Multiplicity of Weak Solutions for a $(p(x), q(x))$-Kirchhoff Equation with Neumann Boundary Conditions

Ahmed AHMED
University of Nouakchott, Faculty of Science and Technology
Mathematics and Computer Sciences Department,
Research Unit Geometry, Algebra, Analysis and Applications
Nouakchott, Mauritania.
E-mail : ahmedmath2001@gmail.com
Mohamed Saad Bouh ELEMINE VALL
University of Nouakchott, Professional University Institute
Department of Industrial Engineering and Applied Mathematics
Nouakchott, Mauritania.
E-mail : saad2012bouh@gmail.com

Abstract

The aim of this study is to investigate the existence of infinitely many weak solutions for the $(p(x), q(x))$-Kirchhoff Neumann problem described by the following equation :

$$\begin{cases} -\left(a_1 + a_2 \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \Delta_{p(\cdot)} u - \left(b_1 + b_2 \int_\Omega \frac{1}{q(x)} |\nabla u|^{q(x)} dx\right) \Delta_{q(\cdot)} u & \\ +\lambda(x)\Big(|u|^{p(x)-2}u + |u|^{q(x)-2}u\Big) = f_1(x,u) + f_2(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

By employing a critical point theorem proposed by B. Ricceri, which stems from a more comprehensive variational principle, we have successfully established the existence of infinitely many weak solutions for the aforementioned problem.

Key words: Nonlinear elliptic equations, Weak solutions to PDEs, Ricceri's variational principle, Double phase problems, Musielak-Orlicz-Sobolev spaces.
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1 Introduction

Studying differential equations with double-phase operators is a novel and fascinating subject. It is caused by factors such as extremely anisotropic materials, Lavrentiev's phenomenon, and nonlinear elasticity theory (see [43, 44, 45, 46]). In recent years, there has been a surge in interest in the study of double-phase problems, with numerous results obtained, see for example [6, 8, 10, 26, 29, 30, 34, 35].

By taking into account the fluctuations in the string's length during vibrations, Kirchhoff's differential equations, as outlined by Kirchhoff [27], extend the classical D'Alembert's wave equation.

$$r\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where L, h, E, P_0 and r are constants.

The Kirchhoff equation (1.1) is characterized by the presence of a non-local component $\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$ which varies depending on average $\frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2$ of the kinetic energy $\frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2$ on the domain $[0, L]$, and as a result, the equation ceases to be a point-wise identity, (see [3, 4, 5, 7, 20, 25, 38]) for related topics.

The aim of this research is to show the existence of infinitely many weak solutions to the following elliptic problem involving double phase operators of Kirchhoff type and a Neumann boundary value condition.

$$\begin{cases} -\left(a_1 + a_2 \displaystyle\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(\cdot)}u - \left(b_1 + b_2 \displaystyle\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\right)\Delta_{q(\cdot)}u & \\ +\lambda(x)\Big(|u|^{p(x)-2}u + |u|^{q(x)-2}u\Big) = f_1(x,u) + f_2(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open domain such that his boundary $\partial\Omega$ is of class C^1, and denoted by ν the outward unit normal to $\partial\Omega$, $p \equiv p(x)$, $q \equiv q(x) \in C_+(\overline{\Omega})$ with

$$N < p^- \leq p^+ < q^- \leq q^+ < +\infty, \tag{1.3}$$

and $a_1,\ a_2,\ b_1,\ b_2 > 0$, $\lambda \in L^\infty(\Omega)$ and there is a positive constant λ_0 satisfying $\lambda_0 \leq \lambda(x)$.

Let $f_1,\ f_2 : \Omega \times \mathbb{R} \longmapsto \mathbb{R}$ are two Carathéodory functions such that for all $r > 0$, we have

$$\sup_{|t|\leq r} |\ f_1(x,t)\ | \in L^1(\Omega) \quad \text{and} \quad \sup_{|t|\leq r} |\ f_2(x,t)\ | \in L^1(\Omega). \tag{1.4}$$

It is not surprising that there have been articles dealing with questions related to this type of operator in the classical Sobolev spaces. We refer the reader to [4, 37, 40] for some examples, where the authors are interested in the Dirichlet problem.

An important generalization of the p-Laplace operator is the $p(\cdot)$-Laplace operator. The $p(\cdot)$-Laplace operator has more complex nonlinearities than the p-Laplace operator.

In variable exponent Sobolev spaces, A. Crespo-Blanco et all in [11] propose a new type of quasi-linear elliptic equations controlled by so-called double phase operators with variable exponents. They prove some properties of the corresponding Musielak-Orlicz Sobolev space and properties of the new double phase operator and show the existence and uniqueness of elliptic equations corresponding to straight sides with dependence into the slope, see also [16, 41] for related topics. In this article, we use Kirchhoff-type operators in an elliptic Neumann problem.

To our knowledge, few papers have been studied dealing with the elliptic problem introducing the Kirchhoff-type operator in the case of the Neumann condition (see [2, 13, 14, 28, 42]). The hypotheses used in this paper, as well as the results, are quite different from the previous results.

But because of its non-homogeneities and the existence of numerous nonlinear elements, the issue (1.2) has a more complex structure if either p or q are non constant functions.

Our earlier work inspired us to expand these conclusions within the Musielak-Orlicz-Sobolev space, a more all-encompassing functional framework that has garnered interest from academics.

The motivation for this work is provided by its physical applications, specifically the issues with equations of the Kolmogorov-type that arise in the theory of diffusion, theory of non-Newtonian fluids with strongly inhomogeneous behavior and a high propensity to increase their viscosity in response to shear rate, electro-rheological fluids electric or magnetic field, and (see references [18, 19, 23, 32, 39]).

We refer to [1, 14, 15, 21, 22, 31] for some more discoveries on elliptic and parabolic problems in Musielak-Orlicz-Sobolev spaces.

The main obstacle with this type of problem is the setting of Sobolev spaces with double phase exponents and the fact that there is a Neumann boundary condition that makes the Theorem 1.1 difficult to apply.

We present a crucial result obtained by B. Ricceri in [36], which is necessary to prove our primary findings.

Theorem 1.1 *(See [17], Theorem 2.2). Consider two Gâteaux differentiable and sequentially weakly lower semi-continuous functionals $\Phi_1, \Phi_2 : E \longrightarrow \mathbb{R}$ on a reflexive real Banach space E and suppose that Φ_2 is continuous with respect on the norm topology and $\lim_{\|u\|_E\to\infty} \Phi_2(u) = +\infty$. For $r > \inf_E \Phi_2$, we set*

$$\varphi(r) = \inf_{u\in\Phi_2^{-1}(]-\infty,r[)} \frac{\Phi_1(u) - \inf_{v\in(\Phi_2^{-1}(]-\infty,r[))_w} \Phi_1(v)}{r - \Phi_2(u)}, \tag{1.5}$$

where $\overline{(\Phi_2^{-1}(]-\infty, r[))_w}$ *denoted the adherence of* $\Phi_2^{-1}(]-\infty, r[)$ *with regards to the topology of weak convergence. Then, the following claims are accurate*

(a) *If we have* $r_0 > \inf_E \Phi_2$ *and* $u_0 \in E$ *such that*

$$\Phi_2(u_0) < r_0, \tag{1.6}$$

and

$$\Phi_1(u_0) - \inf_{v \in \overline{(\Phi_2^{-1}(]-\infty, r_0[))_w}} \Phi_1(v) < r_0 - \Phi_2(u_0), \tag{1.7}$$

then the restriction of $\Phi_1 + \Phi_2$ *to* $\Phi_2^{-1}(]-\infty, r_0[)$ *admits at least global minimum point.*

(b) *If we have two sequences* $(r_n)_n \subset \left(\inf_E \Phi_2, +\infty\right)$ *with* $r_n \to \infty$ *and* $(u_n)_n \subset E$ *such that for any* n

$$\Phi_2(u_n) < r_n, \tag{1.8}$$

and

$$\Phi_1(u_n) - \inf_{v \in \overline{(\Phi_2^{-1}(]-\infty, r_n[))_w}} \Phi_1(v) < r_n - \Phi_2(u_n), \tag{1.9}$$

and also we assume

$$\liminf_{\|u\| \to +\infty} (\Phi_2(u) + \Phi_1(u)) = -\infty. \tag{1.10}$$

Consequently, we can find a sequence $(v_n)_n$ *of local minima of* $\Phi_2 + \Phi_1$ *such that* $\Phi_2(v_n) \to +\infty$ *as* $n \to \infty$.

(c) *If we have two sequences* $(r_n)_n \subset (\inf_E \Phi_2, +\infty)$ *with* $r_n \to \inf_E \Phi_2$ *and* $(u_n)_n \subset E$ *such that for every* n*, the conditions (1.8) and (1.9) are met, and also assume that :*

$$\text{The global minimizers of } \Phi_2 \text{ are not local minimizers of } \Phi_1 + \Phi_2. \tag{1.11}$$

Then, we can find a sequence $(v_n)_n$ *of pairwise different local minimizers of* $\Phi_1 + \Phi_2$ *such that* $\lim_{n\to\infty} \Phi_2(v_n) = \inf_E \Phi_2$*, and* $(v_n)_n$ *weakly converges to a global minimizer of* Φ_2.

The following describes the structure of this paper : In Section 2, we provide some important background information. We outline an improvement in Section 3 (see Theorems 3.1 and 3.2) and support its credibility with examples (see Corollary 3.1 and 3.2).

2 Preliminary results

Consider a smooth bounded open domain $\Omega \subset \mathbb{R}^N$, and let we define

$$\mathcal{C}_+(\overline{\Omega}) = \Big\{ z \in \mathcal{M},\ z(\cdot) : \overline{\Omega} \longrightarrow \mathbb{R}\ :\ 1 < z^- = \operatorname{ess\,inf} \{z(x) : x \in \overline{\Omega}\} \le z^+ = \operatorname{ess\,sup} \{z(x) : x \in \overline{\Omega}\} < \infty \Big\},$$

here $\mathcal{M}$ represents the collection of measurable real functions.

