Combined effects of singular and Hardy nonlinearities in fractional Kirchhoff Choquard equation

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Abstract

The aim of this paper is to investigate the existence and the multiplicity of solutions to the singular Kirchhoff non-local problem with Hardy and Choquard nonlinearities. The problem is defined as follows:

$$\begin{cases} M\Big(\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\Big) - \Delta_p^s u \quad -\alpha\frac{|u|^{p-2}u}{|x|^{sp}} = \lambda f(x)u^{-\gamma} \\ \qquad\qquad\qquad\qquad\qquad\qquad +g(x)\Big(\displaystyle\int_{\Omega}\frac{u^{p^*_{\mu,s}}(y)}{|x-y|^{\mu}}dy\Big)u^{p^*_{\mu,s}-1} \text{ in } \Omega, \\ u>0, \qquad \text{in } \Omega, \\ u=0, \qquad \text{in } \mathbb{R}^N\setminus\Omega, \end{cases}$$

where, $\Omega\subset\mathbb{R}^N$ is a bounded domain, $s\in(0,1)$, $N>sp$, $\gamma\in(0,1)$, α, λ are two positive real parameters $0<\mu<N$, $p^*_s=\frac{Np}{N-sp}$ is the fractional critical Sobolev exponent, while $p_{\mu,s}=\frac{(Np-\mu)}{(N-sp)}$ and $p^*_{\mu,s}=\left(\frac{p}{2}\right).\left(\frac{2N-\mu}{N-sp}\right)$ denote the critical and upper critical exponent in the sense of Hardy Littlewood Sobolev inequality respectively, $M(t)=a+bt^{\theta-1}$, with $a>0, b>0$ and $\theta\in\Big(1,\min\{2p^*_{\mu,s}/p, p^*_{\mu,s}\}\Big)$. Furthermore, f is a non-negative weight and g is a sign-changing weight. The novelty in this work lies in the combination of a fractional framework and a singular term with the Hardy and Choquard nonlinearities. To establish the existence of at least two positive solutions for the problem, the Nehari manifold approach is employed.

Keywords: Kirchhoff problem, Choquard term, Fractional Sobolev spaces, Hardy potential, Singularities, Nehari manifolds.
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1 Introduction

This work is devoted to investigate a Choquard nonlocal problem with Hardy nonlinearity and a singular term. The problem is described by the following equation:

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$$\begin{cases} M\Big(\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\Big) - \Delta_p^s u & -\alpha\frac{|u|^{p-2}u}{|x|^{sp}} = \lambda f(x)u^{-\gamma} \\ & +g(x)\Big(\displaystyle\int_{\Omega} \frac{u^{p^*_{\mu,s}}(y)}{|x-y|^{\mu}}dy\Big)u^{p^*_{\mu,s}-1} \text{ in } \Omega, \\ u>0, \qquad \text{in } \Omega, \\ u=0, \qquad \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.1}$$

where, $\Omega\subset\mathbb{R}^N$ is a bounded domain, $s\in(0,1)$, $N>sp$, $\gamma\in(0,1)$, α, λ are two positive real parameters $0<\mu<N$, $p^*_s=\frac{Np}{N-sp}$ is the fractional critical Sobolev exponents, while $p_{\mu,s}=\frac{(Np-\mu)}{(N-sp)}$ and $p^*_{\mu,s}=\Big(\frac{p}{2}\Big).\Big(\frac{2N-\mu}{N-sp}\Big)$ denote the critical and upper critical exponent in the sense of Hardy Littlewood Sobolev inequality respectively, f is a non-negative weight and g is a sign-changing weight. The continuous function $M:\mathbb{R}^+_0\to\mathbb{R}^+_0$ is defined by $M(t)=a+bt^{\theta-1}$, with $a>0, b>0$ and $\theta\in\Big(1,\min\{2p^*_{\mu,s}/p, p^*_{\mu,s}\}\Big)$, where Ω is a bounded domain of $\mathbb{R}^N$, $s\in(0,1)$, α and λ are positive real parameters, $N>sp$, $\gamma\in(0,1)$, $0<\mu<N$, $p^*_{\mu,s}=\Big(\frac{p}{2}\Big).\Big(\frac{2N-\mu}{N-sp}\Big)$ is the upper critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, f is a positive weight and g is a sign-changing function. The operator $(-\Delta)^s_p$ is a nonlocal operator defined as

$$-\Delta_p^s u(x) := 2\lim_{\epsilon\to 0}\int_{\Omega\setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\mathrm{d}y, \quad x\in\Omega,$$

where $B_\epsilon(x):=\{y\in\Omega : |x-y|<\epsilon\}$. We make the following assumptions regarding the weight functions f and g in the problem:

