UNICITY OF MEROMORPHIC FUNCTIONS CONCERNING DERIVATIVES-DIFFERENCES AND SMALL FUNCTIONS

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ABSTRACT. In this paper, we mainly prove: Let f be a transcendental entire function of finite order with a Borel exceptional entire small function a, and let η be a nonzero finite complex number such that $\Delta_\eta^{n+1} f \not\equiv 0$. If $\Delta_\eta^{n+1} f$ and $\Delta_\eta^n f$ share b CM, where b is a small function of f, then $f(z) = a(z) + Be^{Az}$, where A and B are two nonzero constants and $a(z)$ is a polynomial with $\deg a \leq n-1$. This improves the results due to Chen and Zhang [Ann. Math. Ser.A (Chinese version) 2021] and Liu and Chen [J. Korean Soc. Math. Educ. Ser. B: Pure Apple. Math. 2023]. Meanwhile, we give negative answer to the problems posed by Chen and Xu [Comput. Methods Funct. Theory, 2022], Banerjee and Maity[Bull. Korean Math. Soc., 2021].

1. INTRODUCTION AND MAIN RESULTS

In this paper, we assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory, see [11,23,24]. In the following, a meromorphic function always means meromorphic in the whole complex plane.

By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possible outside of an exceptional set E with finite logarithmic measure $\int_E dr/r < \infty$. A meromorphic function a is said to be a small function of f if it satisfies $T(r, a) = S(r, f)$.

Let f be a nonconstant meromorphic function. The order and the hyper-order of f are defined by

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ T(r,f)}{\log r}, \quad \rho_2(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ \log^+ T(r,f)}{\log r}.$$

If $\rho(f) < \infty$, then the function f is called meromorphic function of finite order.

Let η be a nonzero complex number, and the difference operator is defined as

$$\Delta_\eta f = f(z+\eta) - f(z) \quad \text{and} \quad \Delta_\eta^n f = \Delta_\eta^{n-1}(\Delta_\eta f),$$

where $n(\geq 2)$ is a positive integer.

Let f be a transcendental meromorphic function, and let a be a small function of f. The deficiency of a small function a with respect to f is defined by

$$\delta(a, f) = \underline{\lim_{r\to\infty}} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r,f)} = 1 - \overline{\lim_{r\to\infty}} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)}.$$

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It is easy to see $0 \le \delta(a,f) \le 1$. If $\delta(a,f) > 0$, then a is called a deficient function of f, and if a is a constant, then a is called a deficient value. And we define

$$\lambda(f-a) = \overline{\lim_{r\to\infty}} \frac{\log^+ N\left(r, \frac{1}{f-a}\right)}{\log r}.$$

If $\lambda(f-a) < \rho(f)$ for $\rho(f) > 0$ and $N\left(r, \frac{1}{f-a}\right) = O(\log r)$ for $\rho(f) = 0$, then a is called a Borel exceptional small function of f. If a is a constant, then a is called a Borel exceptional value of f.

Let f and g be two meromorphic functions, and let a either be a small function of both f and g or be a constant. We say that f and g share a CM(IM) if $f-a$ and $g-a$ have the same zeros counting multiplicities(ignoring multiplicities). $N(r,a)$ is a counting function of zeros of both $f-a$ and $g-a$ with the same multiplicity and the multiplicity is counted. If

$$N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{g-a}\right) - 2N(r,a) \le S(r,f) + S(r,g),$$

then we call that f and g share a CM almost. Set $E(a,f) = \{z|f-a=0\}$, where a zero with multiplicity m is counted m times in the set.

Let k be a positive integer, we denote by $N_{k)}\left(r, \frac{1}{f-1}\right)$ the counting function for 1-points of f with multiplicity $\le k$, where multiplicity is counted, and by $\overline{N}_{k)}\left(r, \frac{1}{f-1}\right)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}\left(r, \frac{1}{f-1}\right)$ be the counting function for 1-points of f with multiplicity $\ge k$, where multiplicity is counted, and by $\overline{N}_{(k}\left(r, \frac{1}{f-1}\right)$ the corresponding one for which multiplicity is not counted. Let $E_{k)}(1,f)$ denotes the set of those 1-points of f with multiplicity $\le k$, where a 1-point with multiplicity $m(\le k)$ is counted m times in the set.

Recently many papers studied the uniqueness of transcendental entire function and their higer order difference operators sharing small function, and have get many interesting results, see [14, 17, 18, 20, 21]

In 1926, Nevanlinna [24] proved the following famous five-value theorem.

Theorem A. Let f and g be two nonconstant meromorphic functions, and let a_i $(i = 1,2,3,4,5)$ be five distinct values in the extended complex plane. If f and g share a_i $(i = 1,2,3,4,5)$ IM, then $f \equiv g$.

In 2000, Li and Qiao [16] improved Theorem A as follows.

Theorem B. Let f and g be two nonconstant meromorphic functions, and let a_i $(i = 1,2,3,4,5)$ be five distinct small functions of both f and g. If f and g share a_i $(i = 1,2,3,4,5)$ IM, then $f \equiv g$.

In 2014, Chen and Li [2] proved

Theorem C. Let f be a nonconstant entire function of finite order, let η be a positive integer, and let a be a periodic entire small function of f whose period is η. If $f, \Delta_\eta f, \Delta_\eta^n f$ $(n \ge 2)$ share a CM, then $\Delta_\eta^n f \equiv \Delta_\eta f$.

In 2021, Chen and Zhang [4] proved

Theorem D. Let f be a transcendental entire function of finite order with a Borel exceptional entire small function a satisfying $\rho(a) < 1$, and let η be a nonzero complex number such that $\Delta_\eta^2 f \not\equiv 0$. If $\Delta_\eta^2 f$ and $\Delta_\eta f$ share $\Delta_\eta a$ CM, where $\Delta_\eta a$

is a small function of $\Delta_\eta^2 f$, then

$$f(z) = a(z) + Be^{Az},$$

where A and B are two nonzero constants and $a(z)$ reduces to a constant.

In 2023, Liu and Chen [15] excended Theorem D as follows.