The variable exponent Lebesgue space $L^{z(\cdot)}(\Omega)$ is defined as the set of measurable functions $u : \Omega \longmapsto \mathbb{R}$ satisfying $r_{z(\cdot)}(u) := \int_\Omega |u|^{z(x)} dx < \infty$ endowed with the following norm

$$\|u\|_{L^{z(\cdot)}(\Omega)} = \|u\|_{z(\cdot)} = \inf \{ \sigma > 0 : r_{z(\cdot)}(u/\sigma) \le 1 \},$$

known as the Luxemburg norm. Then, the space $(L^{z(\cdot)}(\Omega), \|\cdot\|_{z(\cdot)})$ is a separable reflexive and uniformly convex Banach space and its dual space is isomorphic to $L^{z'(\cdot)}(\Omega)$, where $\frac{1}{z(\cdot)} + \frac{1}{z'(\cdot)} = 1$.

An crucial instrument for our findings is the following Hölder type inequality

$$\left| \int_\Omega u(x)v(x)dx \right| \le \Big(\frac{1}{z^-} + \frac{1}{(z^-)'} \Big) \|u\|_{z(\cdot)} \|v\|_{z'(\cdot)} \le 2\|u\|_{z(\cdot)} \|v\|_{z'(\cdot)}, \tag{2.1}$$

for all $u \in L^{z(\cdot)}(\Omega)$ and $v \in L^{z'(\cdot)}(\Omega)$.

The modular function $r_{z(\cdot)}$ is a fundamental element in the study of generalized Lebesgue spaces. Furthermore, the following result is presented :

Proposition 2.1 *(See [12, 24]). If $u \in L^{z(\cdot)}(\Omega)$, we have*

(a) $\|u\|_{z(\cdot)} > 1$ *is true precisely when* $\|u\|_{z(\cdot)}^{z^-} < r_{z(\cdot)}(u) < \|u\|_{z(\cdot)}^{z^+}$,

(b) $\|u\|_{z(\cdot)} < 1$ *is true precisely when* $\|u\|_{z(\cdot)}^{z^+} < r_{z(\cdot)}(u) < \|u\|_{z(\cdot)}^{z^-}$.

Provided that $z_1, z_2 \in \mathcal{C}+(\overline{\Omega})$ and $z_1(x) \leq z_2(x)$ for all $x \in \overline{\Omega}$, we can conclude that the following continuous embedding holds :

$$L^{z_2(\cdot)}(\Omega) \hookrightarrow L^{z_1(\cdot)}(\Omega). \tag{2.2}$$

The variable exponent Sobolev space is given by

$$W^{1,z(\cdot)}(\Omega) = \big\{u \in L^{z(\cdot)}(\Omega) \,:\, |\nabla u| \in L^{z(\cdot)}(\Omega)\big\}.$$

The norm for this space is given by the following expression

$$\|u\|_{W^{1,z(\cdot)}(\Omega)} = \|u\|_{1,z(\cdot)} = \|u\|_{z(\cdot)} + \|\nabla u\|_{z(\cdot)}.$$

It is worth mentioning that the space $(W^{1,z(\cdot)}(\Omega), \|\cdot\|_{1,z(\cdot)})$ is a Banach space that is also separable and reflexive. For additional details on this framework, refer to [12].

Remark 2.1 *If $z \in \mathcal{C}_+(\overline{\Omega})$ and $N < z^-$, then the embedding $W^{1,z(\cdot)}(\Omega) \hookrightarrow\hookrightarrow C^0(\overline{\Omega})$ is continuous and compact. Since $W^{1,z(\cdot)}(\Omega)$ is continuously embedded in $W^{1,z^-}(\Omega)$.*

Then we can set by (1.3)

$$C_1 = \sup_{u \in W^{1,p(\cdot)}(\Omega)\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|_{1,p(\cdot)}}, \tag{2.3}$$

and

$$C_2 = \sup_{u \in W^{1,q(\cdot)}(\Omega)\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|_{1,q(\cdot)}}. \tag{2.4}$$

Now, we present the Musielak-Orlicz-Sobolev spaces which is employed in the analysis of our main results. We start by giving the definitions of the Orlicz function and Musielak function.

Definition 2.1 *An Orlicz-type function, marked as $A \in N(\Omega)$, is a function $A : \mathbb{R} \longrightarrow [0, +\infty[$ that is even, continuous, and convex, satisfies $A(0) = 0$ and $0 < A(t)$ for all $t > 0$, and also satisfies :*

$$\lim_{t \to 0^+} \frac{A(t)}{t} = 0, \quad \textit{and} \quad \lim_{t \to +\infty} \frac{A(t)}{t} = +\infty.$$

A function $A : \Omega \times \mathbb{R} \longrightarrow [0, +\infty[$ is said to be a Musielak function, denoted by $A \in \Phi(\Omega)$, if for each $t \geq 0$, $A(\cdot, t) \in \mathcal{M}$ and for almost every $x \in \Omega$, the function $A(x, \cdot)$ is an Orlicz function.

Let $A \in \Phi(\Omega)$, the Musielak-Orlicz space $L_A(\Omega)$ is defined by

$$L_A(\Omega) := \Big\{u \in \mathcal{M}, \text{ and there exist } \sigma > 0 \text{ such that } \int_\Omega A\Big(x, \frac{|u(x)|}{\sigma}\Big)dx < \infty\Big\},$$

having the following norm, recognized as the norm of Luxemburg, given by

$$\|u\|_{L_A(\Omega)} := \inf\Big\{\sigma > 0 : \int_\Omega A\Big(x, \frac{|u(x)|}{\sigma}\Big)dx \leq 1\Big\}.$$

The space $W^1L_A(\Omega)$ is given by the following definition :

$$W^1L_A(\Omega) := \Big\{u \in L_A(\Omega) \text{ with } |\nabla u| \in L_A(\Omega)\Big\},$$

equipped with the following norm

$$\|u\|_{1,A} = \|u\|_A + \|\nabla u\|_A,$$

where $\|\nabla u\|_A = \||\nabla u|\|_A$.

Definition 2.2 *1. A function $A \in \Phi(\Omega)$ is said to be fulfilling the Δ_2-condition noted ($A \in \Delta_2$) when there exists a positive constant $k > 0$ and a non-negative function $b \in L^1(\Omega)$ such that $A(x, 2t) \leq kA(x, t) + b(x)$ for all $x \in \Omega$ and $t \in \mathbb{R}$.*

2. A is said to be locally integrable if $A(\cdot, t_0) \in L^1(\Omega)$ for every $t_0 > 0$.

The function $A'_d(x,t)$ denotes the right-hand derivative of $A(x,\cdot)$ at $t \geq 0$ and is defined as

$$A'_d(x,t) = \lim_{h \to 0^+} \frac{A(x,t+h) - A(x,t)}{h}.$$

If $t < 0$, we define $A'_d(x,t) = -A'_d(x,-t)$. Thus, $A(x,t) = \displaystyle\int_0^{|t|} A'_d(x,s)ds$ for all $t \in \mathbb{R}$ and $x \in \Omega$.

Set $A^* : \Omega \times \mathbb{R} \longrightarrow [0, +\infty[$ by

$$A^*(x,s) = \sup_{t \in \mathbb{R}} \Big(st - A(x,t) \Big) \text{ for each } s \in \mathbb{R} \text{ and } x \in \Omega.$$

According to Young's definition A^* is known as the complementary function to A. It It is commonly understood that A^* meets the criteria for a Musielak function and A also acts as complementary function to A^*.

For the fundamental properties of these spaces, we refer to [9].

We display here some facts that will be used later.

Lemma 2.1 *(See [9]). The following norms are equivalent on $W^1L_A(\Omega)$*

$$\|u\|_{1,A} = \|u\|_A + \||\nabla u|\|_A,$$

$$\|u\|_{2,A} = \max\Big(\|u\|_A, \||\nabla u|\|_A\Big).$$

$$\|u\| = \inf\left\{\sigma > 0 : \int_\Omega \left[A\Big(x, \frac{|u(x)|}{\sigma}\Big) + A\Big(x, \frac{|\nabla u(x)|}{\sigma}\Big)\right] dx \leq 1\right\},$$

Lemma 2.2 *(See [33]). Suppose A and A^* are two complementary Musielak functions satisfying the Δ_2-condition, then we have*

$$1 < a_* \leq \frac{tA'_d(x,t)}{A(x,t)} \leq a^* < \infty, \qquad \text{for any } x \in \Omega, t > 0,$$

and for some constants a_*, a^*.
Furthermore, we have

1. *If $\|u\| \leq 1$,* $\quad \|u\|^{a^*} \leq \displaystyle\int_\Omega \Big[A(x, |u(x)|) + A(x, |\nabla u(x)|)\Big] dx \leq \|u\|^{a_*}$,

2. *If $\|u\| > 1$,* $\quad \|u\|^{a_*} \leq \displaystyle\int_\Omega \Big[A(x, |u(x)|) + A(x, |\nabla u(x)|)\Big] dx \leq \|u\|^{a^*}$,

3. *If $u_n \longmapsto u$ in $W^1L_A(\Omega)$, then*

$$\int_\Omega [A(x, |u_n(x)|) + A(x, |\nabla u_n(x)|)]\, dx \longrightarrow \int_\Omega [A(x, |u(x)|) + A(x, |\nabla u(x)|)]\, dx.$$

Here and in the sequel, consider $p, q \in \mathcal{C}_+(\overline{\Omega})$ two variables exponents satisfy (1.3) and the Musielak function :

$$A(x,t) = t^{p(x)} + t^{q(x)}, \qquad \text{for all } (x,t) \in \Omega \times \mathbb{R}_+^*. \tag{2.5}$$

It is clear that A and its complementary function fulfill the Δ_2-condition. and set

$$W^{1,p(\cdot),q(\cdot)}(\Omega) = W^1L_A(\Omega), \tag{2.6}$$

possessing the norm $\|u\|_{1,p(\cdot),q(\cdot)} = \|u\|_{1,p(\cdot)} + \|u\|_{1,q(\cdot)}$

Proposition 2.2 (*See* [14]). *The space $W^{1,p(\cdot),q(\cdot)}(\Omega)$ embeds continuously into $W^{1,m_0}(\Omega)$ and compactly into $W^{1,m_0}(\Omega)$ under the hypothesis (1.3), then the following embedding $W^{1,p(\cdot),q(\cdot)}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}^0(\overline{\Omega})$ is compact and we put*

$$C_0 = \sup_{u \in W^{1,p(\cdot),q(\cdot)}(\Omega)\setminus\{0\}} \frac{\|u\|_{L^\infty(\Omega)}}{\|u\|_{1,p(\cdot),q(\cdot)}}. \tag{2.7}$$

