Multiplicity and concentration of solutions to a singular Choquard equation with critical Sobolev exponent *

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Abstract: In this paper, we consider a nonautonomous singular Choquard equation with critical exponent

$$\begin{cases} -\Delta u + V(x)u + \lambda(I_\alpha * |u|^p)|u|^{p-2}u = f(x)u^{-\gamma} + |u|^4 u, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3, \end{cases}$$

where I_α is the Riesz potential of order $\alpha \in (0, 3)$ and $1 + \frac{\alpha}{3} \le p < 3$, $0 < \gamma < 1$. Under certain assumptions on V and f, we show the existence and multiplicity of positive solutions for $\lambda > 0$ by using variational method and Nehari type constraint. We also study concentration of solutions as $\lambda \to 0^+$.

Keywords: Singular Choquard equation; Variational method; Concentration; Critical Sobolev exponent

Mathematics Subject Classification: 35J20, 35J75, 35B09, 35B40

1 Introduction

In this paper, we are interested in the nonautonomous Choquard equation

$$\begin{cases} -\Delta u + V(x)u + \lambda(I_\alpha * |u|^p)|u|^{p-2}u = f(x)u^{-\gamma} + |u|^4 u, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3, \end{cases} \qquad (P_\lambda)$$

where $1 + \frac{\alpha}{3} \le p < 3$, $0 < \gamma < 1$, $\lambda > 0$ and I_α with $\alpha \in (0, 3)$ is the Riesz potential defined by $I_\alpha = \frac{\Gamma(\frac{3-\alpha}{2})}{\Gamma(\frac{\alpha}{2})2^\alpha \pi^{3/2}|x|^{3-\alpha}}$, $x \in \mathbb{R}^3 \setminus \{0\}$. Here, Γ denotes the Gamma function. Throughout the paper, we suppose V and f satisfy:

(V_1) $V \in C(\mathbb{R}^3)$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) > V_0 > 0$, where V_0 is a constant.

(V_2) $\text{meas}\{x \in \mathbb{R}^3 : -\infty < V(x) \le \nu\} < +\infty$ for all $\nu \in \mathbb{R}$.

(f_1) $f \in L^{\frac{6}{5+\gamma}}(\mathbb{R}^3)$ is a positive function.

(f_2) There are $\delta_1 > 0$, $\max\{\frac{3+\gamma}{2}, \frac{5+\gamma-2\alpha}{2}\} < \beta_1 < \frac{5+\gamma}{2}$ and $\rho_1 > 0$ such that $f(x) \ge \rho_1|x|^{-\beta_1}$ for $|x| < \delta_1$.

*This work was supported by the National Natural Science Foundation of China (No. 11871152, 11671085), Natural Science Foundation of Fujian Province (No. 2023J01163).

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Recently, many scholars pay attentions to the following more general Choquard equation

$$-\Delta u+V(x)u+\lambda(I_\alpha*|u|^p)|u|^{p-2}u=h(x,u),\quad x\in\mathbb{R}^N, \tag{1.1}$$

where $N\in\mathbb{N}$ and $\alpha\in(0,N)$. Problem (1.1) with $N=3$, $V(x)=1$, $\lambda=-1$, $p=\alpha=2$ and $h(x,u)=0$ was proposed by Pekar [26] to describe the quantum theory of a polaron at rest and as an approximation to Hartree-Fock theory of one component plasma by Choquard (see [19]). Many papers considered problem (1.1) with $\lambda=-1$: when $V(x)=1$, $2\le p\le\frac{N+\alpha}{N-2}$ and $h(x,u)=0$, Ruiz and Van Schaftingen [27] proved that least energy nodal solutions for problem (1.1) have an odd symmetry with respect to a hyperplane when $\alpha\to 0^+$ or $\alpha\to N^-$. Based on [27], Seok [28] further studied limit profiles of ground states as $\alpha\to 0^+$ or $\alpha\to N^-$. When $N\ge 3$,$V(x)=1$ and $p>1$, Seok [29] considered problem (1.1) with a critical local term and showed the existence of radially symmetric nontrivial solution and concentration results as $\alpha\to 0^+$. When $N\ge 3$ and $V(x)=1+\mu g(x)$ is a potential well, Lü [21] obtained the existence of ground state solutions and concentration results as $\mu\to+\infty$ for problem (1.1) with subcritical exponents and $h(x,u)=0$. Li et al. [14] extended the results of Lü [21] to critical case and obtained the existence of ground state solutions and concentration results as $\alpha\to 0$. Ghimenti, Moroz and Van Schaftingen [6] got the existence of least action sign-changing radial solutions for problem (1.1) with $V(x)=1$, $p=2$ and $h(x,u)=0$. The solution is constructed as the limit of least action sign-changing radial solutions when $p\searrow 2$. When $V(x)=1$, Van Schaftingen and Xia [34], Ao [1], Li and Ma [17], Li and Tang [15], Seok [30], Su and Chen [31] further investigated the existence of solutions for problem (1.1) with lower and upper critical exponents. When $V(x)=1+\mu g(x)$ satisfying some conditions and $\mu<0$, Zhong and Tang [42] investigated the existence of ground state sign-changing solutions for problem (1.1) with a critical pure power nonlinearity. As for $\lambda=1$, Mercuri et al. [23] obtained the existence and regularity of ground state solutions and radial solutions for problem (1.1) with $V(x)=0$, $p>1$ and $h(x,u)=|u|^{q-2}u$, $q>1$. When $N\ge 3$, $p\in\left[1+\frac{\alpha}{N},\frac{N}{N-2}\right)$, Wu [36] investigated the existence, multiplicity and asymptotic behavior of positive solution for problem (1.1) with $V(x)$ and $h(x,u)$ satisfying some suitable conditions. Lü [22] and Li et al. [16] discussed the existence and concentration of solutions for problem (1.1) with Kirchhoff term in $\mathbb{R}^3$. We [38] obtained the existence, uniqueness and asymptotical behavior of solutions to problem (1.1) with $N=3$ and $h(x,u)=f(x)u^{-\gamma}$ i.e. a singular nonlinearity. Mukherjee and Sreenadh [25] investigated a nonlinear Choquard equation with upper critical exponent and singularity. For more related topics, we refer to the survey paper [24] and the references therein.

On the bounded domains $\Omega\subset\mathbb{R}^N$ with $N\ge 3$, problem (1.1) without convolution term i.e. $\lambda=0$ is related to the following equation

$$\begin{cases} -\Delta u=\mu f(x)u^{-\gamma}+|u|^{2^*-2}u, & x\in\Omega,\\ u>0, & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.2}$$

where $2^*=\frac{2N}{N-2}$ is a critical Sobolev exponent. When $f(x)=1$, Coclite and Palmieri [4] showed the existence of a solution of (1.2); Yang [37] improved the result of [4] and obtained multiplicity and asymptotic behavior of positive solutions for (1.2); Hirano et al. [8] further established the multiplicity and regularity of positive solutions for (1.2) with $\gamma>0$; Giacomoni and Saoudi [7] proved a multiplicity result for a more general critical and singular problem, involving also a subcritical term and $0<\gamma<3$;

Mukherjee and Sreenadh[25] investigated existence, multiplicity and regularity of positive solutions for a nonlinear singular Choquard equation with upper critical exponent. Consider (1.2) with parameter μ multiplying the critical term, Hirano et al. [9] studied multiplicity of positive solutions for the problem; Wang et al. [35], Sun and Wu [32] obtained existence and multiplicity of positive solutions and an exact estimate result for the problem. We [41] investigated the relation between the number of the maxima of the coefficient function of the critical term and the number of the positive solutions for elliptic equations with singularity in $\mathbb{R}^3$. Both Lei et al. [11] and Liu et al. [20] got two positive solutions for problem (1.2) with Kirchhoff term. When $N = 3$ and $f(x)$ satisfying some suitable conditions, Lei and Liao [12] obtained two positive solutions for problem (1.2) with Poisson term i.e. a singular Schrödinger-Poisson system. Lei, Suo and Chu [13] studied a Schrödinger-Newton system with singularity and critical growth terms in $\mathbb{R}^N$. We [40] obtained existence, uniqueness and asymptotic behaviour of positive solutions for fractional Schrödinger-Poisson system with singularity in $\mathbb{R}^3$.

To the best of our knowledge, many works which considered concentration of solutions for Choquard equations [6, 14, 16, 21, 27, 28, 29, 36, 38, 39, 40] mainly focus on convergence property of one solution such as one ground state positive or sign-changing (nodal) solution and so on, there are few papers investigated convergence property of multiple solutions. Moreover, comparing problem (P_λ) with the previous mentioned works, we need to overcome the lack of compactness as well as the non-differentiability of the functional of the problem and indirect availability of critical point theory due to the presence of singular term.

Define the function space $E = \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3), \|u\|_E < +\infty\}$, where $\|u\|_E = \left(\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)\mathrm{d}x\right)^{1/2}$ and $L^s(\mathbb{R}^3)$ is a Lebesgue space with the norm $\|u\|_s = (\int_{\mathbb{R}^3}|u|^s\mathrm{d}x)^{\frac{1}{s}}$. Then E is a Hilbert space with the inner product $\langle u,\psi\rangle_E = \int_{\mathbb{R}^3}(\nabla u\nabla\psi + V(x)u\psi)\mathrm{d}x$. Obviously, for $s \in [2,6]$, the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous. By [2], we can further get that under assumptions (V_1) and (V_2), the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is compact for any $s \in [2,6)$.

The energy functional corresponding to problem (P_λ) given by

$$J_\lambda(u) = \frac{1}{2}\|u\|_E^2 + \frac{\lambda}{2p}\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p\mathrm{d}x - \frac{1}{1-\gamma}\int_{\mathbb{R}^3}f(x)|u|^{1-\gamma}\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x, \quad (1.3)$$

and a function $u \in E$ is called a solution of problem (P_λ) if $u > 0$ in $\mathbb{R}^3$ and for every $\psi \in E$,

$$\langle u,\psi\rangle_E + \lambda\int_{\mathbb{R}^3}(I_\alpha * u^p)u^{p-2}u\psi\mathrm{d}x - \int_{\mathbb{R}^3}f(x)u^{-\gamma}\psi\mathrm{d}x - \int_{\mathbb{R}^3}u^5\psi\mathrm{d}x = 0. \quad (1.4)$$

By using variational method and Nehari type constraint, our main results on existence, multiplicity and concentration of solutions with respect to the parameter λ for problem (P_λ) can be stated as follows.

Theorem 1.1. *Suppose $\lambda > 0$, $0 < \gamma < 1$, $1 + \frac{\alpha}{3} \le p < 3$ and (V_1), (V_2), (f_1) hold, then there exists $T_0 > 0$ such that for all $0 < \|f\|_{\frac{6}{5+\gamma}} < T_0$, problem (P_λ) admits a positive ground state solution u_λ satisfying u_λ tends to u_0 in E as $\lambda \to 0^+$, where u_0 is a positive ground state solution of the limit problem*

$$\begin{cases} -\Delta u + V(x)u = f(x)u^{-\gamma} + |u|^4u, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3. \end{cases} \quad (P_0)$$

Theorem 1.2. *Suppose $\lambda > 0$, $0 < \gamma < 1$, $1 + \frac{\alpha}{3} \le p < 3$ and (V_1), (V_2), (f_1), (f_2) hold, then there exists $0 < T_{00} < T_0$ such that for all $0 < \|f\|_{\frac{6}{5+\gamma}} < T_{00}$, problem (P_λ) has at least two solutions: a positive ground state solution u_λ and a positive solution v_λ. Moreover, as $\lambda \to 0^+$, these solutions have the following convergence:*

(i) u_λ tends to u_0 in E, where u_0 is a positive ground state solution of problem (P_0);

(ii) v_λ tends to v_0 in E, where v_0 is a positive solution of problem (P_0) and $\|u_0\|_E^2 < \|v_0\|_E^2$.

Throughout the paper, we use the following notations.

- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with the norm $\|\cdot\|^2 = \int_{\mathbb{R}^3} |\nabla \cdot|^2 \mathrm{d}x$.
- Denote $d_\alpha := \frac{\Gamma(\frac{3-\alpha}{2})}{2^\alpha \pi^{3/2}\Gamma(\frac{3+\alpha}{2})}\left(\frac{\Gamma(\frac{3}{2})}{\Gamma(3)}\right)^{\frac{\alpha}{3}}$ and $\mathbb{D}(u) := \int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p \mathrm{d}x$, then it holds

$$\langle \mathbb{D}'(u), \psi\rangle = 2p\int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^{p-2}u\psi \mathrm{d}x, \ \forall \psi \in E.$$

- $B_r(x)$ is a ball centered at x with radius r.
- C and C_i denotes various positive constants, which may vary from line to line.
- $\to$ (resp. $\rightharpoonup$) denotes the strong (resp. weak) convergence.
- $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ for any function u.
- S is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$, namely,

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x\right)^{\frac{1}{3}}} > 0. \tag{1.5}$$

Hence, $\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x \le S^{-3}\|u\|^6 \le S^{-3}\|u\|_E^6$.

