Dynamical behaviors of a discrete two-dimensional competitive system exactly driven by the large centre

Binbin Du
Aviation Foundation College, Naval Aviation University
264001 Yantai, Shandong, P. R. China

Changjian Wu*and Guang Zhang
Institute of Nonlinear Complexity
Zhujiang College, South China Agricultural University
510900 Guangzhou, Guangdong, P. R. China

Xiao-Liang Zhou
School of Mathematics and Statistics
Lingnan Normal University
524048 Zhanjiang, Guangdong, P. R. China

Abstract

In this paper, a new discrete large-sub-center system is obtained by using the Euler and nonstandard discretization methods for the corresponding continuous system. It is surprised that all dynamic behaviors of the discrete system are exactly driven by the large-center equation, for example, the stabilities, the bifurcations, the period-doubling orbits, and the chaotic dynamics, etc. Additionally, the global asymptotical stability, the existence of exact 2-periodic solutions, the flip bifurcation theorem, and the invariant set of the sub-center equation is also given. These results reveal far richer dynamics of the discrete model compared with the continuous model. Through numerical simulation, we can observe some complex dynamic behaviors, such as period-doubling cascade, periodic windows, chaotic dynamics, etc. Especially, our theoretical results are also showed by those numerical simulations.

Keywords: large-sub-center, discrete system, flip bifurcation, center manifold method, chaos.

Mathematics Subject Classification: 39A28, 39A30.

* *Corresponding author:* wchjianyf@163.com (C. J. Wu), qd_gzhang@126.com (G. Zhang).

1 Introduction

In recent years, the discrete dynamical models described by the difference equations have been extensively investigated by a number of authors, for example, many species of insect have no overlap between successive generations, and thus their population evolves in discrete-time steps, see [1-12] for the predator-prey system, [13-18] for the competitive system, and [19-22] for the cooperative system. At the same time, such system is often used in the analysis of dynamic economic systems, for example, economic growth, structural economic change, innovation, economic competition, regional sciences, see Cafagna and Coccorese [20], Ding and Shi [22], Nijkamp and Reggiani [23-25], Askar [26], and Dawid et. al. [27], etc.

The hierarchical structure in the spatial system has been characterized by a discrete system. Especially, Nijkamp and Reggiani [23-25] considered the discrete system of the form

$$\begin{cases} u_{t+1} = au_t(1-u_t), \\ v_{t+1} = rv_t(1-bu_t-v_t), \end{cases} \tag{1}$$

where u represents the size of the large centre and v the size of the sub-centre,

$t \in \{0,1,2,\cdots\} \triangleq \mathbb{Z}^+$, $0 < a \leq 4$, $r > 0$ and $0 < b < 1$. The dynamical behavior of system (1) has been numerically investigated by Nijkamp and Reggiani [23-25].

System (1) can be obtained by using the Euler's method from the continuous system

$$\begin{cases} \frac{dx}{dt} = ax\left(m-x\right), \\ \frac{dy}{dt} = dy\left(n-ex-y\right), \end{cases} \tag{2}$$

where a, m, d, n and e are positive constants. System (2) can be rewritten as

$$\begin{cases} \frac{du_1}{d\tau} = \gamma u_1\left(1-u_1\right), \\ \frac{du_2}{d\tau} = u_2\left(1-u_2-\varepsilon_2 u_1\right), \end{cases} \tag{3}$$

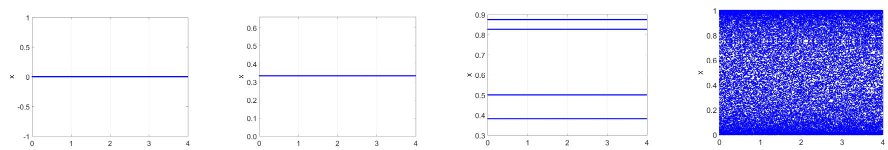
by using a simpler transformation, see [28] or [29], also see (1.2) in [30]. When $0 < \varepsilon_2 < 1$, system (3) has a unique positive steady state $(1, 1-\varepsilon_2)$ which is globally asymptotically stable ([28] or [29]). In this case, we can say that u_1 is the size of the large centre and that u_2 is the size of the sub-centre because the species u_2 has no impact on the evolution of species u_1.

However, we find that the central position of the large centre u is not clear for system (1). Specifically, we do not know the contributions of u and v for the complex behavior of (1). Indeed, the species v has no impact on the evolution of species u in system (1). However, we can see that the species v exists some distinctive dynamical behaviors which can not be driven by the species u. In the following, we will give some explanations.

It is well known that the first equation of system (1)

$$u_{t+1} = au_t\left(1-u_t\right) \tag{4}$$

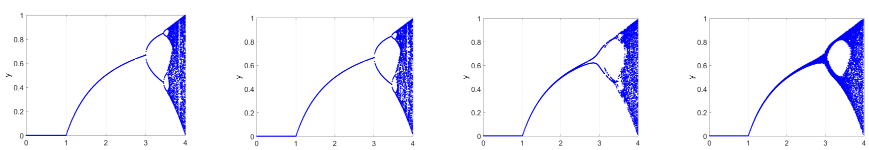
has been extensively discussed by May [31] and subsequently by many other authors, for instance, Baker and Gollub [32], Baumol and Benhabib [33], Frank and Stengos [34], Kelsey [35] and Sharkovsky et. al. [36], so we will not discuss here in any detail the possible evolutionary patterns of u. We will just emphasize that for $a > a^*$ (for example, see Zhang, Jiang and Cheng [37]) a cycle of period 3 appears, beyond which there are cycles in every integer period, as well as an uncountable number of aperiodic trajectories. In view of Li and Yorke [38], this is a typical example of a chaotic region. For example, let $a = 0.5$, 1.5, 3.5 and 4, we can simulate the phase diagrams of (4), see Figure 1.



(a) (b) (c) (d)

Figure 1. The phase diagrams of (4) for $a = 0.5, 1.5, 3.5$ and 4

We observe that the zero solution of (4) is stable for $a = 0.5$, the positive fixed point $\frac{1}{3}$ of (4) is stable for $a = 1.5$, (4) has a stable 4-periodic solution for $a = 3.5$, and (4) is chaos for $a = 4$. We also obtain the bifurcation diagrams of the second equation of (1) for $b = 0.01$ and $a = 0.5$, 1.5, 3.5 and 4, where r is the bifurcation parameter, see Figure 2.



(a) (b) (c) (d)

Figure 2. The bifurcation diagrams of the second equation of (1) for $b = 0.01$ and $a = 0.5, 1.5, 3.5$ and 4.

From Figure 2, we can observe and find the following facts:

(i) When $0 < r < 1$, the zero solution of the second equation of (1) is stable for $a = 0.5$, 1.5, 3.5 and 4;

(ii) For $a = 0.5$ or 1.5, the second equation of (1) undergoes the stability of the zero solution, the stability of the positive fixed points, the period-doubling, and the chaos;

(iii) For $a = 3.5$, the second equation of (1) undergoes the stability of the zero solution, the stability of the positive fixed points, the quasi-fixed point, the quasi-binary period-doubling, and the chaos;

(iv) For $a = 4$, the second equation of (1) undergoes the stability of the zero solution, the stability of the positive fixed points, the quasi-fixed point, the quasi-binary periods, and the chaos.

From the above observations of (i)-(iv), for any $a \in (0,4]$, the second equation of (1) may show the chaotic behaviors when the parameter r is larger. In particular, the second equation of (1) can also cause chaos when $a \in (1,3)$. But, the positive fixed point of the second equation of (1) is also stable when $r \in (0,1)$, even if the large centre equation is chaos for $a = 4$. In this case, the dominance of "the large centre u" has disappeared in fact.