3 Main results

For $u \in W^{1,p(\cdot),q(\cdot)}(\Omega)$ we define the following functionals

$$J_{p(x)}(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx, \quad J_{q(x)}(u) = \int_\Omega \frac{|\nabla u|^{q(x)}}{q(x)} dx,$$

$$J_{p(x)}^{\lambda(x)}(u) = \int_\Omega \lambda(x) \frac{1}{p(x)} |u|^{p(x)} dx, \quad J_{q(x)}^{\lambda(x)}(u) = \int_\Omega \lambda(x) \frac{1}{q(x)} |u|^{q(x)} dx,$$

$$H(u) = \int_\Omega F_2(x,u) dx, \quad \Phi_1(u) = -\int_\Omega F_1(x,u) dx,$$

$$J(u) = a_1 J_{p(x)}(u) + \frac{a_2}{2} \left(J_{p(x)}(u)\right)^2 + b_1 J_{q(x)}(u) + \frac{b_2}{2} \left(J_{q(x)}(u)\right)^2 + J_{p(x)}^{\lambda(x)}(u) + J_{q(x)}^{\lambda(x)}(u), \tag{3.1}$$

and

$$\Phi_2(u) = J(u) - H(u), \tag{3.2}$$

where

$$F_1(x,t) = \int_0^t f_1(x,\rho) d\rho, \quad \text{and} \quad F_2(x,t) = \int_0^t f_2(x,\rho) d\rho. \tag{3.3}$$

Definition 3.1 *A measurable function $u \in W^{1,p(\cdot),q(\cdot)}(\Omega)$ is a weak solution of the Neumann elliptic problem (1.2) if for any $v \in W^{1,p(\cdot),q(\cdot)}(\Omega)$, one has*

$$\begin{aligned} &\left(a_1 + a_2 J_{p(x)}(u)\right) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \left(b_1 + b_2 J_{q(x)}(u)\right) \int_\Omega |\nabla u|^{q(x)-2} \nabla u \nabla v dx \\ &+ \int_\Omega \lambda(x) \left(|u|^{p(x)-2} uv + |u|^{q(x)-2} uv\right) dx = \int_\Omega f_1(x,u) v dx + \int_\Omega f_2(x,u) v dx. \end{aligned} \tag{3.4}$$

Then, it is easy to verify that the weak solutions $u \in W^{1,p(\cdot),q(\cdot)}(\Omega)$ of (1.2) are exactly the critical points of $\Phi_1 + \Phi_2$.

Definition 3.2 *A function $F_1(x,t)$ is said to be of type (S) if for all compact subset of $\mathbb{R}$ noted E, there exists $\varsigma \in E$ such that*

$$F_1(x,\varsigma) = \sup_{t \in E} f_1(x,t) \quad \textit{for almoste every } x \in \Omega. \tag{3.5}$$

Lemma 3.1 *Assume that (1.3) and (1.4) are satisfied. Then, $\Phi_2, \Phi_1 \in C^1(W^{1,p(\cdot),q(\cdot)}(\Omega), \mathbb{R})$ and their Gâteaux derivatives are given by*

$$\begin{aligned} \langle \Phi_2'(u), v \rangle &= \left(a_1 + a_2 J_{p(x)}(u)\right) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \left(b_1 + b_2 J_{q(x)}(u)\right) \int_\Omega |\nabla u|^{q(x)-2} \nabla u \nabla v dx \\ &+ \int_\Omega \lambda(x) \left(|u|^{p(x)-2} uv + |u|^{q(x)-2} uv\right) dx - \int_\Omega f_2(x,u) v dx, \end{aligned}$$

and

$$\langle \Phi_1'(u), v \rangle = -\int_\Omega f_1(x,u) v dx,$$

for any $v, u \in W^{1,p(\cdot),q(\cdot)}(\Omega)$.

Proof We divided this prove into two claims, in the first we prove the Gâteaux differetiability of J and the second focuses on the Gâteaux differentiability of H.
Claim 1: We start by proving that $J_{p(x)}$ is of class $C^1(W^{1,p(\cdot),q(\cdot)}(\Omega), \mathbb{R})$. Fix $x \in \Omega$. We define $\phi : \mathbb{R}^N \to \mathbb{R}$ by $\phi(\zeta) = \frac{|\zeta|^{p(x)}}{p(x)}$. It is clear that, $\phi \in C^1(\mathbb{R}^N, \mathbb{R})$ and $\nabla \phi(\zeta) = |\zeta|^{p(x)-2} \zeta$. Thus, for all $\zeta, \vartheta \in \mathbb{R}^N$, we have

$$\lim_{t \to 0} \frac{\phi(\zeta + t\vartheta) - \phi(\vartheta)}{t} = |\zeta|^{p(x)-2} \zeta \cdot \vartheta.$$

As a consequence, for $u,\ v \in W^{1,p(\cdot),q(\cdot)}(\Omega)$, we obtain

$$\lim_{t\to 0} \frac{1}{p(x)} \left(\frac{|\nabla u + t\nabla v|^{p(x)} - |\nabla u|^{p(x)}}{t} \right) = |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v. \tag{3.6}$$

Applying the mean value theorem, there is a θ in the range $0 < |\theta| < |t|$ such that, for all $t \in \mathbb{R}$ with $0 < |t| < 1$:

$$\begin{aligned} &\frac{1}{p(x)} \left| \frac{|\nabla u + t\nabla v|^{p(x)} - |\nabla u|^{p(x)}}{t} \right| \\ &\quad = |\nabla u + \theta \nabla v|^{p(x)-2} (\nabla u + \theta \nabla v) \cdot \nabla v| \\ &\quad \leq (|\nabla u| + |\nabla v|)^{p(x)-1} |\nabla v|. \end{aligned} \tag{3.7}$$

Since for $u,\ v \in W^{1,p(\cdot),q(\cdot)}(\Omega)$ one has $(|\nabla u| + |\nabla v|)^{p(x)-1} |\nabla v| \in L^1(\Omega)$. Using (3.6) and (3.7) and applying the dominated convergence theorem, we can conclude that:

$$\lim_{t\to 0} \int_\Omega \frac{1}{p(x)} \left(\frac{|\nabla u + t\nabla v|^{p(x)} - |\nabla u|^{p(x)}}{t} \right) dx = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx.$$

It means that $J_{p(x)}$ is Gâteaux differentiable and for $u,\ v \in W^{1,p(\cdot),q(\cdot)}(\Omega)$, we have

$$\langle J'_{p(x)}(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx.$$

By similar arguments, we can show that $J_{q(x)}, J^{\lambda(x)}_{p(x)}$ and $J^{\lambda(x)}_{q(x)}$ are Gâteaux differentiables and for any $v, u \in W^{1,p(\cdot),q(\cdot)}(\Omega)$ we have

$$\begin{aligned} \langle J'_{q(x)}(u), v\rangle &= \int_\Omega |\nabla u|^{q(x)-2} \nabla u \nabla v dx. \\ \langle \left(J^{\lambda(x)}_{p(x)} \right)'(u), v\rangle &= \int_\Omega \lambda(x) |u|^{p(x)-2} u v dx. \\ \langle \left(J^{\lambda(x)}_{q(x)} \right)'(u), v\rangle &= \int_\Omega \lambda(x) |u|^{q(x)-2} u v dx. \end{aligned}$$

It follows that J is Gâteaux differentiable and for $u,\ v \in W^{1,p(\cdot),q(\cdot)}(\Omega)$, we obtain

$$\begin{aligned} \langle J'(u), v\rangle \quad = \quad & \big(a_1 + a_2 J_{p(x)}(u)\big) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \big(b_1 + b_2 J_{q(x)}(u)\big) \int_\Omega |\nabla u|^{q(x)-2} \nabla u \nabla v dx \\ & + \int_\Omega \lambda(x) \left(|u|^{p(x)-2} u v + |u|^{q(x)-2} u v \right) dx. \end{aligned}$$

Next, we prove that $J'_{p(x)} \colon W^{1,p(\cdot),q(\cdot)}(\Omega) \longrightarrow W^{1,p(\cdot),q(\cdot)}(\Omega)^*$ is continuous. To this aim we take a sequence $(u_n)_n$ in $W^{1,p(\cdot),q(\cdot)}(\Omega)$ such that $u_n \longrightarrow u$ in $W^{1,p(\cdot),q(\cdot)}(\Omega)$ as $n \longrightarrow \infty$. By Lemma 2.2, we have $\lim\limits_{n\longrightarrow\infty} \int_\Omega \left|\nabla u_n - \nabla u\right|^{p(x)} dx = 0$. Therefore, after extracting a sub-sequence, we conclude that

$$\lim_{n\to\infty} \nabla u_n = \nabla u \text{ almost everywhere in } \Omega, \tag{3.8}$$

$$|\nabla u_n - \nabla u|^{p(x)} \text{ is dominated by } h(x) \text{ in } L^1(\Omega). \tag{3.9}$$

Since

$$\begin{aligned} |\nabla u_n|^{p(x)} &\leq (|\nabla u| + |\nabla u_n - \nabla u|)^{p(x)} \\ &\leq 2^{p^+-1} \left(|\nabla u|^{p(x)} + |\nabla u_n - \nabla u|^{p(x)} \right) \\ &\leq 2^{p^+-1} \left(|\nabla u|^{p(x)} + h(x) \right). \end{aligned} \tag{3.10}$$

For any $v \in W^{1,p(\cdot),q(\cdot)}(\Omega)$ with $\|v\|_{1,p(\cdot),q(\cdot)} \leq 1$, the Hölder's inequality gives

$$\begin{aligned}\left|\langle J'_{p(x)}(u_n) - J'_{p(x)}(u), v\rangle\right| &= \left|\int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\cdot \nabla v dx\right| \\ &\leq 2\left\||\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u|\right\|_{L^{p'(\cdot)}(\Omega)} \|\nabla v\|_{L^{p(\cdot)}(\Omega)} \\ &\leq 2\left\||\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u|\right\|_{L^{p'(\cdot)}(\Omega)}.\end{aligned}$$

Hence,

$$\|J'_{p(x)}(u_n) - J'_{p(x)}(u)\|_{\left(W^{1,p(\cdot),q(\cdot)}(\Omega)\right)^*} \leq 2\left\||\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u|\right\|_{L^{p'(\cdot)}(\Omega)}. \tag{3.11}$$

It follows from (3.8) that

$$\left||\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right|^{p'(x)} \longrightarrow 0 \quad \text{for a.e.} \quad x \in \Omega.$$