2 Preliminary results

In this and next section, we always assume that all assumptions in Theorem 1.1 hold. It follows from Hardy-Littlewood-Sobolev inequality (see [24]) that

$$\mathbb{D}(u) = \int_{\mathbb{R}^3}(I_\alpha * |u|^p)|u|^p \mathrm{d}x \le d_\alpha\Big(\int_{\mathbb{R}^3} |u|^{\frac{6p}{3+\alpha}} \mathrm{d}x\Big)^{\frac{3+\alpha}{3}}. \tag{2.1}$$

Moreover, since $0 < \gamma < 1$, by Hölder's inequality, (f_1) and (1.5), we have

$$\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x \le \|f\|_{\frac{6}{5+\gamma}}\Big[\int_{\mathbb{R}^3}|u|^6\mathrm{d}x\Big]^{\frac{1-\gamma}{6}} \le \|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\|u\|_E^{1-\gamma}, \tag{2.2}$$

and for any u, $v \in E$, it holds

$$\left|\int_{\mathbb{R}^3} f(x)\Big(|u|^{1-\gamma} - |v|^{1-\gamma}\Big)\mathrm{d}x\right| \le \int_{\mathbb{R}^3} f(x)\,|u-v|^{1-\gamma}\,\mathrm{d}x \le \|f\|_{\frac{6}{5+\gamma}}\Big[\int_{\mathbb{R}^3}|u-v|^6\,\mathrm{d}x\Big]^{\frac{1-\gamma}{6}}. \tag{2.3}$$

In order to prove our results, we first consider the following constrained set:

$$\mathcal{N}_\lambda = \left\{u \in E\setminus\{0\} \,:\, \|u\|_E^2 + \lambda\mathbb{D}(u) - \int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x - \int_{\mathbb{R}^3}|u|^6\mathrm{d}x = 0\right\},$$

and split $\mathcal{N}_\lambda$ as follows

$$\mathcal{N}_\lambda^+ = \left\{ u \in \mathcal{N}_\lambda \,:\, 2\|u\|_E^2 + 2p\lambda\mathbb{D}(u) - (1-\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x > 6\int_{\mathbb{R}^3}|u|^6\mathrm{d}x \right\},$$

$$\mathcal{N}_\lambda^- = \left\{ u \in \mathcal{N}_\lambda \,:\, 2\|u\|_E^2 + 2p\lambda\mathbb{D}(u) - (1-\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x < 6\int_{\mathbb{R}^3}|u|^6\mathrm{d}x \right\},$$

$$\mathcal{N}_\lambda^0 = \left\{ u \in \mathcal{N}_\lambda \,:\, 2\|u\|_E^2 + 2p\lambda\mathbb{D}(u) - (1-\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x = 6\int_{\mathbb{R}^3}|u|^6\mathrm{d}x \right\},$$

for any $\lambda > 0$. One can easily see that for $u \in \mathcal{N}_\lambda$,

$$\begin{aligned}
&2\|u\|_E^2 + 2p\lambda\mathbb{D}(u) - (1-\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x - 6\int_{\mathbb{R}^3}|u|^6\mathrm{d}x \\
=&2\lambda(p-1)\mathbb{D}(u) + (1+\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x - 4\int_{\mathbb{R}^3}|u|^6\mathrm{d}x \\
=&(2-2p)\|u\|_E^2 + (2p-1+\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x - (6-2p)\int_{\mathbb{R}^3}|u|^6\mathrm{d}x \qquad (2.4)\\
=&(1+\gamma)\|u\|_E^2 + \lambda(2p-1+\gamma)\mathbb{D}(u) - (5+\gamma)\int_{\mathbb{R}^3}|u|^6\mathrm{d}x \\
=&-4\|u\|_E^2 - (6-2p)\lambda\mathbb{D}(u) + (5+\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x.
\end{aligned}$$

We also recall the following lemma on the properties of $\mathbb{D}(u)$ from [16, 18], etc.

Lemma 2.1. *For $0 < \alpha < 3$ and $1 + \frac{\alpha}{3} \le p < 3$, assume that $u_n \rightharpoonup u$ in E, then for any $\psi \in E$, we have $\lim\limits_{n\to\infty} \mathbb{D}(u_n) = \mathbb{D}(u)$ and $\lim\limits_{n\to\infty} \langle \mathbb{D}'(u_n), \psi\rangle = \langle \mathbb{D}'(u), \psi\rangle$.*

Set

$$T_1 = \frac{4}{5+\gamma} S^{\frac{1-\gamma}{2}} \left[\frac{(1+\gamma)S^3}{5+\gamma}\right]^{\frac{1+\gamma}{4}}. \qquad (2.5)$$

Lemma 2.2. *Suppose $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, where T_1 is defined in (2.5), then for any $u \in E\backslash\{0\}$, there exist unique $t_{max} = t_{max}(u) > 0$, $t^+ = t^+(u) > 0$ and $t^- = t^-(u) > 0$ with $t^+ < t_{max} < t^-$, such that $t^+u \in \mathcal{N}_\lambda^+$, $t^-u \in \mathcal{N}_\lambda^-$, $J_\lambda(t^+u) = \inf\limits_{0<t\le t^-} J_\lambda(tu)$ and $J_\lambda(t^-u) = \sup\limits_{t\ge t_{max}} J_\lambda(tu)$. Furthermore, $\mathcal{N}_\lambda^0 = \emptyset$ for $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$.*

Proof. For any $u \in E \setminus \{0\}$ and $t > 0$, we have

$$\begin{aligned}
t\frac{\mathrm{d}J_\lambda(tu)}{\mathrm{d}t} =&t^2\|u\|_E^2 + \lambda t^{2p}\mathbb{D}(u) - t^{1-\gamma}\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x - t^6\int_{\mathbb{R}^3}|u|^6\mathrm{d}x \\
=&t^{1-\gamma}\Big[t^{1+\gamma}\|u\|_E^2 - t^{5+\gamma}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x + \lambda t^{2p-1+\gamma}\mathbb{D}(u) - \int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x\Big] \\
\equiv&t^{1-\gamma}\Big[g(t) - \int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x\Big],
\end{aligned} \qquad (2.6)$$

where $g(t)=t^{1+\gamma}\|u\|_E^2-t^{5+\gamma}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x+\lambda t^{2p-1+\gamma}\mathbb{D}(u)$. Rewrite $g'(t)=t^{2p-2+\gamma}g_1(t)$ with

$$g_1(t)=(1+\gamma)t^{2-2p}\|u\|_E^2-(5+\gamma)t^{6-2p}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x+\lambda(2p-1+\gamma)\mathbb{D}(u). \qquad (2.7)$$

Since $\alpha\in(0,3)$ and $1+\frac{\alpha}{3}\le p<3$, we have $\lim_{t\to 0^+}g_1(t)=+\infty$, $\lim_{t\to+\infty}g_1(t)=-\infty$ and

$$g_1'(t)=(1+\gamma)(2-2p)t^{1-2p}\|u\|_E^2-(5+\gamma)(6-2p)t^{5-2p}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x<0,$$

for all $t>0$. Thus, $g(t)$ admits a global maximum point t_{max} which is the unique zero point of $g_1(t)$ and $g(t)$ is increasing on $(0,t_{max})$, decreasing on $(t_{max},+\infty)$. Set $g_2(t)=t^{1+\gamma}\|u\|_E^2-t^{5+\gamma}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x$. Obviously, $g_2(0)=0$, $\lim_{t\to+\infty}g_2(t)=-\infty$ and $g_2(t)$ achieves its maximum at $t_{g_2}=\left[\frac{(1+\gamma)\|u\|_E^2}{(5+\gamma)\int_{\mathbb{R}^3}|u|^6\mathrm{d}x}\right]^{\frac14}$ with

$$\max_{t\in[0,+\infty)}g_2(t)=g_2(t_{g_2})=\frac{4}{5+\gamma}\|u\|_E^2\left[\frac{(1+\gamma)\|u\|_E^2}{(5+\gamma)\int_{\mathbb{R}^3}|u|^6\mathrm{d}x}\right]^{\frac{1+\gamma}{4}}.$$

It follows from (1.5) and (2.2) that

$$\begin{aligned}
&g(t_{max})-\int_{\mathbb{R}^3}f(x)|u|^{1-\gamma}\mathrm{d}x\\
\ge&\max_{t\in(0,+\infty)}g_2(t)-\int_{\mathbb{R}^3}f(x)|u|^{1-\gamma}\mathrm{d}x\\
\ge&\frac{4}{5+\gamma}\|u\|_E^2\left[\frac{(1+\gamma)\|u\|_E^2}{(5+\gamma)\int_{\mathbb{R}^3}|u|^6\mathrm{d}x}\right]^{\frac{1+\gamma}{4}}-\|f\|_{\frac{6}{5+\gamma}}S^{\frac{\gamma-1}{2}}\|u\|_E^{1-\gamma}\\
\ge&\left[\frac{4}{5+\gamma}\left(\frac{1+\gamma}{(5+\gamma)S^{-3}}\right)^{\frac{1+\gamma}{4}}-\|f\|_{\frac{6}{5+\gamma}}S^{\frac{\gamma-1}{2}}\right]\|u\|_E^{1-\gamma}>0,
\end{aligned} \qquad (2.8)$$

since $0<\|f\|_{\frac{6}{5+\gamma}}<T_1$. Consequently, there exist two points $0<t^+<t_{max}<t^-$ such that

$$g(t^+)=g(t^-)=\int_{\mathbb{R}^3}f(x)|u|^{1-\gamma}\mathrm{d}x \text{ and } g'(t^+)>0>g'(t^-).$$

That is $t^+u\in\mathcal{N}_\lambda^+$ and $t^-u\in\mathcal{N}_\lambda^-$. Hence, $\mathcal{N}_\lambda^\pm\ne\emptyset$ when $0<\|f\|_{\frac{6}{5+\gamma}}<T_1$. We can further obtain from (2.6) that $\frac{\mathrm{d}J_\lambda(tu)}{\mathrm{d}t}>0$ for all $t\in(t^+,t^-)$, $\frac{\mathrm{d}J_\lambda(tu)}{\mathrm{d}t}<0$ for all $t\in(0,t^+)$ and $t\in(t^-,\infty)$. Thus, $J_\lambda(t^+u)=\inf_{0<t\le t^-}J_\lambda(tu)$ and $J_\lambda(t^-u)=\sup_{t\ge t_{max}}J_\lambda(tu)$.

Now, we come to show that $\mathcal{N}_\lambda^0=\emptyset$ for $0<\|f\|_{\frac{6}{5+\gamma}}<T_1$. By contradiction, assume that there exists $u_0\in\mathcal{N}_\lambda^0$ and $u_0\ne0$. Similarly to (2.8), we can obtain from (2.4) that

$$\begin{aligned}
0<&\frac{4}{5+\gamma}\|u_0\|_E^2\left[\frac{(1+\gamma)\|u_0\|_E^2}{(5+\gamma)\int_{\mathbb{R}^3}|u_0|^6\mathrm{d}x}\right]^{\frac{1+\gamma}{4}}-\int_{\mathbb{R}^3}f(x)|u_0|^{1-\gamma}dx\\
\le&\frac{4}{5+\gamma}\|u_0\|_E^2-\int_{\mathbb{R}^3}f(x)|u_0|^{1-\gamma}dx\le0,
\end{aligned}$$

which is a contradiction. Hence, $\mathcal{N}_\lambda^0 = \emptyset$ for $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$. $\square$

Lemma 2.3. *Suppose* $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, *then there exists a gap structure in* $\mathcal{N}_\lambda$*:*

$$\|U\|_E > A^* > A_* > \|u\|_E,\ u \in \mathcal{N}_\lambda^+,\, U \in \mathcal{N}_\lambda^-,$$

where

$$A_* = \left(\frac{5+\gamma}{4}\|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\right)^{\frac{1}{1+\gamma}}, \qquad A^* = \left[\frac{(1+\gamma)S^3}{5+\gamma}\right]^{\frac{1}{4}}.$$

Proof. Since $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, we have $\mathcal{N}_\lambda^{\pm} \neq \emptyset$ by Lemma 2.2. For any $u \in \mathcal{N}_\lambda^+$, it follows from (2.2) and (2.4) that

$$\|u\|_E^2 < \frac{5+\gamma}{4}\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x \le \frac{5+\gamma}{4}\|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\|u\|_E^{1-\gamma},$$

which yields $\|u\|_E < A_*$.

For any $U \in \mathcal{N}_\lambda^-$, it follows from (1.5) and (2.4) that

$$(1+\gamma)\|U\|_E^2 < (5+\gamma)\int_{\mathbb{R}^3}|U|^6\mathrm{d}x \le (5+\gamma)S^{-3}\|U\|_E^6,$$

which yields $\|U\|_E > A^*$.

Using $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$ and the definition of T_1, one can further obtain $A_* < \left(\frac{5+\gamma}{4}T_1 S^{\frac{\gamma-1}{2}}\right)^{\frac{1}{1+\gamma}} = A^*$. So the proof is completed. $\square$

Lemma 2.4. *Suppose* $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, *then* $\mathcal{N}_\lambda^-$ *is a closed set in* E.