Clearly, the dynamics of (4) can become “chaotic” for certain parameter values while their “mother-version”

$$\frac{dx}{dt} = rx(1-x)$$

has very simple dynamics. This can be interpreted as “numerical chaos” and such dynamical characteristics have also been called “numerically unstable” [39]. However, there are many situations for which continuous models, i.e., differential equations are the best fit. Thus, we need to be dynamically consistent for the discrete versions of the corresponding differential equations ([39], [40] and [41], and the references therein), such as, stability, bifurcation, and chaos. In the present paper, we will not discuss here in any detail for the dynamical consistency of the discrete versions, and only choose a mixed discretizing method so as to manifest the predominance of the large centre x.

In view of Liu and Elaydi [40], we can obtain a nonstandard discrete system of (3) as

$$\left\{ \begin{array}{l} x_{t+1} = \frac{(1+\varphi_1(h))x_t}{1+\varphi_1(h)x_t}, \\ y_{t+1} = \frac{(1+\varphi_2(h))y_t}{1+\varphi_2(h)(\varepsilon_2 x_t + y_t)}, \end{array} \right. \tag{5}$$

where $t \in \mathbb{Z}^+$,

$$\varphi_1(h) = \frac{e^{\gamma h} - 1}{\gamma} \text{ and } \varphi_2(h) = e^h - 1.$$

The unique positive equilibrium $(1, 1-\varepsilon_2)$ of system (5) is globally asymptotically stable, see Theorems 4 in [40]. In this case, the dynamical behaviors of (3) and (5) are clearly consistent.

On the other hand, our work is also motivated by Kang and Smith [42] and Kang [43]. In [42] and [43], Kang and Smith investigated the global dynamics of a discrete two-dimensional competition model of the form

$$\left\{ \begin{array}{l} x_{t+1} = \frac{r_1 x_t}{a + x_t + y_t}, \\ y_{t+1} = y_t \exp\left(r_2 - x_t - y_t\right), \end{array} \right. \tag{6}$$

where r_1 and r_2 are positive and a is nonnegative. System (6) is called a discrete two-species Lottery-Ricker competition model, where the first equation of (6) is the lottery model and the second equation of (6) is the Ricker model, see Kang and Smith [42] or Kang [43].

The dynamical behaviors for the discrete system of the form

$$\left\{ \begin{array}{l} x_{t+1} = a x_t \left(1 - x_t\right), \\ y_{t+1} = \frac{(1+b)y_t}{1+b(x_t + c y_t)}, \end{array} \right. \tag{7}$$

will be considered in this paper, where $b = e^h - 1 > 0$ and $1 < a \leq 4$. Note that $\varepsilon_2, x_t, \varepsilon_2 x_t \in (0,1)$, thus, $\varepsilon_2 x_t$ is replaced by x_t. For more general applications, we add the coefficient $c > 0$ before y_t. Certainly, c can also be interpreted as the intraspecific acting coefficient. System (7) is also a hybrid discrete system with the logistic model and the lottery model. We will demonstrate that the dynamical behaviors of system (7) is exactly driven by the large centre x.

We have known that the chaos for difference schemes governing discrete population growth is by no means restricted to single-species models, for example, Guckenheomer, Oster and Ipaktchi [47] considered the two-dimensional Leslie model:

$$\begin{cases} x_{t+1} = (b_1 x_t + b_2 y_t) \exp\left(-a\left(x_t + y_t\right)\right), \\ y_{t+1} = s x_t, \end{cases} \tag{8}$$

where b_1, b_2, a and s are positive constants. System (8) possesses for certain choices of the parameters 3-cycles which appear numerically to be globally stable, see Guckenheomer, Oster and Ipaktchi [47]. Thus, Marotto [48] gave an extended version of Li-Yorke's theorem, that is, "Snap-back repellers imply chaos in $\mathbb{R}^n$". Unfortunately, there is a minor technical flaw, see Marotto [49] and the references therein. In [49], Marotto has corrected the flaw, however, the Marotto's theorem is invalid for our system (7) because its positive fixed point is not a repeller. In particular, Liang and Jiang [50] and Huang [51] also investigated the extended versions of Li-Yorke's theorem for the planar monotone or competitive maps. The results in [50] and [51] are also invalid for our system, see Corollary 3 in Huang [51].

Remark 1. We notice that (6) and (8) are coupled systems.

The paper is organized as follows. In Section 2, we will give the local dynamical behaviors of system (7) for its four fixed points

$$E_0 = (0,0), E_1 = \left(\frac{a-1}{a}, 0\right), E_2 = \left(0, \frac{1}{c}\right) \text{ and } E_3 = \left(\frac{a-1}{a}, \frac{1}{ac}\right).$$

By the local analysis of those fixed points, we conjecture that the fixed point E_3 should be globally attractive. Indeed, we prove that any solution $\{(x_t, y_t)\}$ with the initial values $x_0 \in (0,1)$ and $y_0 > 0$ of system (7) satisfies

$$\lim_{t\to\infty} x_t = \frac{a-1}{a} \text{ and } \lim_{t\to\infty} y_t = \frac{1}{ac},$$

when $1 < a < 3$. We observe that such result only require the conditions $b > 0$ and $c > 0$. That is, there is no additional limitation for the time stepsize and the competitive intensity of the sub-centre. In Section 3, we will investigate the bifurcation and the center manifold for $a = 3$. Furthermore, the exact 2-periodic positive solutions of (7) will be considered in Section 4. Some numerical simulations will be given in Section 5. For the convenience of simulation, the invariant set of the sub-center equation is also given in this section. In the final section, some conclusions and discussions will be given.

2 The dynamics about fixed points of (7)

In this section, we will discuss the local dynamical behaviors of system (7) for its four fixed points

$$E_0 = (0, 0), E_1 = \left(\frac{a-1}{a}, 0\right), E_2 = \left(0, \frac{1}{c}\right) \text{ and } E_3 = \left(\frac{a-1}{a}, \frac{1}{ac}\right),$$

where $b, c > 0$, and $1 < a \leq 4$. Specially, the global attractivity of the fixed point E_3 wil also be investigated.

The Jacobian matrix of system (7) at (x, y) is given by

$$J(x, y) = \left(\begin{array}{cc} a - 2ax & 0 \\ \frac{-b(1+b)y}{[1+b(x+cy)]^2} & \frac{(1+b)(1+bx)}{[1+b(x+cy)]^2} \end{array}\right). \tag{9}$$

The characteristic equation of Jacobian matrix can be written as

$$\lambda^2 + p(x, y)\lambda + q(x, y) = 0, \tag{10}$$

where

$$p(x, y) = a(2x - 1) - \frac{(1+b)(1+bx)}{[1+b(x+cy)]^2}$$

and

$$q(x, y) = \frac{2(1+b)(1+bx)(1-2x)}{[1+b(x+cy)]^2}.$$

Let λ_1, λ_2 be two roots of (10), we recall some definitions of topological types for a fixed point (x, y). A fixed point (x, y) of a two-dimension discrete system is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, a sink is locally asymptotic stable. The fixed point (x, y) is called a source when $|\lambda_1| > 1$ and $|\lambda_2| > 1$, a source is locally unstable. (x, y) is called a saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$). And (x, y) is called non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

In the following, we will discuss the local dynamics for the fixed points E_i for $i = 0, 1, 2, 3$. For the fixed point E_0, we have

$$J(E_0) = \left(\begin{array}{cc} a & 0 \\ 0 & 1+b \end{array}\right)$$

which has the eigenvalues $\lambda_1(E_0) = a > 1$ and $\lambda_2(E_0) = 1 + b > 1$, thus, E_0 is unstable and a source or a repelling node.

For E_1, the Jacobian matrix of system (7) is

$$J(E_1) = \left(\begin{array}{cc} 2 - a & 0 \\ 0 & 1 + \frac{b}{a(1+b)-b} \end{array}\right)$$

which has the eigenvalues $\lambda_1(E_1) = 2 - a$ and $|\lambda_1(E_1)| = |2 - a| < 1$ if and only if $1 < a < 3$, and

$$\lambda_2(E_1) = 1 + \frac{b}{a(1+b) - b} > 1.$$

Thus, the fixed point E_1 is a saddle.

