Traveling fronts of a real supercritical quintic Ginzburg-Landau equation coupled by a slow diffusion mode

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Abstract

In this paper, we investigate the existence of traveling front solutions for a class of quintic Ginzburg-Landau equations coupled with a slow diffusion mode. By employing the theory of geometric singular perturbations, we turn the problem into a geometric perturbation problem. We demonstrate the intersection property of the critical manifold and further validate the existence of heteroclinic orbits by computing the zeros of the Melnikov function on the critical manifold. The results demonstrate that under certain parameters, there is 1 or 2 heteroclinic solutions, confirming the existence of traveling front solutions for the considered quintic Ginzburg-Landau equation coupled with a slow diffusion mode.

Keywords: Quintic Ginzburg-Landau equation; Traveling front solution; Heteroclinic solution; Geometric singular perturbation theory; Melnikov function.

1. Introduction

In 1950 V. L. Ginzburg and L. D. Landau [1] introduced the equations that have since been called Ginzburg-Landau equations to describe the quantum phenomenon of superconductivity. This was an extension of Landau′s theory of second-order phase transitions [2]. The Ginzburg-Landau equations and their modified forms have been used to model a wide variety of nonlinear phenomena, such as second-order phase transitions [3, 4], superconductivity, superfluidity [5],

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Bose–Einstein condensates [6, 7] and more. For further reading on these applications, we recommend the work of Kengne et.al. [8].

For different physical backgrounds, there are two main types of the Ginzburg-Landau equations: the real Ginzburg-Landau equation (abbr. RGLE) and the complex Ginzburg-Landau equation (abbr. CGLE). More precisely, the RGLE was first time proposed by Newell and Whitehead [9], and this equation also can be used to explain the Bénard convection [10]. For the CGLE, it was introduced independently by Newell and Whitehead [11], and by DiPrima, Eckhaus and Segel [12], and then this equation is used to describe plane Poiseuille flow [13], chemical reactions [14] and pattern formation [15, 16]. As indicated in [18], the one-dimensional CGLE with cubic nonlinearity is governed by the following equation:

$$\mathcal{A}_t = \kappa_1 \mathcal{A}_{xx} + (\kappa_2 + \tau_1 i)\mathcal{A} + (\kappa_3 + \tau_2 i)|\mathcal{A}|^2\mathcal{A}, \tag{1.1}$$

where $\kappa_i \in \mathbb{R}(i{=}1,2,3)$, $\tau_i \in \mathbb{R}$ $(i{=}1,2)$, $\mathcal{A}(x,t)$: $\mathbb{R}\times\mathbb{R}^+ \to \mathbb{C}$. If $\tau_1{=}\tau_2{=}0$, equation (1.1) reduces to

$$\mathcal{A}_t = \kappa_1 \mathcal{A}_{xx} + \kappa_2\mathcal{A} + \kappa_3|\mathcal{A}|^2\mathcal{A}, \tag{1.2}$$

which describes the onset of stationary periodic solutions in nonlinear stability problems [18].

To prove that unstable pulse solutions become stable under the coupling of the slow $\mathcal{B}$-mode, Doelman et. al. [17] studied a reaction-diffusion system expressed as follows:

$$\begin{cases} \mathcal{A}_t = \mathcal{A}_{xx} - \mathcal{A} + \mathcal{A}^3 + \mu\mathcal{A}\mathcal{B}, \\ \epsilon^2\alpha_1\mathcal{B}_t = \epsilon^{-2}\mathcal{B}_{xx} - \alpha_2\epsilon^2\mathcal{B} + \alpha_3\mathcal{B}_x + \alpha_4\mathcal{A}^2 + \alpha_5\mathcal{A}^2\mathcal{B}, \end{cases} \tag{1.3}$$

where $\epsilon > 0$ is very small, μ, $\alpha_i(i{=}1,2,\cdots,5)$ are real parameters. Doelman et.al. demonstrated the existence of homoclinic pulse solutions of the RGLE

$$\mathcal{A}_t = \mathcal{A}_{xx} - \mathcal{A} + \mathcal{A}^3 + \mu b_0\mathcal{A},$$

as a solution of the whole system (1.3). Additionally, Tu et.al. [18] used the Melnikov function and geometric singular perturbation theory to show that there exist a traveling front in the following system

$$\begin{cases} \mathcal{A}_t = \mathcal{A}_{xx} + \mathcal{A} - \mathcal{A}^3 + \mu\mathcal{A}\mathcal{B}, \\ \epsilon^2\alpha_1\mathcal{B}_t = \epsilon^{-2}\mathcal{B}_{xx} - \alpha_2\epsilon^2\mathcal{B} + \alpha_3\mathcal{B}_x + \alpha_4\mathcal{A}^2 + \alpha_5\mathcal{A}^2\mathcal{B} + c_1 + c_2\mathcal{B} + c_3\mathcal{B}^2, \end{cases} \tag{1.4}$$

wherein c_i $(i{=}1,2,3)$ are $O(1)$ constants. Notably, equation (1.4) introduces three additional terms related to the $\mathcal{B}$ model compared to equation (1.3). As described in [18], Tu et.al., in terms of physics, investigated whether these additional terms would lead to the emergence of new dynamical behaviors in equation (1.4) that would not appear in equation (1.3).

In recent years, many researchers have extensively analyzed the quintic complex Ginzburg-Landau equation (QCGLE) through various algorithms, yielding intriguing findings. For instance, Akhmediev et.al. [19] demonstrated the existence of stable impulsive solutions in a QCGLE. Soheila et.al. [20] employed the homotopy analysis method to solve the generalized QCGLE, while Yao et.al. [21] obtained an iterative solution for the generalized QCGLE using the fractional order natural decomposition method. For the case of one-dimensional, Marcq et.al. [22] showed the presence of solitary wave solutions in QCGLE. Additionally, Rossides et.al. [23] conducted a thorough investigation into the dynamics of multi-pulse interactions in QCGLE, leading to several noteworthy findings. To the best of our knowledge, there are few results in the mathematical community regarding the real supercritical quintic Ginzburg-Landau equation.