Proof. Since $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, by Lemma 2.2, one has $\mathcal{N}_\lambda^- \neq \emptyset$ and $\mathcal{N}_\lambda^0 = \emptyset$. Let $\{U_n\}$ be a sequence in $\mathcal{N}_\lambda^-$ with $U_n \to U_0$ in E, then $U_n \to U_0$ in $L^6(\mathbb{R}^3)$. Since $\mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$, one can obtain from Lemma 2.1, (2.3) and (2.4) that

$$\begin{aligned}\|U_0\|_E^2 = \lim_{n\to\infty}\|U_n\|_E^2 &= \lim_{n\to\infty}\left[\int_{\mathbb{R}^3} f(x)|U_n|^{1-\gamma}\mathrm{d}x + \int_{\mathbb{R}^3}|U_n|^6\mathrm{d}x - \lambda\mathbb{D}(U_n)\right]\\ &= \int_{\mathbb{R}^3} f(x)|U_0|^{1-\gamma}\mathrm{d}x + \int_{\mathbb{R}^3}|U_0|^6\mathrm{d}x - \lambda\mathbb{D}(U_0)\end{aligned}$$

and

$$\begin{aligned}&-4\|U_0\|_E^2 - (6-2p)\lambda\mathbb{D}(U_0) + (5+\gamma)\int_{\mathbb{R}^3} f(x)|U_0|^{1-\gamma}\mathrm{d}x\\ &= \lim_{n\to\infty}\left[-4\|U_n\|_E^2 - (6-2p)\lambda\mathbb{D}(U_n) + (5+\gamma)\int_{\mathbb{R}^3} f(x)|U_n|^{1-\gamma}\mathrm{d}x\right] \le 0,\end{aligned}$$

so $U_0 \in \mathcal{N}_\lambda^- \cup \{0\}$. It follows from $\{U_n\} \subset \mathcal{N}_\lambda^-$ and Lemma 2.3 that

$$\|U_0\|_E^2 = \lim_{n\to\infty}\|U_n\|_E^2 \ge A^* > 0,$$

that is, $U_0 \neq 0$. Hence, $U_0 \in \mathcal{N}_\lambda^-$ and then $\mathcal{N}_\lambda^-$ is a closed set in E. $\square$

Lemma 2.5. *Let $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, given $u \in \mathcal{N}_\lambda^{\pm}$, then there exist $\varepsilon > 0$ and a continuous function $H(w) > 0$, $w \in E$, $\|w\|_E < \varepsilon$ satisfying that*

$$H(0) = 1,\ \ H(w)(u+w) \in \mathcal{N}_\lambda^{\pm},\ \forall w \in E,\ \|w\|_E < \varepsilon.$$

Proof. We only prove the case $u \in \mathcal{N}_\lambda^{+}$. Define $F : E \times \mathbb{R} \to \mathbb{R}$ by

$$F(w,t) = t^2\|u+w\|_E^2 + \lambda t^{2p}\mathbb{D}(u+w) - t^{1-\gamma}\int_{\mathbb{R}^3} f(x)|u+w|^{1-\gamma}\mathrm{d}x - t^6\int_{\mathbb{R}^3}|u+w|^6\mathrm{d}x.$$

In view of $u \in \mathcal{N}_\lambda^{+} \subset \mathcal{N}_\lambda$, we obtain $F(0,1) = 0$ and

$$F_t(0,1) = 2\|u\|_E^2 + 2p\lambda\mathbb{D}(u) - (1-\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x - 6\int_{\Omega}|u|^6\mathrm{d}x > 0.$$

By applying Implicit function Theorem for F at the point $(0,1)$, we get that there exists $\bar{\varepsilon} > 0$ such that for $w \in E$, $\|w\|_E < \bar{\varepsilon}$, the equation $F(w,t) = 0$ has a unique continuous solution $t = H(w) > 0$ satisfying that $H(0) = 1$ and $F(w, H(w)) = 0$ i.e. $H(w)(u+w) \in \mathcal{N}_\lambda$. Moreover, since $F_t(0,1) > 0$ and

$$\begin{aligned}
F_t(w,H(w)) =& 2H(w)\|u+w\|_E^2 + 2p\lambda H^{2p-1}(w)\mathbb{D}(u+w)\\
&- (1-\gamma)H^{-\gamma}(w)\int_{\mathbb{R}^3} f(x)|u+w|^{1-\gamma}\mathrm{d}x - 6H^5(w)\int_{\mathbb{R}^3}|u+w|^6\mathrm{d}x\\
=& H^{-1}(w)\Big[2H^2(w)\|u+w\|_E^2 + 2p\lambda H^{2p}(w)\mathbb{D}(u+w)\\
&- (1-\gamma)H^{1-\gamma}(w)\int_{\mathbb{R}^3} f(x)|u+w|^{1-\gamma}\mathrm{d}x - 6H^6(w)\int_{\mathbb{R}^3}|u+w|^6\mathrm{d}x\Big],
\end{aligned}$$

we can choose $\varepsilon > 0$ possibly small ($\varepsilon < \bar{\varepsilon}$) such that for $w \in E$ and $\|w\|_E < \varepsilon$,

$$\begin{aligned}
&2H^2(w)\|u+w\|_E^2 + 2p\lambda H^{2p}(w)\mathbb{D}(u+w) - (1-\gamma)H^{1-\gamma}(w)\int_{\mathbb{R}^3} f(x)|u+w|^{1-\gamma}\mathrm{d}x\\
&-6H^6(w)\int_{\mathbb{R}^3}|u+w|^6\mathrm{d}x > 0,
\end{aligned}$$

that is

$$H(w)(u+w) \in \mathcal{N}_\lambda^{+},\ \ \text{for any } w \in E,\ \|w\|_E < \varepsilon.$$

This ends the proof of Lemma 2.5. □

Lemma 2.6. *J_λ is coercive and bounded below on $\mathcal{N}_\lambda$. Moreover,*

(i) if $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, then $\inf_{\mathcal{N}_\lambda^{+}\cup\{0\}} J_\lambda = \inf_{\mathcal{N}_\lambda^{+}} J_\lambda < 0$;

(ii) if $0 < \|f\|_{\frac{6}{5+\gamma}} < \dfrac{1-\gamma}{2}T_1$, then $\inf_{\mathcal{N}_\lambda^{-}} J_\lambda \geq \beta_0 > 0$ for some constant $\beta_0 = \beta_0(\gamma, S, \|f\|_{\frac{6}{5+\gamma}})$.

Proof. For any $u \in \mathcal{N}_\lambda$, we can obtain from (1.3), $\lambda > 0$, $0 < \gamma < 1$, $1 + \frac{\alpha}{3} \le p < 3$ and (2.2) that

$$\begin{aligned} J_\lambda(u) =& \Big(\frac{1}{2} - \frac{1}{6}\Big)\|u\|_E^2 + \lambda\Big(\frac{1}{2p} - \frac{1}{6}\Big)\mathbb{D}(u) - \Big(\frac{1}{1-\gamma} - \frac{1}{6}\Big)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x \\ \ge& \frac{1}{3}\|u\|_E^2 - \frac{5+\gamma}{6(1-\gamma)}\|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\|u\|_E^{1-\gamma} \ge \frac{1+\gamma}{3(\gamma-1)}A_*^2, \end{aligned} \tag{2.9}$$

where A_* is defined in Lemma 2.3. Due to $0 < \gamma < 1$, J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.

(i) When $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, $\mathcal{N}_\lambda^\pm \ne \emptyset$ from Lemma 2.2, also $\mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^+ \cup \{0\}$ are two closed sets in E from Lemma 2.4. Hence, $\inf_{\mathcal{N}_\lambda^-} J_\lambda$ and $\inf_{\mathcal{N}_\lambda^+\cup\{0\}} J_\lambda$ are well defined. For any $u \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, we can get from $0 < \gamma < 1$, $1 + \frac{\alpha}{3} \le p < 3$, (2.4) and (2.9) that

$$\begin{aligned} J_\lambda(u) =& \frac{1}{3}\|u\|_E^2 + \lambda\frac{3-p}{6p}\mathbb{D}(u) - \frac{5+\gamma}{6(1-\gamma)}\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x \\ <& \frac{1}{3}\|u\|_E^2 - \frac{2}{3(1-\gamma)}\|u\|_E^2 + \lambda(3-p)\Big(\frac{1}{6p} - \frac{1}{3(1-\gamma)}\Big)\mathbb{D}(u) \\ =& -\frac{1+\gamma}{3(1-\gamma)}\|u\|_E^2 + \lambda(3-p)\frac{1-\gamma-2p}{6p(1-\gamma)}\mathbb{D}(u) < 0, \end{aligned} \tag{2.10}$$

which yields $\inf_{\mathcal{N}_\lambda^+} J_\lambda < 0$. Since $J_\lambda(0) = 0$, we can further get $\inf_{\mathcal{N}_\lambda^+\cup\{0\}} J_\lambda = \inf_{\mathcal{N}_\lambda^+} J_\lambda < 0$.

(ii) Let $u \in \mathcal{N}_\lambda^-$, it follows from Lemma 2.3 that $\|u\|_E > A^*$. Using this and (2.9), $\|f\|_{\frac{6}{5+\gamma}} \in (0, \frac{1-\gamma}{2}T_1)$, we can obtain that

$$\begin{aligned} J_\lambda(u) \ge& \|u\|_E^{1-\gamma}\Big[\frac{1}{3}\|u\|_E^{1+\gamma} - \frac{5+\gamma}{6(1-\gamma)}\|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\Big] \\ \ge& \Big[\frac{(1+\gamma)S^3}{5+\gamma}\Big]^{\frac{1-\gamma}{4}}\Big\{\frac{1}{3}\Big[\frac{(1+\gamma)S^3}{5+\gamma}\Big]^{\frac{1+\gamma}{4}} - \frac{5+\gamma}{6(1-\gamma)}\|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\Big\} > 0, \end{aligned}$$

which implies that there exists a constant $\beta_0 = \beta_0(\gamma, S, \|f\|_{\frac{6}{5+\gamma}})$ such that $\inf_{\mathcal{N}_\lambda^-} J_\lambda \ge \beta_0 > 0$ for $\|f\|_{\frac{6}{5+\gamma}} \in (0, \frac{1-\gamma}{2}T_1)$. □

According to Lemma 2.2 and Lemma 2.4, for $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, $\mathcal{V}_\lambda^- := \mathcal{N}_\lambda^-$ and $\mathcal{V}_\lambda^+ := \mathcal{N}_\lambda^+ \cup \{0\}$ are two closed sets in E, then we can apply Ekeland variational principle to find the minimums of functional J_λ on both $\mathcal{V}_\lambda^+$ and $\mathcal{V}_\lambda^-$. Let $\{u_n\} \subset \mathcal{V}_\lambda^\pm$ be a minimizing sequence for J_λ on $\mathcal{V}_\lambda^\pm$. That is, $\{u_n\} \subset \mathcal{V}_\lambda^\pm$ satisfy

$$\tau_\lambda^\pm < J_\lambda(u_n) < \tau_\lambda^\pm + \frac{1}{n} \tag{2.11}$$

and

$$J_\lambda(z) \ge J_\lambda(u_n) - \frac{1}{n}\|u_n - z\|_E, \forall z \in \mathcal{V}_\lambda^\pm, \tag{2.12}$$

where

$$\tau_\lambda^+ = \inf_{u\in\mathcal{V}_\lambda^+} J_\lambda(u) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u),\ \tau_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u) \text{ and } \tau_\lambda = \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u).$$

From $J_\lambda(|u_n|) = J_\lambda(u_n)$, we could assume that $u_n \geq 0$. Moreover, Lemma 2.6 shows that $\|u_n\|_E \leq C_0$ for some suitable positive constant C_0, so there exists a nonnegative function $u_\lambda \in E$ such that

$$\begin{aligned} u_n &\rightharpoonup u_\lambda, && \text{in } E,\\ u_n &\to u_\lambda, && \text{in } L^s(\mathbb{R}^3),\ s\in[2,6),\\ u_n &\to u_\lambda, && \text{a.e. in } \mathbb{R}^3. \end{aligned} \tag{2.13}$$

By Vitali Convergence Theorem, similarly to the proof of [13, Lemma 2.7], we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} f(x)|u_n|^{1-\gamma}\mathrm{d}x = \int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma}\mathrm{d}x, \tag{2.14}$$

when $\{u_n\}$ is bounded in E. In order to show that all convergence in (2.13) hold true on a strong sense, inspired by [32, 5], we need following Lemmas.