Note that

$$y = \frac{(1+b)\,y}{1+b\,(x+cy)}$$

or

$$1 = \frac{1+b}{1+b\,(x+cy)}$$

if $y \neq 0$. In this case, we have

$$J\,(x,y) = \begin{pmatrix} a-2ax & 0 \\ -\frac{by}{1+b} & \frac{1+bx}{1+b} \end{pmatrix}.$$

Thus, the Jacobian matrix of E_2 is

$$J\,(E_2) = \begin{pmatrix} a & 0 \\ \frac{-b}{c(1+b)} & \frac{1}{1+b} \end{pmatrix},$$

which has the eigenvalues $\lambda_1\,(E_2) = a > 1$ and $\lambda_2\,(E_2) = 1/\,(1+b) < 1$. In this case, the fixed point E_2 is also a saddle.

For the fixed point E_3, similarly, we have

$$J\,(E_3) = \begin{pmatrix} 2-a & 0 \\ \frac{-b}{ac(1+b)} & 1-\frac{b}{a(1+b)} \end{pmatrix}$$

which has two eigenvalues

$$\lambda_1 = 2-a \text{ and } \lambda_2 = 1-\frac{b}{a\,(1+b)}.$$

We find that

$$0 < 1-\frac{b}{a\,(1+b)} < 1$$

for any $a > 1$ and $b > 0$, and that $|\lambda_1\,(E_3)| = |2-a| < 1$ if and only if $1 < a < 3$.

Proposition 1. For any $b, c > 0$ and $1 < a \leq 4$, E_0 is a source and E_1 is a saddle. Additionally, E_2 is also saddle when $a > 1$, and E_3 is local attractive if $1 < a < 3$.

In view of Proposition 1, we can give a simple phase diagram, see Figure 3.

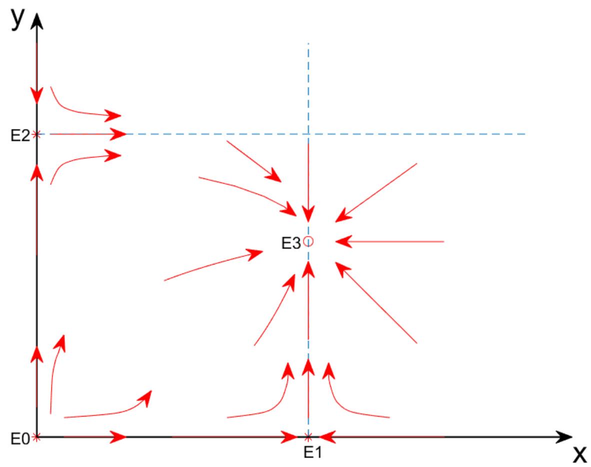


Figure 3. The local phase diagrams of (7) for $1 < a < 3$.

From Figure 3, we can naturally obtain a conjecture that the fixed point E_3 should be globally asymptotically stable for $1 < a < 3$. Thus, we have the following theorem.

Theorem 1. Assume that $b, c > 0$, and $1 < a < 3$. For any $x_0 \in (0, 1)$ and $y_0 > 0$, the solution $\{(x_t, y_t)\}$ of system (7) satisfies

$$\lim_{t\to\infty} x_t = \frac{a-1}{a} \text{ and } \lim_{t\to\infty} y_t = \frac{1}{ac}.$$

Proof. For any $x_0 \in (0, 1)$ and $y_0 > 0$, clearly, we have $x_t \in (0, 1)$ and $y_t > 0$ for $t = 1, 2, ...$, and

$$\lim_{t\to\infty} x_t = \frac{a-1}{a}.$$

Since $y_t > 0$, so

$$\begin{aligned} \frac{1}{y_{t+1}} &= \frac{1 + b\,(x_t + cy_t)}{(1 + b)\, y_t} = \frac{1}{1 + b}\left(\frac{1}{y_t} + \frac{bx_t}{y_t} + bc\right) \\ &= \frac{1}{1 + b}\left[\left(b\left(x_t - \frac{a-1}{a}\right) + 1 + \frac{b\,(a-1)}{a}\right)\frac{1}{y_t} + bc\right], \end{aligned}$$

and

$$\begin{aligned} \frac{1}{y_{t+1}} - ac &= \frac{1}{1 + b}\left[\left(b\left(x_t - \frac{a-1}{a}\right) + 1 + \frac{b\,(a-1)}{a}\right)\frac{1}{y_t} + bc\right] - ac \\ &= \frac{1}{1 + b}\left(b\left(x_t - \frac{a-1}{a}\right) + 1 + \frac{b\,(a-1)}{a}\right)\left(\frac{1}{y_t} - ac\right) \\ &\quad + \frac{abc}{1 + b}\left(x_t - \frac{a-1}{a}\right). \end{aligned}$$

Let

$$X_t = x_t - \frac{a-1}{a}, Y_t = \frac{1}{y_t} - ac,$$

then we have

$$\begin{aligned} Y_{t+1} &= \frac{1}{1+b}\left(bX_t + 1 + \frac{b\,(a-1)}{a}\right)Y_t + \frac{abc}{1+b}X_t \\ &= \left(\frac{b}{1+b}X_t + 1 - \frac{b}{a(1+b)}\right)Y_t + \frac{abc}{1+b}X_t. \end{aligned}$$

Let

$$q = 1 - \frac{b}{2a(1+b)}.$$

Clearly, $0 < q < 1$. Note that $\lim_{t\to\infty} X_t = 0$, thus, for any $\varepsilon > 0$, there exists $N_1 > 0$ such that

$$|X_t| < \min\left(\varepsilon, \frac{1}{2a}\right),$$

and

$$\left|\frac{b}{1+b}X_t\right| < \frac{b}{2a\,(1+b)} \text{ for all } t > N_1.$$

Hence

$$0 < \left|\frac{b}{1+b}X_t\right| + 1 - \frac{b}{a(1+b)} < q < 1 \text{ for all } t > N_1. \tag{11}$$

In view of (11), for $t > N_1$, we get that

$$\begin{aligned} |Y_{t+1}| &= \left|\left(\frac{b}{1+b}X_t + 1 - \frac{b}{a(1+b)}\right)Y_t + \frac{abcX_t}{1+b}\right| \\ &\le \left|\frac{b}{1+b}X_t + 1 - \frac{b}{a(1+b)}\right||Y_t| + \frac{abc}{1+b}|X_t| \\ &\le \left[1 - \frac{b}{2a(1+b)}\right]|Y_t| + \frac{abc}{1+b}\varepsilon = q\,|Y_t| + \frac{abc}{1+b}\varepsilon, \end{aligned}$$

and

$$\begin{aligned} |Y_{N_1+1+t}| &\le q^t\,|Y_{N_1+1}| + \frac{abc}{1+b}\left(1 + q + \cdots + q^{t-1}\right)\varepsilon \\ &\le q^t\,|Y_{N_1+1}| + \frac{abc}{(1+b)\,(1-q)}\varepsilon. \end{aligned}$$

Note that $0 < q < 1$, for any $\varepsilon > 0$, there exists $N_2 > 0$ such that $q^t < \varepsilon$ for $t > N_2$. In particular, for $t > N_1 + N_2 + 1$, we have

$$|Y_t| \le \left[|Y_{N_1+1}| + \frac{abc}{(1+b)\,(1-q)}\right]\varepsilon,$$

which implies that $\lim_{t\to\infty} Y_t = 0$, that is

$$\lim_{t\to\infty} y_t = \frac{1}{ac}.$$

The proof is complete.

Remark 2. We note that the fixed point E_0 is a repeller, however, it is not a snap-back fixed point in view of Theorem 1. Thus, the Marotto's theorem is invalid.

3 Center manifolds and flip bifurcation theorem

Based on the analysis in Section 2, we will discuss the flip bifurcation of the fixed points by using center manifold theorem and bifurcation theory in [44]. To this end, we firstly consider the case $a = 3$, at this time, the fixed points E_1 and E_3 are reduced to

$$E_1 = \left(\frac{2}{3}, 0\right), E_3 = \left(\frac{2}{3}, \frac{1}{3c}\right),$$

respectively. For the fixed E_1, we have

$$J(E_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 + \frac{b}{3+2b} \end{pmatrix},$$

which has two eigenvalues

$$\lambda_1(E_1) = -1 \text{ and } \lambda_2(E_1) = 1 + \frac{b}{3+2b} > 1.$$

Thus, the fixed point E_1 is non-hyperbolic and unstable .