(f) Let $f:\Omega\to\mathbb{R}$ be a wieght such that $f>0$ a.e. in Ω and $f\in L^m(\Omega)$, with $m:=\frac{p^*_s}{p^*_s-1+\gamma}$.

(g) Let $g:\Omega\to\mathbb{R}$ be a sign-changing wieght such that $g\in L^r$, with $r:=\frac{p^*_s}{p_{\mu,s}-p^*_{\mu,s}}$ where $p_{\mu,s}=\frac{(Np-\mu)}{(N-sp)}$ is the critical exponent in the sense Hardy-Littlewood-Sobolev inequality.

To handle the Hardy term in equation (1.1), we utilize the fractional Hardy inequality, which is given as:

$$\mu_0\int_\Omega \frac{|\phi(x)|^p}{|x|^{sp}}\,dx \le \iint_{\mathbb{R}^{2N}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}}\,dx\,dy. \tag{1.2}$$

This inequality allows us to manipulate the Hardy term in the equation. The constant μ_0 is the sharp constant associated with the fractional Hardy inequality. For further details, refer to the reference [8].

Problem (1.1) corresponds to the Choquard-Pekar equation, which has found significant applications in various fields such as quantum mechanics, condensed matter physics, and material science. For more detailed information on this equation, please refer to the references [12, 16]. Moreover, these types of problems have been utilized in the modeling of diverse phenomena, including chaotic dynamics, turbulence, financial dynamics, and plasma physics. To delve deeper into these applications and explore further references, we recommend consulting the works [1, 3] and the references provided therein.

In recent years, there has been significant research on the uniqueness, existence, multiplicity, and regularity of solutions for fractional Choquard problems. For more detailed information, we recommend referring to the following recent articles: Fiscella and Mishra [6], Fiscella and Vaira [7], Gao, Yang, and Yang [9], Goyal and Sharma [10], Muruganandam and Srinivasan [15], Wang, Xiao, and Yang [19], and Yang, Wang, and Wang [20]. These articles, along with their references, provide extensive insights into the analysis of fractional Choquard problems.

Fiscella and Mishra [6] focused on investigating the multiplicity of non-positive solutions using the Nehari approach for problems involving singular and critical nonlinearities with a Hardy term. Their research contributes to our understanding of the existence of multiple non-positive solutions in this context.

In [7], Fiscella and Vaira employed variational methods along with an appropriate truncation argument to establish the existence of two solutions for a critical Kirchhoff-type problem. Their work demonstrates the existence of multiple solutions in this critical setting.

Goyal and Sharma [10] used a fibering map analysis to show the multiplicity of solutions to the fractional weighted Choquard Kirchhoff equation with both Hardy and singular nonlinearities. Their research provides insights into the existence of multiple solutions within this framework.

Furthermore, Wang et al. [19] investigated the multiplicity of non-negative solutions using the Nehari method. Their work contributes to our understanding of the existence of multiple non-negative solutions in the context of fractional Choquard problems.

In this paper, our focus is on a specific type of nonlocal Choquard Kirchhoff problem driven by Hardy and singular nonlinearities, denoted as (1.1). One notable challenge in studying this problem is that the associated energy functional, which characterizes the solutions, is not differentiable throughout the entire space. Consequently, the conventional critical point theory cannot be directly applied to address our problem.

Motivated by the works of Goyal and Sharma [10] and Fiscella and Mishra [6], we adopt the Nehari-manifold technique as a powerful tool to establish the multiplicity of solutions for problem (1.1). This approach allows us to overcome the non-differentiability of the energy functional and explore the existence of multiple solutions.