Furthermore, using (3.10), we can deduce that

$$\begin{aligned}\left||\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right|^{p'(x)} &\leq 2^{p'(x)-1}\left(|\nabla u_n|^{p(x)} + |\nabla u|^{p(x)}\right) \\ &\leq 2^{(p')^+-1}\left(|\nabla u_n|^{p(x)} + |\nabla u|^{p(x)}\right) \\ &\leq 2^{(p')^+ + p^+ -1}\left(|\nabla u|^{p(x)} + h(x)\right).\end{aligned}$$

Since $2^{(p')^+ + p^+ -1}\left(|\nabla u|^{p(x)} + h\right)$ is integrable over Ω, we can apply the dominated convergence theorem to conclude that

$$\int_\Omega \left||\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right|^{p'(x)} dx \to 0 \quad \text{as} \quad n \to \infty.$$

Using Lemma 2.2, we conclude that,

$$\left\||\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u|\right\|_{L^{p'(\cdot)}(\Omega)} \to 0 \quad \text{as} \quad n \to \infty.$$

Combining this with (3.11), gives

$$\|J'_{p(x)}(u_n) - J'_{p(x)}(u)\|_{\left(W^{1,p(\cdot),q(\cdot)}(\Omega)\right)^*} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

This completes the proof that $J'_{p(x)}\colon W^{1,p(\cdot),q(\cdot)}(\Omega) \longrightarrow \left(W^{1,p(\cdot),q(\cdot)}(\Omega)\right)^*$ is continuous, and therefore $J_{p(x)} \in C^1(W^{1,p(\cdot),q(\cdot)}(\Omega), \mathbb{R})$. By similar arguments we can show that $J_{q(x)}$, $J^{\lambda(x)}_{p(x)}$ and $J^{\lambda(x)}_{q(x)}$ are of class C^1 from $W^{1,p(\cdot),q(\cdot)}(\Omega)$ into its dual. Which means that J is of class C^1.

Claim 2: We shall prove that $H \in C^1(W^{1,p(\cdot),q(\cdot)}(\Omega), \mathbb{R})$.

Let $u, v \in W^{1,p(\cdot),q(\cdot)}(\Omega)$ be arbitrary functions, then

$$\langle H'(u), v\rangle = \int_\Omega f_2(x,u)v dx.$$

Applying the mean value theorem again, for $u, v \in W^{1,p(\cdot),q(\cdot)}(\Omega)$ and $t \in \mathbb{R}\setminus\{0\}$, we obtain

$$\frac{F_2(x, u(x) + tv(x)) - F_2(x, u(x))}{t} = v(x)f_2(x, u(x) + \theta v(x)),$$

for some $\theta \in \mathbb{R}$ with $0 < |\theta| < |t|$. Therefore,

$$\frac{F_2(x, u(x) + tv(x)) - F_2(x, u(x))}{t} \longrightarrow v(x)f_2(x, u(x)) \text{ as } t \longrightarrow 0 \text{ for almost every } x \in \Omega. \tag{3.12}$$

Using Proposition 2.2, we see that for $|t| < 1$ there exists $\ell = \|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} > 0$ such that

$$\begin{aligned} \left| \frac{F_2(x, u(x) + tv(x)) - F_2(x, u(x))}{t} \right| &= |v(x)||f_2(x, u(x) + \theta v(x))| \\ &\leq |v(x)| \sup_{|s| \leq \ell} |f_2(x, s)|. \end{aligned} \tag{3.13}$$

From Hölder's inequality and (1.4) we obtain

$$\int_\Omega v(x) \sup_{|s|\leq \ell} |f_2(x, s)| dx \leq 2\|v\|_{L^\infty(\Omega)} \left\| \sup_{|s|\leq\ell} |f_2(x, s)| \right\|_{L^1(\Omega)}.$$

Therefore, the dominated convergence theorem together with (3.12) and (3.13) implies that

$$\lim_{t\to 0} \int_\Omega \frac{F_2(x, u(x) + tv(x)) - F_2(x, u(x))}{t} dx = \int_\Omega f_2(x, u(x))v(x)dx.$$

That is, H admits a Gâteaux derivative and

$$\langle H'(u), v\rangle = \int_\Omega f_2(x, u(x))v(x)dx.$$

For the proof of the continuity of H' in $W^{1,p(\cdot),q(\cdot)}(\Omega)$ we use Remark 2.2 for a sub-sequence still denoted u_n to get $u_n \longrightarrow u$ in $C^0(\overline{\Omega})$. Consequently,

$$(u_n)_n \text{ converges uniformly to } u \text{ in } \Omega, \tag{3.14}$$

$$k := \sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty(\Omega)} < +\infty. \tag{3.15}$$

We obtain that for any $v \in W^{1,p(\cdot),q(\cdot)}(\Omega)$ with $\|u\|_{1,p(\cdot),q(\cdot)} \leq 1$,

$$|\langle H'(u_n) - H'(u), v\rangle| \leq \int_\Omega |f_2(x, u_n(x)) - f_2(x, u(x))||v(x)|dx. \tag{3.16}$$

Therefore,

$$\begin{aligned} |f_2(x, u_n(x)) - f_2(x, u(x))| &\leq 2\big[|f_2(x, u_n)| + |f_2(x, u)|\big] \\ &\leq 2\left[\sup_{|s|\leq k} |f_2(x, s)| + \sup_{|s|\leq k} |f_2(x, s)|\right] \\ &\leq 4 \sup_{|s|\leq k} |f_2(x, s)|. \end{aligned}$$

By (3.16) we have

$$|\langle H'(u_n) - H'(u), v\rangle| \leq 4 \int_\Omega \sup_{|s|\leq k} |f_2(x, s)||v(x)|dx.$$

Note that $\sup_{|s|\leq k} |f_2(x, s)| \in L^1(\Omega)$. Then, the dominated convergence theorem and (3.14), conclude that

$$\lim_{n\to\infty} \int_\Omega |f_2(x, u_n(x)) - f_2(x, u(x))||v(x)|dx = 0.$$

Hence, from (3.16) follows that

$$\lim_{n\to\infty} \|H'(u_n) - H'(u)\|_{W^{1,p(\cdot),q(\cdot)}(\Omega)^*} = 0.$$

This completes the proof that $H' \colon W^{1,p(\cdot),q(\cdot)}(\Omega) \longrightarrow W^{1,p(\cdot),q(\cdot)}(\Omega)^*$ is continuous, and therefore $H \in C^1(W^{1,p(\cdot),q(\cdot)}(\Omega), \mathbb{R})$.

Similarly as above we are able to illustrate that $\Phi_1 \in C^1(W^{1,p(\cdot),q(\cdot)}(\Omega), \mathbb{R})$, and since $\Phi_2 = J - H$, the proof is complete. □

Lemma 3.2 *Assume that* (1.3) *and* (1.4) *hold. Then* Φ_1, Φ_2 *are sequentially weakly lower semi-continuous.*

Proof We divided this prove into two claims, the first concerns the functional J and the second focuses on the functional H.
Claim 1:
For any $u \in W^{1,p(\cdot),q(\cdot)}(\Omega)$ we have

$$J(u) = a_1 J_{p(x)}(u) + \frac{a_2}{2}\left(J_{p(x)}(u)\right)^2 + b_1 J_{q(x)}(u) + \frac{b_2}{2}\left(J_{q(x)}(u)\right)^2 + J^{\lambda(x)}_{p(x)}(u) + J^{\lambda(x)}_{q(x)}(u).$$

Consider a sequence $(u_n)_n$ such that $(u_n)_n$ goes to u weakly in $W^{1,p(\cdot),q(\cdot)}(\Omega)$. Then, by the convexity of $J_{p(x)}$, we have

$$J_{p(x)}(u) \leq J_{p(x)}(u_n) + \langle J'_{p(x)}(u), u - u_n\rangle.$$

When n goes to infinity, the aforementioned inequality, we can find that $J_{p(x)}$ is sequentially weakly lower semi-continuous. Its follows

$$a_1 \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \leq \liminf_{n\to+\infty} a_1 \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx. \tag{3.17}$$

Similarly, we get

$$\frac{a_2}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 \leq \liminf_{n\to+\infty} \frac{a_2}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)^2, \tag{3.18}$$

$$b_1 \int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx \leq \liminf_{n\to+\infty} b_1 \int_\Omega \frac{1}{q(x)}|\nabla u_n|^{q(x)}dx, \tag{3.19}$$

$$\frac{b_1}{2}\left(\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\right)^2 \leq \liminf_{n\to+\infty} \frac{b_1}{2}\left(\int_\Omega \frac{1}{q(x)}|\nabla u_n|^{q(x)}dx\right)^2, \tag{3.20}$$

$$\int_\Omega \lambda(x)\frac{1}{p(x)}|u|^{p(x)}dx \leq \liminf_{n\to+\infty} \int_\Omega \lambda(x)\frac{1}{p(x)}|u_n|^{p(x)}dx, \tag{3.21}$$

and

$$\int_\Omega \lambda(x)\frac{1}{q(x)}|u|^{q(x)}dx \leq \liminf_{n\to+\infty} \int_\Omega \lambda(x)\frac{1}{q(x)}|u_n|^{q(x)}dx. \tag{3.22}$$

Which yields

$$\begin{aligned}
\liminf_{n\to+\infty} J(u_n) &= \liminf_{n\to+\infty}\Bigg[a_1 \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \frac{a_2}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)^2 + b_1\int_\Omega \frac{1}{q(x)}|\nabla u_n|^{q(x)}dx \\
&+ \frac{b_2}{2}\left(\int_\Omega \frac{1}{q(x)}|\nabla u_n|^{q(x)}dx\right)^2 + \int_\Omega \lambda(x)\frac{1}{p(x)}|u_n|^{p(x)}dx + \int_\Omega \lambda(x)\frac{1}{q(x)}|u_n|^{q(x)}dx\Bigg] \\
&\geq \Bigg[\liminf_{n\to+\infty} a_1 \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \liminf_{n\to+\infty}\frac{a_2}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)^2 + \liminf_{n\to+\infty} b_1\int_\Omega \frac{1}{q(x)}|\nabla u_n|^{q(x)}dx \\
&+ \liminf_{n\to+\infty}\frac{b_2}{2}\left(\int_\Omega \frac{1}{q(x)}|\nabla u_n|^{q(x)}dx\right)^2 + \liminf_{n\to+\infty}\int_\Omega \lambda(x)\frac{1}{p(x)}|u_n|^{p(x)}dx + \liminf_{n\to+\infty}\int_\Omega \lambda(x)\frac{1}{q(x)}|u_n|^{q(x)}dx\Bigg] \\
&\geq a_1 J_{p(x)}(u) + \frac{a_2}{2}\left(J_{p(x)}(u)\right)^2 + b_1 J_{q(x)}(u) + \frac{b_2}{2}\left(J_{q(x)}(u)\right)^2 + J^{\lambda(x)}_{p(x)}(u) + J^{\lambda(x)}_{q(x)}(u) \\
&\geq J(u),
\end{aligned} \tag{3.23}$$

which means J is sequentially weakly lower semi-continuous.
Claim 2: Proving that H is weakly-lower semi-continuous.
Proposition 2.2 implies that there exists a sub-sequence of $(u_n)_n$ converging to u uniformly on compact subsets of Ω. Then,

$$(u_n)_n \text{ converges to } u \text{ uniformly in } \Omega,$$
$$k := \sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty(\Omega)} < +\infty.$$