The RGLE in one (unbounded) spatial dimension with quintic nonlinearity can be described by the following equation:

$$\mathcal{A}_t = \kappa_1 \mathcal{A}_{xx} + \kappa_2 \mathcal{A} + \kappa_3 |\mathcal{A}|^4 \mathcal{A}, \tag{1.5}$$

the parameters are the same as those in the earlier RGLE equation. As it was indicated in [18], if $\kappa_1 > 0$, $\kappa_2 < 0$ and $\kappa_3 > 0$ in the equation (1.5), the RGLE with quintic nonlinearity in one spatial dimension is called subcritical. Conversely, if $\kappa_1 > 0$, $\kappa_2 > 0$ and $\kappa_3 < 0$, it is called supercritical. For equation (1.5), the global phase portraits are shown in Fig.1.

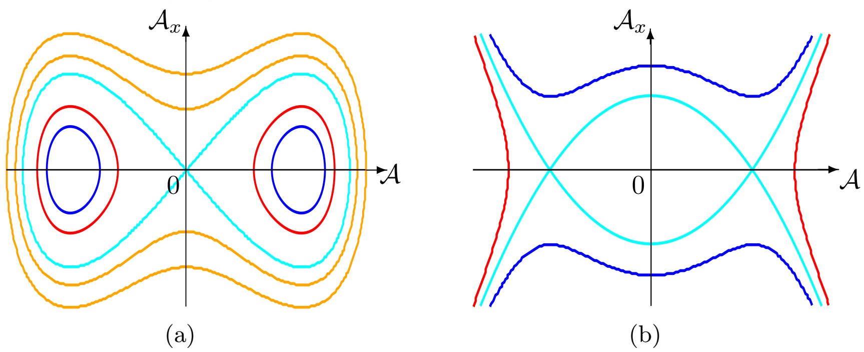

Figure 1: (a) Global phase portraits of the one-dimensional subcritical RGLE, (b) Global phase portraits of the one-dimensional supercritical RGLE.

In this paper, we couple a slow diffusion mode to the RGLE system

$$\mathcal{A}_t = \mathcal{A}_{xx} + \mathcal{A} - \mathcal{A}^5 + \mu \mathcal{A}\mathcal{B},$$

namely,

$$\begin{cases} \mathcal{A}_t = \mathcal{A}_{xx} + \mathcal{A} - \mathcal{A}^5 + \mu \mathcal{A}\mathcal{B}, \\ \epsilon^2 \alpha_1 \mathcal{B}_t = \epsilon^{-2} \mathcal{B}_{xx} - \alpha_2 \epsilon^2 \mathcal{B} + \alpha_3 \mathcal{B}_x + \alpha_4 \mathcal{A}^2 + \alpha_5 \mathcal{A}^2 \mathcal{B} + c_1 + c_2 \mathcal{B} + c_3 \mathcal{B}^2, \end{cases} \tag{1.6}$$

where $\epsilon > 0$ is very small, μ, $\alpha_i (i=1,2,\cdots,5)$, are real parameters, c_i (i=1,2,3) are $O(1)$ constants. Here we focus on discussing the existence of the traveling fronts in system (1.6). From the results obtained by Doelman et.al. [24], we know that this is equivalent to construct a heteroclinic orbit in the corresponding four-dimensional singularly perturbed ordinary differential equations. Therefore, the main purpose of this article is whether, under certain conditions, such heteroclinic orbits can be controlled and approximated to two different saddle points corresponding to ordinary differential equations.

The structure of the article is outlined as follows: In the next section, we first transform system (1.6) into a four-dimensional ordinary differential equation, and then we analyze the dynamics of the layer system. Moreover, we also show our main results (see Theorem 2.1) in this work. In section 3, we elaborate on the expansion of both the stable and unstable manifolds within the slow manifolds, revealing that $W^u(\mathcal{M}_\varepsilon^-)$ intersects $W^s(\mathcal{M}_\varepsilon^+)$ in a transverse manner. We proceed by defining two particular curves that lead to a heteroclinic orbit within system (2.9). More importantly, we give a rigorous proof of Theorem 2.1. Finally, we draw a conclusion for our work.

2. Transforming system (1.6) into ordinary differential equation

Next, we can derive the ordinary differential equation with two fast variables and two slow variables by introducing the transformation $\varphi = x - \tilde{c}t$, where $\tilde{c} = \epsilon^2 c$. This transformation models the traveling fronts of (1.6). i.e.

$$\begin{cases} p' = q, \\ q' = -p + p^5 - \mu p\nu - c\epsilon^2 q, \\ \nu' = \epsilon\tau, \\ \tau' = \epsilon\left[\alpha_2\nu\epsilon^2 - \epsilon\left(\alpha_3 + \alpha_1 c\epsilon^4\right)\tau - \alpha_4 p^2 - \alpha_5\nu p^2 + c_1 + c_2\nu + c_3\nu^2\right], \end{cases} \tag{2.1}$$

in which, $(p, q, \nu, \tau) = (\mathcal{A}, \mathcal{A}_\varphi, \mathcal{B}, \mathcal{B}_\varphi/\epsilon)$ and, the prime denotes the derivative in φ. System (2.1) is called fast system, where p and q are fast variables, while ν and τ are slow variables.

When $\epsilon \to 0$, system (2.1) becomes

$$\begin{cases} p' = q, \\ q' = -(1 + \mu\nu_0)p + p^5, \\ \nu' = 0, \\ \tau' = 0, \end{cases} \tag{2.2}$$

which is layer system, the flow of (2.2) is called the fast flow, where ν_0, ν and τ denote constants.

Taking into account time rescaling $\varphi^* = \epsilon\varphi$, the slow system of (2.1) is governed by

$$\begin{cases} \epsilon\dot{p} = q, \\ \epsilon\dot{q} = -p + p^5 - \mu p\nu - c\epsilon^2 q, \\ \dot{\nu} = \tau, \\ \dot{\tau} = \alpha_2\nu\epsilon^2 - \epsilon\left(\alpha_3 + \alpha_1 c\epsilon^4\right)\tau - \alpha_4 p^2 - \alpha_5\nu p^2 + c_1 + c_2\nu + c_3\nu^2, \end{cases}$$

where the dot denotes the derivative of φ^*. Similarly, when $\epsilon \to 0$ system (2.3) becomes

$$\begin{cases} q = 0, \\ -p + p^5 - \mu p\nu = 0, \\ \dot{\nu} = \tau, \\ \dot{\tau} = \alpha_2\nu\epsilon^2 - \epsilon\left(\alpha_3 + \alpha_1 c\epsilon^4\right)\tau - \alpha_4 p^2 - \alpha_5\nu p^2 + c_1 + c_2\nu + c_3\nu^2, \end{cases} \tag{2.3}$$

