting into $J_k(t)$ and using equation (4.4), we obtain

$$\Pi_k(t) = \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x} \left\{ H(\tau_k - t)[e^{-\kappa t} - e^{-\kappa \tau_k}] + H(\tau_{k-1} - t)[e^{-\kappa \tau_{k-1}} - e^{-\kappa t}] \right\}, \tag{4.9}$$

where $H(t)$ is a Heaviside function. Note that $\Pi_k(0) = \pi_k$, and the complementary probability $\Lambda_k(t)$ that the particle is captured by the kth target before time t is given by

$$\Lambda_k(t) = \int_0^t J_k(t')dt' = \int_0^\infty J_k(t')dt' - \int_t^\infty J_k(t')dt' = \pi_k - \Pi_k(t). \tag{4.10}$$

Therefore, the conditional MFPT can be expressed as

$$T_k = \int_0^\infty \frac{\Pi_k(t)}{\pi_k} dt = \frac{\widetilde{\Pi}_k(0)}{\pi_k},$$

where $\widetilde{\Pi}_k(s)$ denotes the Laplace transform of $\Pi_k(\tau)$:

$$\begin{aligned}\widetilde{\Pi}_k(s) = \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x} \Bigg\{ & \frac{1}{s+\kappa}(1 - e^{-(s+\kappa)\tau_k}) - \frac{1}{s}(1 - e^{-s\tau_k})e^{-\kappa\tau_k} \\ & - \frac{1}{s+\kappa}(1 - e^{-(s+\kappa)\tau_{k-1}}) + \frac{1}{s}(1 - e^{-s\tau_k})e^{-\kappa\tau_{k-1}} \Bigg\} . \end{aligned} \quad (4.11)$$

Hence,

$$\pi_k T_k = \widetilde{\Pi}_k(0) = \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x} \left[\frac{\pi_k}{\kappa} + \tau_k e^{-\kappa\tau_k} - \tau_{k-1} e^{-\kappa\tau_{k-1}} \right] . \quad (4.12)$$

4.2. Target capture with a return phase: probabilistic renewal method

To incorporate the effects of stochastic resetting (or reflection at $x = L$ in Model A) and delays, we introduce the discrete random variable $K(t) \in \{0, 1, ..., N\}$, which determines whether the particle has been captured by the kth target ($K(t) = k, k \neq 0$) or has not been absorbed by any target ($K(t) = 0$) on the time interval $[0, t]$. Consider the following set of FPTs;

$$\begin{aligned} \mathcal{T}_k &= \inf\{t > 0; (k-1)l \leqslant X(t) \leqslant kl, K(t) = k\}, \\ \mathcal{S} &= \inf\{t > 0; X(t) = 0, K(t) = 0\}, \\ \mathcal{R}_k &= \inf\{t > 0; (k-1)l \leqslant X(t + \mathcal{S} + \mathcal{N}) \leqslant kl, K(t + \mathcal{S} + \mathcal{N}) = k\}. \end{aligned}$$

Here, $\mathcal{T}_k$ denotes the FPT for finding the kth target irrespective of the number of return phases; $\mathcal{S}$ denotes the FPT for the first return to the origin given that no target has captured the particle; $\mathcal{N}$ denotes the first refractory period; $\mathcal{R}_k$ denotes the FPT for finding the kth target given that at least one return phase has occurred. Next, we introduce the sets

$$\Omega_k = \{\mathcal{T}_k < \infty\}, \quad \Gamma_k = \{\mathcal{S} < \mathcal{T}_k < \infty\} \subset \Omega_k,$$

where Ω_k is the set of all events for which the particle is eventually absorbed by the kth target, and Γ_k is the subset of Ω_k for which the particle returns to the origin at least once. It then follows that

$$\Omega_k \backslash \Gamma_k = \{\mathcal{T}_k < \mathcal{S} = \infty\}.$$

In other words, $\Omega_k\backslash\Gamma_k$ is the set of all events for which the particle is captured by the kth target without any returns to the origin via resetting (or reflection at $x = L$ in the case of Model A). Next, we generalize the probabilistic approach of Section 3.4 to calculate the splitting probability $\pi_k^{(r)}$ and MFPT $T_k^{(r)}$ in the presence of resetting ($r > 0$).

The splitting probability $\pi_k^{(r)}$ can be decomposed as

$$\pi_k^{(r)} := \mathbb{P}[\Omega_k] = \mathbb{P}[\Omega_k\backslash\Gamma_k] + \mathbb{P}[\Gamma_k]. \tag{4.13}$$

Note that the probability that the particle is captured by the kth target in the interval $[\tau, \tau + d\tau]$ without any returns to the origin is $e^{-r\tau}J_k(\tau)d\tau$ with $J_k(\tau)$ given by equation (4.4). Hence,

$$\mathbb{P}[\Omega_k\backslash\Gamma_k] = \int_0^\infty e^{-r\tau}J_k(\tau)d\tau = -\int_0^\infty e^{-r\tau}\frac{d\Pi_k(\tau)}{d\tau}d\tau = -r\widetilde{\Pi}_k(r) + \pi_k = r\widetilde{\Lambda}_k(r). \tag{4.14}$$