Thus, almost everywhere in Ω, we have $\lim_{n\to\infty} F_2(x, u_n(x)) = F_2(x, u(x))$ and $|F_2(x, u_n(x))| \leq k \sup_{|s|\leq k} |f_2(x, s)|$ for all n. Since $\sup_{|s|\leq k} |f_2(x, s)| \in L^1(\Omega)$ by (1.4). Thus, the dominated convergence theorem gives that $\lim_{n\to\infty} H(u_n) = H(u)$. The weak semi-continuity of the functional H implies that Φ_2 is sequentially weak with lower semi-continuity, which means that Φ_1 is sequentially weakly continuous. □

We will now demonstrate that Φ_2 is coercive. For brevity, we will write c_i for some positive constant throughout.

Proposition 3.1 *Assume that G satisfies exactly one of the following two conditions:*

1. *There exist $\tau > 0$, $0 < \varepsilon < \dfrac{p^+ \min(a_1, b_1, \lambda_0)}{2^{p^- -1} q^+}$ and $\theta_1,\ \theta_2,\ \theta_3 \in L^1(\Omega)$ with $\theta_1 \neq 0$ and $\theta_2 \neq 0$ such that*

$$|F_2(x,t)| \leq \varepsilon \left(\frac{\theta_1(x)}{p^+ C_1^{p^-} \|\theta_1\|_{L^1(\Omega)}} |t|^{p^-} + \frac{\theta_2(x)}{q^+ C_2^{q^-} \|\theta_2\|_{L^1(\Omega)}} |t|^{q^-} \right) + \theta_3(x), \tag{3.24}$$

   for almost all $x \in \Omega$ and all $t \geq \tau$.

2. *There exist $\tau > 0$, $0 < \varepsilon < \dfrac{p^-}{q^+} \dfrac{\lambda_0}{\|\lambda\|_{L^\infty(\Omega)}}$ and $\theta_4 \in L^1(\Omega)$ such that*

$$|F_2(x,t)| \leq \varepsilon \left(\frac{\lambda(x)}{p(x)} |t|^{p(x)} + \frac{\lambda(x)}{q(x)} |t|^{q(x)} \right) + \theta_4(x), \tag{3.25}$$

   for almost all $x \in \Omega$ and all $t \geq \tau$.

Then, Φ_2 is coercive.

Proof Suppose (3.24) holds. Then, without loss of generality, we have

$$\begin{aligned}
\Phi_2(u) &= a_1 J_{p(x)}(u) + \frac{a_2}{2} \left(J_{p(x)}(u)\right)^2 + b_1 J_{q(x)}(u) + \frac{b_2}{2} \left(J_{q(x)}(u)\right)^2 + J^{\lambda(x)}_{p(x)}(u) + J^{\lambda(\cdot)}_{q(x)}(u) - \int_\Omega F_2(x,u)dx \\
&\geq a_1 \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + b_1 \int_\Omega \frac{1}{q(x)} |\nabla u|^{q(x)} dx + \int_\Omega \lambda(x) \left(\frac{1}{p(x)} |u|^{p(x)} + \frac{1}{q(x)} |u|^{q(x)} \right) dx \\
&\quad - \int_\Omega \left[\frac{\varepsilon \theta_1(x) |u|^{p^-}}{p^+ C_1^{p^-} \|\theta_1\|_{L^1(\Omega)}} + \frac{\varepsilon \theta_2(x) |u|^{q^-}}{q^+ C_2^{q^-} \|\theta_2\|_{L^1(\Omega)}} + \theta_3(x) \right] dx \\
&\geq \frac{\min(a_1, b_1)}{q^+} \left(\int_\Omega |\nabla u|^{p(x)} dx + \int_\Omega |\nabla u|^{q(x)} dx \right) + \frac{\lambda_0}{q^+} \int_\Omega \left(|u|^{p(x)} + |u|^{q(x)} \right) dx \\
&\quad - \left[\|\theta_3\|_{L^1(\Omega)} + \frac{\varepsilon \|u\|^{p^-}_{L^\infty(\Omega)}}{p^+ C_1^{p^-}} + \frac{\varepsilon \|u\|^{q^-}_{L^\infty(\Omega)}}{q^+ C_2^{q^-}} \right] \\
&\geq \frac{\min(a_1, b_1)}{q^+} \left(\|\nabla u\|^{p^-}_{p(\cdot)} + \|\nabla u\|^{q^-}_{q(\cdot)} - 2 \right) + \frac{\lambda_0}{q^+} \left(\|u\|^{p^-}_{p(\cdot)} + \|u\|^{q^-}_{q(\cdot)} - 2 \right) \\
&\quad - \left[\|\theta_3\|_{L^1(\Omega)} + \frac{\varepsilon \|u\|^{p^-}_{1,p(\cdot)}}{p^+} + \frac{\varepsilon \|u\|^{q^-}_{1,q(\cdot)}}{q^+} \right] \\
&\geq \frac{\min(a_1, b_1, \lambda_0)}{2^{p^- -1} q^+} \|u\|^{p^-}_{1,p(\cdot)} + \frac{\min(a_1, b_1, \lambda_0)}{2^{q^- -1} q^+} \|u\|^{q^-}_{1,q(\cdot)} - \varepsilon \left[\frac{\|u\|^{p^-}_{1,p(\cdot)}}{p^+} + \frac{\|u\|^{q^-}_{1,q(\cdot)}}{q^+} \right] - c_1 \\
&\geq \left(\frac{\min(a_1, b_1, \lambda_0)}{2^{p^- -1} q^+} - \frac{\varepsilon}{p^+} \right) \|u\|^{p^-}_{1,p(\cdot)} + \left(\frac{\min(a_1, b_1, \lambda_0)}{2^{q^- -1} q^+} - \frac{\varepsilon}{q^+} \right) \|u\|^{q^-}_{1,q(\cdot)} - c_1 \\
&\geq c_2 \left(\|u\|^{p^-}_{1,p(\cdot)} + \|u\|^{q^-}_{1,q(\cdot)} \right) - c_1 \\
&\geq c_2 \left(\|u\|^{p^-}_{1,p(\cdot)} + \|u\|^{p^-}_{1,q(\cdot)} - 1 \right) - c_3 \\
&\geq \frac{c_2}{2^{p^- -1}} \|u\|^{p^-}_{1,p(\cdot),q(\cdot)} - c_3.
\end{aligned} \tag{3.26}$$

Under assumption (3.25) by using similar arguments as above, we get

$$\begin{aligned}
\Phi_2(u) &\geq \frac{\min(a_1,b_1)}{q^+}\left(\|\nabla u\|_{p(\cdot)}^{p^-}+\|\nabla u\|_{q(\cdot)}^{q^-}-2\right)+\frac{\lambda_0}{q^+}\left(\|u\|_{p(\cdot)}^{p^-}+\|u\|_{q(\cdot)}^{q^-}-2\right)\\
&\quad -\int_\Omega\left(\frac{\varepsilon\lambda(x)}{p(x)}|u|^{p(x)}+\frac{\varepsilon\lambda(x)}{q(x)}|u|^{q(x)}+\theta_4(x)\right)dx\\
&\geq \frac{\min(a_1,b_1)}{q^+}\left(\|\nabla u\|_{p(\cdot)}^{p^-}+\|\nabla u\|_{q(\cdot)}^{q^-}-2\right)+\frac{\lambda_0}{q^+}\left(\|u\|_{p(\cdot)}^{p^-}+\|u\|_{q(\cdot)}^{q^-}-2\right)\\
&\quad -\varepsilon\frac{\|\lambda\|_{L^\infty(\Omega)}}{p^-}\left(\|u\|_{p(\cdot)}^{p^-}+\|u\|_{q(\cdot)}^{q^-}-2\right)-c_4\\
&\geq \frac{\min(a_1,b_1)}{q^+}\left(\|\nabla u\|_{p(\cdot)}^{p^-}+\|\nabla u\|_{q(\cdot)}^{q^-}\right)+\left(\frac{\lambda_0}{q^+}-\varepsilon\frac{\|\lambda\|_{L^\infty(\Omega)}}{p^-}\right)\left(\|u\|_{p(\cdot)}^{p^-}+\|u\|_{q(\cdot)}^{q^-}\right)-c_5\\
&\geq \min\left(\frac{\min(a_1,b_1)}{q^+},\left(\frac{\lambda_0}{q^+}-\varepsilon\frac{\|\lambda\|_{L^\infty(\Omega)}}{p^-}\right)\right)\left(\|\nabla u\|_{p(\cdot)}^{p^-}+\|u\|_{p(\cdot)}^{p^-}+\|\nabla u\|_{q(\cdot)}^{q^-}+\|u\|_{q(\cdot)}^{q^-}\right)-c_5\\
&\geq \frac{c_6}{2^{q^--1}}\left(\|u\|_{1,p(\cdot)}^{p^-}+\|u\|_{1,q(\cdot)}^{q^-}\right)-c_5\\
&\geq \frac{c_6}{2^{q^--1}}\left(\|u\|_{1,p(\cdot)}^{p^-}+\|u\|_{1,q(\cdot)}^{p^-}\right)-c_7\\
&\geq \frac{c_6}{2^{q^--1}\times 2^{p^--1}}\|u\|_{1,p(\cdot),q(\cdot)}^{p^-}-c_7.
\end{aligned} \tag{3.27}$$