Theorem E. Let f be a transcendental entire function of finite order with a Borel exceptional entire small function a satisfying $\rho(a) < 1$, let n be a positive integer, and let η be a nonzero complex number such that $\Delta_\eta^{n+1} f \not\equiv 0$. If $\Delta_\eta^{n+1} f$ and $\Delta_\eta^n f$ share $\Delta_\eta^n a$ CM, where $\Delta_\eta^n a$ is a small function of $\Delta_\eta^{n+1} f$, then

$$f(z) = a(z) + Be^{Az},$$

where A and B are two nonzero constants and $a(z)$ reduces to a constant.

By Theorems A–E, we natural pose the following problem.

Problem 1. Whether "$\rho(a) < 1$" can be deleted or not in Theorems D and E?

In this paper, we give a positive answer to Problem 1 and prove the following result.

Theorem 1. Let f be a transcendental entire function of finite order with a Borel exceptional entire small function a, let n be a positive integer, and let η be a nonzero finite complex number such that $\Delta_\eta^{n+1} f \not\equiv 0$. If $\Delta_\eta^{n+1} f$ and $\Delta_\eta^n f$ share b CM, where b is a small function of f, then

$$f(z) = a(z) + Be^{Az},$$

where A and B are two nonzero constants and $a(z)$ is a polynomial with $\deg a \leq n - 1$.

Remark 1. In Theorem E and Theorem 1, "$a(z)$ reduces to a constant" is not valid.

Example 1. Let $f = a(z) + Be^{Az}$, where $a(z) = z^{n-1}$ and A, B are nonzero finite complex numbers satisfying $e^{A\eta} = 2$, and let $b = 0$. Obviously, $\Delta_\eta^{n+1} f(z) = B(e^{A\eta} - 1)^{n+1} e^{Az} = B(e^{A\eta} - 1)^n e^{Az} = \Delta_\eta^n f(z)$. Hence $\Delta_\eta^n f(z)$ and $\Delta_\eta^{n+1} f(z)$ share b CM, but $a(z)$ is not a constant.

In 2011, Heittokangas et al. [9] started to consider the uniqueness of meromorphic function with its shifts and proved

Theorem F. Let f be a nonconstant entire function of finite order, and let η be a nonzero finite complex number. If $f(z)$ and $f(z+\eta)$ share two distinct finite values a, b IM, then $f(z) \equiv f(z+\eta)$.

In 2020, Qi et al. [19] proved

Theorem G. Let f be a nonconstant meromorphic function of finite order, and let a, η be two nonzero finite complex numbers. If $f'(z)$ and $f(z+\eta)$ share a CM, and $E\left(0, f(z+\eta)\right) \subset E\left(0, f'(z)\right)$, $E\left(\infty, f'(z)\right) \subset E\left(\infty, f(z+\eta)\right)$, then $f'(z) \equiv f(z+\eta)$.

In 2022, Chen and Xu [5] proved

Theorem H. Let f be a nonconstant meromorphic function with $\rho_2(f) < 1$, let η be a nonzero finite complex number, and let k be a positive integer. If $f^{(k)}(z)$ and $f(z+\eta)$ share $0, \infty$ CM and 1 IM, then $f^{(k)}(z) \equiv f(z+\eta)$.

Chen and Xu [5] posed the following problem.

Problem 2. Let f be a nonconstant meromorphic function with $\rho_2(f) < 1$, and let η be a nonzero finite complex number. If $f^{(k)}$ and $f(z+\eta)$ share $0, \infty$ CM and $E_{1)}\left(1, f^{(k)}(z)\right) = E_{1)}\left(1, f(z+\eta)\right)$, then $f^{(k)}(z) \equiv f(z+\eta)$?

In this paper, we give a negative answer to Problem 2.

Example 2. Let $f(z)=\sin z$, $\eta=\pi$, $k=4$. Obviously $\rho(f)=1$. By a simple calculation, we know that $f^{(4)}(z)=\sin z$ and $f(z+\eta)=-\sin z$. In this case, we have $f^{(4)}(z)$ and $f(z+\eta)$ share $0,\infty$ CM, and $E_{1)}(1,f^{(4)}(z))=E_{1)}(1,f(z+\eta))=\varnothing$, but $f^{(4)}(z)\not\equiv f(z+\eta)$.

In addition, we further studied this problem and have proved

Theorem 2. Let f be a nonconstant meromorphic function with $\rho_2(f)<1$, let η be a nonzero finite complex number, and let k be a positive integer. If $E\left(0,f(z+\eta)\right)\subset E\left(0,f^{(k)}(z)\right)$, $E\left(\infty,f^{(k)}(z)\right)\subset E\left(\infty,f(z+\eta)\right)$, $E_{2)}(1,f^{(k)}(z))=E_{2)}\left(1,f(z+\eta)\right)$, then $f^{(k)}(z)\equiv f(z+\eta)$.

In the following,

$$L_\eta f(z)=\sum_{j=0}^{k} b_j f(z+j\eta),\quad L_\eta^b f(z)=\sum_{j=0}^{k} b_j f(z+j\eta),$$

where $b_j\in\mathbb{C}$, $b_k\neq 0$ and $b=\sum_{j=0}^{k} b_j$.

In 2021, Banerjee and Maity [1] proved the following results.

Theorem I. Let f be a nonconstant meromorphic function with $\rho_2(f)<1$, let η be a nonzero complex number, and let a be a small periodic function of f whose period is η. If $L_\eta^0 f\not\equiv 0$, and $E(0,f)\subset E(0,L_\eta^0 f)$, $E(a,f)\subset E(a,L_\eta^0 f)$, $E(\infty,L_\eta^0 f)\subset E(\infty,f)$, then $L_\eta^0 f\equiv f$.

Theorem J. Let f be a nonconstant meromorphic function of finite order, and let η, b, a_1, a_2 be nonzero complex numbers with $a_1\neq a_2$. If $L_\eta^0 f\not\equiv 0$, and $L_\eta^0 f, f$ share a_1,a_2,∞ CM, then $L_\eta^0 f\equiv f$.