From the definition of FPTs, we have

$$\mathbb{P}[\Gamma_k] = \mathbb{P}[\mathcal{S} < \infty]\mathbb{P}[\mathcal{R}_k < \infty], \tag{4.15}$$

and memoryless return to the origin implies that $\mathbb{P}[\mathcal{R}_k < \infty] = \pi_k^{(r)}$. In the case of Model B, we obtain

$$\begin{aligned}\mathbb{P}[\mathcal{S} < \infty] &= \int_0^\infty re^{-r\tau}\left[1 - \sum_{k=1}^N \Lambda_k(\tau)\right]d\tau \\ &= 1 - r\sum_{k=1}^N \widetilde{\Lambda}_k(r) = r\sum_{k=1}^N \widetilde{\Pi}_k(r) + \frac{1+\lambda}{2(v_+ + \lambda)}e^{\frac{2(v_+ + \lambda)}{1+\lambda}x - \kappa L/v_+}.\end{aligned} \tag{4.16}$$

Here, we used the fact that the probability of first switching to the shrinking phase in the time interval $[\tau, \tau + d\tau]$ is equal to the product of the reset probability $re^{-r\tau}d\tau$ and the probability $1 - \sum_{k=1}^N \Lambda_k(\tau)$ that the particle has not been captured by a target up to time τ. The term $\frac{1+\lambda}{2(v_+ + \lambda)}e^{\frac{2(v_+ + \lambda)}{1+\lambda}x - \kappa L/v_+}$ in the final expression arises from the normalization condition (4.6), which denotes the probability that the particle reaches the end of the array at $x = L$, after which it continues in the anterograde state until the first reset. It turns out that $\mathbb{P}[\mathcal{S} < \infty]$ is the same for Model A. Next, we integrate only the resetting time over the interval $\tau \in [0, L/v_+]$, after which the particle returns to the origin with probability one:

$$\mathbb{P}[\mathcal{S} < \infty] = \int_0^{L/v_+} re^{-r\tau}\left[1 - \sum_{k=1}^N \Lambda_k(\tau)\right]d\tau + e^{-rL/v_+}\frac{1+\lambda}{2(v_+ + \lambda)}e^{\frac{2(v_+ + \lambda)}{1+\lambda}x - \kappa L/v_+}.$$

The second term on the right-hand side is the probability that the particle reaches the end of the array without resetting or being captured by the target. Using the fact that $\sum_{k=1}^N \Pi_k(t) = 0$ for $t > L/v_+$, we have

$$\int_0^{L/v_+} re^{-r\tau}\left[1-\sum_{k=1}^N \Lambda_k(\tau)\right]d\tau = \int_0^{L/v_+} re^{-r\tau}\left[\sum_{k=1}^N \Pi_k(\tau) + \frac{1+\lambda}{2(v_+ +\lambda)}e^{\frac{2(v_+ +\lambda)}{1+\lambda}x-\kappa L/v_+}\right]d\tau$$
$$= r\sum_{k=1}^N \widetilde{\Pi}_k(r) + \frac{1+\lambda}{2(v_+ +\lambda)}e^{\frac{2(v_+ +\lambda)}{1+\lambda}x-\kappa L/v_+}(1-e^{-rL/v_+}),$$

and we recover equation (4.16). Hence, for both models, equation (4.15) becomes

$$\mathbb{P}[\Gamma_k] = \pi_k^{(r)}\left[r\sum_{k=1}^N \widetilde{\Pi}_k(r) + \frac{1+\lambda}{2(v_+ +\lambda)}e^{\frac{2(v_+ +\lambda)}{1+\lambda}x-\kappa L/v_+}\right]. \tag{4.17}$$

Combining equations (4.14) and (4.17), we obtain an implicit expression of the form

$$\pi_k^{(r)} = r\widetilde{\Lambda}_k(r) + \left[1-r\sum_{k=1}^N \widetilde{\Lambda}_k(r)\right]\pi_k^{(r)},$$

and rearranging this expression, we obtain the following expressions, which hold for both models:

$$\pi_k^{(r)} = \frac{r\widetilde{\Lambda}_k(r)}{r\sum_{l=1}^N \widetilde{\Lambda}_l(r)} = \frac{\pi_k - r\widetilde{\Pi}_k(r)}{1-r\sum_{l=1}^N \widetilde{\Pi}_l(r) - \frac{1+\lambda}{2(v_+ +\lambda)}e^{\frac{2(v_+ +\lambda)}{1+\lambda}x-\kappa L/v_+}}. \tag{4.18}$$

Summing both sides of equation (4.18) and using equation (4.6), we have $\sum_{k=1}^N \pi_k^{(r)} = 1$. In other words, in the presence of reset, the particle is captured by one of the targets with probability one. Consequently, using the fact that

$$\lim_{r\to 0} r\widetilde{\Pi}_k(r) = \Pi_k(\infty) = 0,$$

we have

$$\lim_{r\to 0}\pi_k^{(r)} = \frac{\pi_k}{1-\frac{1+\lambda}{2(v_+ +\lambda)}e^{\frac{2(v_+ +\lambda)}{1+\lambda}x-\kappa L/v_+}} = \widehat{\pi}_k.$$

Note that the splitting probability $\pi_k^{(r)}$ is independent of the refractory rate η and the retrograde speed v_-. However, implicit in the calculation of $\pi_k^{(r)}$ is the assumption that $v_-, \eta > 0$; otherwise, resetting would not allow the particle to return to the origin and escape from the refractory state in a finite time.