Lemma 2.7. *Assume $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$. Suppose $\{u_n\} \subset \mathcal{N}_\lambda^\pm$ satisfy (2.13) with $u_\lambda \not\equiv 0$, then there exists a constant $C_1 > 0$ such that for n large enough, the following alternative holds true:*

(i) if $\{u_n\} \subset \mathcal{N}_\lambda^+$, we have

$$(1+\gamma)\|u_n\|_E^2 + \lambda(2p-1+\gamma)\mathbb{D}(u_n) - (5+\gamma)\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x \geq C_1;$$

(ii) if $\{u_n\} \subset \mathcal{N}_\lambda^-$, we have

$$(1+\gamma)\|u_n\|_E^2 + \lambda(2p-1+\gamma)\mathbb{D}(u_n) - (5+\gamma)\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x \leq -C_1.$$

Proof. We only prove (i), since (ii) follows similarly. Using $u_n \in \mathcal{N}_\lambda^+$, (2.4), Lemma 2.1, (2.14) and $u_\lambda \not\equiv 0$, it is enough to show that

$$(5+\gamma)\int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma}\mathrm{d}x - (6-2p)\lambda\mathbb{D}(u_\lambda) > \liminf_{n\to\infty}\left[4\|u_n\|_E^2\right]. \tag{2.15}$$

Arguing by contradiction, assume that

$$(5+\gamma)\int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma}\mathrm{d}x - (6-2p)\lambda\mathbb{D}(u_\lambda) = \liminf_{n\to\infty}\left[4\|u_n\|_E^2\right]. \tag{2.16}$$

Since $u_n \in \mathcal{N}_\lambda^+$, one has

$$(5+\gamma)\int_{\mathbb{R}^3} f(x)|u_n|^{1-\gamma}\mathrm{d}x - (6-2p)\lambda\mathbb{D}(u_n) > 4\|u_n\|_E^2.$$

According to (2.14) and Lemma 2.1, we can further obtain

$$(5+\gamma)\int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma}\mathrm{d}x-(6-2p)\lambda\mathbb{D}(u_\lambda)\geq \limsup_{n\to\infty}\left[4\|u_n\|_E^2\right]\geq \liminf_{n\to\infty}\left[4\|u_n\|_E^2\right]. \tag{2.17}$$

It follows from (2.16) and (2.17) that

$$(5+\gamma)\int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma}\mathrm{d}x-(6-2p)\lambda\mathbb{D}(u_\lambda)=\lim_{n\to\infty}\left[4\|u_n\|_E^2\right]. \tag{2.18}$$

Since $u_n\in\mathcal{N}_\lambda^+\subset\mathcal{N}_\lambda$, i.e. $\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x=\|u_n\|_E^2+\lambda\mathbb{D}(u_n)-\int_{\mathbb{R}^3}f(x)|u_n|^{1-\gamma}\mathrm{d}x$, passing to the limit as $n\to\infty$ and using (2.14), (2.18) and Lemma 2.1 lead to

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x=\frac{1+\gamma}{4}\int_{\mathbb{R}^3}f(x)|u_\lambda|^{1-\gamma}\mathrm{d}x+\frac{p-1}{2}\lambda\mathbb{D}(u_\lambda). \tag{2.19}$$

Therefore, it follows from (2.18), (2.19), $\lambda>0$ and $u_\lambda\not\equiv 0$ that

$$\lim_{n\to\infty}\frac{(1+\gamma)\|u_n\|_E^2}{(5+\gamma)\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x}<1. \tag{2.20}$$

Similarly to (2.8), for $0<\|f\|_{\frac{6}{5+\gamma}}<T_1$, one can get from (2.2), (2.14), (2.18) and (2.20) that

$$\begin{aligned}
0\leq&\left[\frac{4}{5+\gamma}\left(\frac{1+\gamma}{(5+\gamma)S^{-3}}\right)^{\frac{1+\gamma}{4}}-\|f\|_{\frac{6}{5+\gamma}}S^{\frac{\gamma-1}{2}}\right]\lim_{n\to\infty}\|u_n\|_E^{1-\gamma}\\
\leq&\lim_{n\to\infty}\frac{4}{5+\gamma}\|u_n\|_E^2\left(\frac{(1+\gamma)\|u_n\|_E^2}{(5+\gamma)\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x}\right)^{\frac{1+\gamma}{4}}-\lim_{n\to\infty}\int_{\mathbb{R}^3}f(x)u_n^{1-\gamma}dx\\
<&\frac{4}{5+\gamma}\lim_{n\to\infty}\|u_n\|_E^2-\int_{\mathbb{R}^3}f(x)|u_\lambda|^{1-\gamma}\mathrm{d}x=-\frac{6-2p}{5+\gamma}\lambda\mathbb{D}(u_\lambda)<0,
\end{aligned}$$

which is clearly impossible. So (2.15) holds and this ends the proof. □

For any $0\leq\psi\in E$, we apply Lemma 2.5 with $u=u_n\in\mathcal{N}_\lambda^\pm$ (n large enough such that $\frac{(1-\gamma)C_0}{n}<C_1$) and $w=\eta\psi$, $\eta>0$ small enough, we can find $h_{n,\psi}(\eta)=H(\eta\psi)$ such that $h_{n,\psi}(0)=1$ and $h_{n,\psi}(\eta)(u_n+\eta\psi)\in\mathcal{N}_\lambda^\pm$. However, we have no idea whether or not $h_{n,\psi}(\eta)$ is differentiable. For the sake of proof, we set

$$h'_{n,\psi}(0)=\lim_{\eta\to 0^+}\frac{h_{n,\psi}(\eta)-1}{\eta}\in[-\infty,+\infty].$$

If the above limit does not exist, we choose $\eta_k\to 0$ (instead of $\eta\to 0$) with $\eta_k>0$ such that $h'_{n,\psi}(0)=\lim_{k\to\infty}\frac{h_{n,\psi}(\eta_k)-1}{\eta_k}\in[-\infty,+\infty]$.

Lemma 2.8. *Assume* $0<\|f\|_{\frac{6}{5+\gamma}}<T_1$. *Suppose* $\{u_n\}\subset\mathcal{N}_\lambda^\pm$ *satisfy* (2.12) *and* (2.13) *with* $u_\lambda\not\equiv 0$, *then* $h'_{n,\psi}(0)$ *is uniformly bounded for any* $0\leq\psi\in E$ *and* n *large enough.*

Proof. We only consider that u_n, $h_{n,\psi}(\eta)(u_n+\eta\psi)\in\mathcal{N}_\lambda^+$ since the situation on $\mathcal{N}_\lambda^-$ can be proved similarly. By u_n, $h_{n,\psi}(\eta)(u_n+\eta\psi)\in\mathcal{N}_\lambda^+\subset\mathcal{N}_\lambda$, we have

$$\|u_n\|_E^2+\lambda\mathbb{D}(u_n)-\int_{\mathbb{R}^3}f(x)u_n^{1-\gamma}\mathrm{d}x=\int_{\mathbb{R}^3}u_n^6\mathrm{d}x,$$
$$h_{n,\psi}^2(\eta)\|u_n+\eta\psi\|_E^2+\lambda h_{n,\psi}^{2p}\mathbb{D}(u_n+\eta\psi)-h_{n,\psi}^{1-\gamma}(\eta)\int_{\mathbb{R}^3}f(x)(u_n+\eta\psi)^{1-\gamma}\mathrm{d}x$$
$$=h_{n,\psi}^6(\eta)\int_{\mathbb{R}^3}(u_n+\eta\psi)^6\mathrm{d}x.$$

Using $0<\gamma<1$ and $\lambda>0$, the above two equalities yield

$$\begin{aligned}0=&\Big[h_{n,\psi}^2(\eta)-1\Big]\|u_n+\eta\psi\|_E^2+\lambda\Big[h_{n,\psi}^{2p}(\eta)-1\Big]\mathbb{D}(u_n+\eta\psi)\\&-\Big[h_{n,\psi}^{1-\gamma}(\eta)-1\Big]\int_{\mathbb{R}^3}f(x)(u_n+\eta\psi)^{1-\gamma}\mathrm{d}x-\Big[h_{n,\psi}^6(\eta)-1\Big]\int_{\mathbb{R}^3}(u_n+\eta\psi)^6\mathrm{d}x\\&+\Big[\|u_n+\eta\psi\|_E^2-\|u_n\|_E^2\Big]+\lambda\Big[\mathbb{D}(u_n+\eta\psi)-\mathbb{D}(u_n)\Big]\\&-\int_{\mathbb{R}^3}f(x)\Big[(u_n+\eta\psi)^{1-\gamma}-u_n^{1-\gamma}\Big]\mathrm{d}x-\int_{\mathbb{R}^3}\Big[(u_n+\eta\psi)^6-u_n^6\Big]\mathrm{d}x\\\le&\Big[h_{n,\psi}(\eta)-1\Big]\bigg\{\Big[h_{n,\psi}(\eta)+1\Big]\|u_n+\eta\psi\|_E^2+\lambda\frac{h_{n,\psi}^{2p}(\eta)-1}{h_{n,\psi}(\eta)-1}\mathbb{D}(u_n+\eta\psi)\\&-\frac{h_{n,\psi}^{1-\gamma}(\eta)-1}{h_{n,\psi}(\eta)-1}\int_{\mathbb{R}^3}f(x)(u_n+\eta\psi)^{1-\gamma}\mathrm{d}x-\frac{h_{n,\psi}^6(\eta)-1}{h_{n,\psi}(\eta)-1}\int_{\mathbb{R}^3}(u_n+\eta\psi)^6\mathrm{d}x\bigg\}\\&+\Big[\|u_n+\eta\psi\|_E^2-\|u_n\|_E^2\Big]+\lambda\Big[\mathbb{D}(u_n+\eta\psi)-\mathbb{D}(u_n)\Big].\end{aligned}$$

Dividing by $\eta>0$ and passing to the limit as $\eta\to0^+$, it follows from (2.4) and the continuity of $h_{n,\psi}(\eta)$ that

$$\begin{aligned}0\le&h_{n,\psi}'(0)\Big\{2\|u_n\|_E^2+2p\lambda\mathbb{D}(u_n)-(1-\gamma)\int_{\mathbb{R}^3}f(x)u_n^{1-\gamma}\mathrm{d}x-6\int_{\mathbb{R}^3}u_n^6\mathrm{d}x\Big\}\\&+2\langle u_n,\psi\rangle_E+\lambda\langle\mathbb{D}'(u_n),\psi\rangle\\=&h_{n,\psi}'(0)\Big\{(1+\gamma)\|u_n\|_E^2+\lambda(2p-1+\gamma)\mathbb{D}(u_n)-(5+\gamma)\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x\Big\}\\&+2\langle u_n,\psi\rangle_E+\lambda\langle\mathbb{D}'(u_n),\psi\rangle,\end{aligned}\tag{2.21}$$

which implies that $h_{n,\psi}'(0)\ne-\infty$ according to Lemma 2.7 and the boundedness of $\{u_n\}$. Now we show that $h_{n,\psi}'(0)\ne+\infty$. Arguing by contradiction, we assume that $h_{n,\psi}'(0)=+\infty$ and so $h_{n,\psi}(\eta)>1$ for n sufficiently large and $\eta>0$ small. Applying condition (2.12) with $z=h_{n,\psi}(\eta)(u_n+\eta\psi)$ leads to

$$\begin{aligned}\frac{1}{n}[h_{n,\psi}(\eta)-1]\|u_n\|_E+\frac{\eta}{n}h_{n,\psi}(\eta)\|\psi\|_E\ge&\frac{1}{n}\|u_n-h_{n,\psi}(\eta)(u_n+\eta\psi)\|_E\\\ge&J_\lambda(u_n)-J_\lambda[h_{n,\psi}(\eta)(u_n+\eta\psi)].\end{aligned}\tag{2.22}$$

Since $u_n \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, then one can get from (1.3) and (2.22) that

$$
\begin{aligned}
\frac{\|\psi\|_E}{n} h_{n,\psi}(\eta) \geq & \frac{h_{n,\psi}(\eta)-1}{\eta}\Big\{ -\frac{\|u_n\|_E}{n} - \Big(\frac{1}{2}-\frac{1}{1-\gamma}\Big)[h_{n,\psi}(\eta)+1]\|u_n+\eta\psi\|_E^2 \\
& - \lambda\Big(\frac{1}{2p}-\frac{1}{1-\gamma}\Big)\frac{h_{n,\psi}^{2p}(\eta)-1}{h_{n,\psi}(\eta)-1}\mathbb{D}(u_n+\eta\psi) \\
& + \Big(\frac{1}{6}-\frac{1}{1-\gamma}\Big)\frac{h_{n,\psi}^{6}(\eta)-1}{h_{n,\psi}(\eta)-1}\int_{\mathbb{R}^3}(u_n+\eta\psi)^6 \mathrm{d}x\Big\} \\
& - \Big(\frac{1}{2}-\frac{1}{1-\gamma}\Big)\frac{\|u_n+\eta\psi\|^2-\|u_n\|^2}{\eta} \\
& + \Big(\frac{1}{6}-\frac{1}{1-\gamma}\Big)\int_{\mathbb{R}^3}\frac{(u_n+\eta\psi)^6-u_n^6}{\eta}\mathrm{d}x \\
& - \lambda\Big(\frac{1}{2p}-\frac{1}{1-\gamma}\Big)\frac{\mathbb{D}(u_n+\eta\psi)-\mathbb{D}(u_n)}{\eta}.
\end{aligned}
$$

Letting $\eta \to 0^+$, using the continuity of $h_{n,\psi}(\eta)$, Lemma 2.7 and $\|u_n\|_E \leq C_0$, we obtain

$$
\begin{aligned}
\frac{\|\psi\|_E}{n} \geq & h'_{n,\psi}(0)\Big\{ -\frac{\|u_n\|_E}{n} - \Big(1-\frac{2}{1-\gamma}\Big)\|u_n\|_E^2 - \lambda\Big(1-\frac{2p}{1-\gamma}\Big)\mathbb{D}(u_n) \\
& + \Big(1-\frac{6}{1-\gamma}\Big)\int_{\mathbb{R}^3}u_n^6\mathrm{d}x\Big\} - \Big(1-\frac{2}{1-\gamma}\Big)\langle u_n,\psi\rangle_E \\
& + \Big(1-\frac{6}{1-\gamma}\Big)\int_{\mathbb{R}^3}u_n^5\psi\mathrm{d}x - \lambda\Big(\frac{1}{2p}-\frac{1}{1-\gamma}\Big)\langle \mathbb{D}'(u_n),\psi\rangle \\
= & h'_{n,\psi}(0)\Big\{ -\frac{\|u_n\|_E}{n} + \frac{1}{1-\gamma}\Big[(\gamma+1)\|u_n\|_E^2 + \lambda(2p-1+\gamma)\mathbb{D}(u_n) \\
& - (5+\gamma)\int_{\mathbb{R}^3}u_n^6\mathrm{d}x\Big]\Big\} - \Big(1-\frac{2}{1-\gamma}\Big)\langle u_n,\psi\rangle_E \\
& + \Big(1-\frac{6}{1-\gamma}\Big)\int_{\mathbb{R}^3}u_n^5\psi\mathrm{d}x - \lambda\Big(\frac{1}{2p}-\frac{1}{1-\gamma}\Big)\langle \mathbb{D}'(u_n),\psi\rangle \\
\geq & h'_{n,\psi}(0)\Big(-\frac{C_0}{n}+\frac{C_1}{1-\gamma}\Big) - \Big(1-\frac{2}{1-\gamma}\Big)\langle u_n,\psi\rangle_E \\
& + \Big(1-\frac{6}{1-\gamma}\Big)\int_{\mathbb{R}^3}u_n^5\psi\mathrm{d}x - \lambda\Big(\frac{1}{2p}-\frac{1}{1-\gamma}\Big)\langle \mathbb{D}'(u_n),\psi\rangle
\end{aligned}
\tag{2.23}
$$

which is impossible because $h'_{n,\psi}(0) = +\infty$ and $-\frac{C_0}{n} + \frac{C_1}{1-\gamma} > 0$ for n large enough. Hence, $h'_{n,\psi}(0) \neq +\infty$. To sum up, $|h'_{n,\psi}(0)| < +\infty$. Moreover, Lemma 2.7, (2.21) and (2.23) with $\|u_n\| \leq C_0$ also imply that

$$
|h'_{n,\psi}(0)| \leq C_2, \tag{2.24}
$$

for n sufficiently large and a suitable positive constant C_2. □

Lemma 2.9. *Assume* $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$. *Suppose* $\{u_n\} \subset \mathcal{N}_\lambda^{\pm}$ *satisfy* (2.12) *and* (2.13) *with* $u_\lambda \not\equiv 0$, *then for any* $\psi \in E$, *we have as* $n \to \infty$,

$$\langle u_n, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle - \int_{\mathbb{R}^3} f(x) u_n^{-\gamma}\psi \mathrm{d}x - \int_{\mathbb{R}^3} u_n^5 \psi \mathrm{d}x = o(1). \tag{2.25}$$