Theorem K. Let f be a nonconstant meromorphic function with $\rho_2(f)<1$, let η be a nonzero complex number, and let a_1,a_2 be two distinct periodic small functions of f whose period are η. If $L_\eta^1 f\not\equiv 0$, and $E(a_1,f)\subset E(a_1,L_\eta^1 f)$, $E(a_2,f)\subset E(a_2,L_\eta^1 f)$, $E(\infty,L_\eta^1 f)\subset E(\infty,f)$, then $L_\eta^1 f\equiv f$.

Banerjee and Maity [1] posed the following problem.

Problem 3. Are Theorems I–K valid or not for $L_\eta^b f$ where $b\neq 0,1$ or $L_\eta f$?

In this paper, we give a negative answer to Problem 3.

Example 3. Let $f(z)=\frac{e^{2z}+1}{e^{2z}-1}$, and let $L_\eta f(z)=f(z)+f(z+\eta)-f(z+2\eta)-f(z+3\eta)-f(z+4\eta)=-\frac{e^{2z}+1}{e^{2z}-1}$, where $\eta=\pi i$. Obviously, $f(z)\neq\pm1$, $L_\eta f(z)\neq\pm1$. Hence, $f(z)$ and $L_\eta f(z)$ share $1,-1,\infty$ CM, but $f(z)\not\equiv L_\eta f(z)$.

2. Some Lemmas

In order to prove our results, we need the following lemmas.

Lemma 1. [12] Let f be a nonconstant entire function of finite order. If a is a Borel exceptional entire function of f, then $\delta(a,f)=1$.

Lemma 2. [11] Let f be a nonconstant meromorphic function, and let k be a positive integer. Then

$$m\left(r,\frac{f^{(k)}}{f}\right)=S(r,f).$$

Lemma 3. [6,10] Let f be a nonconstant meromorphic function with $\rho_2(f)<1$, and let η be a nonzero finite complex number. Then

$$m\left(r,\frac{f(z+\eta)}{f(z)}\right)=S(r,f),\quad m\left(r,\frac{f(z)}{f(z+\eta)}\right)=S(r,f).$$

Especially, if $\rho(f) < +\infty$, then for any $\varepsilon > 0$, we have

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = O\left(r^{\rho(f)-1+\varepsilon}\right).$$

Lemma 4. [6,10] Let f be a nonconstant meromorphic function with $\rho_2(f) < 1$, and let η be a nonzero finite complex number. Then

$$N\left(r, f(z+\eta)\right) = N(r, f(z)) + S(r,f),$$
$$\overline{N}\left(r, \frac{1}{f(z+\eta)}\right) = \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r,f).$$

Lemma 5. [13] Let η be a nonzero finite complex number, let n be a positive integer, and let f be a transcendental meromorphic function of finite order satisfying $\delta(a,f) = 1, \delta(\infty,f) = 1$, where a is a small function of f. If $\Delta_\eta^n f \not\equiv 0$, then

(1) $T(r, \Delta_\eta^n f) = T(r,f) + S(r,f)$,

(2) $\delta\left(\Delta_\eta^n a, \Delta_\eta^n f\right) = \delta\left(\infty, \Delta_\eta^n f\right) = 1$.

Lemma 6. [11] Let f be a nonconstant meromorphic function, and let a, b be two distinct small functions of f. Then

$$T(r,f) \leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + S(r,f).$$

Lemma 7. [23] Let f be a meromorphic function. If $f \neq 0, \infty$, then there exists an entire function α such that $f(z) = e^{\alpha(z)}$.

Lemma 8. [3] Let a be a finite complex number, let f be a transcendental meromorphic function of finite order with two Borel exceptional values a, ∞, and let η be a nonzero finite complex number such that $\Delta_\eta f \not\equiv 0$. If f and $\Delta_\eta f$ share a, ∞ CM, then $a = 0, f(z) = e^{Az+B}$, where $A(\neq 0), B$ are two finite constants.

Lemma 9. [22, 23] Let $n \geq 3$ be a positive integer, let $f_j (j = 1, \cdots, n)$ be meromorphic functions which are not constants except for f_n, and let $\sum_{j=1}^n f_j \equiv 1$. If $f_n \not\equiv 0$, and

$$\sum_{j=1}^{n} N\left(r, \frac{1}{f_j}\right) + (n-1)\sum_{j=1}^{n} \overline{N}(r, f_j)\,(\lambda + o(1))\,T(r, f_k),$$

where I is a set of $r \in (0, \infty)$ with infinite linear measure, $r \in I, k = 1, 2, \cdots, n-1, \lambda < 1$, then $f_n \equiv 1$.

Lemma 10. [8, 23] Let f and g be two nonconstant meromorphic functions satisfying

$$\delta(0,f) = \delta(\infty,f) = 1, \quad \delta(0,g) = \delta(\infty,g) = 1.$$

If f and g share 1 CM almost, then either $f \equiv g$ or $fg \equiv 1$.

Lemma 11. [13] Let f be a meromorphic function of finite order, and let η, c, d be three nonzero finite complex numbers. If $f(z+\eta) = cf(z)$, then either $T(r,f) \geq dr$ for sufficiently large r or f is a constant.

Lemma 12. [7] Let f be a meromorphic function with $\rho(f) < 1$, and let η be a nonzero finite complex number. Then for each given $\varepsilon > 0$, and a positive integer n, there exists a set $E \subset (1, \infty)$ that depends on f, and it has finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \bigcup [0,1]$, we have

$$\left|\frac{\Delta_\eta^n f(z)}{f(z)}\right| \leq |z|^{n(\rho(f)-1)+\varepsilon}.$$

Lemma 13. Let α be an entire function with $\rho(\alpha) \leq 1$, let n be a positive integer, and let η, d be two nonzero finite complex numbers. If $\Delta_\eta^n \alpha \equiv 0$, then either $T(r, \alpha) \geq dr$ for sufficiently large r or α is a polynomial with $\deg \alpha \leq n-1$.

Proof. We prove the lemma by mathematical induction. In the following, d denote a positive number, not necessarily the same at each occurrence. For $n = 1$ we have

$$\alpha(z+\eta) = \alpha(z). \tag{2.1}$$

By Lemma 11 and (2.1) we know that Lemma 13 is valid for $n = 1$.