the flow of (2.3) is called the slow flow. Direct computation shows that the critical manifold is

$$C_0 = \left\{(p_0, q_0, \nu_0, \tau_0) \mid p - p^5 + \mu p\nu_0 = 0, q_0 = 0\right\},$$

i.e.,

$$\mathcal{M}_0^+ = \left\{(p_0, q_0, \nu_0, \tau_0) \mid p_0 = (1 + \mu\nu_0)^{\frac{1}{4}}, q_0 = 0\right\},$$
$$\mathcal{M}_0^- = \left\{(p_0, q_0, \nu_0, \tau_0) \mid p_0 = -(1 + \mu\nu_0)^{\frac{1}{4}}, q_0 = 0\right\},$$

and

$$\mathcal{M}_0^0 = \left\{(p_0, q_0, \nu_0, \tau_0) \mid p_0 = 0, q_0 = 0\right\},$$

which are two-dimensional manifolds in the four-dimensional phase space. Points on the critical manifold correspond the equilibria of the layer system (2.2), where τ_0 is a constant. The (slow) dynamics of the reduced system (2.3) are restricted to the critical manifolds $\mathcal{M}_0^+$, $\mathcal{M}_0^-$ and $\mathcal{M}_0^0$. Therefore, the layer system and reduced system form a two-dimensional system in the four-dimensional phase space, then the dynamics of these systems can be analyzed easily.

Firstly, we will analyze the dynamics of system (2.2). In the following discussion, we are in the restriction of

$$1 + \mu\nu_0 > 0. \tag{2.4}$$

The layer system (2.2) is a two-dimensional system in the four-dimensional phase space,

$$\begin{cases} p' = q, \\ q' = -p + p^5 - \mu p\nu_0. \end{cases} \tag{2.5}$$

Note that system (2.5) is a two-parameter (ν_0 and τ_0) families of planar integrable systems. These equations describe the leading order behaviour of the amplitude $\mathcal{A}$ in the full system

(1.6), with the constants ν_0 and τ_0 representing the approximate values that ν and τ take in the full system (2.1) during an $O(1)$ time interval in φ, during which the amplitude $\mathcal{A}$ forms its pulse, i.e. jumps away from and returns to $|\mathcal{A}| = 0$.

A direct computation shows that system (2.5) has two normally hyperbolic manifolds $\mathcal{M}_0^+ = \{p_0 = (1+\mu\nu_0)^{\frac{1}{4}}, q_0 = 0\}, \mathcal{M}_0^- = \{p_0 = -(1+\mu\nu_0)^{\frac{1}{4}}, q_0 = 0\}$ and a non-hyperbolic $\mathcal{M}_0^0 = \{p_0 = q_0 = 0\}$. This indicates that the layer system (2.5) has a Hamiltonian structure, and its Hamiltonian function is

$$H(p, q, \nu_0, \tau_0) = \frac{q^2}{2} + \frac{1}{2}(1+\mu\nu_0)p^2 - \frac{1}{6}p^6.$$

The phase diagram of this function is shown below.

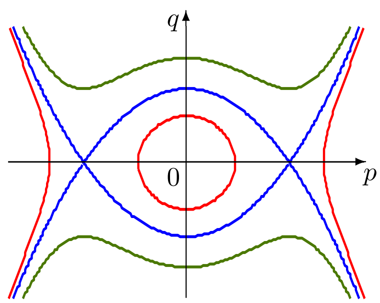

Figure 2: The plane portrait of the Hamiltonian system (2.5).

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For any point $((1+\mu\nu_0)^{\frac{1}{4}}, 0, \nu_0, \tau_0) \in \mathcal{M}_0^+$, there will always be a corresponding point $(-(1+\mu\nu_0)^{\frac{1}{4}}, 0, 0, 0) \in \mathcal{M}_0^-$, creating a heteroclinic connection between the two points, and vice versa. These heteroclinic orbits will form the three-dimensional stable and unstable manifolds of the two-dimensional critical manifold $\mathcal{M}_0^+$ and $\mathcal{M}_0^-$, denoted as $W_0^s\left(\mathcal{M}_0^+\right)$, $W_0^u\left(\mathcal{M}_0^+\right)$, as well as $W_0^s\left(\mathcal{M}_0^-\right)$, $W_0^u\left(\mathcal{M}_0^-\right)$. According to geometric singular perturbation theory of Fenichel [25], if $\epsilon > 0$ is very small, the normally hyperbolic critical manifolds $\mathcal{M}_0^+$ and $\mathcal{M}_0^-$ of system (2.5) as well as their stable and unstable manifolds, exist. Therefore, the two-dimensional slow manifold $\mathcal{M}_\epsilon^+$ as well as its three-dimensional stable manifold $W_\epsilon^s\left(\mathcal{M}_\epsilon^+\right)$ and unstable manifold $W_\epsilon^u\left(\mathcal{M}_\epsilon^+\right)$, persist, and are $O(\epsilon)$-close and diffeomorphism to their counterparts $W_0^s\left(\mathcal{M}_0^+\right)$ and $W_0^u\left(\mathcal{M}_0^+\right)$, respectively. Similarly, we can obtain the same conclusion for the two-dimensional slow manifold $\mathcal{M}_\epsilon^-$ of system (2.1).