Proof. For any $0 \leq \psi \in E$, applying condition (2.12) with $z = h_{n,\psi}(\eta)(u_n + \eta\psi)$ leads to

$$\begin{aligned}
&\frac{|1 - h_{n,\psi}(\eta)|}{\eta}\frac{\|u_n\|_E}{n} + \frac{\|\psi\|_E}{n} h_{n,\psi}(\eta)\\
\geq&\frac{1}{n\eta}\|u_n - h_{n,\psi}(\eta)(u_n + \eta\psi)\|_E \geq \frac{1}{\eta}\big\{J_\lambda(u_n) - J_\lambda[h_{n,\psi}(\eta)(u_n + \eta\psi)]\big\}\\
=&\frac{h_{n,\psi}(\eta) - 1}{\eta}\Big\{ - \frac{h_{n,\psi}(\eta) + 1}{2}\|u_n + \eta\psi\|_E^2 - \frac{\lambda[h_{n,\psi}^{2p}(\eta) - 1]}{2p[h_{n,\psi}(\eta) - 1]}\mathbb{D}(u_n + \eta\psi)\\
&+ \frac{h_{n,\psi}^{1-\gamma}(\eta) - 1}{(1-\gamma)[h_{n,\psi}(\eta) - 1]}\int_{\mathbb{R}^3} f(x)(u_n + \eta\psi)^{1-\gamma}\mathrm{d}x + \frac{h_{n,\psi}^6(\eta) - 1}{6[h_{n,\psi}(\eta) - 1]}\int_{\mathbb{R}^3}(u_n + \eta\psi)^6\mathrm{d}x\Big\}\\
&- \frac{1}{2}\frac{\|u_n + \eta\psi\|_E^2 - \|u_n\|_E^2}{\eta} - \frac{\lambda[\mathbb{D}(u_n + \eta\psi) - \mathbb{D}(u_n)]}{2p\eta}\\
&+ \frac{1}{6}\int_{\mathbb{R}^3}\frac{(u_n + \eta\psi)^6 - u_n^6}{\eta}\mathrm{d}x + \frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)\frac{(u_n + \eta\psi)^{1-\gamma} - u_n^{1-\gamma}}{\eta}\mathrm{d}x.
\end{aligned}$$

Passing to the liminf as $\eta \to 0^+$ and using the continuity of $h_{n,\psi}(\eta)$, Fatou's Lemma, $0 < \gamma < 1$ lead to

$$\begin{aligned}
&\frac{|h'_{n,\psi}(0)| \cdot \|u_n\|_E}{n} + \frac{\|\psi\|_E}{n}\\
\geq&h'_{n,\psi}(0)\Big\{ - \|u_n\|_E^2 + \int_{\mathbb{R}^3} f(x)u_n^{1-\gamma}\mathrm{d}x - \lambda\mathbb{D}(u_n) + \int_{\mathbb{R}^3} u_n^6\mathrm{d}x\Big\} - \langle u_n, \psi\rangle_E\\
&- \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle + \int_{\mathbb{R}^3} u_n^5\psi\mathrm{d}x + \liminf_{\eta\to 0^+}\frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)\frac{(u_n + \eta\psi)^{1-\gamma} - u_n^{1-\gamma}}{\eta}\mathrm{d}x\\
\geq& - \langle u_n, \psi\rangle_E - \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle + \int_{\mathbb{R}^3} u_n^5\psi\mathrm{d}x + \int_{\mathbb{R}^3}\frac{f(x)}{1-\gamma}\liminf_{\eta\to 0^+}\frac{(u_n + \eta\psi)^{1-\gamma} - u_n^{1-\gamma}}{\eta}\mathrm{d}x\\
=& - \langle u_n, \psi\rangle_E - \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle + \int_{\mathbb{R}^3} u_n^5\psi\mathrm{d}x + \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi\mathrm{d}x,
\end{aligned}$$

since $u_n \in \mathcal{N}_\lambda^{\pm} \subset \mathcal{N}_\lambda$. Hence, for n large, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi\mathrm{d}x \leq&\frac{|h'_{n,\psi}(0)| \cdot \|u_n\|_E}{n} + \frac{\|\psi\|_E}{n} + \langle u_n, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle - \int_{\mathbb{R}^3} u_n^5\psi\mathrm{d}x\\
\leq&\frac{C_0 \cdot C_2 + \|\psi\|_E}{n} + \langle u_n, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle - \int_{\mathbb{R}^3} u_n^5\psi\mathrm{d}x,
\end{aligned}$$

thanks to $\|u_n\|_E \le C_0$ and $|h'_{n,\varphi}(0)| \le C_2$ by (2.24). Thus, for any $0 \le \psi \in E$, we can get as $n \to \infty$,

$$\langle u_n, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle - \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi \mathrm{d}x - \int_{\mathbb{R}^3} u_n^5\psi \mathrm{d}x \ge o(1). \tag{2.26}$$

Now, we come to show that (2.26) holds for every $\psi \in E$. For any $\psi \in E$ and $\varepsilon > 0$, set $\psi_\varepsilon = u_n + \varepsilon\psi$ and $\Omega_\varepsilon = \{x \in \mathbb{R}^3 : \psi_\varepsilon \le 0\}$. Since $u_n \in \mathcal{N}_\lambda$, by applying inequality (2.26) with $\psi = \psi_\varepsilon^+$, we have

$$\begin{aligned}
o(1) \le& \frac{1}{\varepsilon}\Big\{\langle u_n, \psi_\varepsilon^+\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi_\varepsilon^+\rangle - \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi_\varepsilon^+ \mathrm{d}x - \int_{\mathbb{R}^3} u_n^5\psi_\varepsilon^+ \mathrm{d}x\Big\}\\
=& \frac{1}{\varepsilon}\int_{\mathbb{R}^3\setminus\Omega_\varepsilon} \Big\{\nabla u_n\nabla\psi_\varepsilon + V(x)u_n\psi_\varepsilon + \lambda(I_\alpha * u_n^p)u_n^{p-2}u_n\psi_\varepsilon - f(x)u_n^{-\gamma}\psi_\varepsilon - u_n^5\psi_\varepsilon\Big\}\,\mathrm{d}x\\
=& \frac{1}{\varepsilon}\Big\{\|u_n\|_E^2 + \lambda\mathbb{D}(u_n) - \int_{\mathbb{R}^3} f(x)u_n^{1-\gamma}\mathrm{d}x - \int_{\mathbb{R}^3} u_n^6\mathrm{d}x\Big\}\\
&+ \Big\{\langle u_n, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle - \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi \mathrm{d}x - \int_{\mathbb{R}^3} u_n^5\psi \mathrm{d}x\Big\}\\
&- \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \Big\{\nabla u_n\nabla\psi_\varepsilon + V(x)u_n\psi_\varepsilon + \lambda(I_\alpha * u_n^p)u_n^{p-2}u_n\psi_\varepsilon - f(x)u_n^{-\gamma}\psi_\varepsilon - u_n^5\psi_\varepsilon\Big\}\,\mathrm{d}x\\
=& \Big\{\langle u_n, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle - \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi \mathrm{d}x - \int_{\mathbb{R}^3} u_n^5\psi \mathrm{d}x\Big\}\\
&- \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \Big[|\nabla u_n|^2 + V(x)u_n^2 + \lambda(I_\alpha * u_n^p)u_n^p\Big]\,\mathrm{d}x\\
&- \int_{\Omega_\varepsilon} \Big[\nabla u_n\nabla\psi + V(x)u_n\psi + \lambda(I_\alpha * u_n^p)u_n^{p-2}u_n\psi\Big]\,\mathrm{d}x\\
&+ \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \Big[f(x)u_n^{-\gamma}\psi_\varepsilon + u_n^5\psi_\varepsilon\Big]\,\mathrm{d}x\\
\le& \Big\{\langle u_n, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle - \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi \mathrm{d}x - \int_{\mathbb{R}^3} u_n^5\psi \mathrm{d}x\Big\}\\
&- \int_{\Omega_\varepsilon} \Big[\nabla u_n\nabla\psi + V(x)u_n\psi + \lambda(I_\alpha * u_n^p)u_n^{p-2}u_n\psi\Big]\,\mathrm{d}x.
\end{aligned} \tag{2.27}$$

Letting $\varepsilon \to 0^+$ to the above inequality and using the fact that $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0^+$, we have

$$\langle u_n, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle - \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi \mathrm{d}x - \int_{\mathbb{R}^3} u_n^5\psi \mathrm{d}x \ge o(1),\ \forall\psi \in E.$$

This inequality also holds for $-\psi$, hence we conclude that (2.25) holds for every $\psi \in E$. □

Lemma 2.10. *Assume* $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$. *Suppose* $\{u_n\} \subset \mathcal{N}_\lambda^\pm$ *satisfy* (2.12), (2.13) *and*

$$J_\lambda(u_n) \to c < c_*,\ \text{as } n \to \infty, \tag{2.28}$$

where $c \neq 0$ and $c_* = \dfrac{1}{3}S^{\frac{3}{2}} - D_*\|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}}$ *with* $D_* = \dfrac{1+\gamma}{2}\left[\dfrac{2S}{3(1-\gamma)}\right]^{\frac{\gamma-1}{\gamma+1}}\left[\dfrac{5+\gamma}{6(1-\gamma)}\right]^{\frac{2}{\gamma+1}}$, *then $u_\lambda \not\equiv 0$ and $\{u_n\}$ possesses a subsequence strongly convergent to u_λ in E.*

Proof. We claim that $u_\lambda \not\equiv 0$. Arguing by contradiction, $u_\lambda \equiv 0$. Then, by $u_n \in \mathcal{N}_\lambda^{\pm} \subset \mathcal{N}_\lambda$, Lemma 2.1 and (2.14), we have

$$\|u_n\|_E^2 = \int_{\mathbb{R}^3} |u_n|^6 \mathrm{d}x + o(1). \tag{2.29}$$

It follows from (2.29) and $J_\lambda(u_n) \to c \neq 0$ that

$$c = J_\lambda(u_n) + o(1) = \frac{1}{3}\|u_n\|_E^2 + o(1). \tag{2.30}$$

If $c < 0$, we get a contradiction from the last relation. If $c > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|u_n\|_E^2 \geq c$ for $n \geq n_0$. This together with (1.5) and (2.29) leads to $\lim\limits_{n\to\infty} \|u_n\|_E^2 \geq S^{\frac{3}{2}}$. Then, by (2.28), (2.30) and the fact of the above relation, we obtain that

$$c < c_* = \frac{1}{3}S^{\frac{3}{2}} - D_*\|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}} < \frac{1}{3}S^{\frac{3}{2}} \leq \frac{1}{3}\lim_{n\to\infty}\|u_n\|_E^2 = c,$$

which is a contradiction. Therefore $u_\lambda \not\equiv 0$. By Brézis-Lieb's Lemma, we have

$$\begin{aligned} \|u_n\|_E^2 &= \|u_\lambda\|_E^2 + \|u_n - u_\lambda\|_E^2 + o(1), \\ \int_{\mathbb{R}^3} |u_n|^6 \mathrm{d}x &= \int_{\mathbb{R}^3} |u_\lambda|^6 \mathrm{d}x + \int_{\mathbb{R}^3} |u_n - u_\lambda|^6 \mathrm{d}x + o(1). \end{aligned} \tag{2.31}$$

For any $\psi \in E$, set

$$Q(u_n, \psi) = (u_n, \psi)_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle - \int_{\mathbb{R}^3} f(x) u_n^{-\gamma}\psi \mathrm{d}x - \int_{\mathbb{R}^3} u_n^5 \psi \mathrm{d}x.$$