Suppose that for $n = k-1$ the lemma is valid. Next we consider the case $n = k$. From $\Delta_\eta^k \alpha \equiv 0$ and above discussion we deduce that either $T\left(r, \Delta_\eta^{k-1}\alpha\right) \geq dr$ for sufficiently large r or $\Delta_\eta^{k-1}\alpha$ is a constant.

If $T\left(r, \Delta_\eta^{k-1}\alpha\right) \geq dr$ for sufficiently large r, then by $\rho(\alpha) \leq 1$, Lemma 3 (setting $\varepsilon = \frac{1}{2}$) and for sufficiently large r, we obtain

$$\begin{aligned} T\left(r, \Delta_\eta^{k-1}\alpha\right) &= m\left(r, \Delta_\eta^{k-1}\alpha\right) \leq m(r,\alpha) + m\left(r, \frac{\Delta_\eta^{k-1}\alpha}{\alpha}\right) + O(1) \\ &\leq T(r,\alpha) + Mr^{\frac{1}{2}} \leq T(r,\alpha) + \frac{1}{2}dr, \end{aligned} \tag{2.2}$$

where M is a positive number. Since $T\left(r, \Delta_\eta^{k-1}\alpha\right) \geq dr$, then by (2.2) we have $T(r,\alpha) \geq d_0 r$, where $d_0 = \frac{d}{2}$.

If $\Delta_\eta^{k-1}\alpha \equiv C$, where C is a constant, then $p(z) = \frac{C}{(k-1)!\eta} z^{k-1}$ is a solution of $\Delta_\eta^{k-1}\alpha \equiv C$. Let $\beta(z)$ be any solution of $\Delta_\eta^{k-1}\alpha \equiv 0$. Then we know that either $T(r,\beta) \geq dr$ for sufficiently large r or β is a polynomial with $\deg \beta \leq k-2$. From above argument we have either $T(r, \beta+p) \geq T(r,\beta) - T(r,p) \geq \frac{d}{2}r$ or $\beta + p$ is a polynomial with $\deg(\beta+p) \leq k-1$. It follows that either $T(r,\alpha) \geq dr$ for sufficiently large r or α is a polynomial with $\deg \alpha \leq k-1$.

Thus Lemma 13 is proved.

Lemma 14. [7] Let f be a meromorphic function of finite order, and let η be a nonzero finite complex number. Then for each positive integer k, $\rho\left(\Delta_\eta^k f\right) \leq \rho(f)$.

Lemma 15. [24] Let f be a meromorphic function. Then $\rho(f) = \rho(f')$.

3. Proof of Theorem 1

First, we claim $\rho(f) > 0$. Suppose on the contrary that $\rho(f) = 0$. Set $F(z) = f(z) - a(z)$. Since a is a Borel exceptional entire small function of f, we obtain

$$N\left(r, \frac{1}{F}\right) = N\left(r, \frac{1}{f-a}\right) = O(\log r).$$

Hence F has finitely many zeros. We assume that $a_1, a_2, \cdots, a_n$ are all zeros of F, where n is a positive integer.

From $\rho(f) = 0$, we deduce $\frac{F}{(z-a_1)(z-a_2)\cdots(z-a_n)} = e^h$, where h is a constant. Then we have $F(z) = c(z-a_1)(z-a_2)\cdots(z-a_n)$, where $c = e^h$. It follows that

$$T(r,F) = n\log r + O(1). \tag{3.1}$$

By (3.1) we deduce that f is a nonzero polynomial. Since b is a small function of f, then we know that b is a constant, which contradicts with $\Delta_\eta^{n+1} f$ and $\Delta_\eta^n f$ share b CM. Hence $\rho(f) > 0$.

Since a is a Borel exceptional entire small function of f, then by Lemma 1, we obtain $\delta(a, f) = 1$. Obviously, $\delta(\infty, f) = 1$. It follows from Lemma 5 that

$$\delta(\Delta_\eta^n a, \Delta_\eta^n f) = 1, \quad \delta(\Delta_\eta^{n+1} a, \Delta_\eta^{n+1} f) = 1, \tag{3.2}$$

$$\delta(\infty, \Delta_\eta^n f) = 1, \quad \delta(\infty, \Delta_\eta^{n+1} f) = 1. \tag{3.3}$$

Now, we consider three cases

Case 1. $b \equiv \Delta_\eta^{n+1} a$.

Case 1.1. $\Delta_\eta^{n+1} a \not\equiv \Delta_\eta^n a$.

Since $\Delta_\eta^n f$ and $\Delta_\eta^{n+1} f$ share b CM, then by (3.2), (3.3), Lemma 5 and Lemma 6 we have

$$\begin{aligned} T(r, f) &= T(r, \Delta_\eta^n f) + S(r, f) \\ &\le \overline{N}(r, \Delta_\eta^n f) + \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - \Delta_\eta^n a}\right) + \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - b}\right) + S(r, f) \\ &\le \overline{N}\left(r, \frac{1}{\Delta_\eta^{n+1} f - b}\right) + S(r, f) \le S(r, f), \end{aligned}$$

a contradiction.

Case 1.2. $\Delta_\eta^{n+1} a \equiv \Delta_\eta^n a$.

Set

$$G = \Delta_\eta^n f - \Delta_\eta^n a. \tag{3.4}$$

Then we have

$$\Delta_\eta G = \Delta_\eta^{n+1} f - \Delta_\eta^n a.$$

Since $\Delta_\eta^n f$ and $\Delta_\eta^{n+1} f$ share $b (\equiv \Delta_\eta^n a)$ CM, we obtain that G and $\Delta_\eta G$ share $0, \infty$ CM.