The conditional MFPT $\mathbb{E}[\mathcal{T}_k 1_{\Omega_k}] = \pi_k^{(r)} T_k^{(r)}$ can be analyzed following a similar approach to splitting probability by introducing the decomposition

$$\mathbb{E}[\mathcal{T}_k 1_{\Omega_k}] = \mathbb{E}[\mathcal{T}_k 1_{\Omega_k \setminus \Gamma_k}] + \mathbb{E}[\mathcal{T}_k 1_{\Gamma_k}]. \tag{4.19}$$

The first expectation can be evaluated by noting that it is the MFPT captured by the kth target without any resetting, and the probability density function of such an event is $e^{-r\tau} J_k(\tau) d\tau$. Hence,

$$\mathbb{E}[\mathcal{T}_k 1_{\Omega_k \setminus \Gamma_k}] = -\int_0^\infty \tau e^{-r\tau} \frac{d\Pi_k(\tau)}{d\tau} d\tau = \left[1 + r\frac{d}{dr}\right] \widetilde{\Pi}_k(r). \tag{4.20}$$

The second expectation can be further decomposed as

$$\begin{aligned} \mathbb{E}[\mathcal{T}_k 1_{\Gamma_k}] &= \mathbb{E}[(\mathcal{S} + \widehat{\tau} + \mathcal{R}_k) 1_{\Gamma_k}] = \mathbb{E}[\mathcal{S} 1_{\Gamma_k}] + \frac{1}{\eta}\mathbb{P}[\Gamma_k] + \mathbb{E}[\mathcal{R}_k 1_{\Gamma_k}] \\ &= \mathbb{E}[\mathcal{S} 1_{\Gamma_k}] + \left(\frac{1}{\eta} + T_k^{(r)}\right)\mathbb{P}[\Gamma_k], \end{aligned} \tag{4.21}$$

with $\mathbb{P}[\Gamma_k]$ given by equation (4.17). Here, $\mathcal{N}$ denotes the random time spent in the refractory state at $x = 0$ before switching back to the search phase, with $\mathbb{E}[\mathcal{N}] = \eta^{-1}$; we used the result $\mathbb{E}[\mathcal{R}_k 1_\Gamma] = T_k^{(r)} \mathbb{P}[\Gamma_k]$. The latter follows from the fact that return to the origin restarts the stochastic process without any memory.

To calculate $\mathbb{E}[\mathcal{S} 1_{\Gamma_k}]$, it is necessary to incorporate the time to return to the origin following the first return even, which differs for Models A and B. In the case of Model A, the first return is initiated before reaching the end $x = L$ with probability $re^{-r\tau} \sum_k \Pi_k(\tau) d\tau$ in the interval $[\tau, \tau + d\tau]$. At time τ the particle is at position $v_+\tau$ and takes an additional time $v_+\tau/v_-$ to return to $x = 0$. On the other hand, the particle reaches $x = L$ with probability $\frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda} x - \kappa L/v_+}$ after the time $\tau = L/v_+$ and then returns to the origin over a time interval equal to L/v_-. Thus, we have

$$\begin{aligned} \mathbb{E}[\mathcal{S} 1_{\Gamma_k}] &= \pi_k^{(r)} \Bigg\{ \int_0^\infty re^{-r\tau}\left(\tau + \frac{v_+\tau}{v_-}\right)\left[\sum_{k=1}^N \Pi_k(\tau)\right] d\tau \\ &\qquad + \left(\frac{L}{v_+} + \frac{L}{v_-}\right) \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda} x - \kappa L/v_+} \Bigg\} \\ &= \pi_k^{(r)} \Bigg\{ \left(\frac{L}{v_+} + \frac{L}{v_-}\right) \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda} x - \kappa L/v_+} - r\left(1 + \frac{v_+}{v_-}\right)\left[\sum_{k=1}^N \frac{d\widetilde{\Pi}_k(r)}{dr}\right] \Bigg\}. \end{aligned} \tag{4.22a}$$

Here, we used $\mathbb{P}[\mathcal{R}_k < \infty] = \pi_k^{(r)}$ and the fact that $\sum_{k=1}^N \Pi_k(\tau) = 0$ for $\tau > L/v_+$. In the case of Model B, resetting can occur any time after the particle passes beyond the array; thus, we have

$$\begin{aligned}\mathbb{E}[\mathcal{S}1_{\Gamma_k}] = \pi_k^{(r)} \Bigg\{ & \int_0^\infty re^{-r\tau}\left(\tau + \frac{v_+\tau}{v_-}\right)\left[\sum_{k=1}^N \Pi_k(\tau)\right] d\tau \\ & + \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x - \kappa L/v_+} \int_{L/v_+}^\infty re^{-r\tau}\left(\tau + \frac{v_+\tau}{v_-}\right) d\tau \Bigg\} \\ = \pi_k^{(r)} \Bigg\{ & \frac{1+\lambda}{2(v_+ + \lambda)} \left(1 + \frac{v_+}{v_-}\right)\left(\frac{1}{r} + \frac{L}{v_+}\right) e^{\frac{2(v_+ + \lambda)}{1+\lambda}x - (\kappa + r)L/v_+} \\ & - r\left(1 + \frac{v_+}{v_-}\right)\left[\sum_{k=1}^N \frac{d\widetilde{\Pi}_k(r)}{dr}\right]\Bigg\}. \end{aligned} \tag{4.22b}$$