The $W_0^s\left(\mathcal{M}_0^+\right)$ $\left(resp. W_0^u\left(\mathcal{M}_0^+\right)\right)$ and $W_0^u\left(\mathcal{M}_0^-\right)$ $\left(resp. W_0^s\left(\mathcal{M}_0^-\right)\right)$ coincide with each other, forming a three-dimensional heteroclinic manifold in the four-dimensional phase space. Assuming that (2.4) holds, this heteroclinic manifold can be seen as the union of a two-parameter families of heteroclinic orbits, denoted as

$$p_0 = \sqrt{2}\tanh\left(\sqrt{1+\mu\nu_0}\varphi\right)\sqrt{-\frac{\sqrt{1+\mu\nu_0}}{\tanh(\sqrt{1+\mu\nu_0}\varphi)^2-3}}, \tag{2.6}$$

and

$$q_0 = p_0' = 3\sqrt{2}\,\mathrm{sech}\left(\sqrt{1+\mu\nu_0}\varphi\right)^2\left[-\frac{\sqrt{1+\mu\nu_0}}{-3+\tanh\left(\sqrt{1+\mu\nu_0}\varphi\right)^2}\right]^{\frac{3}{2}}. \tag{2.7}$$

Under small perturbations (i.e. $0<\epsilon\ll 1$), most of the points in $\mathcal{M}_\epsilon^+\cup\mathcal{M}_\epsilon^-$ are no longer saddles of system (2.1). Therefore, we choose

$$c_1 + c_2\mathcal{B} + c_3\mathcal{B}^2 = (\sqrt{1+\mu\nu})(\alpha_4+\alpha_5\nu), \tag{2.8}$$

such that system (2.1) has two saddles $S_1 = ((1+\mu\nu)^{\frac{1}{4}},0,0,0)\in\mathcal{M}_\epsilon^+$ and $S_2 = (-(1+\mu\nu)^{\frac{1}{4}},0,0,0)\in\mathcal{M}_\epsilon^-$, where v is a constant.

From (2.8), system (2.1) becomes

$$\begin{cases} p' = q, \\ q' = -p + p^5 - \mu p\nu - c\epsilon^2 q, \\ \nu' = \epsilon\tau, \\ \tau' = \epsilon\left[\alpha_2\nu\epsilon^2 - \epsilon\left(\alpha_3+\alpha_1 c\epsilon^4\right)\tau - \alpha_4 p^2 - \alpha_5\nu p^2 + (\sqrt{1+\mu\nu})(\alpha_4+\alpha_5\nu)\right]. \end{cases} \tag{2.9}$$

Finally, we present the main results of this article, which will be proved in the next section.

Theorem 2.1. *If $\mu\neq 0$, $\alpha_2>0$, $\alpha_i\in\mathbb{R}\,(i=3,4,5)$. For every $\epsilon>0$ sufficiently small, ν_0 satisfying (2.4), then there exists a unique*

$$\tilde{c} = \epsilon^2 c = \epsilon^2 c_0 + O(\epsilon^3),$$

such that system (2.9) has a unique solution $\mathcal{L}_h((p_h(\varphi), q_h(\varphi), \nu_h(\varphi), \tau_h(\varphi))$ that is heteroclinic to $S_1=(1,0,0,0)\in\mathcal{M}_\epsilon^+$ and $S_2=(-1,0,0,0)\in\mathcal{M}_\epsilon^-$ as $\varphi\to\pm\infty$, this solution satisfies

$$|p_h(\varphi;\nu_0)-p_0(\varphi;\nu_0)| = O(\epsilon), |q_h(\varphi;\nu_0)-q_0(\varphi;\nu_0)| = O(\epsilon),$$

where ν_0 is given below. The specific expressions for p_0 and q_0 are given in (2.6) and (2.7) respectively. The heteroclinic solutions to system (2.9) are corresponding to the fronts of system (1.6) with $(\mathcal{A}_h(\varphi),\mathcal{B}_h(\varphi)) = (p_h(\varphi), q_h(\varphi))$ and $\lim_{\varphi\to\pm\infty}(\mathcal{A}_h(\varphi),\mathcal{B}_h(\varphi)) = (\pm1,0)$, and, there is one case to consider:

$$c_0 = \frac{\alpha_3\nu_0\mu}{2(1+\mu\nu_0)}, \nu_0 = -\frac{2\sqrt{3}\alpha_4\,\mathrm{arctanh}\left(\frac{\sqrt{3}}{3}\right)}{2\sqrt{3}\alpha_5\,\mathrm{arctanh}\left(\frac{\sqrt{3}}{3}\right)+\sqrt{\alpha_3^2+4\alpha_2}}.$$

3. Existence of traveling fronts in system (2.9)

The flows on $\mathcal{M}_\epsilon^-$ and $\mathcal{M}_\epsilon^+$ for the full system (2.9) can be determined by substituting $\left(\pm(1+\mu\nu_0)^{\frac{1}{4}}, 0\right)$ into (p,q), then system (2.9) reduces

$$\begin{cases} \nu' = \epsilon\tau, \\ \tau' = \epsilon\left[\alpha_2\nu\epsilon^2 - \epsilon\left(\alpha_3 + \alpha_1 c\epsilon^4\right)\tau\right]. \end{cases} \tag{3.1}$$

By applying the results on [17, 18], if $\alpha_2 > 0$, we have two saddles $S_1 = (1,0,0,0) \in \mathcal{M}_\epsilon^+$ and $S_2 = (-1,0,0,0) \in \mathcal{M}_\epsilon^-$ of the system (2.9), as well as the two equations

$$l^u = \left\{(\nu,\tau) \mid \tau = \left(\frac{\left(-\alpha_3 + \sqrt{\alpha_3^2 + 4\alpha_2}\right)}{2}\epsilon + O\left(\epsilon^5\right)\right)\nu\right\},$$

and

$$l^s = \left\{(\nu,\tau) \mid \tau = \left(\frac{\left(-\alpha_3 - \sqrt{\alpha_3^2 + 4\alpha_2}\right)}{2}\epsilon + O\left(\epsilon^5\right)\right)\nu\right\},$$

representing the unstable and stable manifolds of the saddle S_1 on $\mathcal{M}_\epsilon^+$, respectively. It is worth noting that S_2 on the critical manifolds $\mathcal{M}_\epsilon^-$ has the same expressions.

3.1. Transversal intersection between $W^u\left(\mathcal{M}_\epsilon^-\right)$ and $W^s\left(\mathcal{M}_\epsilon^+\right)$

Currently, we plan to combine the fast isocline heteroclinic solution in (2.5) with partial curves $l^{u,s}$ representing the slow orbits in $\mathcal{M}_\epsilon^+$ and $\mathcal{M}_\epsilon^-$. In this way, if all the manifolds intersected transversally [25, 26], we can construct a global singularity structure that corresponds to a heteroclinic orbit in the full system (2.9).