It follows from (3.2) and (3.3) that

$$\delta(0, G) = 1, \quad \delta(0, \Delta_\eta G) = 1, \tag{3.5}$$

$$\delta(\infty, G) = 1, \quad \delta(\infty, \Delta_\eta G) = 1. \tag{3.6}$$

By $\delta(a, f) = 1$, $\delta(\infty, f) = 1$ and Lemma 5, we obtain

$$T(r, G) = T(r, f) + S(r, f). \tag{3.7}$$

Since a is a Borel exceptional function of f, then by $\rho(f) > 0$ we have

$$\overline{\lim_{r\to\infty}} \frac{\log^+ N\left(r, \frac{1}{f-a}\right)}{\log r} < \rho(f). \tag{3.8}$$

By Lemma 3 and Nevanlinna's first fundamental theorem we have

$$m\left(r, \frac{1}{f-a}\right) \le m\left(r, \frac{1}{\Delta_\eta^n (f-a)}\right) + m\left(r, \frac{\Delta_\eta^n (f-a)}{f-a}\right) + S(r, f),$$

$$T(r, f-a) - N\left(r, \frac{1}{f-a}\right) \le T\left(r, \Delta_\eta^n (f-a)\right) - N\left(r, \frac{1}{\Delta_\eta^n (f-a)}\right) + S(r, f).$$

Hence, by Lemma 5 we have

$$N\left(r, \frac{1}{\Delta_\eta^n (f-a)}\right) \le N\left(r, \frac{1}{f-a}\right) + S(r,f). \tag{3.9}$$

By Lemma 3 (setting $\varepsilon = \frac{1}{2}$), we obtain

$$S(r,f) \le M r^{\rho(f)-\frac{1}{2}}, \tag{3.10}$$

where M is a positive number.

It follows from (3.8) that

$$N\left(r, \frac{1}{f-a}\right) \le r^{\frac{\rho(f)+\lambda(f-a)}{2}}. \tag{3.11}$$

By (3.10) and (3.11) we have

$$N\left(r, \frac{1}{f-a}\right) + S(r,f) \le (1+M) r^{M_1}, \tag{3.12}$$

where $M_1 = \max\left\{\rho(f) - \frac{1}{2}, \frac{\rho(f)+\lambda(f-a)}{2}\right\}$.

It follows from (3.8), (3.9) and (3.12) that

$$\frac{\log^+ N\left(r, \frac{1}{\Delta_\eta^n (f-a)}\right)}{\log r} \le \frac{\log (1+M) r^{M_1}}{\log r} \le M_1 + \frac{\log(1+M)}{\log r}.$$

Then we have

$$\overline{\lim_{r\to\infty}} \frac{\log^+ N\left(r, \frac{1}{G}\right)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log^+ N\left(r, \frac{1}{\Delta_\eta^n (f-a)}\right)}{\log r} \le M_1 < \rho(f). \tag{3.13}$$

By (3.7) and (3.13) we deduce that 0 is a Borel exceptional value of G. It follows from Lemma 8 that $G = e^{A_1 z + B_1}$, where $A_1 (\neq 0), B_1$ are two constants.

From (3.4) we get

$$\Delta_\eta^n \left(f(z) - a(z)\right) = e^{A_1 z + B_1}. \tag{3.14}$$

By Hadamard's factorization theorem, we obtain

$$f(z) - a(z) = \beta(z) e^{p(z)}, \tag{3.15}$$

where $\beta(z)$ is an entire function such that $\rho(\beta) = \lambda(\beta) < \rho(f)$, and $p(z)$ is a nonconstant polynomial with $\deg p = \rho(f)$. Hence we have

$$T(r,\beta) = S(r, e^p). \tag{3.16}$$

It follows from (3.14) and (3.15) that $\Delta_\eta^n \left(\beta(z) e^{p(z)}\right) = e^{A_1 z + B_1}$. That is

$$\sum_{i=0}^{n} (-1)^i C_n^i \beta\left(z + (n-i)\eta\right) e^{p(z+(n-i)\eta)} = e^{A_1 z + B_1}. \tag{3.17}$$

Next, we consider two subcases.

Case 1.2.1. $\deg p \ge 2$.

By (3.17) we have

$$\sum_{i=0}^{n} (-1)^i C_n^i \frac{\beta\left(z + (n-i)\eta\right)}{e^{A_1 z + B_1}} e^{p(z+(n-i)\eta)} \equiv 1. \tag{3.18}$$

If $n=1$, then by (3.18) we have

$$\frac{\beta\left(z+\eta\right)}{e^{A_1z+B_1}}e^{p(z+\eta)}-\frac{\beta(z)}{e^{A_1z+B_1}}e^{p(z)}\equiv 1. \tag{3.19}$$

Obviously, $T\left(r,e^{A_1z+B_1}\right)=S(r,e^p)$. Then by (3.16), (3.19) and Nevanlinna's second fundamental theorem we have

$$\begin{aligned}T(r,e^p)&\leq T\left(r,\frac{\beta(z)}{e^{A_1z+B_1}}e^p\right)+S(r,e^p)\leq \overline{N}\left(r,\frac{\beta(z)}{e^{A_1z+B_1}}e^p\right)\\&+\overline{N}\left(r,\frac{1}{\frac{\beta(z)}{e^{A_1z+B_1}}e^p}\right)+\overline{N}\left(r,\frac{1}{\frac{\beta(z)}{e^{A_1z+B_1}}e^p+1}\right)+S\left(r,\frac{\beta(z)}{e^{A_1z+B_1}}e^p\right)\leq S(r,e^p),\end{aligned}$$

a contradiction.

If $n\geq 2$, then by (3.18) and Lemma 9 we get a contradiction.

Case 1.2.2. $\deg p=1$.

Set $p(z)=kz+t$, where $k(\neq 0),t$ are two finite complex numbers. Next we consider two subcases.

Case 1.2.2.1. $A_1\neq k$.

Then by (3.17) we have

$$\sum_{i=0}^{n}(-1)^iC_n^id_i\beta\left(z+(n-i)\eta\right)e^{(k-A_1)z}\equiv 1, \tag{3.20}$$

where $d_i=e^{(n-i)k\eta+t-B_1}$.

By (3.20) and $A_1\neq k$ we have $\sum_{i=0}^{n}(-1)^iC_n^id_i\beta(z+(n-i)\eta)\neq 0,\infty$. From Lemma 7 and $\rho(\beta)<\rho(f)=1$ we know that there exists a polynomial $\gamma(z)$ such that $\sum_{i=0}^{n}(-1)^iC_n^id_i\beta(z+(n-i)\eta)=e^{\gamma(z)}$. Since $\rho(\beta)<\rho(f)=1$, we know that $\gamma(z)$ is a constant. Combining with (3.20) we deduce that $e^{(k-A_1)z}$ is a constant, a contradiction.