For the fixed E_3, we have

$$J(E_3) = \begin{pmatrix} -1 & 0 \\ \frac{-b}{3c(1+b)} & 1 - \frac{b}{3(1+b)} \end{pmatrix}$$

and

$$\lambda_1(E_3) = -1 \text{ and } \lambda_2(E_3) = 1 - \frac{b}{3(1+b)} < 1.$$

Let

$$u_t = x_t - \frac{2}{3} \text{ or } x_t = u_t + \frac{2}{3}.$$

From the first equation of (7), we directly get that the center manifold of the form

$$u_{t+1} = -u_t - 3u_t^2. \tag{12}$$

The zero solution of (12) is locally asymptotically stable. In view of the center manifold theorem (see Theorem 3.2.2 in Guckenheimer and Holmes [44]), the fixed point

$$E_3 = \left(\frac{2}{3}, \frac{1}{3c}\right)$$

of (7) is also locally asymptotically stable.

The generic one-parameter family has a two-dimensional center manifold (including the parameter direction) on which it is topologically equivalent to the saddle-node family defined by the first equation of (7), see Guckenheimer and Holmes [44]. Now, we assume that a is a parameter and rewrite (7) as

$$\begin{pmatrix} x \\ y \\ \eta \end{pmatrix} \rightarrow \begin{pmatrix} ax(1-x) \\ \frac{(1+b)y}{1+b(x+cy)} \\ \eta \end{pmatrix}$$

Let $\eta = a - 3$, $u = x - \frac{2}{3}$ and $v = y - \frac{1}{3c}$. We have

$$\begin{aligned} u' &= (\eta+3)\left(u+\frac{2}{3}\right)\left(1-u-\frac{2}{3}\right)-\frac{2}{3} \qquad (13) \\ &= -u+\frac{2}{9}\eta-3u^2-\frac{1}{3}u\eta-u^2\eta \triangleq -u+\frac{2}{9}\eta+f_\eta(u,v). \end{aligned}$$

Similarly, we can obtain that

$$\begin{pmatrix} u \\ v \\ \eta \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & \frac{2}{9} \\ \frac{-b}{3c(1+b)} & \frac{3+2b}{3+3b} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} + \begin{pmatrix} f_\eta(u,v) \\ g_\eta(u,v) \\ 0 \end{pmatrix}.$$

The coefficient matrix

$$\begin{pmatrix} -1 & 0 & \frac{2}{9} \\ \frac{-b}{3c(1+b)} & \frac{3+2b}{3+3b} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

has the eigenvalues

$$-1, \frac{2b+3}{3b+3} \text{ and } 1$$

and the corresponding eigenvectors

$$\text{col}\left(\frac{1}{b}(6c+5bc),1,0\right), \text{col}(0,1,0), \text{ and } \text{col}\left(\frac{1}{9},-\frac{1}{9c},1\right).$$

Let

$$T = \begin{pmatrix} \frac{1}{b}(6c+5bc) & 0 & \frac{1}{9} \\ 1 & 1 & -\frac{1}{9c} \\ 0 & 0 & 1 \end{pmatrix},$$

which implies that

$$T^{-1}=\left(\begin{array}{ccc} \frac{b}{6c+5bc} & 0 & -\frac{b}{54c+45bc} \\ -\frac{b}{6c+5bc} & 1 & \frac{2b+2}{18c+15bc} \\ 0 & 0 & 1 \end{array}\right).$$

We assume that

$$\left(\begin{array}{c} u \\ v \\ \eta \end{array}\right)=T\left(\begin{array}{c} w \\ z \\ \delta \end{array}\right),$$

or

$$\left(\begin{array}{c} w \\ z \\ \delta \end{array}\right)=T^{-1}\left(\begin{array}{c} u \\ v \\ \eta \end{array}\right).$$

Thus, we have

$$T\left(\begin{array}{c} w \\ z \\ \delta \end{array}\right)\rightarrow\left(\begin{array}{ccc} -1 & 0 & \frac{2}{9} \\ \frac{-b}{3c(1+b)} & \frac{3+2b}{3+3b} & 0 \\ 0 & 0 & 1 \end{array}\right)T\left(\begin{array}{c} w \\ z \\ \delta \end{array}\right)+\left(\begin{array}{c} F(w,z,\delta) \\ G(w,z,\delta) \\ 0 \end{array}\right),$$

$$\left(\begin{array}{c} w \\ z \\ \delta \end{array}\right)\rightarrow T^{-1}\left(\begin{array}{ccc} -1 & 0 & \frac{2}{9} \\ \frac{-b}{3c(1+b)} & \frac{3+2b}{3+3b} & 0 \\ 0 & 0 & 1 \end{array}\right)T\left(\begin{array}{c} w \\ z \\ \delta \end{array}\right)+T^{-1}\left(\begin{array}{c} F(w,z,\delta) \\ G(w,z,\delta) \\ 0 \end{array}\right),$$

or

$$\left(\begin{array}{c} w \\ z \\ \delta \end{array}\right)\rightarrow\left(\begin{array}{ccc} -1 & 0 & 0 \\ 0 & \frac{2b+1}{3b+3} & 0 \\ 0 & 0 & 1 \end{array}\right)\left(\begin{array}{c} w \\ z \\ \delta \end{array}\right)+T^{-1}\left(\begin{array}{c} F(w,z,\delta) \\ G(w,z,\delta) \\ 0 \end{array}\right),$$

where

$$\begin{aligned} F(w,z,\delta) &= -(3+\delta)\left(\frac{c(6+5b)}{b}w+\frac{1}{9}\delta\right)^2-\frac{1}{3}\left(\frac{c(6+5b)}{b}w+\frac{1}{9}\delta\right)\delta \\ &= -c^2\left(75+\frac{180}{b}+\frac{108}{b^2}\right)w^2-\frac{2}{27}\delta^2-c\left(5-\frac{6}{b}\right)w\delta \\ &\quad -c\left(\frac{10}{9}+\frac{4}{3b}\right)w\delta^2-c^2\left(25+\frac{60}{b}+\frac{36}{b^2}\right)w^2\delta-\frac{1}{81}\delta^3. \end{aligned}$$

In the following, we give the flip bifurcation theorem which can be seen in [44].

Lemma 1. Let $f_\mu:\mathbb{R}\rightarrow\mathbb{R}$ be a one-parameter family of mappings such that f_{μ_0} has a fixed point x_0 with eigenvalue -1. Assume

$$\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x\partial \mu}=\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2}-\left(\frac{\partial f}{\partial x}-1\right)\frac{\partial^2 f}{\partial x\partial \mu}\neq 0 \tag{14}$$

and

$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)^2+\frac{1}{3}\left(\frac{\partial^3 f}{\partial x^3}\right)\neq 0 \tag{15}$$

at (x_0,μ_0). Then there is a smooth curve of fixed points of f_μ, passing through (x_0,μ_0), the stability of which changes at (x_0,μ_0). There is also a smooth curve γ passing through (x_0,μ_0) so that $\gamma-\{(x_0,\mu_0)\}$ is a union of hyperbolic period 2 orbits. The curve γ has quadratic tangency with the line $\mathbb{R}\times\{\mu_0\}$ at (x_0,μ_0).

Theorem 2. For $a=3$, system (7) undergoes a flip bifurcation and the bifurcated 2-periodic points are stable.