The fast field with a Hamiltonian structure, as we are all aware, is associated with a Melnikov integral, whose simple zeroes correspond to transversal intersections of $W^u\left(\mathcal{M}_\epsilon^-\right)$ and $W^s\left(\mathcal{M}_\epsilon^+\right)$. Thus, we can ascertain whether $W^u\left(\mathcal{M}_\epsilon^-\right)$ and $W^s\left(\mathcal{M}_\epsilon^+\right)$ intersect by employing the Melnikov integral. Robinson et.al. [27] proposed that the transversal intersection between $W^u\left(\mathcal{M}_\epsilon^-\right)$ and $W^s\left(\mathcal{M}_\epsilon^+\right)$ can be determined by

$$\Delta\left(\nu_0,\tau_0\right) = \int_{-\infty}^{+\infty} c_0 q_0^2\left(\varphi;\nu_0\right)\epsilon + \mu\tau_0 p_0\left(\varphi;\nu_0\right) q_0\left(\varphi;\nu_0\right)\varphi d\varphi, \tag{3.2}$$

where the explicit expressions of $p_0\left(\varphi;\nu_0\right)$ and $q_0\left(\varphi;\nu_0\right)$ are given in (2.6) and (2.7), respectively. They are the heteroclinic orbits $\mathcal{L}_0$ of the layer system (2.5) . Hence, we substitute (2.6) and

(2.7) into (3.2) , and then we have

$$\begin{aligned}\Delta\left(\nu_0,\tau_0\right) &= \int_{-\infty}^{+\infty} c_0 q_0^2\left(\varphi;\nu_0\right)\epsilon+\mu\tau_0 p_0\left(\varphi;\nu_0\right) q_0\left(\varphi;\nu_0\right)\varphi d\varphi, \\ &= c_0\epsilon\int_{-\infty}^{+\infty} q_0^2\left(\varphi;\nu_0\right) d\varphi+\mu\tau_0\int_{-\infty}^{+\infty} p_0(\varphi;\nu_0) q_0\left(\varphi;\nu_0\right)\varphi d\varphi \\ &= -\frac{\sqrt{3}\epsilon c_0\left(1+u\nu_0\right)}{2}\ln(2-\sqrt{3})+\frac{\sqrt{3}\ln(2+\sqrt{3})}{2}u\tau_0,\end{aligned} \tag{3.3}$$

which permit two zeros,

$$\tau_0=-\frac{\epsilon c_0\left(1+\mu\nu_0\right)}{\mu},\mu\neq 0, \tag{3.4}$$

and

$$c_0=0,\mu=0. \tag{3.5}$$

For the case of (3.5), when $c_0 = 0$ this wave becomes stationary and it is impossible for system (1.6) to have traveling fronts. Therefore, we will only consider the case (3.4) for the rest of the article. From the analysis above, we can conclude that $W^u\left(\mathcal{M}_\epsilon^-\right)$ and $W^s\left(\mathcal{M}_\epsilon^+\right)$ intersect transversally. As a result, any heteroclinic orbit from $\mathcal{M}_\epsilon^+$ to $\mathcal{M}_\epsilon^-$ should satisfy $\tau_0=-\frac{\epsilon c_0(1+\mu\nu_0)}{\mu}$ at leading order.

3.2. Take-Off and Touch-Down curves

As $\varphi\to\pm\infty$, each heteroclinic orbit $\mathcal{L}_0$ within the heteroclinic manifold asymptotically approaches the points on $\mathcal{M}_0^+$ and $\mathcal{M}_0^-$, known as base points. Let ζ_0 represents the initial value of the flow $\mathcal{L}\left(\varphi,\zeta_0\right)$ of system (2.9), and let ζ_0^- and ζ_0^+ be the base points of the flows on $\mathcal{M}_\epsilon^-$ and $\mathcal{M}_\epsilon^+$. According to Doelman and Tu et.al. in [17, 18], for any orbit $\mathcal{L}\left(\varphi,\zeta_0\right)$ with $\zeta_0=\mathcal{L}\left(0,\zeta_0\right)\in W^u\left(\mathcal{M}_\epsilon^-\right)\cap W^s\left(\mathcal{M}_\epsilon^+\right)\cap\{q=0\}$, geometric singular perturbation theory states that there exist two orbits respectively, $\mathcal{L}^-=\mathcal{L}^-\left(\varphi,\zeta_0^-\right)\subset\mathcal{M}_\epsilon^-$ and $\mathcal{L}^+=\mathcal{L}^+\left(\varphi,\zeta_0^+\right)\subset\mathcal{M}_\epsilon^+$, such that $\left\|\mathcal{L}\left(\varphi,\zeta_0\right)-\mathcal{L}^+\left(\varphi,\zeta_0^+\right)\right\|$ is exponentially small in ϵ when $\varphi\geq O\left(\frac{1}{\epsilon}\right)$ and $\left\|\mathcal{L}\left(\varphi,\zeta_0\right)-\mathcal{L}^-\left(\varphi,\zeta_0^-\right)\right\|$ is exponentially small in ϵ when $-\varphi\geqslant O\left(\frac{1}{\epsilon}\right)$.

The take-off curve and the touch-down curve on $\mathcal{M}_\epsilon^\pm$ play a crucial role in determining whether any of the constructed family of orbits $\mathcal{L}(\varphi;\nu_0)$ is a heteroclinic orbit connecting the two saddle points $S_{1,2}$. This means that the heteroclinic orbit settling on $W^s\left(\mathcal{M}_\epsilon^+\right)\cap W^u\left(\mathcal{M}_\epsilon^-\right)$ can be connected respectively by the stable manifold of the saddle $S_1\in\mathcal{M}_\epsilon^+$ and the unstable manifold of the saddle $S_2\in\mathcal{M}_\epsilon^-$ transversally. By applying the results from [18, 28], we define the take-off curve $\mathcal{T}_o^-\subset\mathcal{M}_\epsilon^-$

$$\mathcal{T}_o^-=\bigcup_{\zeta_0}\left\{\zeta_0^-=\mathcal{L}^-\left(0,\zeta_0^-\right)\right\},$$

and the touch-down curve $\mathcal{T}_d^+ \subset \mathcal{M}_\epsilon^+$

$$\mathcal{T}_d^+ = \bigcup_{\zeta_0} \left\{ \zeta_0^+ = \mathcal{L}^+ \left(0, \zeta_0^+\right) \right\}.$$