Combining equations (4.20) and (4.21) with either (4.22a) or (4.22b), we obtain an implicit expression of the form:

$$\begin{aligned}\pi_k^{(r)} T_k^{(r)} = & \left[1 + r\frac{d}{dr}\right]\widetilde{\Pi}_k(r) + \left\{\mathcal{A} - r\left(1 + \frac{v_+}{v_-}\right)\left[\sum_{k=1}^N \frac{d\widetilde{\Pi}_k(r)}{dr}\right]\right\}\pi_k^{(r)} \\ & + \left(\frac{1}{\eta} + T_k^{(r)}\right)\pi_k^{(r)}\left[r\sum_{k=1}^N \widetilde{\Pi}_k(r) + \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x - \kappa L/v_+}\right], \end{aligned} \tag{4.23}$$

where

$$\mathcal{A} = \begin{cases} \left(\frac{L}{v_+} + \frac{L}{v_-}\right)\frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x - \kappa L/v_+} & \text{(Model A)}, \\ \left(1 + \frac{v_+}{v_-}\right)\left(\frac{1}{r} + \frac{L}{v_+}\right)\frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x - (\kappa + r)L/v_+} & \text{(Model B)}. \end{cases} \tag{4.24}$$

Rearranging equation (4.23), we obtain the following result for the conditional MFPT:

$$T_k^{(r)} = \frac{\mathbb{L}\widetilde{\Pi}_k(r) + \mathcal{B}\pi_k^{(r)}}{\pi_k^{(r)}\left[1 - r\sum_{k=1}^N \widetilde{\Pi}_k(r) - \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x - \kappa L/v_+}\right]}. \tag{4.25}$$

Here

$$\mathbb{L}\widetilde{\Pi}_k(r) = \left[1 + r\frac{d}{dr}\right]\widetilde{\Pi}_k(r) - r\left(1 + \frac{v_+}{v_-}\right)\left[\sum_{k=1}^N \frac{d\widetilde{\Pi}_k(r)}{dr}\right]\pi_k^{(r)}, \tag{4.26}$$

and

$$\mathcal{B} = \frac{1}{\eta}\left[r\sum_{k=1}^N \widetilde{\Pi}_k(r) + \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x - \kappa L/v_+}\right] + \mathcal{A}. \tag{4.27}$$

The MFPTs of Models A and B exhibit different behavior as $r \to 0$. In particular, it is clear from equation (4.24) that for Model B, we have $\mathcal{A} \to \infty$ as $r \to 0$, which implies $T_k^{(r)} \to \infty$ as $r \to 0$. This reflects the fact that the MFPTs of Model B are infinite without resetting. In contrast, $\mathcal{A}$ is independent of r for Model A; thus, we obtain

$$\lim_{r\to 0} T_k^{(r)} = \frac{1}{1 - \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x - \kappa L/v_+}} \left\{ \frac{\widetilde{\Pi}_k(0)[1 - \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x - \kappa L/v_+}]}{\pi_k} \right.$$
$$\left. + \left[\frac{1}{\eta} + L\left(\frac{1}{v_+} + \frac{1}{v_-} \right) \right] \frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x - \kappa L/v_+} \right\}. \quad (4.28)$$

This expression has an intuitive interpretation. As $r \to 0$, the particle can only return to the origin by reflecting at the end $x = L$, which occurs with probability $\frac{1+\lambda}{2(v_+ + \lambda)} e^{\frac{2(v_+ + \lambda)}{1+\lambda}x - \kappa L/v_+}$ during one search phase. The first term in square brackets represents the conditional MFPT without any returns to the origin, and the second term represents the additional time taken for the particle to reach the end and return to the origin once before being captured by the target.

5 Results

In this section, we illustrate the parameter dependence of the splitting probability $\pi_k^{(r)}$ and conditional MFPT $T_k^{(r)}$, which are given in equations (4.18) and (4.25), respectively. We fix the units of time and length by setting the capture rate $\kappa = 1$ and the target size $l = 1$.

Figure 2 shows the plots of the splitting probability $\pi_k^{(r)}$ as a function of k for an array of $N = 10$ targets with various speeds v_+. Several observations can be made. First, the splitting probability is a monotonically decreasing function of k. As one might expect, targets closer to the origin are more likely to capture the particle. Second, increasing v_+ tends to alleviate this impact, leading to a more even distribution of splitting probabilities. In particular, there is a crossover of the plots for different speeds. Moreover, increasing the particle position x can increase the splitting probability. Additionally, the splitting probability can decrease for a large resetting rate r. Figure 3 shows analogous plots of the conditional MFPT $T_k^{(r)}$ for Model A. In contrast to the splitting probability, the MFPT