Case 1.2.2.2. $A_1=k$.

Thus by (3.17) we have

$$\sum_{i=0}^{n}(-1)^iC_n^i\beta\left(z+(n-i)\eta\right)e^{k\eta(n-i)}\equiv e^{B_1-t}. \tag{3.21}$$

If $\beta'\equiv 0$, we know that β is a constant. It follows from (3.15) that $f(z)=a(z)+Be^{Az}$ where A,B are two nonzero constants.

Since $b=\Delta_\eta^n a$, then by $\Delta_\eta^{n+1}a=\Delta_\eta^n a$ we have

$$\Delta_\eta b=b.$$

It follows that $b(z+\eta)=2b(z)$. By Lemma 11 we know that either $T(r,b)>dr$ for sufficiently large r or b is a constant, then by b is a small function of f, we know that b is a constant. Obviously $\Delta_\eta^n a(z)=b=0$.

From a is a Borel exceptional entire small function of f, we have $\rho(a)\leq 1$. It follows from Lemma 13 that a is a polynomial with $\deg a\leq n-1$. Therefore, $f(z)=a(z)+Be^{Az}$, where A,B are two nonzero constants and $a(z)$ is a polynomial with $\deg a\leq n-1$.

If $\beta'\not\equiv 0$, then by (3.21) we have

$$\sum_{i=0}^{n}(-1)^iC_n^i\frac{\beta'\left(z+(n-i)\eta\right)}{\beta'(z)}e^{k\eta(n-i)}\equiv 0. \tag{3.22}$$

We now rewrite equation (3.22) in the form

$$\left(e^{k\eta}\right)^n \frac{\Delta_\eta^n \beta'(z)}{\beta'(z)} + D_{n-1}\frac{\Delta_\eta^{n-1}\beta'(z)}{\beta'(z)} + \cdots + D_1\frac{\Delta_\eta \beta'(z)}{\beta'(z)} = D_0, \tag{3.23}$$

where $D_{n-1}, \cdots, D_1, D_0$ are constants.

By Lemma 15 we know that $\rho(\beta') = \rho(\beta) < \rho(f) = 1$. Now we choose ε such that $0 < \varepsilon < 1 - \rho(\beta')$. Then by Lemma 12 we know that there exists a set $E \subset (1, \infty)$ with finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \bigcup [0, 1]$, and for $1 \le j \le n$, we have

$$\frac{\Delta_\eta^j \beta'(z)}{\beta'(z)} = o(1). \tag{3.24}$$

Let $|z| = r \notin E \bigcup [0, 1]$ and $|z| \to \infty$. By (3.23) and (3.24) we have $D_0 = 0$. Thus we have

$$\left(e^{k\eta}\right)^n \Delta_\eta^n \beta'(z) + D_{n-1}\Delta_\eta^{n-1}\beta'(z) + \cdots + D_1 \Delta_\eta \beta'(z) = 0. \tag{3.25}$$

Case a. $\Delta_\eta \beta' \equiv 0$.

By Lemma 13 we deduce that either $T(r, \beta') > dr$ for sufficiently large r or β' is a constant, then by $\beta \not\equiv 0$ and $\rho(\beta') = \rho(\beta) < 1$ we know that β' is a nonzero constant.

By (3.22) we have

$$\sum_{i=0}^{n} (-1)^i C_n^i e^{k\eta(n-i)} = 0.$$

Hence $\left(e^{k\eta} - 1\right)^n = 0$, which yields $e^{k\eta} = 1$.

Set $\beta(z) = c_0 z + c_1$ where $c_0 (\neq 0), c_1$ are two constants. By (3.15) and $A_1 = k$ we have $f(z) = a(z) + (c_0 z + c_1)\, e^{kz + B_1}$. Thus,

$$\Delta_\eta^n f(z) = \Delta_\eta^n a(z) + \Delta_\eta^n \left((c_0 z + c_1) e^{kz + B_1}\right). \tag{3.26}$$

If $n = 1$, then by (3.26), $e^{k\eta} = 1$ and $b = \Delta_\eta^{n+1} a = \Delta_\eta^n a$ we have

$$\begin{aligned}\Delta_\eta f(z) &= \Delta_\eta a(z) + (c_0 z + c_0 \eta + c_1) e^{k(z+\eta) + B_1} - (c_0 z + c_1) e^{kz + B_1} \\ &= \Delta_\eta a(z) + c_0 \eta e^{kz + B_1} = b + c_0 \eta e^{kz + B_1},\end{aligned}$$

and

$$\begin{aligned}\Delta_\eta^2 f(z) &= \Delta_\eta \left(\Delta_\eta a(z) + c_0 \eta e^{kz + B_1}\right) \\ &= \Delta_\eta^2 a(z) + c_0 \eta e^{k(z+\eta) + B_1} - c_0 \eta e^{kz + B_1} \\ &= \Delta_\eta^2 a(z) = b. \end{aligned} \tag{3.27}$$

Hence by $\Delta_\eta f(z)$ and $\Delta_\eta^2 f(z)$ share b CM, we get a contradiction.

If $n \ge 2$, then by a is a polynomial with $\deg a \le n - 1$ and (3.27) we have $\Delta_\eta^{n+1} f(z) = \Delta_\eta^{n+1} a(z) \equiv 0$, a contradiction.

Case b. $\Delta_\eta \beta'(z) \not\equiv 0$.