The sets of $\mathcal{T}_o^-$ and $\mathcal{T}_d^+$ are determined by the accumulated change in ν and τ during half the jump through the fast field. The calculation of the curves $\mathcal{T}_{o,d}$ implicitly provides information about the possibility of a jump from $\zeta_0^- \in \mathcal{M}_\epsilon^-$ to $\zeta_0^+ \in \mathcal{M}_\epsilon^+$. The accumulated change in τ over a full jump through the fast field is represented by

$$\begin{aligned}
\Delta\tau &= \int_{k\log\epsilon}^{-k\log\epsilon} \epsilon \left[\alpha_2\epsilon^2\nu - \epsilon\left(\alpha_3 + \alpha_1\epsilon^4 c\right)\tau - \alpha_4 p^2 - \alpha_5\nu p^2 + (\sqrt{1+\mu\nu})(\alpha_4 + \alpha_5\nu)\right] d\varphi \\
&= -\epsilon \int_{k\log\epsilon}^{-k\log\epsilon} \alpha_4 p^2 + \alpha_5\nu p^2 - (\sqrt{1+\mu\nu})(\alpha_4+\alpha_5\nu) d\varphi + O\left(\epsilon^2|\log\epsilon|\right) \\
&= -\epsilon \int_{k\log\epsilon}^{-k\log\epsilon} (\alpha_4+\alpha_5\nu_0)\left(p_0^2 - (\sqrt{1+\mu\nu})\right) d\varphi + O\left(\epsilon^2|\log\epsilon|\right) \\
&= -\epsilon\left(\alpha_4+\alpha_5\nu_0\right)\int_{-\infty}^{\infty}\left(p_0^2 - (\sqrt{1+\mu\nu})\right) d\varphi + O\left(\epsilon^{1+2k}\right) + O\left(\epsilon^2|\log\epsilon|\right) \\
&= -\epsilon\left(\alpha_4+\alpha_5\nu_0\right)\int_{-\infty}^{\infty}\left(\frac{-3}{2+\cosh(2\sqrt{1+\mu\nu}\varphi)}\right)\sqrt{1+\mu\nu} d\varphi + O\left(\epsilon^2|\log\epsilon|\right) \\
&= 3\epsilon\left(\alpha_4+\alpha_5\nu_0\right)\sqrt{(1+\mu\nu_0)}\frac{1}{2\sqrt{1+\mu\nu}}\frac{4\sqrt{3}\operatorname{arctanh}(\frac{\sqrt{3}}{3})}{3} + O\left(\epsilon^2|\log\epsilon|\right) \\
&= 2\sqrt{3}\epsilon\left(\alpha_4+\alpha_5\nu_0\right)\operatorname{arctanh}(\frac{\sqrt{3}}{3}) + O\left(\epsilon^2|\log\epsilon|\right),
\end{aligned} \tag{3.6}$$

in the same way, we have

$$\begin{aligned}
\Delta\nu &= \int_{k\log\epsilon}^{-k\log\epsilon} \nu' \mid_{\mathcal{L}(\varphi,\zeta_0)} d\varphi \\
&= \epsilon\int_{k\log\epsilon}^{-k\log\epsilon} (\tau_0 + O(\epsilon)) d\varphi \\
&= -2k\epsilon\log\epsilon(\tau_0 + O(\epsilon)),
\end{aligned} \tag{3.7}$$

where k denotes a positive constant, $\mathcal{L}(\varphi, \zeta_0)$ can be approximated by the unperturbed heteroclinic orbit $(p_0(\varphi;\nu_0), q_0(\varphi;\nu_0), \nu_0, \tau_0) = \mathcal{L}_0(\varphi;\zeta_0)$ given by (2.6)-(2.7). So the take-off and touch-down curves are respectively

$$\mathcal{T}_o^- = \left\{\nu_0^-, \tau_0^- = (\nu_0, \tau_0) \mid \tau_0^- = \left(\tau_0 - \sqrt{3}\epsilon\left(\alpha_4+\alpha_5\nu_0\right)\operatorname{arctanh}(\frac{\sqrt{3}}{3})\right)\right\},$$

and

$$\mathcal{T}_d^+ = \left\{ \nu_0^+, \tau_0^+ = (\nu_0, \tau_0) \mid \tau_0^+ = \left(\tau_0 + \sqrt{3}\epsilon\,(\alpha_4 + \alpha_5\nu_0)\,\text{arctanh}(\frac{\sqrt{3}}{3}) \right) \right\},$$

with $\tau_0 = -\frac{\epsilon c_0(1+\mu\nu_0)}{\mu}$. The existence of heteroclinic orbits approaching to the saddles $S_1 = (1,0,0,0) \in \mathcal{M}_\epsilon^+$ and $S_2 = (-1,0,0,0) \in \mathcal{M}_\epsilon^-$ respectively as $\varphi \to \pm\infty$ requires that $\mathcal{T}_o^- \cap l^u$ and $\mathcal{T}_d^+ \cap l^s$ intersect transversally. In fact, the ν-coordinates of the base points ζ_0^- and ζ_0^+ of a heteroclinic orbit have to be equal at leading order since $\Delta\nu = O(\epsilon^2|\log\epsilon|)$ during an excursion through the fast field, see (3.7).

If the two equations

$$-\frac{\epsilon c_0\,(1+\mu\nu_0)}{\mu} - \sqrt{3}\epsilon\,(\alpha_4+\alpha_5\nu_0)\,\text{arctanh}\left(\frac{\sqrt{3}}{3}\right) = \frac{\epsilon\left(-\alpha_3 + \sqrt{\alpha_3^2+4\alpha_2}\right)}{2}\nu_0, \tag{3.8}$$

and

$$-\frac{\epsilon c_0\,(1+\mu\nu_0)}{\mu} + \sqrt{3}\epsilon\,(\alpha_4+\alpha_5\nu_0)\,\text{arctanh}\left(\frac{\sqrt{3}}{3}\right) = \frac{\epsilon\left(-\alpha_3 - \sqrt{\alpha_3^2+4\alpha_2}\right)}{2}\nu_0, \tag{3.9}$$

in which (ν_0, c_0) can be seen as two unknowns, have non-degenerate zeroes, then $\mathcal{T}_o^- \cap l^u$ and $\mathcal{T}_d^+ \cap l^s$ intersect transversally.