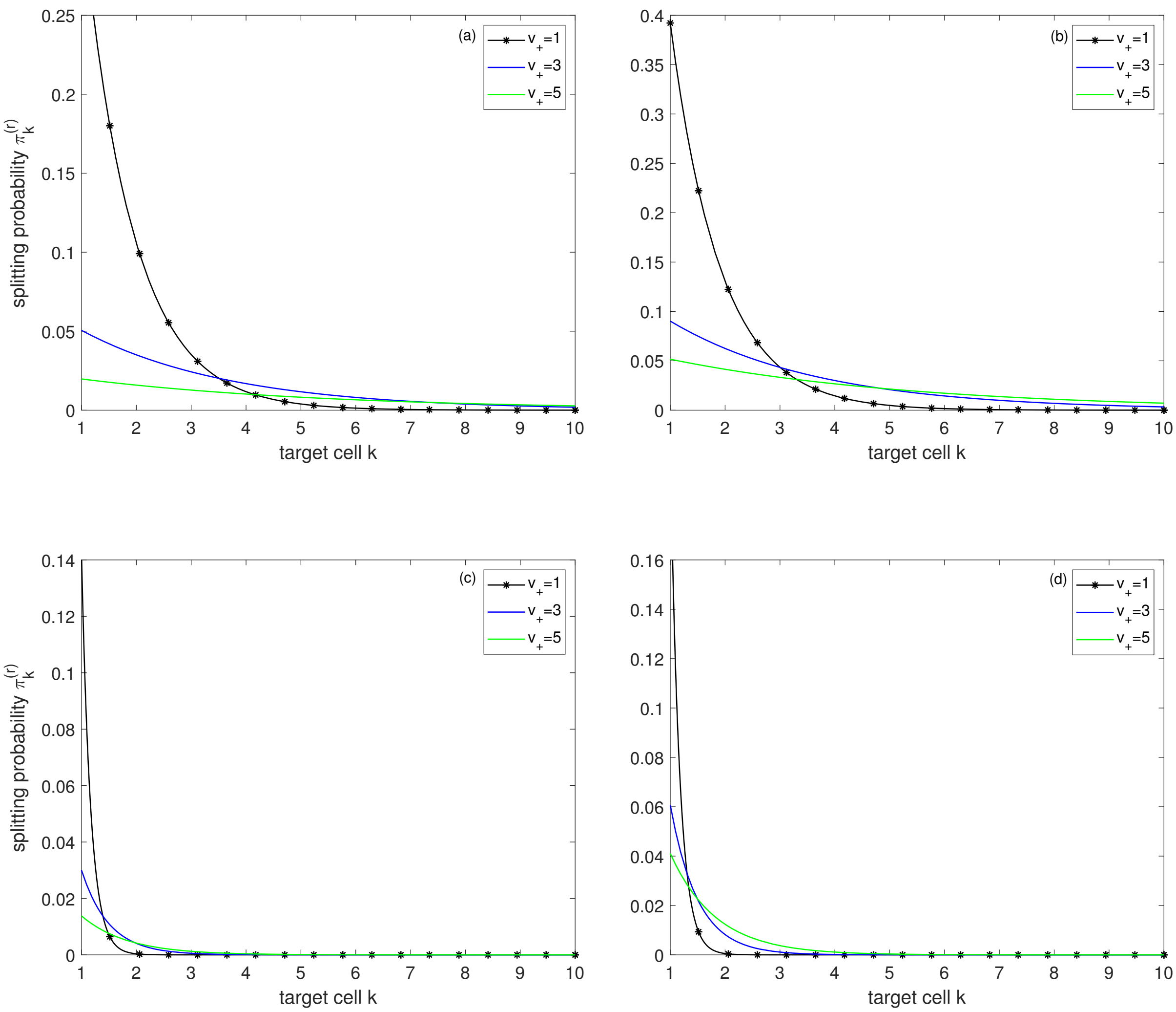

Figure 2: Plots of the splitting probability $\pi_k^{(r)}$ as a function of target site k, $k = 1, ..., 10$, for various speeds v_+. (a) $r = 0.1$, $x = 0$, (b) $r = 0.1$, $x = 0.1$, (c) $r = 5$, $x = 0$, and (d) $r = 5$, $x = 0.1$. Another parameter is $\lambda = 0.1$.