Thanks to (3.26)-(3.27) and as a result of the coercivity of Φ_2, it follows that there are constants α_1 and α_2 satisfy:

$$\Phi_2(u)\geq \alpha_1\|u\|_{1,p(\cdot),q(\cdot)}^{p^-}\quad \text{for any } \|u\|_{1,p(\cdot),q(\cdot)}\geq\alpha_2. \tag{3.28}$$

□

Lemma 3.3 *Assume the hypothesis (1.3) and one of the assumptions (3.24) and (3.25). Hence, there are positives constants δ_1, δ_2 and δ_3 such that*

$$\int_\Omega\lambda(x)\left(\frac{|\varsigma|^{p(x)}}{p(x)}+\frac{|\varsigma|^{q(x)}}{q(x)}\right)dx-\int_\Omega f_2(x,\varsigma)dx\leq\delta_1|\varsigma|^{q^+}+\delta_2,\ \textit{for any } \varsigma\in\mathbb{R}. \tag{3.29}$$

Proof Under the assumptions (1.3) and (3.24) implies

$$\begin{aligned}
&\int_\Omega\lambda(x)\left(\frac{|\varsigma|^{p(x)}}{p(x)}+\frac{|\varsigma|^{q(x)}}{q(x)}\right)dx-\int_\Omega F_2(x,\varsigma)dx\\
&\leq\|\lambda\|_{L^\infty(\Omega)}|\Omega|\left(\frac{1}{p^-}|\varsigma|^{p^+}+\frac{1}{q^-}|\varsigma|^{q^+}\right)+\varepsilon\int_\Omega\frac{\theta_1(x)}{p^-C_1^{p^-}\|\theta_1\|_{L^1(\Omega)}}|\varsigma|^{p^+}dx+\varepsilon\int_\Omega\frac{\theta_2(x)}{q^-C_2^{q^-}\|\theta_2\|_{L^1(\Omega)}}|\varsigma|^{q^+}dx+\int_\Omega\theta_3(x)dx\\
&\leq\|\lambda\|_{L^\infty(\Omega)}|\Omega|\left(\frac{1}{p^-}|\varsigma|^{q^+}+\frac{1}{p^-}|\varsigma|^{q^+}\right)+\frac{\varepsilon}{p^-C_1^{p^-}}\int_\Omega|\varsigma|^{q^+}dx+\frac{\varepsilon}{p^-C_2^{q^-}}\int_\Omega|\varsigma|^{q^+}dx+\|\theta_3\|_{L^1(\Omega)}\\
&\leq\left(\frac{2\|\lambda\|_{L^\infty(\Omega)}|\Omega|}{p^-}+\frac{\varepsilon|\Omega|}{p^-}\left(\frac{1}{C_1^{p^-}}+\frac{1}{C_2^{q^-}}\right)\right)|\varsigma|^{q^+}+\|\theta_3\|_{L^1(\Omega)},\ \text{for sufficiently large }\varsigma.
\end{aligned}$$

Then, we have (3.29) for $\delta_1=\left(\frac{2\|\lambda\|_{L^\infty(\Omega)}|\Omega|}{p^-}+\frac{\varepsilon|\Omega|}{p^-}\left(\frac{1}{C_1^{p^-}}+\frac{1}{C_2^{q^-}}\right)\right)$ and $\delta_2=\|\theta_3\|_{L^1(\Omega)}$.

On the other hand, assumptions (1.3) and (3.25) implies

$$\begin{aligned}
&\int_\Omega \lambda(x)\left(\frac{1}{p(x)}|\varsigma|^{p(x)}+\frac{1}{q(x)}|\varsigma|^{q(x)}\right)dx-\int_\Omega F_2(x,\varsigma)dx\\
&\le \|\lambda\|_{L^\infty(\Omega)}|\Omega|\left(\frac{1}{p^-}|\varsigma|^{q^+}+\frac{1}{p^-}|\varsigma|^{q^+}\right)+\varepsilon\int_\Omega\lambda(x)\left(\frac{1}{p(x)}|t|^{p(x)}+\frac{1}{q(x)}|t|^{q(x)}\right)dx+\int_\Omega\theta_4(x)dx\\
&\le \frac{2\|\lambda\|_{L^\infty(\Omega)}|\Omega|}{p^-}|\varsigma|^{q^+}+\frac{2\varepsilon\|\lambda\|_{L^\infty(\Omega)}|\Omega|}{p^-}|\varsigma|^{q^+}+\|\theta_4\|_{L^1(\Omega)}\\
&\le \frac{2(1+\varepsilon)\|\lambda\|_{L^\infty(\Omega)}|\Omega|}{p^-}|\varsigma|^{q^+}+\|\theta_4\|_{L^1(\Omega)},\ \text{for } \varsigma \text{ large enough.}
\end{aligned}$$

Therefore, we establish (3.29) with δ_1 and δ_2 being $\dfrac{2(1+\varepsilon)\|\lambda\|_{L^\infty(\Omega)}|\Omega|}{p^-}$ and $\|\theta_4\|_{L^1(\Omega)}$, respectively, for ς sufficiently large. □

The following theorem is our first main result.

Theorem 3.1 *Assuming that (1.3) and (1.4) are satisfied, and either (3.24) or (3.25) hold, and that F satisfies condition (3.5). Additionally, we assume that F_1 satisfies the following condition*

$$\int_\Omega \lambda(x)\left(\frac{1}{p(x)}|\varsigma|^{p(x)}+\frac{1}{q(x)}|\varsigma|^{q(x)}\right)dx-\int_\Omega\Big(F_2(x,\varsigma)+F_1(x,\varsigma)\Big)dx=-\infty. \tag{3.30}$$

Moreover, there exist positive sequences $(a_n)_n$ and $(b_n)_n$ such that

$$\lim_{n\to\infty} b_n=+\infty,\quad \lim_{n\to\infty}\frac{a_n^{q^+}}{b_n^{p^-}}=0. \tag{3.31}$$

Finally, we assume the existence of a positive integrable function with $\|h\|_{L^1(\Omega)}=1$ and some positive constants $\delta_1,\delta_2,\delta_3>0$ such that

$$F_1(x,a_n)+h(x)\left(\alpha_1\left(\frac{b_n}{C_0}\right)^{p^-}-\delta_1|a_n|^{q^+}-\delta_2\right)\ge \sup_{t\in[a_n,b_n]}F_1(x,t), \tag{3.32}$$

$$F_1(x,-a_n)+h(x)\left(\alpha_1\left(\frac{b_n}{C_0}\right)^{p^-}-\delta_1|a_n|^{q^+}-\delta_2\right)\ge \sup_{t\in[-b_n,-a_n]}F_1(x,t), \tag{3.33}$$

for any n we have for almost all x in Ω, where α_1 is the coercivity constant defined in (3.28), $\delta_1=\left(\dfrac{2\|\lambda\|_{L^\infty(\Omega)}|\Omega|}{p^-}+\dfrac{\varepsilon|\Omega|}{p^-}\left(\dfrac{1}{C_1^{p^-}}+\dfrac{1}{C_2^{q^-}}\right)\right)$ and $\delta_2=\|\theta_3\|_{L^1(\Omega)}$ if we assume (3.24) and $\delta_1=\dfrac{2(1+\varepsilon)\|\lambda\|_{L^\infty(\Omega)}|\Omega|}{p^-}$ and $\delta_2=\|\theta_4\|_{L^1(\Omega)}$ if we assume (3.25). The last inequalities (3.32) and (3.33) are strict on a non-negligible subset of Ω.

Then, we can construct a sequence $(v_n)_n$ of local minima of $\Phi_1+\Phi_2$ such that $\lim\limits_{n\to\infty}\Phi_2(v_n)=\infty$. Consequently, the problem (1.2) admits an unbounded sequence of weak solutions.

Proof For $r>\inf\limits_{u\in W^{1,p(\cdot),q(\cdot)}(\Omega)}\Phi_2(u)$, we define

$$\Theta(r)=\inf\Big\{\kappa>0:\Phi_1^{-1}(]-\infty,r[)\subset\overline{\mathbf{B}(0,\kappa)}\Big\}, \tag{3.34}$$

where $\mathbf{B}(0,\kappa)$ denoted the ball centered at 0 with radius κ in $W^{1,p(\cdot),q(\cdot)}(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(\cdot),q(\cdot)}$, and $\overline{\mathbf{B}(0,\kappa)}$ denote its closure in $W^{1,p(\cdot),q(\cdot)}(\Omega)$. Since Φ_2 is coercive, we have $\Theta(r)\in]0,+\infty[$ for all $r>\inf\limits_{u\in W^{1,p(\cdot),q(\cdot)}(\Omega)}\Phi_2(u)$. Owing of (3.28), we have

$$\text{if}\quad \Phi_2(u)<\alpha_1\|u\|_{1,p(\cdot),q(\cdot)}^{p^-},\quad \text{then}\quad \|u\|_{1,p(\cdot),q(\cdot)}<\alpha_2.$$

With the help of (3.34), one may observe that $\Phi_2^{-1}(]-\infty, r[) \subset \overline{\mathbf{B}(0,\Theta(r))}$ yields $\overline{(\Phi_2^{-1}(]-\infty, r[))_w} \subset \overline{\mathbf{B}(0,\Theta(r))}$. By using (2.7), we get,

$$\overline{\mathbf{B}(0,\Theta(r))} \subset \left\{u \in \mathcal{C}(\overline{\Omega}) : \|u\|_{L^\infty(\Omega)} \le C_0\Theta(r)\right\},$$

which yields

$$\inf_{v\in\overline{(\Phi_2^{-1}(]-\infty,r[))_w}} \Phi_1(v) \ge \inf_{\|v\|_{1,p(\cdot),q(\cdot)}\le\Theta(r)} \Phi_1(v) \ge \inf_{\|v\|_{L^\infty(\Omega)}\le C_0\Theta(r)} \Phi_1(v). \tag{3.35}$$