Proof. Note that

$$f(w,\delta)=-w+\frac{b}{6c+5bc}F(w,z,\delta).$$

Thus, we have

$$\frac{\partial f}{\partial \delta}\left|_{(0,0)}\right.=0,\ \frac{\partial f}{\partial w}\left|_{(0,0)}\right.=-1,$$
$$\frac{\partial^2 f}{\partial \delta\partial w}\left|_{(0,0)}\right.=-\frac{c(5b-6)}{b}\frac{b}{6c+5bc}=-\frac{5b-6}{5b+6},$$
$$\frac{\partial^2 f}{\partial w^2}\left|_{(0,0)}\right.=\frac{-2c^2b}{6c+5bc}\ \text{and}\ \frac{\partial^3 f}{\partial w^3}\left|_{(0,0)}\right.=0.$$

$$\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x\partial \mu}=-2\frac{5b-6}{5b+6},$$
$$\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2}-\left(\frac{\partial f}{\partial x}-1\right)\frac{\partial^2 f}{\partial x\partial \mu}=-2\frac{5b-6}{5b+6}$$

and

$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)^2+\frac{1}{3}\left(\frac{\partial^3 f}{\partial x^3}\right)=\frac{1}{2}\times\left(\frac{-2c^2b}{6c+5bc}\right)^2=\frac{2c^4b^2}{(6c+5bc)^2}.$$

In view of Lemma 1, we finish the proof of theorem.

Remark 3. In view of Theorems 1 and 2, we can see that the Marotto's theorem is invalid, see Marotto [48] and [49]. On the other hand, when the parameter condition $a=3$ hold, a flip bifurcation occurs at fixed point E_3. This reflects in the market competition that there will be a stable 2-period cycle between two firms when $a>3$ and $a-3$ is enough small. As a approaches 3 from above, the period-two cycle "shrinks" and disappears. A flip corresponds to a pitchfork bifurcation of the second iterate. Please see Figure 4a and Figure 4c in Section 5.

4 Two periodic positive solutions

In view of Theorem 2, we have known that system (7) has a stable 2-periodic solution when $a > 3$ is near 3. In the present section, we will consider its exact 2-periodic solutions. A real sequence $\{x_t\}$ is 2-periodic if and only if

$$x_t = a_0 + a_1 (-1)^t, \tag{16}$$

where $a_0, a_1 \in \mathbb{R}$ and $a_1 \neq 0$, see Zhang, Jiang and Cheng [37]. To find our desired solutions, we substitute (16) into the first equation of (7) and obtain

$$\begin{aligned} & a_0 - a_1 (-1)^t - a \left(a_0 + a_1 (-1)^t\right) + a \left(a_0 + a_1 (-1)^t\right)^2 \\ = \quad & a_0 - aa_0 + aa_0^2 + aa_1^2 + (-a_1 - aa_1 + 2aa_0a_1)(-1)^t = 0 \end{aligned}$$

which implies that

$$\begin{cases} a_0 - aa_0 + aa_0^2 + aa_1^2 = 0, \\ -a_1 - aa_1 + 2aa_0a_1 = 0. \end{cases} \tag{17}$$

By solving (17), we obtain the non-trivial roots

$$a_0 = \frac{a+1}{2a} \text{ and } a_1^{(\pm)} = \pm \frac{1}{2a} \sqrt{a^2 - 2a - 3}. \tag{18}$$

In order that a_1 is a nonzero real number, we need

$$a^2 - 2a - 3 > 0$$

which implies that $a > 3$.

Let $y_t = b_0 + b_1 (-1)^t$. Similarly, we also have

$$\begin{aligned} 0 \quad = \quad & \left(b_0 - b_1 (-1)^t\right) \left(1 + b \left(a_0 + a_1 (-1)^t + c \left(b_0 + b_1 (-1)^t\right)\right)\right) \\ & - (b+1) \left(b_0 + b_1 (-1)^t\right) \\ = \quad & ba_0b_0 - bb_0 + bcb_0^2 + ba_1b_1 + bcb_1^2 \\ & + (-bb_1 + ba_0b_1 + ba_1b_0 + 2bcb_0b_1)(-1)^t. \end{aligned}$$

Let

$$\begin{cases} ba_0b_0 - bb_0 + bcb_0^2 + ba_1b_1 + bcb_1^2 = 0, \\ -bb_1 + ba_0b_1 + ba_1b_0 + 2bcb_0b_1 = 0. \end{cases} \tag{19}$$

By solving (19), we obtain the non-trivial roots

$$b_0 = -\frac{1}{2c}(1 - a_0 + a_1) \text{ and } b_1 = -\frac{1}{2c}(1 - a_0 + a_1),$$

$$b_0 = \frac{1}{2c}(1 - a_0 + a_1) \text{ and } b_1 = -\frac{1}{2c}(1 - a_0 + a_1),$$

or

$$b_0 = \frac{1-a_0}{c}, b_1 = -\frac{a_1}{c}.$$

We easily prove that

$$b_0 = -\frac{1}{2c}\left(1-a_0+a_1\right) \text{ and } b_1 = -\frac{1}{2c}\left(1-a_0+a_1\right)$$

and

$$b_0 = \frac{1}{2c}\left(1-a_0+a_1\right) \text{ and } b_1 = -\frac{1}{2c}\left(1-a_0+a_1\right)$$

are invalid and they will be omitted.

In the following, we will discuss

$$b_0 = \frac{1-a_0}{c} \text{ and } b_1 = -\frac{a_1}{c}.$$

To this end, we assume that

$$\begin{cases} x_t = a_0 + a_1\left(-1\right)^t, \\ y_t = b_0 + b_1\left(-1\right)^t, \end{cases}$$

is a 2-periodic positive solution of (7), where

$$a_0 = \frac{a+1}{2a},\ a_1^{(\pm)} = \pm\frac{1}{2a}\sqrt{a^2-2a-3},$$

$$b_0 = \frac{1-a_0}{c} \text{ and } b_1 = -\frac{a_1}{c}.$$

In fact, we have

$$1-a_0 = 1-\frac{a+1}{2a} = \frac{a-1}{2a},$$

$$\frac{a-1}{2ac} - \frac{1}{2a}\sqrt{a^2-2a-3} = \frac{1}{2a}\left(\frac{a-1}{c} - \sqrt{a^2-2a-3}\right),$$

$$\begin{aligned} \frac{a-1}{c} - \sqrt{a^2-2a-3} &= \frac{\left(\frac{a-1}{c}\right)^2 - \left(a^2-2a-3\right)}{\frac{a-1}{c} + \sqrt{a^2-2a-3}} \\ &= \frac{\left(1-c^2\right)\left(a-1\right)^2 + 4c^2}{c\left(a-1\right) + c^2\sqrt{a^2-2a-3}}, \end{aligned}$$

thus, we need to discuss

$$\left(1-c^2\right)\left(a-1\right)^2 + 4c^2 > 0. \tag{20}$$

When $0 < c \le 1$, clearly, (20) is true. In the following, we assume that $c > 1$. In this case, (20) can be rewritten by

$$\left(a-1\right)^2 - c^2\left[\left(a-1\right)^2 - 4\right] > 0$$

or

$$c < \sqrt{\frac{a^2 - 2a + 1}{a^2 - 2a - 3}}.$$

To sum up, we obtain the following result.

Theorem 3. Assume that $b > 0$, $3 < a \leq 4$ and

$$c < \sqrt{\frac{a^2 - 2a + 1}{a^2 - 2a - 3}},$$

then system (7) has a positive 2-periodic solution of the form

$$\begin{cases} x_t = a_0 + a_1 (-1)^t, \\ y_t = b_0 + b_1 (-1)^t, \end{cases}$$

where

$$a_0 = \frac{a+1}{2a}, a_1^{(\pm)} = \pm \frac{1}{2a} \sqrt{a^2 - 2a - 3},$$

$$b_0 = \frac{1 - a_0}{c} \text{ and } b_1 = -\frac{a_1}{c}.$$

Remark 4. For $3 < a \leq 4$, we have

$$\frac{a^2 - 2a + 1}{a^2 - 2a - 3} = 1 + \frac{4}{(a-1)^2 - 4} \geq \frac{9}{5},$$

and

$$\lim_{a \to 3^+} \frac{a^2 - 2a + 1}{a^2 - 2a - 3} = +\infty.$$

Thus, the parameters b and c can be chosen for any positive numbers.