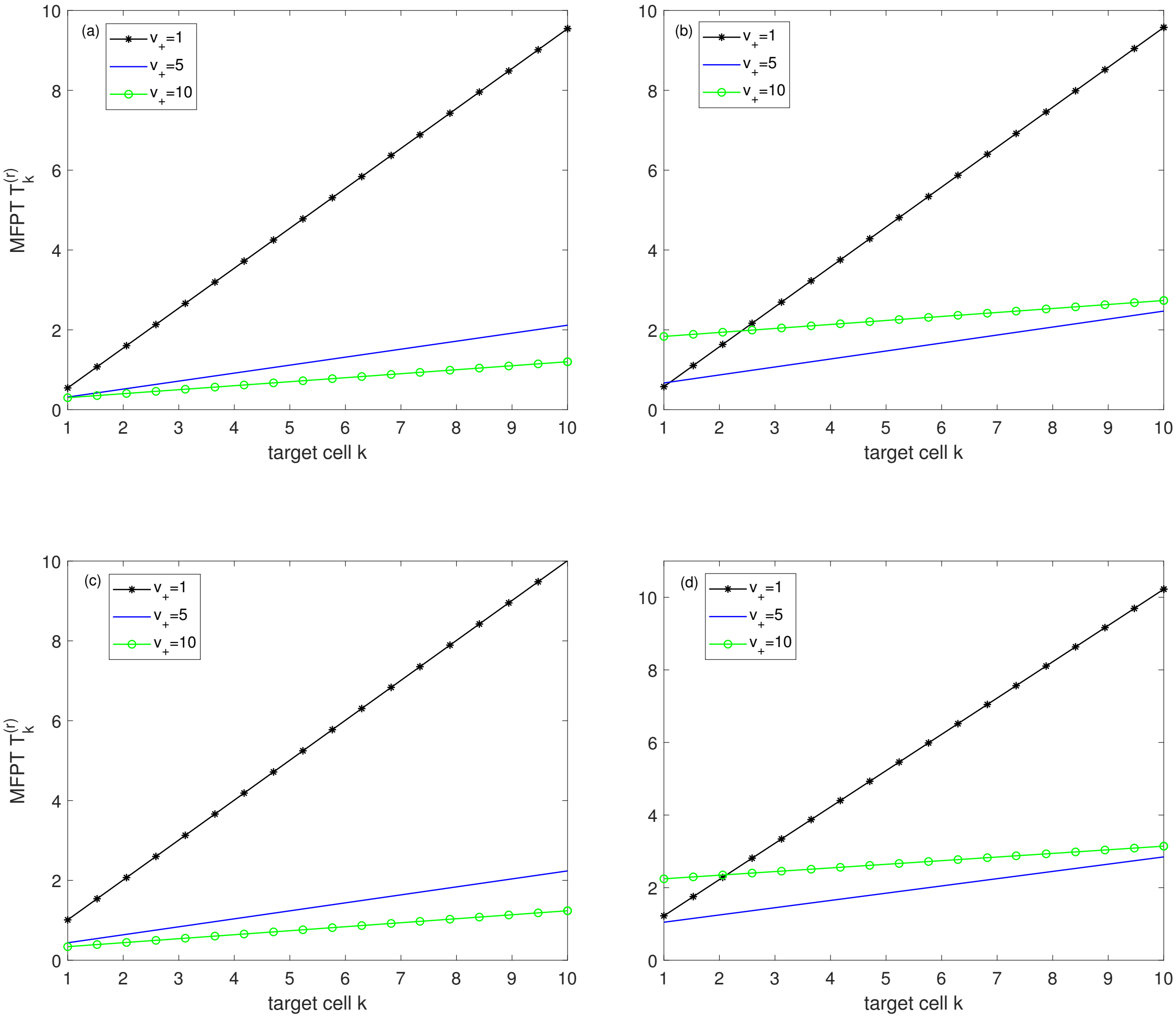

Figure 3: Plots of conditional MFPT $T_k^{(r)}$ as a function of target site k, $k = 1, ..., 10$, for various speeds v_+ (Model A). (a) $r = 0.1$, $x = 0$, (b) $r = 0.1$, $x = 0.1$, (c) $r = 1$, $x = 0$, and (d) $r = 1$, $x = 0.1$. Other parameters are $v_- = 1$, $\eta = 1$ and $\lambda = 0.1$.

depends on the refractory rate η and the retrograde speed v_-. For illustration, we set $\eta = v_- = 1$. One would expect the conditional MFPT to increase for more distal targets, which is confirmed in Figure 3. It can be seen that $T_k^{(r)}$ increases approximately linearly with k. Similar to the splitting probability, increasing v_+ tends to flatten the curves so that the MFPT is a weaker function of k, and there is a crossover of plots for different speeds. Additionally, increasing the particle position x can increase conditional MFPT. The conditional MFPT can increase for a large resetting rate r. This is because frequent returns to the starting point reduce the particle's chance of finding the target, as the particle is often reset before it can reach the target.

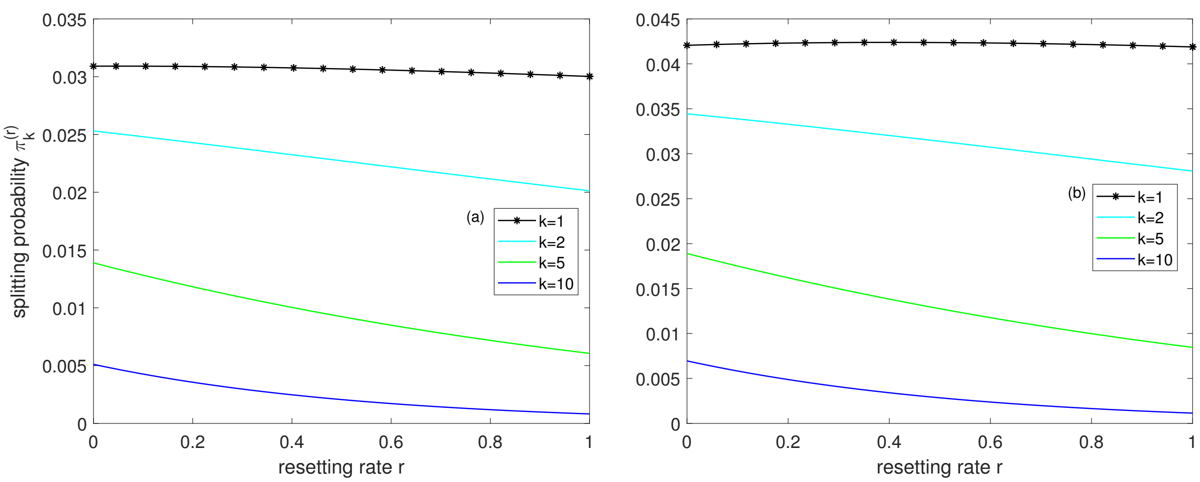

Figure 4: Plots of the splitting probability $\pi_k^{(r)}$ as a function of the resetting rate r for various targets k. (a) $x = 0$, (b) $x = 0.05$. Other parameters are $v_+ = 5$, $v_- = 1$, $\eta = 1$ and $\lambda = 1$.