Then,

$$\begin{aligned} J_\lambda(u_n) - \frac{1}{6}Q(u_n, u_n) =& \frac{1}{3}\|u_n\|_E^2 + \lambda(\frac{1}{2p} - \frac{1}{6})\mathbb{D}(u_n) - (\frac{1}{1-\gamma} - \frac{1}{6})\int_{\mathbb{R}^3} f(x) u_n^{1-\gamma}\mathrm{d}x \\ =& \frac{1}{3}\|u_n - u_\lambda\|_E^2 + \frac{1}{3}\|u_\lambda\|_E^2 + \lambda(\frac{1}{2p} - \frac{1}{6})\mathbb{D}(u_n) \\ & - (\frac{1}{1-\gamma} - \frac{1}{6})\int_{\mathbb{R}^3} f(x) u_n^{1-\gamma}\mathrm{d}x + o(1). \end{aligned} \tag{2.32}$$

Applying (2.25) with $\psi = u_\lambda$ and using $u_n \in \mathcal{N}_\lambda^{\pm} \subset \mathcal{N}_\lambda$, (2.13), (2.14), (2.31), Lemma 2.1 lead to

$$\begin{aligned} o(1) =& -\langle u_n, u_\lambda\rangle_E - \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), u_\lambda\rangle + \int_{\mathbb{R}^3} f(x) u_n^{-\gamma} u_\lambda \mathrm{d}x + \int_{\mathbb{R}^3} u_n^5 u_\lambda \mathrm{d}x \\ =& \|u_n\|_E^2 - \langle u_n, u_\lambda\rangle_E + \lambda\mathbb{D}(u_n) - \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), u_\lambda\rangle - \int_{\mathbb{R}^3} f(x) u_n^{1-\gamma}\mathrm{d}x \\ & + \int_{\mathbb{R}^3} f(x) u_n^{-\gamma} u_\lambda \mathrm{d}x - \int_{\mathbb{R}^3} u_n^6 \mathrm{d}x + \int_{\mathbb{R}^3} u_n^5 u_\lambda \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
=&\|u_n\|_E^2 - \|u_\lambda\|_E^2 - \int_{\mathbb{R}^3} f(x)u_\lambda^{1-\gamma}\mathrm{d}x + \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}u_\lambda\mathrm{d}x - \int_{\mathbb{R}^3} u_n^6\mathrm{d}x \\
&+ \int_{\mathbb{R}^3} u_\lambda^6\mathrm{d}x + o(1) \\
=&\|u_n - u_\lambda\|_E^2 - \int_{\mathbb{R}^3} f(x)u_\lambda^{1-\gamma}\mathrm{d}x + \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}u_\lambda\mathrm{d}x - \int_{\mathbb{R}^3} |u_n - u_\lambda|^6\mathrm{d}x + o(1).
\end{aligned}
$$

Therefore,

$$
\lim_{n\to\infty} \|u_n - u_\lambda\|_E^2 - \int_{\mathbb{R}^3} f(x)u_\lambda^{1-\gamma}\mathrm{d}x + \lim_{n\to\infty}\int_{\mathbb{R}^3} f(x)u_n^{-\gamma}u_\lambda\mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}^3} |u_n - u_\lambda|^6\mathrm{d}x. \tag{2.33}
$$

By Fatou's Lemma, we can obtain

$$
\int_{\mathbb{R}^3} f(x)u_\lambda^{1-\gamma}\mathrm{d}x \le \liminf_{n\to\infty}\int_{\mathbb{R}^3} f(x)u_n^{-\gamma}u_\lambda\mathrm{d}x. \tag{2.34}
$$

We can get from (2.33) and (2.34) that

$$
\lim_{n\to\infty} \|u_n - u_\lambda\|_E^2 \le \lim_{n\to\infty}\int_{\mathbb{R}^3} |u_n - u_\lambda|^6\mathrm{d}x. \tag{2.35}
$$

Set $\lim\limits_{n\to\infty} \|u_n - u_\lambda\|_E^2 = l$, then it follows from (1.5) and (2.35) that $l \le S^{-3}l^3$, which implies that either $l = 0$ or $l \ge S^{\frac{3}{2}}$. Suppose $l \ge S^{\frac{3}{2}}$, then one can obtain from (2.28), (2.32), (2.25), (2.14), (2.2), Lemma 2.1 and Young inequalities that

$$
\begin{aligned}
c_* > c =&\frac{l}{3} + \frac{1}{3}\|u_\lambda\|_E^2 + \lambda(\frac{1}{2p} - \frac{1}{6})\mathbb{D}(u_\lambda) - (\frac{1}{1-\gamma} - \frac{1}{6})\int_{\mathbb{R}^3} f(x)u_\lambda^{1-\gamma}\mathrm{d}x \\
\ge&\frac{1}{3}S^{\frac{3}{2}} + \frac{1}{3}\|u_\lambda\|_E^2 - (\frac{1}{1-\gamma} - \frac{1}{6})\|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\|u_\lambda\|^{1-\gamma} \\
\ge&\frac{1}{3}S^{\frac{3}{2}} - \frac{1+\gamma}{2}\left[\frac{2S}{3(1-\gamma)}\right]^{\frac{\gamma-1}{\gamma+1}}\left[\frac{5+\gamma}{6(1-\gamma)}\right]^{\frac{2}{\gamma+1}}\|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}} = c_*,
\end{aligned}
$$

which is a contradiction. So $l = 0$ and $u_n \to u_\lambda$ strongly in E. □

3 Existence of a first solution in $\mathcal{N}_\lambda^+$

In this section, we want to prove Theorem 1.1 by a minimization argument on $\mathcal{N}_\lambda^+$.
Proof of Theorem 1.1. Fix $0 < \|f\|_{\frac{6}{5+\gamma}} < T_0 = \min\{T_1, T_2\}$, where T_1 is defined in (2.5) and $T_2 = \dfrac{4}{(5+\gamma)S^{\frac{\gamma-1}{2}}}\left[\dfrac{S^{\frac{3}{2}}(1-\gamma)}{1+\gamma}\right]^{\frac{\gamma+1}{2}}$, then $c_* > 0$. Due to Lemma 2.2, Lemma 2.4 and Ekeland variational principle, we can obtain a minimizing sequence $\{u_n\} \subset \mathcal{V}_\lambda^+ = \mathcal{N}_\lambda^+ \cup \{0\}$ satisfying $(2.11)^+$, $(2.12)^+$ and (2.13). According to $(2.11)^+$ and Lemma 2.6 (i), we have

$$
J_\lambda(u_n) \to \tau_\lambda^+ < 0 < c_*,
$$

so $\{u_n\} \subset \mathcal{N}_\lambda^+$ and applying Lemma 2.10 with $c = \tau_\lambda^+$ results in $u_\lambda \not\equiv 0$ and $u_n \to u_\lambda$ in E, up to a subsequence.

Step 1. *u_λ is a solution of problem (P_λ).*

One can further obtain from the above relation, $u_n \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, Lemma 2.1 and Lemma 2.7 (i) that $u_\lambda \in \mathcal{N}_\lambda$ and

$$(1+\gamma)\|u_\lambda\|_E^2 + \lambda(2p-1+\gamma)\mathbb{D}(u_\lambda) - (5+\gamma)\int_{\mathbb{R}^3} |u_\lambda|^6 \mathrm{d}x > 0.$$

Hence, $u_\lambda \in \mathcal{N}_\lambda^+$. Furthermore, passing to the limit as $n \to \infty$ in (2.25) and using Fatou's Lemma, Lemma 2.1 and (2.13) lead to

$$\int_{\mathbb{R}^3} f(x)u_\lambda^{-\gamma}\psi \mathrm{d}x \leq \liminf_{n\to\infty}\int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi \mathrm{d}x = \langle u_\lambda, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_\lambda), \psi\rangle - \int_{\mathbb{R}^3} u_\lambda^5\psi \mathrm{d}x, \tag{3.1}$$

for any $0 \leq \psi \in E$. We can repeat the arguments used in (2.26)-(2.27) to derive that (3.1) holds for any $\psi \in E$. Thus, u_λ verifies (1.4) by the arbitrariness of $\psi \in E$ in (3.1). Similar to the proof of [33, Theorem 1], we have $u_\lambda \in C_{loc}^2(\mathbb{R}^3)$. Since $u_\lambda \geq 0$, $u_\lambda \not\equiv 0$ and u_λ satisfies (1.4), the strong maximum principle implies $u_\lambda > 0$ in $\mathbb{R}^3$ and then u_λ is a solution of problem (P_λ).

Step 2. *u_λ is a ground state solution of problem (P_λ).*

For any $u \in \mathcal{N}_\lambda^-$, according to Lemma 2.2, there exists unique $0 < t^+(u) < t_{max} < t^-(u)$ such that $t^+(u)u \in \mathcal{N}_\lambda^+$, $t^-(u)u \in \mathcal{N}_\lambda^-$, $J_\lambda(t^+(u)u) = \inf_{0<t\leq t^-(u)} J_\lambda(tu)$ and $J_\lambda(t^-(u)u) = \sup_{t\geq t_{max}} J_\lambda(tu)$. Then $t^-(u) = 1$ and there exists $\bar{t}(u) \in (t_{max}, t^-(u))$ such that $J_\lambda(t^+(u)u) < J_\lambda(\bar{t}(u)u)$. So

$$\tau_\lambda^+ \leq J_\lambda(t^+(u)u) < J_\lambda(\bar{t}(u)u) \leq J_\lambda(t^-(u)u) = J_\lambda(u).$$

By the arbitrariness of $u \in \mathcal{N}_\lambda^-$ and the definition of $\tau_\lambda^\pm$ and τ_λ, we have $\tau_\lambda^+ < \tau_\lambda^-$ and so $\tau_\lambda = \tau_\lambda^+$ thanks to $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ by Lemma 2.2. Therefore, $J_\lambda(u_\lambda) = \tau_\lambda^+ = \tau_\lambda$ and thus u_λ is a ground state solution of problem (P_λ).

Step 3. *For any vanishing sequence $\{\lambda_n\} \subset (0,1)$, $u_{\lambda_n} \to u_0$ strongly in E where u_0 is a positive solution of problem (P_0) .*

For any vanishing sequence $\{\lambda_n\} \subset (0,1)$, since $\{u_{\lambda_n}\} \subset \mathcal{N}_{\lambda_n}^+$ is a ground state solution sequence to problem (P_{λ_n}) provided by Step 2, then $J_{\lambda_n}(u_{\lambda_n}) = \tau_{\lambda_n}^+ = \tau_{\lambda_n}$ and

$$\langle u_{\lambda_n}, \psi\rangle_E + \frac{\lambda_n}{2p}\langle \mathbb{D}'(u_{\lambda_n}), \psi\rangle = \int_{\mathbb{R}^3} f(x)u_{\lambda_n}^{-\gamma}\psi \mathrm{d}x + \int_{\mathbb{R}^3} u_{\lambda_n}^5\psi \mathrm{d}x, \tag{3.2}$$

for every $\psi \in E$ and $n \in \mathbb{N}$. By Lemma 2.3, (2.9) and (2.10), we have $\|u_{\lambda_n}\|_E < A_*$ and $\frac{1+\gamma}{3(\gamma-1)}A_*^2 \leq \tau_{\lambda_n} < 0$. Thus, there exists a subsequence of $\{\lambda_n\}$, still denoted by $\{\lambda_n\}$, such that as $n \to \infty$, $\tau_{\lambda_n} \to \mu_1 \leq 0$ and

$$\begin{aligned} u_{\lambda_n} &\rightharpoonup u_0, && \text{in } E,\\ u_{\lambda_n} &\to u_0, && \text{in } L^s(\mathbb{R}^3),\ s \in [2,6),\\ u_{\lambda_n} &\to u_0, && \text{a.e. in } \mathbb{R}^3, \end{aligned} \tag{3.3}$$

where u_0 is a nonnegative function in E. According to (2.10), Lemma 2.1 and weak lower semicontinuity of the norm, we can further get

$$\begin{aligned}\mu_1 =& \liminf_{n\to\infty} J_{\lambda_n}(u_{\lambda_n})\\ \le& \liminf_{n\to\infty}\left[-\frac{1+\gamma}{3(1-\gamma)}\|u_{\lambda_n}\|_E^2+\lambda_n(3-p)\frac{1-\gamma-2p}{6p(1-\gamma)}\mathbb{D}(u_{\lambda_n})\right]\\ \le& -\frac{1+\gamma}{3(1-\gamma)}\|u_0\|_E^2<0.\end{aligned}\tag{3.4}$$

This together with $c_*>0$ leads to $J_{\lambda_n}(u_{\lambda_n})\to\mu_1<0<c_*$. Using (3.2) and the statement in the proof of Lemma 2.10, one can similarly obtain that $u_0\not\equiv 0$ and $u_{\lambda_n}\to u_0$ strongly in E. Then, according to $\|u_{\lambda_n}\|_E<A_*$ and $\{u_{\lambda_n}\}\subset\mathcal{N}^+_{\lambda_n}\subset\mathcal{N}_{\lambda_n}$, we have $\|u_0\|_E\le A_*$ and $u_0\in\mathcal{N}_0$. Passing to the lim as $n\to\infty$ in (3.2) and repeating the arguments used in Step 1, for every $\psi\in E$, we have

$$\langle u_0,\psi\rangle_E=\int_{\mathbb{R}^3} f(x)u_0^{-\gamma}\psi\mathrm{d}x+\int_{\mathbb{R}^3}u_0^5\psi\mathrm{d}x,\tag{3.5}$$

and u_0 is a positive solution of problem (P_0). Hence, $J_0(u_0)=\mu_1\ge\tau_0$ where $\tau_0=\inf_{u\in\mathcal{N}_0}J_0(u)$.