Based on the above analysis, we can conclude that a singular heteroclinic orbit connecting S_1 to S_2 is formed, consisting of one of the fast heteroclinic orbits (2.6)-(2.7) and the slow trajectories $l^{u,s} \subset \mathcal{M}_\epsilon^\pm$. This singular structure persists for $0 < \epsilon \ll 1$, as the slow manifolds $\mathcal{M}_0^\pm$ are normally hyperbolic and all the involved manifold intersect transversally. Therefore, according to geometric singular perturbation theory, the heteroclinic orbits $\mathcal{L}(\varphi, \zeta_0)$ can be controlled and will approach the two different saddles of system (2.9) as $\varphi \to \pm\infty$, resulting in the existence of traveling fronts in system (1.6), as illustrated in the figure 3.

Before proving Theorem 2.1, we first give the following result.

Lemma 3.1. *If $\mu \neq 0$, $\alpha_2 > 0$, $\alpha_i \in \mathbb{R}\,(i = 3,4,5)$, and (2.4) holds, then when $\epsilon > 0$ is small enough, it is possible that there is one type of solutions (ν_0, c_0) to equation (3.8)-(3.9), namely,*

$$c_0 = \frac{\alpha_3\nu_0\mu}{2\,(1+\mu\nu_0)}, \nu_0 = -\frac{2\sqrt{3}\alpha_4\,\text{arctanh}\left(\frac{\sqrt{3}}{3}\right)}{2\sqrt{3}\alpha_5\,\text{arctanh}\left(\frac{\sqrt{3}}{3}\right) + \sqrt{\alpha_3^2+4\alpha_2}}.$$

i.e. there is one kind of base point pairs ζ_0^- and ζ_0^+ that can give rise to a heteroclinic connection between S_1 and S_2. This means that the corresponding intersections of $\mathcal{T}_o^- \cap l^u$ and $\mathcal{T}_d^+ \cap l^s$ are transversal.

In the figure on the right, the green curve and red curve represent the singular heteroclinic orbit and the heteroclinic orbit of the entire system (2.9) generated by the singular heteroclinic orbit respectively.

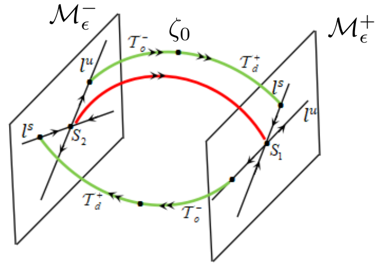

Figure 3: Schematic diagram of heteroclinic solutions for the system-wide (2.9).

Proof. The sum of equations (3.8) and (3.9) is

$$\frac{-2\epsilon c_0 (1 + \mu\nu_0)}{\mu} = -\epsilon\alpha_3\nu_0,$$

then we have $2c_0 + (2c_0 - \alpha_3)\,\mu\nu_0 = 0$, namely,

$$c_0 = \frac{\alpha_3\nu_0\mu}{2(1 + \mu\nu_0)}. \tag{3.10}$$

Plugging (3.10) into (3.8) or (3.9), we get

$$\nu_0 = -\frac{2\sqrt{3}\alpha_4 \operatorname{arctanh}\left(\frac{\sqrt{3}}{3}\right)}{2\sqrt{3}\alpha_5 \operatorname{arctanh}\left(\frac{\sqrt{3}}{3}\right) + \sqrt{\alpha_3^2 + 4\alpha_2}}.$$

Therefore, the proof of Lemma is complete. □

Now we turn to prove Theorem 2.1.

Proof of Theorem 2.1. Upon revisiting Section 3.1, 3.2 and Lemma 3.1, it becomes evident that when $0 < \epsilon \ll 1$, the three-dimensional stable manifold $W_\epsilon^s(\mathcal{M}_\epsilon^+)$ and unstable manifold $W_\epsilon^u(\mathcal{M}_\epsilon^-)$, take-off curve $\mathcal{T}_o^-$ and unstable manifold l^u of the saddle $S_2 = (-1, 0, 0, 0)$ intersect transversely as well as the touch-down curve $\mathcal{T}_d^+$ and stable manifold l^s of the saddle $S_1 = (1, 0, 0, 0)$. Otherwise the three-dimensional stable manifold $W_\epsilon^s(\mathcal{M}_\epsilon^+)$ and unstable manifold $W_\epsilon^u(\mathcal{M}_\epsilon^-)$, the take-off curve $\mathcal{T}_o^-$ and unstable manifold l^u of the saddle $S_1 = (1, 0, 0, 0)$ intersect transversely as well as the touch-down curve $\mathcal{T}_d^+$ and stable manifold l^s of the saddle $S_2 = (-1, 0, 0, 0)$. Therefore the conclusions of Theorem 2.1 are easily obtained. □

4. Conclusion

Using the Fenichel's geometric singular perturbation theory and Melnikov's methods, we investigate the traveling fronts of a quintic Ginzburg-Landau equation (1.6) with slow diffusion. Firstly, we transform system (1.6) into a four-dimensional ordinary differential equation (2.9) using a specific transformation, and employ geometric singular perturbation theory to carry out fast and slow separation to obtain the layer system and reduced system, as well as their dynamics. We then measure the transversal intersection of the stable and unstable manifolds of the slow manifolds $W^u(\mathcal{M}_\epsilon^-)$ and $W^s(\mathcal{M}_\epsilon^+)$ using the Melnikov function. We define the take-off curve $\mathcal{T}_o^-$ and touch-down curve $\mathcal{T}_d^+$ to intersect transversally with the stable and unstable manifolds of the two different saddle points $S_1 \in \mathcal{M}_\epsilon^+$ and $S_2 \in \mathcal{M}_\epsilon^-$, respectively. This allows us to obtain the existence of the heteroclinic orbit located at two different saddle points of the system near the singular heteroclinic orbit. Furthermore, we establish the existence of the traveling fronts solutions of the quintic Ginzburg-Landau equation (1.6). Finally, under certain parameter conditions, we demonstrate that the coupled quintic Ginzburg-Landau system (1.6) may possess one or two heteroclinic solutions.