Figures 4 and 5 show the splitting probability $\pi_k^{(r)}$ and conditional MFPT $T_k^{(r)}$ (Model A) as a function of r for various targets k and fixed speed $v_+ = 5$. It can be seen that the splitting probability decreases with r, whereas increasing the resetting rate r leads to an approximately linear increase in $T_k^{(r)}$. Additionally, the size of the splitting probability and conditional MFPTs depend on the particle position x. Therefore, resetting in Model A tends to have a detrimental effect on the conditional MFPTs because the particle returns to the origin without resetting. As shown in Figure 6, conditional MFPTs are nonmonotonic functions of r (Model B). This is because conditional MFPTs tend to

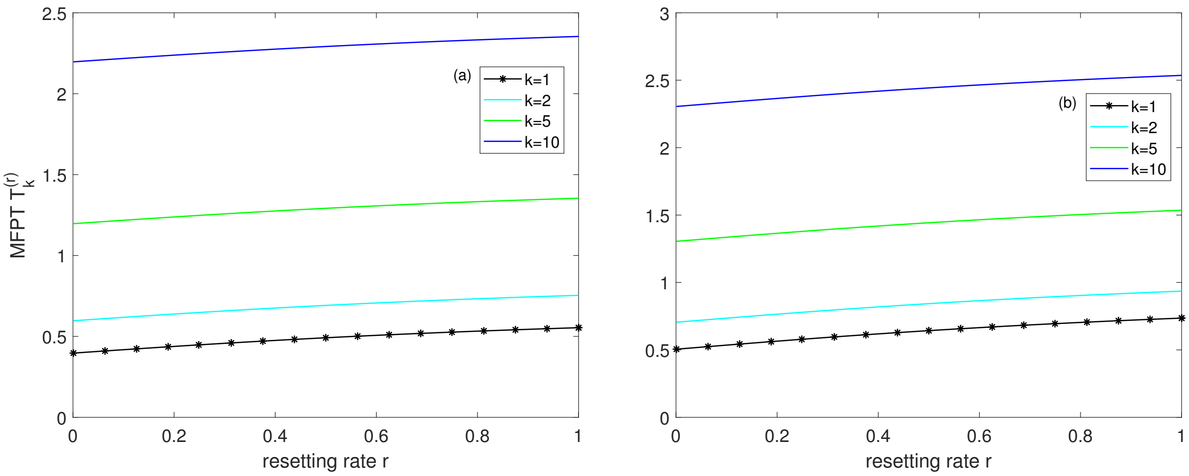

Figure 5: Plots of conditional MFPT $T_k^{(r)}$ as a function of the resetting rate r for various targets k. (a) $x = 0$, (b) $x = 0.05$. Other parameters are $v_+ = 5$, $v_- = 1$, $\eta = 1$ and $\lambda = 1$.

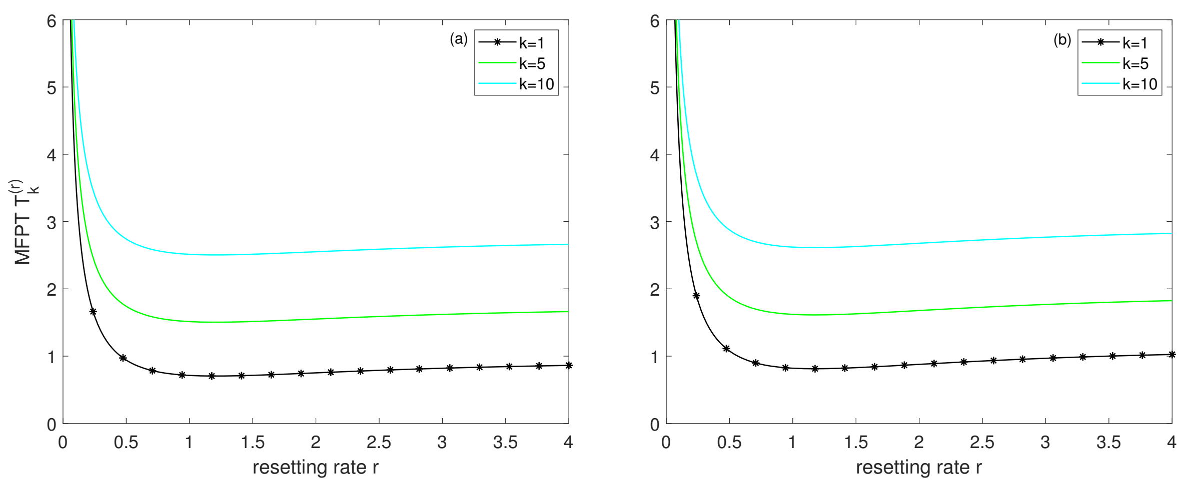

Figure 6: Model B. Plots of conditional MFPT $T_k^{(r)}$ as a function of the resetting rate r for various targets k. (a) $x = 0$, (b) $x = 0.05$. Other parameters are $v_+ = 5$, $v_- = 1$, $\eta = 1$ and $\lambda = 15$.

infinity as $r \to 0$. Therefore, there exists an optimal resetting rate that minimizes $T_k^{(r)}$ for a given k.

6 Discussion

In this paper, we analyzed a directed search-and-capture model driven by Lévy motion with stochastic resetting, refractory periods, and finite return times. We used a probabilistic renewal method to determine the splitting probability $\pi_k^{(r)}$ and MFPT $T_k^{(r)}$ for a particle captured by the kth contiguous target in a one-dimensional array, which are derived from equations (4.18) and (4.25), respectively. We also compared the search for the bounded domain $[0, L]$ (Model A) with a partially bounded search on the half-line (Model B). Our findings showed that the probability of target capture without resetting is one in Model A and less than one in Model B. In contrast, the probability of target capture is the same in both models (4.18), but with different conditional MFPTs (4.25). According to equation (4.25), the conditional MFPTs for the bounded search were monotonically increasing functions of r, whereas the corresponding MFPTs on the half-line were nonmonotonic with respect to r, and there exists an optimal resetting rate that minimizes the conditional MFPTs in this case.