Step 4. *u_0 is a ground state solution of problem (P_0).*

In order to show u_0 is a ground state solution of problem (P_0), it is enough to prove that $J_0(u_0)=\tau_0$. Noticing that $\lambda=0$ is allowed in Step 1 and Step 2, then problem (P_0) admits a ground state solution w_0 satisfying $0<w_0\in\mathcal{N}_0$ and $J_0(w_0)=\tau_0$. By Lemma 2.2, for all $n\in\mathbb{N}$, there exists $0<t^+_{\lambda_n}<t^-_{\lambda_n}$ such that $t^\pm_{\lambda_n}w_0\in\mathcal{N}^\pm_{\lambda_n}\subset\mathcal{N}_{\lambda_n}$ and $J_{\lambda_n}(t^+_{\lambda_n}w_0)=\inf_{0<t\le t^-_{\lambda_n}}J_{\lambda_n}(tw_0)$. We claim that $\{t^-_{\lambda_n}\}$ is bounded. Suppose to the contrary that there exists a subsequence of $\{t^-_{\lambda_n}\}$, still denoted by $\{t^-_{\lambda_n}\}$ such that $t^-_{\lambda_n}\to+\infty$ as $n\to\infty$. Then, by $t^-_{\lambda_n}w_0\in\mathcal{N}^-_{\lambda_n}\subset\mathcal{N}_{\lambda_n}$ and (2.4), we have

$$\frac{1}{(t^-_{\lambda_n})^4}\|w_0\|_E^2+\frac{\lambda_n}{(t^-_{\lambda_n})^{6-2p}}\mathbb{D}(w_0)=\frac{1}{(t^-_{\lambda_n})^{5+\gamma}}\int_{\mathbb{R}^3}f(x)|w_0|^{1-\gamma}\mathrm{d}x+\int_{\mathbb{R}^3}|w_0|^6\mathrm{d}x,\tag{3.6}$$

and

$$-(6-2p)(t^-_{\lambda_n})^{2p}\lambda_n\mathbb{D}(w_0)+(5+\gamma)(t^-_{\lambda_n})^{1-\gamma}\int_{\mathbb{R}^3}f(x)|w_0|^{1-\gamma}\mathrm{d}x<4(t^-_{\lambda_n})^2\|w_0\|_E^2.\tag{3.7}$$

Moreover, $w_0\in\mathcal{N}_0$ means

$$\|w_0\|_E^2=\int_{\mathbb{R}^3}f(x)|w_0|^{1-\gamma}\mathrm{d}x+\int_{\mathbb{R}^3}|w_0|^6\mathrm{d}x.\tag{3.8}$$

Subtracting (3.6) with (3.8) provides

$$\left[1-\frac{1}{(t^-_{\lambda_n})^4}\right]\|w_0\|_E^2-\frac{\lambda_n}{(t^-_{\lambda_n})^{6-2p}}\mathbb{D}(w_0)=\left[1-\frac{1}{(t^-_{\lambda_n})^{5+\gamma}}\right]\int_{\mathbb{R}^3}f(x)|w_0|^{1-\gamma}\mathrm{d}x.\tag{3.9}$$

Passing to the limit in the above equality, we have

$$\|w_0\|_E^2=\int_{\mathbb{R}^3}f(x)|w_0|^{1-\gamma}\mathrm{d}x,$$

a contradiction to (3.8). Therefore, $\{t^-_{\lambda_n}\}$ is bounded. Up to a subsequence, suppose that $t^-_{\lambda_n} \to t^-_0$. We claim that $t^-_0 \geq 1$. Arguing by contradiction suppose that $0 < t^-_0 < 1$, then it follows from (3.9) and (3.7) that

$$\left[1 - \frac{1}{(t^-_0)^4}\right] \|w_0\|^2_E = \left[1 - \frac{1}{(t^-_0)^{5+\gamma}}\right] \int_{\mathbb{R}^3} f(x)|w_0|^{1-\gamma} \mathrm{d}x, \tag{3.10}$$

and

$$(5+\gamma)(t^-_0)^{1-\gamma} \int_{\mathbb{R}^3} f(x)|w_0|^{1-\gamma} \mathrm{d}x \leq 4(t^-_0)^2 \|w_0\|^2_E. \tag{3.11}$$

Combining (3.10) with (3.11), we can deduce that

$$4(t^-_0)^{5+\gamma} - (5+\gamma)(t^-_0)^4 + 1 + \gamma \leq 0,$$

which is impossible since $4t^{5+\gamma} - (5+\gamma)t^4 + 1 + \gamma > 0$ for all $t \in (0,1)$. Therefore, $t^-_0 \geq 1$. If $t^-_0 > 1$, then $t^-_{\lambda_n} > 1$ for some n large enough. This together with $J_{\lambda_n}(t^+_{\lambda_n} w_0) = \inf_{0<t\leq t^-_{\lambda_n}} J_{\lambda_n}(tw_0)$ leads to $J_{\lambda_n}(w_0) \geq J_{\lambda_n}(t^+_{\lambda_n} w_0)$ for some n large enough. If $t^-_0 = 1$, then $t^-_{\lambda_n} \to 1$. For some n large enough with $t^-_{\lambda_n} \geq 1$, we have $J_{\lambda_n}(w_0) \geq J_{\lambda_n}(t^+_{\lambda_n} w_0)$ by the similar statement above. For some n large enough with $t^-_{\lambda_n} < 1$, according to Lemma 2.2, there exists $t_{\lambda_n} \in (t^+_{\lambda_n}, t^-_{\lambda_n})$ such that $J_{\lambda_n}(w_0) = J_{\lambda_n}(t_{\lambda_n} w_0) \geq J_{\lambda_n}(t^+_{\lambda_n} w_0)$. Follows from above two cases, we get $J_{\lambda_n}(w_0) \geq J_{\lambda_n}(t^+_{\lambda_n} w_0)$ for some n large enough when $t^-_0 = 1$. To sum up, $J_{\lambda_n}(w_0) \geq J_{\lambda_n}(t^+_{\lambda_n} w_0)$ for some n large enough. Hence, we can obtain from $t^+_{\lambda_n} w_0 \in \mathcal{N}^+_{\lambda_n}$ and $\tau^+_{\lambda_n} = \tau_{\lambda_n}$ that

$$\begin{aligned} \tau_0 =& J_0(w_0) = J_{\lambda_n}(w_0) - \frac{\lambda_n}{2p}\mathbb{D}(w_0) \geq J_{\lambda_n}(t^+_{\lambda_n} w_0) - \frac{\lambda_n}{2p}\mathbb{D}(w_0) \\ \geq& \tau^+_{\lambda_n} - \frac{\lambda_n}{2p}\mathbb{D}(w_0) = \tau_{\lambda_n} - \frac{\lambda_n}{2p}\mathbb{D}(w_0), \end{aligned}$$

for some n large enough and so

$$\limsup_{n\to+\infty} \tau_{\lambda_n} \leq \tau_0. \tag{3.12}$$

Using (3.12), one can further get

$$\tau_0 \leq J_0(u_0) = \limsup_{n\to+\infty} J_{\lambda_n}(u_{\lambda_n}) = \limsup_{n\to+\infty} \tau_{\lambda_n} \leq \tau_0.$$

This shows that $J_0(u_0) = \tau_0$ and so u_0 is a ground state solution of problem (P_0). The proof is completed. □

4 Existence of a second solution in $\mathcal{N}^-_\lambda$

It is well known that S can be attained by the function

$$U_\varepsilon(x) = \frac{(3\varepsilon)^{\frac{1}{4}}}{(\varepsilon + |x|^2)^{\frac{1}{2}}}, \ \varepsilon > 0, \ x \in \mathbb{R}^3, \tag{4.1}$$

and $\|U_\varepsilon\|^2 = \|U_\varepsilon\|_6^6 = S^{\frac{3}{2}}$. Let $\eta(x) \in C_0^\infty(\mathbb{R}^3)$ be a radially symmetric function such that $0 \le \eta \le 1$, $\eta|_{B_{\frac{\delta}{2}}}(0) \equiv 1$ and supp$\eta \subset B_\delta(0)$ for some $\delta > 2\delta_1$ where δ_1 is given in (f_2). Moreover, set $w_\varepsilon(x) = \eta(x)U_\varepsilon(x)$, then for $\varepsilon > 0$ small enough, we have (see [3])

$$\int_{\mathbb{R}^3} |\nabla w_\varepsilon|^2 \mathrm{d}x = K_1 + O(\varepsilon^{\frac{1}{2}}), \quad \int_{\mathbb{R}^3} |w_\varepsilon|^6 \mathrm{d}x = K_2 + O(\varepsilon^{\frac{3}{2}}), \tag{4.2}$$

and

$$\int_{\mathbb{R}^3} |w_\varepsilon|^s \mathrm{d}x = \begin{cases} O(\varepsilon^{\frac{s}{4}}), & s \in [2,3), \\ O(\varepsilon^{\frac{s}{4}}|\ln\varepsilon|), & s = 3, \\ O(\varepsilon^{\frac{6-s}{4}}), & s \in (3,6), \end{cases} \tag{4.3}$$

where $K_1,\ K_2$ are positive constants and $\dfrac{K_1}{K_2^{\frac{1}{3}}} = S$. Using (4.2), we can further get

$$\frac{\int_{\mathbb{R}^3} |\nabla w_\varepsilon|^2 \mathrm{d}x}{(\int_{\mathbb{R}^3} |w_\varepsilon|^6 \mathrm{d}x)^{\frac{1}{3}}} = S + O(\varepsilon^{\frac{1}{2}}) \tag{4.4}$$

Lemma 4.1. *Assume (V_1), (V_2), (f_1) and (f_2) hold, then there exists $0 < T_{00} < T_0$ where T_0 is defined in proof of Theorem 1.1, such that for $0 < \|f\|_{\frac{6}{5+\gamma}} < T_{00}$ and $\varepsilon > 0$ small, we have*

$$\tau_\lambda^- \le \sup_{t \ge 0} J_\lambda(tw_\varepsilon) < c_*, \ \forall \lambda > 0,$$

where c_* is given in Lemma 2.10.

Proof. For $0 < \|f\|_{\frac{6}{5+\gamma}} < \dfrac{1-\gamma}{2}T_1$, by Lemma 2.2 and Lemma 2.6 (ii), there exists $t_\varepsilon > t_{max} > 0$ such that $t_\varepsilon w_\varepsilon \in \mathcal{N}_\lambda^-$ and $J_\lambda(t_\varepsilon w_\varepsilon) = \sup_{t\ge 0} J_\lambda(tw_\varepsilon) \ge \beta_0 > 0$. We can get from this and $J_\lambda(tw_\varepsilon) \to -\infty$ as $t \to +\infty$ that there exist positive constants t_{00}, t_0 independent of ε such that $t_{00} \le t_\varepsilon \le t_0$. Motivated by [10, 42], let $J_\lambda(t_\varepsilon w_\varepsilon) = A(\varepsilon) + B(\varepsilon) + C(\varepsilon) - D(\varepsilon)$, where

$$A(\varepsilon) = \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3} |\nabla w_\varepsilon|^2 \mathrm{d}x - \frac{t_\varepsilon^6}{6}\int_{\mathbb{R}^3} |w_\varepsilon|^6 \mathrm{d}x, \qquad B(\varepsilon) = \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3} V(x)|w_\varepsilon|^2 \mathrm{d}x$$
$$C(\varepsilon) = \lambda\frac{t_\varepsilon^{2p}}{2p}\mathbb{D}(w_\varepsilon), \qquad D(\varepsilon) = \frac{t_\varepsilon^{1-\gamma}}{1-\gamma}\int_{\mathbb{R}^3} f(x)|w_\varepsilon|^{1-\gamma}\mathrm{d}x.$$

For the purpose of proof, set $g_3(t) = \dfrac{t^2}{2}\displaystyle\int_{\mathbb{R}^3} |\nabla w_\varepsilon|^2 \mathrm{d}x - \dfrac{t^6}{6}\displaystyle\int_{\mathbb{R}^3} |w_\varepsilon|^6 \mathrm{d}x$, then one can easily get that $g_3(t)$ achieves its maximum at T_{max} with $T_{max} = \left(\dfrac{\int_{\mathbb{R}^3} |\nabla w_\varepsilon|^2 \mathrm{d}x}{\int_{\mathbb{R}^3} |w_\varepsilon|^6 \mathrm{d}x}\right)^{\frac{1}{4}}$. Thus, it follows from (4.4) that

$$A(\varepsilon) = g_3(t_\varepsilon) \le g_3(T_{max}) = \frac{1}{3}\frac{(\int_{\mathbb{R}^3} |\nabla w_\varepsilon|^2 \mathrm{d}x)^{\frac{3}{2}}}{(\int_{\mathbb{R}^3} |w_\varepsilon|^6 \mathrm{d}x)^{\frac{1}{2}}} = \frac{1}{3}S^{\frac{3}{2}} + O(\varepsilon^{\frac{1}{2}}). \tag{4.5}$$

Since $t_{00} \le t_\varepsilon \le t_0$, one can get from $V \in C(\mathbb{R}^3)$, the definition of w_ε and (4.3) that

$$B(\varepsilon) = \frac{t_\varepsilon^2}{2} \int_{B_\delta(0)} V(x)|w_\varepsilon|^2 \mathrm{d}x \le \max_{x \in B_\delta(0)} V(x) \cdot \frac{t_0^2}{2} \int_{B_\delta(0)} |w_\varepsilon|^2 \mathrm{d}x = O(\varepsilon^{\frac{1}{2}}). \tag{4.6}$$