5 Numerical simulation

In this section, we will present some numerical simulations to support and extend the theoretical results obtained in the former sections. For $1 < a \leq 4$, we see that the interval $(0, 1)$ is a invariant set of $x_{t+1} = ax_t (1 - x_t)$. For the convenience of simulation, we will seek a invariant set of y. Note that

$$\frac{1}{y_{t+1}} = \frac{1 + b (x_t + cy_t)}{(1 + b) y_t} \geq \frac{bc}{1 + b},$$

or

$$y_{t+1} \leq \frac{1 + b}{bc}.$$

We therefore obtain the following result.

Proposition 2. The set

$$(0,1) \times \left(0, \frac{1+b}{bc}\right]$$

is a invariant set of system (7) when $b, c > 0$ and $1 < a \leq 4$.

In view of Proposition 2, for fixed a, b and c, the initial value (x_0, y_0) can be immediately chosen by

$$(x_0, y_0) \in (0,1) \times \left(0, \frac{1+b}{bc}\right].$$

In the following, we will present the bifurcation diagrams and the maximum Lyapunov exponents for system (7) to confirm the above theoretical analysis and show some new interesting complex dynamical behaviors by using numerical simulations. Here, the parameters b and c are fixed at $b = c = 0.5$, and $a \in (1, 4)$ is the bifurcation parameter, see Figure 4.

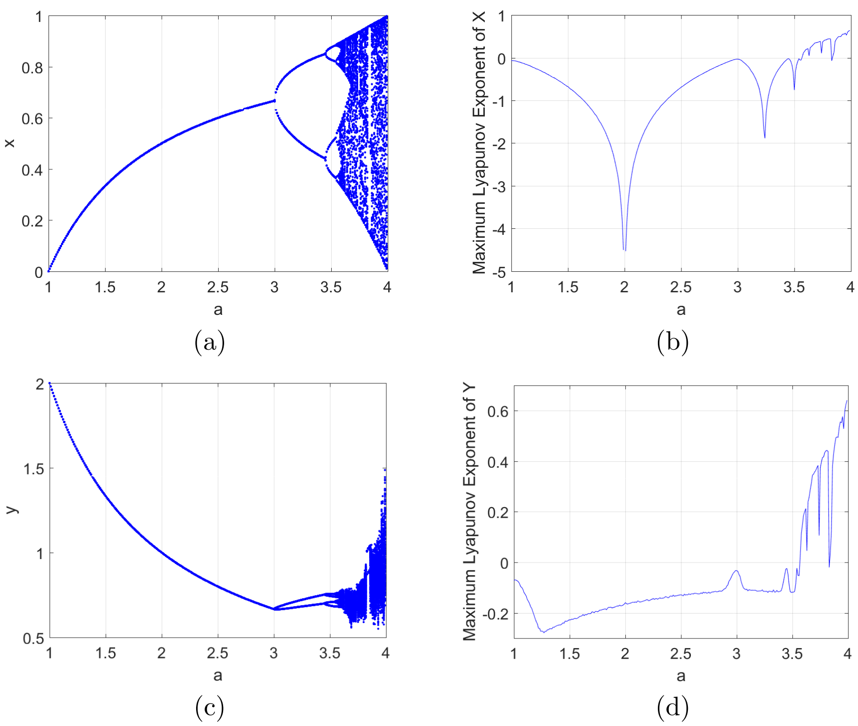


Figure 4. $b = c = 0.5$, and $a \in (1, 4)$ is the bifurcation parameter.

Figure 4a is the bifurcation diagram about x for system (7), Figure 4b is the corresponding Lyapunov exponent diagram, and Figure 4c and Figure 4d are similar for the variable y. The bifurcation diagrams and the Lyapunov exponent diagrams may describe the dynamical behaviors of x and y in different ways, for example, the stabilities, the bifurcations, the period-doubling orbits, the chaos, and the periodic windows.

From Figure 4, we observe the following:

(i) $1 < a < 3$: The unique positive fixed E_3 is stable, therefore, Theorem 1 is ture.

(ii) $3 < a < 3.57$: A cascade of sudden changes provokes the oscillation of the population in cycles of stable period 2^n, where n increases from 1. When a is close to 3, system (7) exists a stable 2-periodic solution, Theorems 2 and 3 are showed by the numerical simulations. In the following, the period-doubling cascade is appeared. On the other hand, denote $f(x) = ax(1-x)$, we can obtain the Schwarzian of $f(x)$ as

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left[\frac{f''}{f'}\right]^2 < 0$$

when $3 < a < 3.57$ and $x \in [0, 1]$ with $x \neq 1/2$. If a_n is the bifurcation value of the parameter corresponding to the appearance of a cycle B of period 2^n, then the cycle B is attracting for $a_n < a < a_{n+1}$, see Sharkovsky, Kolyada, Sivak and Fedorenko [36].

(iii) $3.57 < a < 3.828$: When the parameter moves, the system alternates between periodic behaviors with high periods on parameter interval windows and chaotic regimes for parameter values not located in intervals. The population can not be predictable although the system is deterministic.

(iv) $3.828 < a < 3.85$: The orbit of period 3 appears for $a = 3.828$ after a regime where unpredictable bursts, named intermittences, have become rarer until their disappearance in the three-periodic time signal. It is well known that "Period three implies chaos", see Li and Yorke [38], also see Sharkovskii [45] or [46]. In [37], Zhang, Jiang and Cheng obtained a necessary and sufficient condition of existence of 3-periodic solution for the first equation of system (7), see Theorem 2 ($a > 1 + 2\sqrt{2} \approx 3.828$) in Zhang, Jiang and Cheng [37]. Hence, any periodic cycles which include period 2^n are instable.

(v) $3.85 < a \leq 4$: Chaotic behavior with periodic windows is observed in this interval. At this time, their maximum Lyapunov exponents are positive.

In the following, we will give some phase diagrams of (7). Here, the parameters b and c are still fixed at $b = c = 0.5$, and $a \in (1, 4)$ is chose the different values, see Figures 5 and 6.

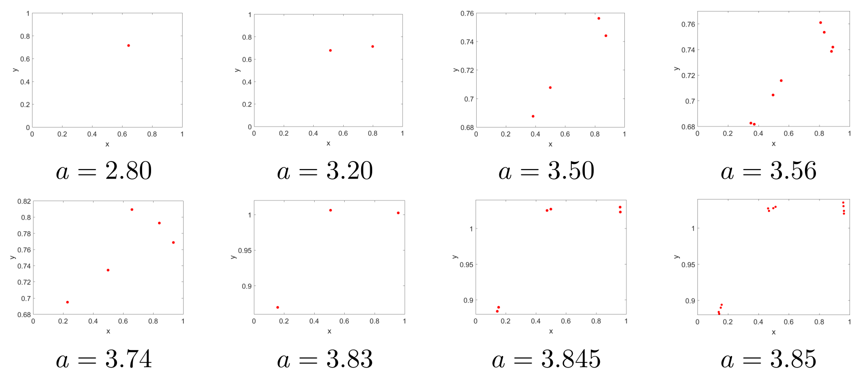


Figure 5. Different choices at $a \in (1, 4)$.

In Figure 5, we can observe the fixed point and the period-doubling orbits of 2, 4 and 8 for $a = 2.80$, 3.20, 3.50 and 3.56, respectively. Furthermore, we also observe another periodic orbits of 5, 3, 6 and 12 for $a = 3.74$, 3.83, 3.845 and 3.85, respectively. To emerge the change process of (7) from a fixed point to chaos, in the final of this section, we again add Figure 6 for a varying from 3.58 to 3.9.