The classical search-and-capture model with stochastic resetting mainly considers the Brownian motion as a stochastic factor. However, the process of vesicle transport to synaptic targets in the neuronal system is often burst and unpredictable; Lévy motion as a classical non-Gaussian process, can more realistically and accurately depict these phenomena. Furthermore, the search-and-capture process can be more efficient after adding stochastic resetting. Moreover, Lévy motion has important application prospects in biology. For example, it can simulate protein synthesis during gene regulation. When stochastic resetting is added, the probability of the synthesis process can change: a higher survival probability and shorter MFPT increase the likelihood successful protein synthesis. This paper considers Lévy motion, which is a standard Brownian motion combined with a Poisson process with parameter λ; however, for general Lévy motion, it will be a problem for further research. From a biological perspective, it is crucial to understand the practical benefits of stochastic resetting. Specifically, it is important to investigate what happens when stochastic resetting occurs during high-dimensional search processes. These issues are of great value for discussion.

References

[1] M.Coppey, O.Benichou, R.Voituriez, M.Moreau. Kinetics of Target Site Localization of a Protein on DNA: A Stochastic Approach. *Biophys.J.* **2004**, *87*, 1640-1649.

[2] E.Gelenbe. Search in unknown random environments. *Phys.Rev.E.* **2010**, *82*, 061112.

[3] O.Benichou, M.Coppey, M.Moreau, PH.Suet, R.Voituriez. Optimal search strategies for hidden targets. *Phys.Rev.Lett.* **2005**, *94*, 198101.

[4] M.Lomholt, T.Koren, R.Metzler, J.Klafter. Lévy strategies in intermittent search processes are advantageous. *Proc.Natl Acad.Sci.USA.* **2008**, *105*, 11055-11059.

[5] M.R.Evans, S.N.Majumdar. Diffusion with Stochastic Resetting. *Phys.Rev.Lett.* **2011**, *106*, 160601.

[6] S.C.Manrubia, D.H.Zanette. Stochastic multiplicative processes with reset events. *Phys.Rev.E.* **1999**, *59*, 4945.

[7] S.Reuveni, M.Urbakh, J.Klafter. Role of substrate unbinding in Michaelis-Menten enzymatic reactions. *Proc. Natl. Acad. Sci.* **2014**, *11*, 4391.

[8] T.Rotbart, S.Reuveni, M.Urbakh. Michaelis-Menten reaction scheme as a unified approach towards the optimal restart problem. *Phys.Rev.E.* **2015**, *92*. 060101.

[9] T.Rotbart, S.Reuveni, M.Urbakh. Single-molecule theory of enzymatic inhibition. *Nat. Commun.* **2018**, *9*, 779.

[10] E.Roldan, A.Lisica, D.Sanchez-Taltavull, S.W.Grill. Stochastic resetting in backtrack recovery by RNA polymerases with Riemann-Liouville fractional derivative. *Phys.Rev.E.* **2016**, *93*, 062411.

[11] A.Montanari, R.Zecchina. Optimizing searches via rare events. *Phys.Rev.Lett.* **2002**, *88*, 178701.

[12] F.Bartumeus, J.Catalan. Optimal search behaviour and classic foraging theory. *J.Phys.A:Math.Theor.* **2009**, *42*, 434002.

[13] H.C.Berg. Random Walks in Biology. *Princeton University Press.* New York, **1983**.

[14] L.Kusmierz, S.N.Majumdar, S.Sabhapandit, G.Schehr. First Order Transition for the Optimal Search Time of Lévy Flights with Resetting. *Phys.Rev.Lett.* **2014**, *113*, 220602.

[15] L.Kusmierz, E.Gudowska-Nowak. Optimal first-arrival times in Lévy flights with resetting. *Phys.Rev.E.* **2015**, *92*, 052127.

[16] M.Montero, J.Villarroel. Continuous-time random walks with drift and stochastic reset events. *Phys.Rev.E.* **2013**, *87*, 012116.

[17] V.Mendez, D.Campos. Characterization of stationary states in random walks with stochastic resetting. *Phys.Rev.E.* **2016**, *93*, 022106.

[18] V.P.Shkilev. Continuous-time random walk under time-dependent resetting. *Phys.Rev.E.* **2017**, *96*, 012126.

[19] A.S.Bodrova, A.V.Chechkin, I.M.Sokolov. Nonrenewal resetting of scaled Brownian motion. *Phys.Rev.E.* **2019**, *100*, 012119.

[20] A.S.Bodrova, A.V.Chechkin, I.M.Sokolov. Scaled Brownian motion with renewal resetting. *Phys.Rev.E.* **2019**, *100*, 012120.

[21] Sun X, Duan J. Fokker-Planck equations for nonlinear dynamical systems driven by non-Gaussian Lévy processes. *Journal of Mathematical Physics.* **2012**, *53*.

[22] M.R.Evans, S.N.Majumdar, G.Schehr. Stochastic resetting and applications. *J.Phys.A* **2019**, 1751–8121.

[23] P.C.Bressloff, H.Kim. Search-and-capture model of cytoneme-mediated morphogen gradient formation. *Phys.Rev.E* **2019**, *99*, 052401.

[24] H.T.Zhu, G.K.ER, IU Vai Pan, Kou Kun Pang. PDE solution to nonlinear stochastic dynamic systems driven by Poisson white noise. *Proceedings of the 6th National Civil Engineering Graduate Academic Forum.*