By (2.1), (4.3) and $1 + \frac{\alpha}{3} \le p < 3$, we also have

$$C(\varepsilon) \le \lambda \frac{t_0^{2p}}{2p} d_\alpha \Big(\int_{\mathbb{R}^3} |w_\varepsilon|^{\frac{6p}{3+\alpha}} \mathrm{d}x \Big)^{\frac{3+\alpha}{3}} = \begin{cases} O(\varepsilon^{\frac{p}{2}}), & \frac{3+\alpha}{3} \le p < \frac{3+\alpha}{2}, \\ O(\varepsilon^{\frac{p}{2}} |\mathrm{ln}\varepsilon|^{\frac{3+\alpha}{3}}), & p = \frac{3+\alpha}{2}, \\ O(\varepsilon^{\frac{3+\alpha-p}{2}}), & \frac{3+\alpha}{2} < p < 3, \end{cases} \tag{4.7}$$

Similarly, by (f_2) and $\frac{3+\gamma}{2} < \beta_1 < \frac{5+\gamma}{2} < 3$, for any ε satisfying $0 < \varepsilon \le \delta_1^2$, we have

$$\begin{aligned} D(\varepsilon) &= \frac{t_\varepsilon^{1-\gamma}}{1-\gamma} \int_{|x|<\delta} f(x)|w_\varepsilon|^{1-\gamma} \mathrm{d}x \\ &= \frac{t_\varepsilon^{1-\gamma}}{1-\gamma} \left[\int_{|x|<\delta_1} f(x)|w_\varepsilon|^{1-\gamma} \mathrm{d}x + \int_{\delta_1 \le |x| < \delta} f(x)|w_\varepsilon|^{1-\gamma} \mathrm{d}x \right] \\ &\ge \frac{t_{00}^{1-\gamma}}{1-\gamma} \int_{|x|<\delta_1} \frac{\rho_1 |x|^{-\beta_1} (3\varepsilon)^{\frac{1-\gamma}{4}}}{(\varepsilon + |x|^2)^{\frac{1-\gamma}{2}}} \mathrm{d}x = C_3 \varepsilon^{\frac{1-\gamma}{4}} \int_0^{\delta_1} \frac{r^2}{r^{\beta_1} (\varepsilon + r^2)^{\frac{1-\gamma}{2}}} \mathrm{d}r \\ &= C_3 \varepsilon^{\frac{\gamma+5-2\beta_1}{4}} \int_0^{\frac{\delta_1}{\sqrt{\varepsilon}}} \frac{r^2}{r^{\beta_1} (1 + r^2)^{\frac{1-\gamma}{2}}} \mathrm{d}r \ge C_3 \varepsilon^{\frac{\gamma+5-2\beta_1}{4}} \int_0^1 \frac{r^2}{2^{\frac{1-\gamma}{2}} r^{\beta_1}} \mathrm{d}r = C_4 \varepsilon^{\frac{\gamma+5-2\beta_1}{4}}. \end{aligned} \tag{4.8}$$

Case 1. $\frac{3+\alpha}{3} \le p \le \frac{3+\alpha}{2}$.

For $\frac{3+\alpha}{3} \le p \le \frac{3+\alpha}{2}$, using the fact that $\lim\limits_{\varepsilon \to 0^+} \varepsilon^{\frac{p-1}{2}} |\mathrm{ln}\varepsilon|^{\frac{3+\alpha}{3}} = 0$ and $\frac{p}{2} \ge \frac{3+\alpha}{6} > \frac{1}{2}$, we can obtain from (4.7) that $C(\varepsilon) = O(\varepsilon^{\frac{1}{2}})$. Combining this with (4.5), (4.6) and (4.8) leads to

$$J_\lambda(t_\varepsilon w_\varepsilon) \le \frac{1}{3} S^{\frac{3}{2}} + O(\varepsilon^{\frac{1}{2}}) - C_4 \varepsilon^{\frac{\gamma+5-2\beta_1}{4}} \le \frac{1}{3} S^{\frac{3}{2}} + C_5 \varepsilon^{\frac{1}{2}} - C_4 \varepsilon^{\frac{\gamma+5-2\beta_1}{4}}.$$

Set $\varepsilon = \|f\|_{\frac{6}{5+\gamma}}^{\frac{4}{1+\lambda}}$ and $T_* = \left(\frac{C_4}{C_5 + D_*} \right)^{\frac{1+\gamma}{2\beta_1 - 3 - \gamma}}$ where D_* is given in Lemma 2.10, since $\frac{3+\gamma}{2} < \beta_1 < \frac{5+\gamma}{2}$, we have

$$C_5 \varepsilon^{\frac{1}{2}} - C_4 \varepsilon^{\frac{\gamma+5-2\beta_1}{4}} = \|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}} \Big(C_5 - C_4 \|f\|_{\frac{6}{5+\gamma}}^{\frac{\gamma+3-2\beta_1}{1+\gamma}} \Big) < -D_* \|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}},$$

and so

$$\sup_{t \ge 0} J_\lambda(t w_\varepsilon) = J_\lambda(t_\varepsilon w_\varepsilon) < \frac{1}{3} S^{\frac{3}{2}} - D_* \|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}} = c_*,$$

for all $\|f\|_{\frac{6}{5+\gamma}}$ sufficiently small with $\|f\|_{\frac{6}{5+\gamma}} < T_3 = \min\{\frac{1-\gamma}{2} T_1, T_2, T_*\}$.

Case 2. $\frac{3+\alpha}{2} < p < 3$.

When $p - \alpha \le 2$ i.e. $\frac{3+\alpha-p}{2} \ge \frac{1}{2}$, similarly to Case 1, we can obtain $\sup_{t \ge 0} J_\lambda(t w_\varepsilon) <$

c_*. Hence, we only consider the situation when $p-\alpha>2$. It follows from $p-\alpha>2$ and $\frac{3+\alpha}{2}<p<3$ that $\frac{\alpha}{2}<\frac{3+\alpha-p}{2}<\frac{1}{2}$. Hence, one can get from (4.5)-(4.8) that

$$J_\lambda(t_\varepsilon w_\varepsilon)\le \frac{1}{3}S^{\frac{3}{2}}+O(\varepsilon^{\frac{1}{2}})+O(\varepsilon^{\frac{3+\alpha-p}{2}})-C_4\varepsilon^{\frac{\gamma+5-2\beta_1}{4}}\le \frac{1}{3}S^{\frac{3}{2}}+C_6\varepsilon^{\frac{\alpha}{2}}-C_4\varepsilon^{\frac{\gamma+5-2\beta_1}{4}}.$$

Set $\varepsilon=\|f\|_{\frac{6}{5+\gamma}}^{\frac{4}{\alpha(1+\gamma)}}$ and $T_{**}=\left(\frac{C_4}{C_6+D_*}\right)^{\frac{\alpha(1+\gamma)}{2\beta_1+2\alpha-5-\gamma}}$, since $\frac{5+\gamma-2\alpha}{2}<\beta_1<\frac{5+\gamma}{2}$, we have

$$C_6\varepsilon^{\frac{\alpha}{2}}-C_4\varepsilon^{\frac{\gamma+5-2\beta_1}{4}}=\|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}}\left(C_6-C_4\|f\|_{\frac{6}{5+\gamma}}^{\frac{\gamma+5-2\beta_1-2\alpha}{\alpha(1+\gamma)}}\right)<-D_*\|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}},$$

and so for all $\|f\|_{\frac{6}{5+\gamma}}$ sufficiently small with $\|f\|_{\frac{6}{5+\gamma}}<T_4=\min\{\frac{1-\gamma}{2}T_1,\,T_2,\,T_{**}\}$, we have

$$\sup_{t\ge 0}J_\lambda(tw_\varepsilon)=J_\lambda(t_\varepsilon w_\varepsilon)<\frac{1}{3}S^{\frac{3}{2}}-D_*\|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}}=c_*.$$

To sum up, set $T_{00}=\min\{T_3,T_4\}$, then for all $\|f\|_{\frac{6}{5+\gamma}}$ sufficiently small with $\|f\|_{\frac{6}{5+\gamma}}<T_{00}$, we have

$$\tau_\lambda^-\le J_\lambda(t_\varepsilon w_\varepsilon)=\sup_{t\ge 0}J_\lambda(tw_\varepsilon)<c_*,$$

since $t_\varepsilon w_\varepsilon\in\mathcal{N}_\lambda^-$ and this ends the proof. □

Proof of Theorem 1.2. Fix $0<\|f\|_{\frac{6}{5+\gamma}}<T_{00}$, according to Theorem 1.1, we only need to show the existence and asymptotic behavior of another solution v_λ which is different with the first solution u_λ. Since $\mathcal{N}_\lambda^-$ is a closed set in E by Lemma 2.4, applying the Ekeland variational principle to construct a minimizing sequence $\{u_n\}\subset\mathcal{N}_\lambda^-$ satisfying $(2.11)^-$, $(2.12)^-$ and (2.13) with weak limit v_λ, to not confuse with u_λ obtained in Section 2 and Section 3.

Step 1. *v_λ is a solution of problem (P_λ).*

We can get from $(2.11)^-$, Lemma 2.6 (ii) and Lemma 4.1 that

$$\tau_\lambda^-\ge\beta_0>0\ \text{ and }\ J_\lambda(u_n)\to\tau_\lambda^-<c_*,$$

so Lemma 2.10 with $c=\tau_\lambda^-$ results in $v_\lambda\not\equiv 0$ and $u_n\to v_\lambda$ in E, up to a subsequence. Then, $J_\lambda(v_\lambda)=\tau_\lambda^-$. Moreover, $u_n\in\mathcal{N}_\lambda^-\subset\mathcal{N}_\lambda$ and $u_n\to v_\lambda$ further lead to $v_\lambda\in\mathcal{N}_\lambda$. Similarly, one can get from Lemma 2.1 and Lemma 2.7 (ii) that

$$(1+\gamma)\|v_\lambda\|_E^2+\lambda(2p-1+\gamma)\mathbb{D}(v_\lambda)-(5+\gamma)\int_{\mathbb{R}^3}|v_\lambda|^6\mathrm{d}x<0,$$

therefore, $v_\lambda\in\mathcal{N}_\lambda^-$. Following the argument used for the first solution u_λ in Section 3, we see that v_λ is also a positive solution of problem (P_λ). Moreover, since $u_\lambda\in\mathcal{N}_\lambda^+$ and $v_\lambda\in\mathcal{N}_\lambda^-$, we get from Lemma 2.3 that $\|v_\lambda\|_E>\|u_\lambda\|_E$. So u_λ and v_λ are distinct.

Step 2. *For any vanishing sequence $\{\lambda_n\}\subset(0,1)$, $v_{\lambda_n}\to v_0$ strongly in E where v_0 is a positive solution of problem (P_0) .*

For any vanishing sequence $\{\lambda_n\}\subset(0,1)$, since $\{v_{\lambda_n}\}\subset\mathcal{N}_{\lambda_n}^-$ is a positive solution sequence to problem (P_{λ_n}) provided by Step 1, then $\beta_0\le J_{\lambda_n}(v_{\lambda_n})=\tau_{\lambda_n}^-<c_*$, $\|v_{\lambda_n}\|_E>A^*>0$ and

$$(v_{\lambda_n},\psi)_E+\frac{\lambda_n}{2p}\langle\mathbb{D}'(v_{\lambda_n}),\psi\rangle=\int_{\mathbb{R}^3}f(x)v_{\lambda_n}^{-\gamma}\psi\mathrm{d}x+\int_{\mathbb{R}^3}v_{\lambda_n}^5\psi\mathrm{d}x, \tag{4.9}$$

for every $\psi \in E$ and $n \in \mathbb{N}$. Since $v_{\lambda_n} \in \mathcal{N}_{\lambda_n}$ and $J_{\lambda_n}(v_{\lambda_n}) < c_*$, then $\{v_{\lambda_n}\}$ is bounded in E by (2.9). Thus, there exists a subsequence of $\{\lambda_n\}$, still denoted by $\{\lambda_n\}$, such that as $n \to \infty$, $\tau^-_{\lambda_n} \to \mu_2$ and

$$\begin{aligned} v_{\lambda_n} &\rightharpoonup v_0, && \text{in } E, \\ v_{\lambda_n} &\to v_0, && \text{in } L^s(\mathbb{R}^3),\ s \in [2,6), \\ v_{\lambda_n} &\to v_0, && \text{a.e. in } \mathbb{R}^3, \end{aligned} \tag{4.10}$$

where v_0 is nonnegative in E. Hence, $\mu_2 \geq \beta_0 > 0$ and $J_{\lambda_n}(v_{\lambda_n}) \to \mu_2 < c_*$. Using (4.9) and the statement in the proof of Lemma 2.10, one can similarly obtain that $v_0 \not\equiv 0$ and $v_{\lambda_n} \to v_0$ strongly in E. Then, $\|v_0\|_E \geq A^*$ follows from $\|v_{\lambda_n}\|_E > A^*$. Passing to the lim as $n \to \infty$ in (4.9) and repeating the arguments used in Step 1 in the proof of Theorem 1.1, we have that v_0 is a positive solution of problem (P_0). It follows from $\|u_0\|_E \leq A_*$, $\|v_0\|_E \geq A^*$ and $A_* < A^*$ in Lemma 2.3 that $\|u_0\|_E < \|v_0\|_E$. The proof is completed. □

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