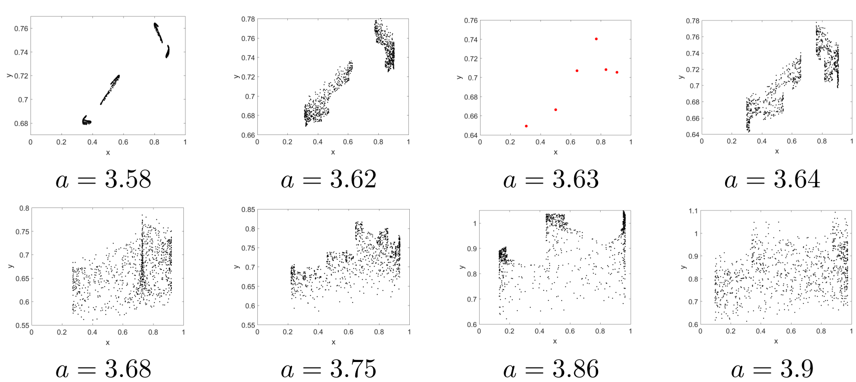


Figure 6. Different choices at $a \in (1, 4)$.

When a is varying from 3.58 to 3.62, we observe that two point clouds are gradually formed. When $a = 3.63$, two point clouds disappear and a period-6 orbit appears. When $a = 3.64$, two point clouds reappear and have a larger scope. When a is varying from 3.68 to 3.73, there is one point cloud. When $a = 3.74$, the point cloud disappear and a period-5 orbit appears (see Figure 5). When a is varying from 3.75 to 3.82, we observe one point cloud. When $a = 3.83$, 3.845 and 3.85, we observe periodic orbits of 3, 6 and 12 (see Figure 5). Period three implies chaos, when $a = 3.86$, we observe that a larger range of point cloud reappears and a clear outline emerges above the point clouds. When $a = 3.9$, the range of point clouds is larger and the boundaries become blurry.

6 Conclusions and discussions

In this final section, we will induce the obtained conclusions in this paper, at the same time, we also hope to explain the responses of the sub-centre by using the numerical simulations when the sub-centre attends in the logistic equation.

In this paper, we obtain a new discrete large-sub-center system by using the Euler and nonstandard discretization method for the corresponding continuous system. The theoretical analysis and the numerical simulations exhibit that all dynamic behaviors of the discrete system are exactly driven by the large-center equation, for example, the stabilities, the bifurcations, the period-doubling orbits, and the chaotic dynamics, etc. Thus, such system may better describe the hierarchical structure in the spatial systems. According to the studies, we draw the following conclusions:

(i) By using the characteristic roots of Jacobian matrix, the local dynamics of fixed points for the system are given, see Proposition 1.

(ii) When $1 < a < 3$, a global stability result is obtained, see Theorem 1. At the same time, the invariant set of system is also sought out when $1 < a \leq 4$, see Proposition 2**.**

(iii) A flip bifurcation theorem is proved, see Theorem 2. It ensures that the period-doubling orbits and the chaotic dynamics are facts. By observing Figures. 4-6, the flip bifurcation will present orbits with periods of 2, 4 and 8. These periodic points are stable, see Sharkovsky, Kolyada, Sivak and Fedorenko [36].

(iv) The exact 2-periodic orbits are also solved, see Theorem 3.

(v) The chaotic dynamics are observed in Figures 4, 5 and 6. At this time, any periodic cycles which include period 2^n are also instable, see Li and Yorke [38].

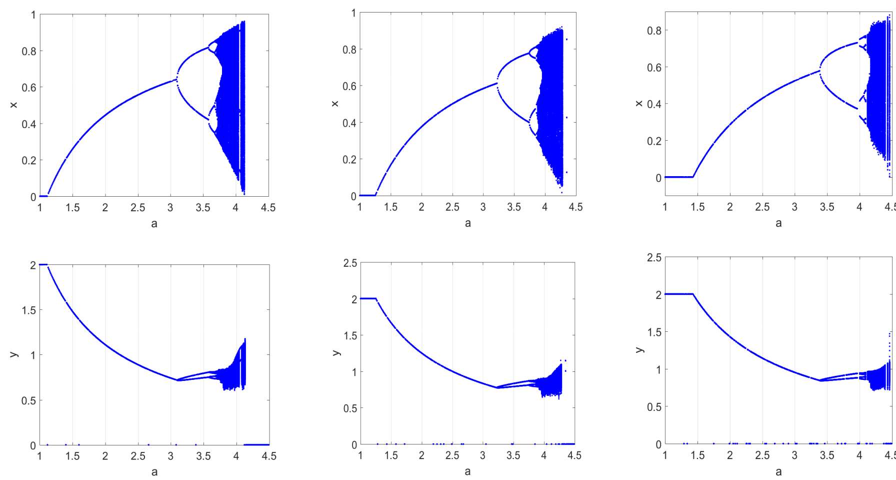
(vi) All dynamic behaviors of system (7) are exactly driven by its first equation.

Remark 5. In this paper, the main purpose is to study the dynamical consistency of the second equation for system (7). The other additional informations of the first equation for (7) can be seen in May [31], Baker and Gollub [32], Baumol and Benhabib [33], Frank and Stengos [34] and Kelsey [35], and the references therein.

We still have a question, can the sub-centre change the dynamical behaviors of the large centre when the sub-centre attends in the logistic equation? For example, we consider the system of the form

$$\begin{cases} x_{t+1} = ax_t\left(1 - x_t - \varepsilon y_t\right), \\ y_{t+1} = \frac{(1+b)y_t}{1+b(x_t+cy_t)}. \end{cases} \tag{21}$$

For the fixed $b = c = 0.5$, we give the following bifurcation diagrams of x and y for $\varepsilon = 0.05$, 0.1 and 0.15, see Figure 7, where $a \in [1, 4.5]$ is the bifurcation parameter.



$\varepsilon = 0.05$ $\varepsilon = 0.1$ $\varepsilon = 0.15$

Figure 7. $b = c = 0.5$ and $a \in [1, 4.5]$ is the bifurcation parameter.

From Figure 7, we can see that the competition of the sub-centre can indeed change the dynamical behaviors of the large centre. We will consider this in another paper.

CRediT authorship contribution statement

Binbin Du: Conceptualization, Methodology, Formal analysis, Investigation, Supervision, Writing-review & editing, Visualization. **Changjian Wu:** Conceptualization, Methodology, Software, Investigation, Writing-original draft, Writing-review & editing, Visualization. **Guang Zhang:** Conceptualization, Methodology, Formal analysis, Investigation, Writing-original draft, Writing-review & editing, Visualization. **Xiaoliang Zhou:** Conceptualization, Methodology, Investigation, Writing-original draft, Writing-review & editing, Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

The authors would like to thank the anonymous referees for their comments and suggestions on the manuscript. We also thank the help of Professor Yubin Yan for reading the paper carefully, who is from Department of Mathematical and Physical Science, Faculty of Science & Engineering, University of

Chester, Thornton Science Park, Pool Lane, Ince, Chester CH24NU, UK. This work was supported by Natural Science Foundation of Guangdong Province(No. 2022A1515010964).

References

[1] C. Damgaard, Dynamics in a discrete two-species competition model: coexistence and over-compensation, Journal of Theoretical Biology, 227(2004), 197-203.

[2] X. L. Liu and D. M. Xiao, Complex dynamic behaviors of a discrete-time predator-prey system, Chaos, Solitons and Fractals, 32(2007), 80-94.

[3] L. G. Yuan and Q. G. Yang, Bifurcation, invariant curve and hybrid control in a discrete-time predator-prey system, Appl. Math. Model., 39(2015), 2345-2362.

[4] A. Q. Khan, J. Y. Ma and D. M. Xiao, Bifurcations of a two-dimensional discrete time plant-herbivore system, Communications in Nonlinear Science and Numerical Simulation, 39(2016), 185-198.

[5] M. Biswas and N. Bairagi, On the dynamic consistency of a two-species competitive discrete system with toxicity: Local and global analysis, Journal of Computational and Applied Mathematics 363(2020), 145-155.

[6] A. Q. Khan, Bifurcation analysis of a discrete-time two-species model, Discrete Dynamics in Nature and Society, Volume 2020, Article ID 2954059, 12 pages.

[7] M. S. Li and X. L. Zhou, Dynamic properties of a discrete population model with diffusion, Advances in Difference Equations, (2020) 2020:580, https://doi.org/10.1186/s13662-020-03033-w.