Suppose $\kappa \ge \alpha_1\alpha_2^{p^-}$ and let $u \in W^{1,p(\cdot),q(\cdot)}(\Omega)$ satisfy $\Phi_2(u) < \kappa$. If $\|u\|_{1,p(\cdot),q(\cdot)} \ge \alpha_2$, then by (3.28), we have

$$\kappa > \Phi_2(u) \ge \alpha_1\|u\|^{p^-}_{1,p(\cdot),q(\cdot)},$$

this shows that $\|u\|_{1,p(\cdot),q(\cdot)} \le \left(\dfrac{\kappa}{\alpha_1}\right)^{\frac{1}{p^-}}$. If $\|u\|_{1,p(\cdot),q(\cdot)} < \alpha_2$, it is easy to see that $\|u\|_{1,p(\cdot),q(\cdot)} \le \left(\dfrac{\kappa}{\alpha_1}\right)^{\frac{1}{p^-}}$. By the definition of $\Theta(\kappa)$, we have

$$\Theta(\kappa) \le \left(\frac{\kappa}{\alpha_1}\right)^{\frac{1}{p^-}}. \tag{3.36}$$

Since $F_1(x,\cdot)$ satisfies condition (3.5), for each n, there exists $\varsigma_n \in [-a_n, a_n]$ such that

$$F_1(x,\varsigma_n) = \sup_{t\in[-a_n,a_n]} F_1(x,t) \quad \text{for almost all } x \text{ in } \quad \Omega. \tag{3.37}$$

In order to satisfy (b) of Theorem 1.1, we consider u_n as the constant function with value ς_n and $r_n = \alpha_1\left(\dfrac{b_n}{C_0}\right)^{p^-}$, which leads to $\lim\limits_{n\to\infty} r_n \to +\infty$. Using (3.36), we conclude that

$$\Theta(r_n) \le \frac{b_n}{C_0} \qquad \text{then} \qquad C_0\Theta(r_n) \le b_n. \tag{3.38}$$

By (3.29), one has

$$\begin{aligned} m_n = \int_\Omega \lambda(x)\left(\frac{1}{p(x)}|\varsigma|^{p(x)} + \frac{1}{q(x)}|\varsigma|^{q(x)}\right)dx - \int_\Omega F_2(x,\varsigma)dx &\le \delta_1|\varsigma|^{q^+} + \delta_2 \\ &\le \delta_1|a_n|^{q^+} + \delta_2. \end{aligned}$$

For n large enough (3.31) can be write

$$\delta_1|a_n|^{q^+} + \delta_2 < \alpha_1\left(\frac{b_n}{C_0}\right)^{p^-} = r_n,$$

consequently we find $m_n < r_n$ which means (1.8) holds. Without loss of generality, we can assume that (1.8) holds for all n.

From (3.32)-(3.33) and (3.37), we may find the following inequality

$$F_1(x,\varsigma_n) + h(x)(r_n - m_n) \ge \sup_{|t|\le b_n} F_1(x,t) \;\; \text{a.e. in} \quad \Omega, \tag{3.39}$$

which is strict on a non-negligible subdomain of Ω. Using (3.38) and (3.39), we obtain (1.9) and (1.10) follows directly from (3.30).

Then, hypotheses of Theorem 1.1 (b) hold true which completes the proof of Theorem 3.1. □

Now, we present an example to illustrate the results cited in the previous theorem.

Corolary 3.1 *Let* $\Omega =]0,1[^2$ *then* $N = 2$ *and let* $p(x) = \dfrac{5}{2} + \dfrac{1}{4}|\sin(x+y)|$, *and* $q(x) = 3 + \dfrac{1}{2}|\cos(x+y)|$, *then* $p^- = \dfrac{5}{2}$, $p^+ = \dfrac{11}{4}$, $q^- = 3$ *and* $q^+ = \dfrac{7}{2}$. *Consider the functions* $f_2(x,t) = \dfrac{13}{100}\left(\dfrac{p^-\theta_1(x)}{p^+C_1^{p^-}}|t|^{p^--1} + \dfrac{q^-\theta_2(x)}{q^+C_2^{q^-}}|t|^{q^--1}\right)$

where $\theta_1, \theta_2 \in L^1(\Omega)$ are positives functions with $\|\theta_1\|_{L^1(\Omega)} = \|\theta_2\|_{L^1(\Omega)} = 1$, and consider $f_1(x,t) \equiv \alpha(x)g(t)$, with a positive function $\alpha \in L^1(\Omega)$ such that $\|\alpha\|_{L^1(\Omega)} = 1$ and a continuous function g such that $g(t) = G'(t)$ and $G(-t) = G(t)$. Then, the following nonlinear elliptic double phase Kirchhoff-type problem

$$\begin{cases} -\left(1 + 2\int_\Omega \frac{1}{\left(\frac{5}{2} + \frac{1}{4}|\sin(x+y)|\right)} |\nabla u|^{\left(\frac{5}{2}+\frac{1}{4}|\sin(x+y)|\right)} dx\right) \Delta_{\left(\frac{5}{2}+\frac{1}{4}|\sin(x+y)|\right)} u \\ -\left(1 + 2\int_\Omega \frac{1}{\left(3 + \frac{1}{2}|\cos(x+y)|\right)} |\nabla u|^{\left(3+\frac{1}{2}|\cos(x+y)|\right)} dx\right) \Delta_{\left(3+\frac{1}{2}|\cos(x+y)|\right)} u \\ +\frac{2+x^2+y^2}{1+x^2+y^2}\left(|u|^{\left(\frac{1}{2}+\frac{1}{4}|\sin(x+y)|\right)} u + |u|^{\left(1+\frac{1}{2}|\cos(x+y)|\right)} u\right) \\ = \alpha(x)g(t) + \frac{13}{100}\left(\frac{10\,\theta_1(x)}{11 C_1^{\frac{5}{2}}}|t|^{\frac{3}{2}} + \frac{6\theta_2(x)}{7C_2^3}|t|^2\right) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.40}$$

has a sequence of weak solutions $(u_n)_n$ in $W^{1,p(\cdot),q(\cdot)}(\Omega)$ with unbounded norm.

Proof It is clear that $0 < \varepsilon = \frac{13}{100} < \frac{p^+ \min(a_1, b_1, \lambda_0)}{2^{p^- - 1} q^+} = 0.27$, then (3.24) is satisfied. Furthermore, we have $F_1(x,t) = \alpha(x)G(t)$, and we can choose two positive sequences $(a_n)_n$ and $(b_n)_n$ such that $a_1 \geq 1$, $b_n^{\frac{5}{2}} = 2n^2 a_n^3$ and $a_{n+1} > b_n$ for every n. Define $G(a_n) = a_n^4$ and $G(b_n)$ such that

$$G(a_n) < G(b_n) < \left(\alpha_1\left(\frac{b_n}{C_0}\right)^{\frac{5}{2}} - \delta_1|a_n|^3 - \delta_2\right) + G(a_n), \tag{3.41}$$

where $\delta_1 = \frac{4}{5}|\Omega| + \frac{13}{250}|\Omega|\left(\frac{1}{C_1^{\frac{5}{2}}} + \frac{1}{C_2^3}\right)$ and $\delta_2 = \|\theta_3\|_{L^1(\Omega)}$.

Put $r_n = \alpha_1\left(\frac{b_n}{C_0}\right)^{p^-}$ and $\varsigma_n = a_n$.

Since

$$\int_\Omega \frac{\lambda(x)}{p(x)}|a_n|^{p(x)}dx + \int_\Omega \frac{\lambda(x)}{q(x)}|a_n|^{q(x)}dx - \int_\Omega F_2(x,a_n)dx - \int_\Omega F_1(x,a_n)dx$$
$$\leq \delta_1|a_n|^{q^+} + \delta_2 - \|\alpha\|_{L^1(\Omega)} a_n^{q^+ + 1} \to -\infty,$$

as $n \to \infty$, then the conditions (3.30)-(3.31) hold true. Taking $h(x) = \alpha(x)$, then (3.53) implies (3.32)-(3.33). As a result, all the assumptions of Theorem 3.1 are satisfied, then problem (3.40) has a sequence of weak solutions $(u_n)_n$ in $W^{1,p(\cdot),q(\cdot)}(\Omega)$ with unbounded norm. □

The second main result is cited in the following theorem.

Theorem 3.2 *Assume the assumptions (1.3) and (1.4) and the following hypothesis :*

$$F_2(x,t) \text{ is non-positive for almost every } x \in \Omega \text{ and for all } t \in \mathbb{R}. \tag{3.42}$$

There exists $\delta, \epsilon > 0$ such that

$$-F_2(x,t) \leq \delta|t|^{p^-} \text{ for almost every } x \in \Omega \text{ and for } |t| \leq \epsilon. \tag{3.43}$$

In addition, assume that the functional F_1 satisfies the condition (3.5) and

$$\limsup_{|\varsigma| \to 0} \frac{\int_\Omega F_1(x,\varsigma)dx + \int_\Omega F_2(x,\varsigma)dx}{|\varsigma|^{p^-}} > \int_\Omega \lambda(x)\left(\frac{1}{p(x)}|\varsigma|^{p(x)} + \frac{1}{q(x)}|\varsigma|^{q(x)}\right)dx. \tag{3.44}$$

Assume that $(a_n)_n$ and $(b_n)_n$ are positive sequences satisfying

$$\lim_{n\to\infty} b_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{a_n^{p^-}}{b_n^{q^+}} = 0. \tag{3.45}$$

Additionally, there exists a positive function $h \in L^1(\Omega)$, where $\|h\|_{L^1(\Omega)} = 1$, such that for every n and all x in Ω, we obtain

$$F_1(x, a_n) + h(x)\left(\delta_3\left(\frac{b_n}{C_0}\right)^{q^+} - \delta_4 a_n^{p^-}\right) \geq \sup_{t\in[a_n,b_n]} F_1(x,t), \tag{3.46}$$

$$F_1(x, -a_n) + h(x)\left(\delta_3\left(\frac{b_n}{C_0}\right)^{q^+} - \delta_4 a_n^{p^-}\right) \geq \sup_{t\in[-b_n,-a_n]} F_1(x,t), \tag{3.47}$$

with $\delta_3 = \dfrac{\min(a_1, b_1, \lambda_0)}{2^{2q^+-2}q^+}$ and $\delta_4 = \left(\dfrac{2\|\lambda\|_{L^\infty(\Omega)}}{p^-} + \delta\right)|\Omega|$, the relations (3.46) and (3.47) are strict on a non negligible subdomain of Ω. Accordingly we can construct a sequence $(v_n)_n$ of separate local minima of $\Phi_1 + \Phi_2$ where v_n tends to 0 in $W^{1,p(\cdot),q(\cdot)}(\Omega)$. Hence, a sequence of non-zero weak solutions to problem (1.2) exists and converges strongly to 0 in $W^{1,p(\cdot),q(\cdot)}(\Omega)$.