It follows from Lemmas 14, 15 that $\rho\left(\Delta_\eta \beta'\right) \le \rho\left(\beta'\right) = \rho(\beta) < 1$. Therefore by (3.25) and Lemma 12 we have $D_1 = 0$. Now we suppose that $D_l \neq 0$, where $2 \le l \le n$, and $D_{l-1} = \cdots = D_1 = 0$. Then by (3.25) we have

$$\left(e^{k\eta}\right)^n \Delta_\eta^n \beta'(z) + D_{n-1}\Delta_\eta^{n-1}\beta'(z) + \cdots + D_l \Delta_\eta^l \beta'(z) = 0.$$

We claim $\Delta_\eta^l \beta'(z) \equiv 0$. Otherwise, we have

$$\left(e^{k\eta}\right)^n \frac{\Delta_\eta^n \beta'(z)}{\Delta_\eta^l \beta'(z)} + D_{n-1} \frac{\Delta_\eta^{n-1} \beta'(z)}{\Delta_\eta^l \beta'(z)} + \cdots + D_{l+1} \frac{\Delta_\eta^{l+1} \beta'(z)}{\Delta_\eta^l \beta'(z)} = -D_l. \tag{3.28}$$

By (3.28) and Lemma 12 we have $D_l = 0$, a contradiction. Hence $\Delta_\eta^l \beta'(z) \equiv 0$.

It follows from Lemma 13 that either $T(r, \beta') > dr$ for sufficiently large r or β' is a polynomial with $\deg \beta' \leq l - 1$, then by $\rho(\beta') = \rho(\beta) < 1$ we know that β' is a polynomial with $\deg \beta' \leq l - 1$. From (3.22) we have $\sum_{i=0}^n (-1)^i C_n^i e^{k\eta(n-i)} = 0$, which yields $e^{k\eta} = 1$.

By (3.21) we deduce that $\sum_{i=0}^n (-1)^i C_n^i \beta\left(z + (n-i)\eta\right) \equiv e^{B_1 - t}$. That is $\Delta_\eta^n \beta \equiv C_1$, where $C_1 = e^{B_1 - t}$. By (3.15) we have $f(z) = a(z) + \beta(z) e^{kz + B_1}$. Thus, by $e^{k\eta} = 1$ and $b = \Delta_\eta^{n+1} a = \Delta_\eta^n a$ we have

$$\begin{aligned}
\Delta_\eta^n f(z) &= \Delta_\eta^n a(z) + \Delta_\eta^n \left(\beta(z) e^{kz + B_1}\right) \\
&= \Delta_\eta^n a(z) + \sum_{i=0}^n (-1)^i C_n^i \beta(z + (n-i)\eta) e^{k(z + (n-i)\eta) + B_1} \\
&= \Delta_\eta^n a(z) + \sum_{i=0}^n (-1)^i C_n^i \beta(z + (n-i)\eta) e^{kz + B_1} \\
&= \Delta_\eta^n a(z) + \Delta_\eta^n \beta(z) e^{kz + B_1} \\
&= \Delta_\eta^n a(z) + C_1 e^{kz + B_1} = b + C_1 e^{kz + B_1},
\end{aligned}$$

and

$$\begin{aligned}
\Delta_\eta^{n+1} f(z) &= \Delta_\eta \left(\Delta_\eta^n a(z) + C_1 e^{kz + B_1}\right) \\
&= \Delta_\eta^{n+1} a(z) + C_1 e^{k(z+\eta) + B_1} - C_1 e^{kz + B_1} \\
&= \Delta_\eta^{n+1} a(z) = b.
\end{aligned}$$

Hence by $\Delta_\eta^n f(z)$ and $\Delta_\eta^{n+1} f(z)$ share b CM, we get a contradiction.

Case 2. $b \equiv \Delta_\eta^n a$.

Case 2.1. $\Delta_\eta^{n+1} a \not\equiv \Delta_\eta^n a$.

Since $\Delta_\eta^n f$ and $\Delta_\eta^{n+1} f$ share b CM, then by (3.2), (3.3), Lemma 5 and Lemma 6 we have

$$\begin{aligned}
T(r, f) &= T(r, \Delta_\eta^{n+1} f) + S(r, f) \\
&\leq \overline{N}(r, \Delta_\eta^{n+1} f) + \overline{N}\left(r, \frac{1}{\Delta_\eta^{n+1} f - \Delta_\eta^{n+1} a}\right) + \overline{N}\left(r, \frac{1}{\Delta_\eta^{n+1} f - b}\right) + S(r, f) \\
&\leq \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - b}\right) + S(r, f) \leq S(r, f),
\end{aligned}$$

a contradiction.

Case 2.2. $\Delta_\eta^{n+1} a \equiv \Delta_\eta^n a$.

Using the same argument as used in Case 1.2, we get $f(z) = a(z) + B e^{Az}$, where A, B are two nonzero constants and $a(z)$ is a polynomial with $\deg a \leq n - 1$.

Case 3. $b \not\equiv \Delta_\eta^n a$ and $b \not\equiv \Delta_\eta^{n+1} a$.

Set

$$F_1 = \frac{\Delta_\eta^n f - \Delta_\eta^n a}{b - \Delta_\eta^n a}, \quad F_2 = \frac{\Delta_\eta^{n+1} f - \Delta_\eta^{n+1} a}{b - \Delta_\eta^{n+1} a}. \tag{3.29}$$

It follows from (3.2), (3.3) and (3.29) that

$$\delta(0, F_1) = \delta(\infty, F_1) = 1, \tag{3.30}$$
$$\delta(0, F_2) = \delta(\infty, F_2) = 1. \tag{3.31}$$

From $\Delta_\eta^n f$ and $\Delta_\eta^{n+1} f$ share b CM, we know that F_1 and F_2 share 1 CM almost. It follows from (3.30), (3.31) and Lemma 10 that either $F_1F_2 \equiv 1$ or $F_1 \equiv F_2$.

If $F_1F_2 \equiv 1$, from (3.29) we obtain

$$\left(\Delta_\eta^n f - \Delta_\eta^n a\right)\left(\Delta_\eta^{n+1} f - \Delta_\eta^{n+1} a\right) = \left(b - \Delta_\eta^n a\right)\left(b - \Delta_\eta^{n+1} a\right). \tag{3.32}$$

By (3.32), $\delta(a, f) = 1$, Lemma 3 and Nevanlinna's first fundamental theorem we have

$$\begin{aligned} 2T(r,f) &\le T\left(r, \frac{1}{(f-a)^2}\right) + S(r,f) \\ &\le m\left(r, \frac{1}{(f-a)^2}\right) + S(r,f) \\ &\le m\left(r, \frac{\Delta_\eta^n f - \Delta_\eta^n a}{f-a}\right) + m\left(r, \frac{\Delta_\eta^{n+1} f - \Delta_\eta^{n+1} a}{f-a}\right) \\ &\quad + m\left(r, \frac{1}{\left(b - \Delta_\eta^n a\right)\left(b - \Delta_\eta^{n+1} a\right)}\right) + S(r,f) \le S(r,f), \end{aligned}$$

a contradiction. Therefore $F_1 \equiv F_2$.