By employing the Nehari-manifold technique, we aim to provide insights into the existence and multiplicity of solutions for the considered nonlocal Choquard Kirchhoff problem with Hardy and singular nonlinearities, as described by (1.1).

To present the main result of this work, we shall introduce the following notations: Let

$$\Lambda_1 := \Big(\frac{p+\gamma-1}{2p^*_{\mu,s}-p}\Big)^{\frac{p+\gamma-1}{2p^*_{\mu,s}-p}} \Big(\frac{d_{a,\alpha}(2p^*_{\mu,s}-p)}{2p^*_{\mu,s}+\gamma-1}\Big)^{\frac{2p^*_{\mu,s}+\gamma-1}{2p^*_{\mu,s}-p}} S^{\frac{2p^*_{\mu,s}+\gamma-1}{p^*_{\mu,s}-p}} \frac{1}{\|f\|_m} \Big(\frac{1}{C_g(N,\mu)}\Big)^{\frac{p+\gamma-1}{2p^*_{\mu,s}-p}}, \quad (1.3)$$

where $C_g(N,\mu)$, S, $d_{a,\alpha}$ will be defined in (2.6), (2.7) and (2.12) respectively.

$$\Lambda_2 := \frac{S^{\frac{(\theta+1)(2p^*_{\mu,s}+\gamma-1)}{2(2p^*_{\mu,s}-\theta-1)}}}{\Big[C_g(N,\mu)\Big]^{\frac{\theta+\gamma}{2p^*_{\mu,s}-\theta-1}}}\Big[\frac{2\sqrt{(1+\gamma)(2\theta+\gamma-1)d_{a,\alpha}b}}{2p^*_{\mu,s}+\gamma-1}\Big]^{\frac{\theta+\gamma}{2p^*_{\mu,s}-\theta-1}} \times \Big[\frac{2\sqrt{(2p^*_{\mu,s}-2)(2p^*_{\mu,s}-2\theta)d_{a,\alpha}b}}{\Big(2p^*_{\mu,s}+\gamma-1\Big)\|f\|_m}\Big] \tag{1.4}$$

and

$$\Lambda_* := \min\{\Lambda_1, \Lambda_2\}.$$

Our main result is the following theorem.

Theorem 1.1. *Let $N > sp$ with $s \in (0,1)$, $\alpha \in (0, a\mu_0)$, $a > 0, b > 0$ and $\theta \in \Big(1, \min\{2p^*_{\mu,s}/p, p^*_{\mu,s}\}\Big)$. Assume that the assumptions* (f), (g) *hold. Then, there exists Λ_*, which depends on α, such that problem* (1.1) *has at least two non-negative solutions for all $\lambda \in (0, \Lambda_*)$.*

This paper is structured as follows:

Section 2 provides the necessary background information, including basic definitions and notations that will be used throughout the paper. In Section 3, we introduce and discuss the application of the Nehari manifold to our specific problem, as described by problem (1.1). This section presents the key concept and technique used in our analysis. Section 4 is dedicated to proving important results related to the compactness of the functional energy associated with our problem. These results are essential for the subsequent analysis and proof. In Section 5, we establish the existence of a non-negative solution within the Nehari manifold $\mathcal{N}^+_{\alpha,\lambda}$, demonstrating the existence of one solution with a specific property. Section 6 focuses on proving the existence of a non-negative solution within the Nehari manifold $\mathcal{N}^-_{\alpha,\lambda}$, which completes the proof of our main results.

2 Preliminaries

In this section, we introduce some fundamental notations and definitions related to fractional Sobolev spaces and Choquard equations, which will be utilized in the subsequent parts of the paper.

We begin by defining the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$, which consists of functions u in $L^p(\mathbb{R}^N)$ satisfying a certain regularity condition. Specifically, we have

$$W^{s,p}(\mathbb{R}^N) := \Big\{u \in L^p(\mathbb{R}^N) : \frac{u(x)-u(y)}{|x-y|^{\frac{N}{p+s}}} \in L^p(\mathbb{R}^N \times \mathbb{R}^N)\Big\},$$

where $s \in (0,1)$ and p is a fixed exponent. The fractional Sobolev space is equipped with the norm

$$||u||_{W^{s,p}(\mathbb{R}^N)} := ||u||^p_{L^p(\mathbb{R}^N)} + \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy\Big)^{1/p}. \tag{2.5}$$

This norm measures the regularity and decay properties of functions in the fractional Sobolev space.