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Appendix

Appendix A

The derivation process for the explicit expressions of equations (2.6) and (2.7) is as follows: The first integral of equation (2.5) is

$$H(p,q,\nu_0,\tau_0)=\frac{q^2}{2}+\frac{1}{2}(1+\mu\nu_0)p^2-\frac{1}{6}p^6. \tag{5.1}$$

By performing a simple calculation, we determine that equation (2.5) has two hyperbolic saddle points, denoted as $(p = \pm(1+\mu\nu_0)^{\frac{1}{4}}, q = 0)$. Substituting $(p = \pm(1+\mu\nu_0)^{\frac{1}{4}}, q = 0)$ into equation (5.1) yields

$$H(p, q, \nu_0, \tau_0) = \frac{1}{3}(1+\mu\nu_0)^{\frac{3}{2}},$$

then we have

$$\frac{1}{2}q^2 + \frac{1}{2}(1+\mu\nu_0)p^2 - \frac{1}{6}p^6 = \frac{1}{3}(1+\mu\nu_0)^{\frac{3}{2}},$$

therefore

$$q = \pm\sqrt{\frac{1}{3}\left(2(1+\mu\nu_0)^{\frac{3}{2}} - 3(1+\mu\nu_0)p^2 + p^6\right)}.$$

i.e.

$$\begin{aligned}\frac{dp}{d\varphi} &= \pm\sqrt{\frac{1}{3}\left(2(1+\mu\nu_0)^{\frac{3}{2}} - 3(1+\mu\nu_0)p^2 + p^6\right)}\\ &= \pm\sqrt{\frac{1}{3}\left(\left(p-(1+\mu\nu_0)^{\frac{1}{4}}\right)^2\left(p+(1+\mu\nu_0)^{\frac{1}{4}}\right)^2\left(p^2+2(1+\mu\nu_0)^{\frac{1}{2}}\right)\right)},\end{aligned}$$

then we have

$$\pm\sqrt{3}\frac{dp}{(p-(1+\mu\nu_0)^{\frac{1}{4}})(p+(1+\mu\nu_0)^{\frac{1}{4}})\sqrt{p^2+2(1+\mu\nu_0)^{\frac{1}{2}}}} = d\varphi,$$

by integrating both sides and using symmetry of the phase portrait of system (5.1) with respect to p and q, we have

$$\sqrt{3}\int -\frac{dp}{(p-(1+\mu\nu_0)^{\frac{1}{4}})(p+(1+\mu\nu_0)^{\frac{1}{4}})\sqrt{p^2+2(1+\mu\nu_0)^{\frac{1}{2}}}} = \varphi + c. \tag{5.2}$$

By substituting the initial point $(0, 0)$ into the above equation, we get $c = 0$. Taking the auxiliary calculation of Mathematica and referencing the integral table, we obtain

$$\text{ArcTanh}\left[\frac{\sqrt{3}p}{\sqrt{p^2+2\sqrt{1+\mu\nu_0}}}\right] = \sqrt{1+\mu\nu_0}\varphi,$$

thus, we obtain equation (2.6). The derivative of variable φ in equation (2.6) gives equation (2.7).

Appendix B

Here are the calculation details for the Melnikov integral (3.3).

According to equation (2.6) and equation (2.7), we have

$$p_0 \cdot q_0 = \frac{6\,\mathrm{sech}(\sqrt{1+\mu\nu_0}\varphi)^2(1+\mu\nu_0)\tanh(\sqrt{1+\mu\nu_0}\varphi)}{(-3+\tanh(\sqrt{1+\mu\nu_0}\varphi)^2)^2},$$

and

$$q_0^2 = -\frac{18\,\mathrm{sech}(\sqrt{1+\mu\nu_0}\varphi)^4(1+\mu\nu_0)^{\frac{3}{2}}}{(-3+\tanh(\sqrt{1+\mu\nu_0}\varphi)^2)^3}.$$

Substituting them into equation (3.2), we get

$$\begin{aligned}
&\int_{-\infty}^{+\infty} c_0 q_0^2\left(\varphi;\nu_0\right)\epsilon + \mu\tau_0 p_0\left(\varphi;\nu_0\right) q_0\left(\varphi;\nu_0\right)\varphi d\varphi \\
=&c_0\epsilon\int_{-\infty}^{+\infty} q_0^2\left(\varphi;\nu_0\right)d\varphi + \mu\tau_0\int_{-\infty}^{+\infty} p_0(\varphi;\nu_0)q_0\left(\varphi;\nu_0\right)\varphi d\varphi \\
=&c_0\epsilon\int_{-\infty}^{+\infty} -\frac{18\,\mathrm{sech}(\sqrt{1+\mu\nu_0}\varphi)^4(1+\mu\nu_0)^{\frac{3}{2}}}{(-3+\tanh(\sqrt{1+\mu\nu_0}\varphi)^2)^3}d\varphi \\
+&\mu\tau_0\int_{-\infty}^{+\infty}\frac{6\,\mathrm{sech}(\sqrt{1+\mu\nu_0}\varphi)^2(1+\mu\nu_0)\tanh(\sqrt{1+\mu\nu_0}\varphi)}{(-3+\tanh(\sqrt{1+\mu\nu_0}\varphi)^2)^2}d\varphi \\
=&-18(1+\mu\nu_0)c_0\epsilon\int_{-\infty}^{+\infty}\frac{\mathrm{sech}(\sqrt{1+\mu\nu_0}\varphi)^4}{(-3+\tanh(\sqrt{1+\mu\nu_0}\varphi)^2)^3}d(\sqrt{1+\mu\nu_0}\varphi) \\
+&6\mu\tau_0\int_{-\infty}^{+\infty}\frac{\mathrm{sech}(\sqrt{1+\mu\nu_0}\varphi)^2\tanh(\sqrt{1+\mu\nu_0}\varphi)(\sqrt{1+\mu\nu_0}\varphi)}{(-3+\tanh(\sqrt{1+\mu\nu_0}\varphi)^2)^2}d(\sqrt{1+\mu\nu_0}\varphi),
\end{aligned} \tag{5.3}$$

taking the auxiliary calculation of Mathematica and referencing the integral table, we obtain

$$\int_{-\infty}^{+\infty}\frac{\mathrm{sech}\,t^4}{\left(-2-\mathrm{sech}\,t^2\right)^3}dt = \frac{\sqrt{3}\ln(2-\sqrt{3})}{36},$$

and

$$\int_{-\infty}^{+\infty}\frac{\mathrm{sech}\,t^2\tanh t\cdot t}{\left(-3+\tan t^2\right)^2} = \frac{\sqrt{3}\ln(2+\sqrt{3})}{12},$$

substituting the above two equations into equation (5.3), we obtain equation (3.3).