[8] S. Akhtar, R. Ahmed, M. Batool, N. A. Shah and J. D. Chung, Stability, bifurcation and chaos control of a discretized Leslie prey-predator model, Chaos, Solitons and Fractals 152(2021), 111345.

[9] J. Chen, Y. Chen, Z. Zhu and F. Chen, Stability and bifurcation of a discrete predator-prey system with Allee effect and other food resource for the predators, Journal of Applied Mathematics and Computing, https://doi.org/10.1007/s12190-022-01764-5.

[10] X. Li and Y. Liu, Transcritical bifurcation and flip bifurcation of a new discrete ratio-dependent predator-prey system, Qualitative Theory of Dynamical Systems, 21(2022),122, https://doi.org/10.1007/s12346-022-00646-2.

[11] X. L. Han and C. Y. Lei, Bifurcation and turing instability analysis for a space- and time-discrete predator–prey system with Smith growth function, Chaos, Solitons and Fractals, 166(2023), 112910.

[12] W. B. Yao and X. Y. Li, Bifurcation difference induced by different discrete methods in a discrete predator-prey model, Journal of Nonlinear Modeling and Analysis, 4(1)(2022), 64-79.

[13] S. Rocklin and G. Oster, Competition between Phenotypes, Journal of Mathematical Biology, 3(1976), 225-261.

[14] H. Jiang and T. D. Rogers, The discrete dynamics of symmetric competition in the plane, Journal of Mathematical Biology, 25(1987), 573-596.

[15] R. M. Nisbet and L. C. Onyiah, Population dynamic consequences of competition within and between age classes, Journal of Mathematical Biology, 32(1994), 329-344.

[16] D. M. Chan and J. E. Franke, Probabilities of extinction, weak extinction, permanence, and mutual exclusion in discrete, competitive, Lotka-Volterra systems that involve lnvading species, Mathematical and Computer Modelling, 40(2004), 809-821.

[17] C. W. Shih and J. P. Tseng, Global consensus for discrete-time competitive systems, Chaos, Solitons & Fractals, 41(1)(2009), 302-310.

[18] L. R. Ricardo and F. P. Danièle, Periodic and chaotic events in a discrete model of logistic type for the competitive interaction of two species, Chaos, Solitons & Fractals, 41(1)(2009), 334-347.

[19] M. S. Khan, M. Samreen, J. F. Gómez-Aguilar and E. Pérez-Careta, On the qualitative study of a discrete-time phytoplankton-zooplankton model under the effects of external toxicity in phytoplankton population, Heliyon, 8(2022), e12415.

[20] V. Cafagna and P. Coccorese, Dynamical systems and the arising of cooperation in a Cournot duopoly, Chaos, Solitons and Fractals, 25(3)(2005), 655-664.

[21] K. Yuan and J. D. Cao, Periodic oscillatory solution in delayed competitive–cooperative neural networks: A decomposition approach, Chaos, Solitons & Fractals, 27(1)(2006), 223-231.

[22] Z. W. Ding and G. P. Shi, Cooperation in a dynamical adjustment of duopoly game with incomplete information, Chaos, Solitons and Fractals, 40(2)(2009), 989-993.

[23] P. Nijkamp and A. Reggiani, Spatial competition and ecologically based socioeconomic models, Socio-Spatial Dynamics 3(2)(1992), 89-109.

[24] P. Nijkamp and A. Reggiani, Space-Time Dynamics, Spatial Competition and the Theory of Chaos, Structure and Change in the Space-Economv (T.R. Lakshmanan and P. Nijkamp, eds.), Springer, Berlin 1993.

[25] P. Nijkamp and A. Reggiani, Non-linear evolution of dynamic spatial systems: The relevance of chaos and ecologically-based models, Regional Science and Urban Economics, 25(1995), 183-210.

[26] S. S. Askar, A competition of duopoly game whose players are public: Dynamic investigations, Communications in Nonlinear Science and Numerical Simulation, 111(2022), 106486.

[27] H. Dawid, M. Y. Keoula, M. Kopel and P. M. Kort, Dynamic investment strategies and leadership in product innovation, European Journal of Operational Research, 306(1)(2023), 431-447.

[28] K. Gopalsamy, Stability and oscillations in delay differential equations of population dynamics, Kluwer Academic Publishers, Dordrecht, 1992.

[29] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, New York, 1993.

[30] X. Jin and X. Y. Li, Dynamics of a discrete two-species competitive model with Michaelies-Menten type harvesting in the first species, Journal of Nonlinear Modeling and Analysis, 5(3)(2023), 494-523.

[31] R. M. May, Simple mathematical models with very complicated dynamics, Nature 261(1976), 459-467.

[32] G. L. Baker and J. P. Gollub, Chaotic dynamics: An introduction, Cambridge University Press, Cambridge, 1990.

[33] W. J. Baumol and J. Benhabib, Chaos: Significance, mechanism and economic applications, Journal of Economic Perspectives, 3(1)(1989), 77-105.

[34] M. Frank and T. Stengos, Chaotic dynamics in economic time-series, Journal of Economic Surveys 2(2)(1988), 103-133.

[35] D. Kelsey, The economics of chaos or the chaos of economics?, Oxford Economic Papers 40(1988), 1-3.

[36] A. N. Sharkovsky, S. F. Kolyada, A. G. Sivak and V. V. Fedorenko, Dynamies of One-Dimensional Maps, Naukova Dumka, Kiev, 1989.

[37] G. Zhang, D. M. Jiang and S. S. Cheng, 3-periodic traveling wave solutions for a dynamical coupled map lattice, Nonlinear Dyn., 50(2007), 235-247.

[38] T. Y. Li and J. A. Yorke, Period three implies chaos, American Mathematical Monthly, 82(10)(1975), 985-992.

[39] R. E. Mickens, Genesis of elementary numerical instabilities in finite-difference models of ordinary differential equations, in: G.S. Ladde, M. Sambandham (Ed.), Proceedings of Dynamics Systems and Applications, vol. 1, Dynamics publisher, USA 1994, pp. 251-258.

[40] P. Z. Liu and S. N. Elaydi, Discrete competitive and cooperative Lotka–Volterra type, J. Computational Analysis and Applications, 3(1)(2001), 53-73.

[41] A. Voroshilova and J. Wafubwa, Discrete competitive Lotka–Volterra model with controllable phase volume, Systems, 2020, 8, 17, doi:10.3390/systems8020017.

[42] Y. Kang and H. Smith, Global dynamics of a discrete two-species Lottery-Ricker competition model, Journal of Biological Dynamics, 6(2)(2012), 358-376.

[43] Y. Kang, Pre-images of invariant sets of a discrete-time two-species competition model, Journal of Difference Equations and Applications, 18(10)(2012), 1709-1733.

[44] J. Guckenheimer , P. Holmes , Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer, New York, 1983.

[45] A. N. Sharkovskii, Coexistence of cycles of a continuous mapping of the line into itself, Ukrainian Math. J., 16(1964), 61–71.

[46] A. N. Sharkovskii, Coexistence of cycles of a continuous mapping of the line into itself, International J. Bifurcation and Chaos, 5(5)(1995), 1263-1273(which is translated by J. Tolosa).

[47] J. Guckenheomer, G. Oster, and A. Ipaktchi, The dynamics of density dependent population models, J. Math. Biology, 4(1977), 101-147.

[48] F. R. Marotto, Snap-back repellers imply chaos in $\mathbb{R}^n$, J. Math. Anal. Appl., 63(1)(1978), 199-223.

[49] F. R. Marotto, On redefining a snap-back repeller, Chaos, Solitons and Fractals, 25(2005), 25-28.

[50] X. Liang and J. F. Jiang, On the topological entropy, nonwandering set and chaos of monotone and competitive dynamical systems, Chaos, Solitons and Fractals, 14(2002), 689-696.

[51] Y. Huang, The Marotto theorem on planar monotone or competitive maps, Chaos, Solitons and Fractals, 19(2004), 1105-1112.