It follows that

$$\frac{\Delta_\eta^n f - \Delta_\eta^n a}{b - \Delta_\eta^n a} \equiv \frac{\Delta_\eta^{n+1} f - \Delta_\eta^{n+1} a}{b - \Delta_\eta^{n+1} a}. \tag{3.33}$$

By (3.33) we have

$$\frac{\Delta_\eta^{n+1} f - b}{\Delta_\eta^n f - b} \equiv \frac{b - \Delta_\eta^{n+1} a}{b - \Delta_\eta^n a}. \tag{3.34}$$

Since f is a transcendental entire function of finite order and $\Delta_\eta^n f$ and $\Delta_\eta^{n+1} f$ share b CM, then by Lemma 7 we know that there exists a polynomial $\mu(z)$ satisfying $\deg \mu \le \rho(f)$ such that

$$\frac{\Delta_\eta^{n+1} f - b}{\Delta_\eta^n f - b} \equiv e^{\mu(z)}. \tag{3.35}$$

It follows from (3.33)-(3.35) that

$$\frac{\Delta_\eta^{n+1} f - \Delta_\eta^{n+1} a}{\Delta_\eta^n f - \Delta_\eta^n a} = e^{\mu(z)}. \tag{3.36}$$

By $G = \Delta_\eta^n f - \Delta_\eta^n a$ and (3.36) we have

$$\Delta_\eta G = e^{\mu(z)} G.$$

Using the same argument as used in Case 1.2, we get $f(z) = a(z) + Be^{Az}$, where A and B are two nonzero constants and $a(z)$ is a polynomial with $\deg a \le n - 1$.

Thus Theorem 1 is proved.

4. Proof of Theorem 2

Set

$$\varphi(z)=\frac{f^{(k)}(z)}{f(z+\eta)}. \tag{4.1}$$

By Lemma 2 and Lemma 3 we have

$$m(r,\varphi)=S(r,f). \tag{4.2}$$

Since $E\left(0,f(z+\eta)\right)\subset E\left(0,f^{(k)}(z)\right)$, $E\left(\infty,f^{(k)}(z)\right)\subset E\left(\infty,f(z+\eta)\right)$, then by (4.1) we deduce that $N\left(r,\varphi\right)=S(r,f)$ and $\varphi(z)\not\equiv 0$. Hence by (4.2) we have

$$T(r,\varphi)=S(r,f). \tag{4.3}$$

We claim $\varphi(z)\equiv 1$. Otherwise we suppose that $\varphi(z)\not\equiv 1$.

From $E\left(\infty,f^{(k)}(z)\right)\subset E\left(\infty,f(z+\eta)\right)$, we have

$$N(r,f^{(k)}(z))\leq N\left(r,f(z+\eta)\right). \tag{4.4}$$

It follows that $N(r,f^{(k)})=N(r,f)+k\overline{N}(r,f)$, Lemma 4 and (4.4) that

$$\overline{N}(r,f)=S(r,f). \tag{4.5}$$

By (4.1), (4.3), $E_{2)}\left(1,f^{(k)}(z)\right)=E_{2)}\left(1,f(z+\eta)\right)$ and Nevanlinna's first fundamnetal theorem we have

$$\overline{N}_{2)}\left(r,\frac{1}{f^{(k)}-1}\right)=\overline{N}_{2)}\left(r,\frac{1}{f(z+\eta)-1}\right)\leq N\left(r,\frac{1}{\varphi-1}\right)\leq S(r,f). \tag{4.6}$$

By (4.1) we have

$$f^{(k)}-\varphi=\varphi\left[f(z+\eta)-1\right]. \tag{4.7}$$

It follows from (4.3) and (4.7) that

$$T\left(r,f^{(k)}\right)=T(r,f)+S(r,f). \tag{4.8}$$

Thus, we have $S(r,f)=S\left(r,f^{(k)}\right)$.

By (4.3), (4.6), (4.7), Lemma 4 and Nevanlinna's first fundamental theorem we obtain

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f^{(k)}-\varphi}\right)&=\overline{N}\left(r,\frac{1}{\varphi}\right)+\overline{N}\left(r,\frac{1}{f(z+\eta)-1}\right)\\&\leq\overline{N}_{2)}\left(r,\frac{1}{f(z+\eta)-1}\right)+\overline{N}_{(3}\left(r,\frac{1}{f(z+\eta)-1}\right)+S(r,f)\\&\leq\frac{1}{3}N_{(3}\left(r,\frac{1}{f(z+\eta)-1}\right)+S(r,f)\\&\leq\frac{1}{3}T(r,f)+S(r,f).\end{aligned} \tag{4.9}$$

Hence, by (4.5), (4.6), (4.8), (4.9) and Lemma 6 we have

$$T\left(r,f^{(k)}\right)\leq\overline{N}\left(r,f^{(k)}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-\varphi}\right)+S\left(r,f^{(k)}\right)$$

$$\leq\overline{N}\left(r,f\right)+\overline{N}_{2)}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_{(3}\left(r,\frac{1}{f^{(k)}-1}\right)+\frac{1}{3}T(r,f)+S\left(r,f^{(k)}\right)$$

$$\leq \frac{1}{3}T\left(r, f^{(k)}\right) + \frac{1}{3}T(r, f) + S\left(r, f^{(k)}\right) \leq \frac{2}{3}T\left(r, f^{(k)}\right) + S\left(r, f^{(k)}\right).$$

It follows that $T\left(r, f^{(k)}\right) \leq S\left(r, f^{(k)}\right)$, a contradiction.

Thus Theorem 2 is proved.

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