Proof We now proceed to demonstrate that Theorem 1.1 (c) holds by verifying all the assumptions. Taking into account the inequality (3.42), for $\|u\|_{1,p(\cdot),q(\cdot)} \leq 1$ one has

$$\begin{aligned}
\Phi_2(u) &= J(u) - \int_\Omega F_2(x,u)dx \\
&\geq a_1 J_{p(x)}(u) + \frac{a_2}{2}\left(J_{p(x)}(u)\right)^2 + b_1 J_{q(x)}(u) + \frac{b_2}{2}\left(J_{q(x)}(u)\right)^2 + J_{p(x)}^{\lambda(x)}(u) + J_{q(x)}^{\lambda(x)}(u) \\
&\geq \frac{\min(a_1,b_1)}{q^+}\left(\|\nabla u\|_{p(\cdot)}^{p^+} + \|\nabla u\|_{q(\cdot)}^{q^+}\right) + \frac{\lambda_0}{q^+}\left(\|u\|_{p(\cdot)}^{p^+} + \|u\|_{q(\cdot)}^{q^+}\right) \\
&\geq \frac{\min(a_1,b_1,\lambda_0)}{2^{q^+-1}q^+}\|u\|_{1,p(\cdot)}^{q^+} + \frac{\min(a_1,b_1,\lambda_0)}{2^{q^+-1}q^+}\|u\|_{1,q(\cdot)}^{q^+} \\
&\geq \frac{\min(a_1,b_1,\lambda_0)}{2^{2q^+-2}q^+}\|u\|_{1,p(\cdot),q(\cdot)}^{q^+} \\
&\geq \delta_3 \|u\|_{1,p(\cdot),q(\cdot)}^{q^+},
\end{aligned}$$

with $\delta_3 = \dfrac{\min(a_1,b_1,\lambda_0)}{2^{2q^+-2}q^+}$. Consequently, Φ_2 is coercive, $\inf_{W^{1,p(\cdot),q(\cdot)}(\Omega)} \Phi_2 = \Phi_2(0) = 0$ and 0 is the exclusive global minimizer of Φ_2. Owing to (3.44) one has

$$\begin{aligned}
&\limsup_{|\varsigma|\to 0}\left\{\Phi_2(\varsigma) + \Phi_1(\varsigma)\right\} \\
&= \limsup_{|\varsigma|\to 0}\left\{\int_\Omega \lambda(x)\left(\frac{|\varsigma|^{p(x)}}{p(x)} + \frac{|\varsigma|^{q(x)}}{q(x)}\right)dx - \int_\Omega F_1(x,\varsigma)dx - \int_\Omega F_2(x,\varsigma)\,dx\right\} \\
&\leq \limsup_{|\varsigma|\to 0}\left\{\int_\Omega \lambda(x)\left(\frac{|\varsigma|^{p^-}}{p(x)} + \frac{|\varsigma|^{p^-}}{q(x)}\right)dx - \int_\Omega F_1(x,\varsigma)dx - \int_\Omega F_2(x,\varsigma)\,dx\right\} < 0,
\end{aligned}$$

that is, at 0, $\Phi_1 + \Phi_2$ does not reach a local minimum, as a result (1.11) is fulfilled.

Let r be small enough so that if $\Phi_2(u) < r$, then $\|u\|_{1,p(\cdot),q(\cdot)}$ is no greater than $\left(\dfrac{r}{\delta_3}\right)^{\frac{1}{q^+}}$. It follows that $\Theta(r)$ is less than or equal to $\left(\dfrac{r}{\delta_3}\right)^{\frac{1}{q^+}}$.

Now put $r_n = \delta_3\left(\dfrac{b_n}{C_0}\right)^{q^+}$ If we set u_0 and u_n to be the constant functions ς_0 and ς_n, respectively, as prescribed by Theorem 1.1, then

$$C_0\Theta(r_n) \leq b_n. \tag{3.48}$$

The inequalities (3.43) and (1.3) affirm the existence of a sequence $(\varsigma_n)_n \subset \mathbb{R}$ such that $\varsigma_n \in [-a_n, a_n]$ such that for any a_n small enough,

$$\begin{aligned} m_n &= \int_\Omega \lambda(x)\left(\frac{|\varsigma_n|^{p(x)}}{p(x)} + \frac{|\varsigma_n|^{q(x)}}{q(x)}\right)dx - \int_\Omega F_2(x,\varsigma_n)dx \\ &\leq \|\lambda\|_{L^\infty(\Omega)}\left(\frac{1}{p^-}|\varsigma_n|^{p^-} + \frac{1}{p^-}|\varsigma_n|^{p^-}\right) + \delta|\Omega||\varsigma_n|^{p^-} \\ &\leq \left(\frac{2\|\lambda\|_{L^\infty(\Omega)}}{p^-} + \delta\right)|\Omega||\varsigma_n|^{p^-} \\ &\leq \delta_4|a_n|^{p^-}, \end{aligned} \tag{3.49}$$

where $\delta_4 = \left(\frac{2\|\lambda\|_{L^\infty(\Omega)}}{p^-} + \delta\right)|\Omega|$.

If we choose n to be sufficiently large, then it follows from (3.45) that

$$\delta_4|a_n|^{p^-} < \delta_3\left(\frac{b_n}{C_0}\right)^{q^+} = r_n.$$

Then (1.8) is obtained.
Because $F_1(x,\cdot)$ satisfies condition (3.5), for any n, we can find $\varsigma_n \in [-a_n, a_n]$ with

$$F_1(x,\varsigma_n) = \sup_{t\in[-a_n,a_n]} F_1(x,t) \quad \text{a.e. in} \quad \Omega. \tag{3.50}$$

Then, thanks to (3.46) and (3.47) we find the following inequality

$$\sup_{|t|\leq b_n} F_1(x,t) \leq F_1(x,\varsigma_n) + h(x)(r_n - m_n)\text{a.e. in} \quad \Omega, \tag{3.51}$$

which is strict on a non negligible subset of Ω. Then, the inequality (1.9) acquires immediately from (3.48) and (3.51). Consequently, Theorem 1.1 (c) holds, since the necessary hypotheses have been fulfilled.

As a result, we can conclude that a sequence $(v_n)_n$ of separate local minima of $\Phi_1 + \Phi_2$ exists, and satisfies $\lim_{n\mapsto+\infty} \Phi_2(v_n) = 0$. This means that $\lim_{n\mapsto+\infty} \|v_n\|_{1,p(\cdot),q(\cdot)} = 0$, thereby completing the proof. □

We introduce now an example to illustrate the results cited in the second theorem.

Corolary 3.2 *For* $N \geq 1$ *we set* $\Omega = \left\{x\in\mathbb{R}^N : \sum_{i=1}^N x_i^2 < 1\right\}$ *and let* $p(x) = N + 1 + \frac{1}{2}\sum_{i=1}^N x_i^2$, *and* $q(x) = N + 2 + \frac{1}{2N\left(1+\sum_{i=1}^N x_i^2\right)}$, *then* $p^- = N+1$, $p^+ = N + \frac{3}{2}$, $q^- = N+2$ *and* $q^+ = N + 2 + \frac{1}{2N}$. *Consider the functions* $F_2(x,t) = -\frac{N+1}{2N(N+2)+1}\left(\frac{2+\left|\sin\left(\sum_{i=1}^N x_i^2\right)\right|}{N+1+\frac{1}{2}\sum_{i=1}^N x_i^2}|t|^{p(x)} + \frac{2+\left|\sin\left(\sum_{i=1}^N x_i^2\right)\right|}{N+2+\frac{1}{2N\left(1+\sum_{i=1}^N x_i^2\right)}}|t|^{q(x)}\right)$ *and let* $f_1(x,t) \equiv h(x)g(t)$, *with* $h \in L^1(\Omega)$ *be a positive function such that* $\|h\|_{L^1(\Omega)} = 1$ *and* g *a continuous function with* $g(t) = G'(t)$ *and let* $(a_n)_n$ *and* $(b_n)_n$ *are two positives sequences with* $a_n^{p^-} = \frac{1}{n}b_n^{q^+}$ *and* $b_{n+1} < a_n$, *with* $G(0) = 0$, $G(a_n) = a_n^{p^-+1}$ *and*

$$G(a_n) < G(b_n) < \left(\delta_3\left(\frac{b_n}{C_0}\right)^{q^+} - \delta_4|a_n|^{p^-}\right) + G(a_n). \tag{3.52}$$

Then, the following nonlinear Kirchhoff problem

$$\begin{cases} -\left(\frac{1}{2}+3\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(\cdot)}u-\left(1+\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\right)\Delta_{q(\cdot)}u \\ +\left(2+\left|\sin\left(\sum_{i=1}^N x_i^2\right)\right|\right)\left(|u|^{p(x)-2}u+|u|^{q(x)-2}u\right) \\ = h(x)f_1(t)-\frac{N+1}{2N(N+2)+1}\left(2+\left|\sin\left(\sum_{i=1}^N x_i^2\right)\right|\right)\left(|u|^{p(x)-2}u+|u|^{q(x)-2}u\right) \ \ in\ \Omega, \\ \frac{\partial u}{\partial \nu}=0 \quad on \quad \partial\Omega, \end{cases} \tag{3.53}$$

has weak solutions formed by a sequence $(u_n)_n$ in $W^{1,p(\cdot),q(\cdot)}(\Omega)$ such that

$$\lim_{n\to\infty}\|u_n\|_{1,p(\cdot),q(\cdot)}=\infty.$$

Proof Put $r_n=\delta_3\Big(\frac{b_n}{C_0}\Big)^{N+2+\frac{1}{2N}}$ and $\varsigma_n=a_n$, one has

$$\begin{aligned}&\int_\Omega F_1(x,a_n)dx+\int_\Omega F_2(x,a_n)dx-\int_\Omega\left(2+\left|\sin\left(\sum_{i=1}^N x_i^2\right)\right|\right)\left(\frac{1}{p(x)}+\frac{1}{q(x)}\right)dx|a_n|^{q^+}\\ &>|a_n|^{p^-+1}-|a_n|^{p^-}-\frac{\lambda|\Omega|}{q^+}|a_n|^{p^-}\longrightarrow 0\ \ as\ \ n\to\infty,\end{aligned}$$

therefore (3.42)-(3.44) hold true. Using (3.52) we obtain the conditions (3.46)-(3.47).

Therefore, we can conclude that the assumptions required by Theorem 3.2 are satisfied, thus completing the proof. □

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