In our analysis, we will consider the space X_0, defined as

$$X_0 = \Big\{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\Big\},$$

where Ω is a given domain. The norm in X_0 is given by

$$\|u\|_{X_0} = \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy\Big)^{1/p} = \|u\|,$$

which is equivalent to the norm defined in Eq. (2.5). This space allows us to consider functions that vanish outside the domain Ω.

Now, we state the following important inequality.

Proposition 2.1 (Proposition 2.1 of [15])**.** *For $u,\, v \in L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$. Then, we have*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^q|v(y)|^q}{|x-y|^\mu} dxdy \le \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu} dxdy\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)|^q|v(y)|^q}{|x-y|^b} dxdy\Big)^{\frac{1}{2}},$$

where $\mu \in (0,N)$ and $q \in [\frac{p(2N-\mu)}{2N}, p^*_{\mu,s}]$.

Thus from Proposition 2.1, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)|u(x)|^{p^*_{\mu,s}}|v(y)|^{p^*_{\mu,s}}}{|x-y|^\mu} dxdy \le C_g(N,\mu)\|u\|^{2p^*_{\mu,s}}_{L^{p^*_s}(\mathbb{R}^N)} \tag{2.6}$$

where $C_g(N,\mu)$ is a suitable constant. Define

$$S := \inf_{u \in X_0(\Omega)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y}{\Big(\displaystyle\int_{\mathbb{R}^N} |u(x)|^{\frac{Np}{N-sp}} \mathrm{d}x\Big)^{\frac{N-sp}{N}}}. \tag{2.7}$$

Using eq. (2.6), we define

$$\mathbb{S}_{C,\mu} = \inf_{u \in X_0(\Omega)\setminus\{0\}} \frac{\|u\|^p}{\Big(\displaystyle\int_\Omega |u(x)|^{p^*_{\mu,s}}\Big(\int_\Omega \frac{|u(y)|^{p^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\Big)\mathrm{d}x\Big)^{\frac{p}{2p^*_{\mu,s}}}}. \tag{2.8}$$

To overcome the singularities of $u^{-\gamma}$ and obtain a non-negative solution for problem (1.1), we introduce the following modified problem

$$\begin{aligned} M\Big(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy\Big)(-\Delta)^s_p u - \alpha\frac{|u|^{p-2}u}{|x|^{sp}} &= \lambda f(x)(u^+)^{-\gamma} \\ &+ g(x)\left(\int_\Omega \frac{(u^+)^{p^*_\mu}(y)}{|x-y|^\mu} dy\right)(u^+)^{p^*_\mu-1} \text{ in } \Omega, \\ u &> 0 \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{2.9}$$

Here $M(t) = a + bt^{\theta-1}$ and $u^+ = \max\{u, 0\}$. Therefore, we say that $u \in X_0$ is a weak solution of the problem (2.9), if $f(x)(u)^{-\gamma}\phi \in L^1(\Omega)$, and the following equation holds:

$$\left(a + b\|u\|^{p\theta-p}\right)\langle u, \phi\rangle - \alpha \int_\Omega \frac{u^{p-1}}{|x|^{sp}}\phi(x)\,dx \tag{2.10}$$

$$-\lambda \int_\Omega f(x)(u^+)^{-\gamma}\phi(x)dx - \int_\Omega\int_\Omega g(x)\frac{(u(y)^+)^{p^*_{\mu,s}}}{|x-y|^\mu}(u(x)^+)^{p^*_{\mu,s}-1}\phi(x)\,dx\,dy = 0, \tag{2.11}$$

for any $\phi \in X_0$, where

$$\langle u, \phi\rangle = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}dxdy.$$

Note that it is straightforward to see that if $u > 0$ is a solution to problem (2.9), then it is also a solution to problem (1.1). Note that, it is very simple to see that if $u > 0$ is a solution to problem (2.9), then it is also a solution to problem (1.1). Problem (2.9) has a variational structure, and the functional energy $E_{\alpha,\lambda}\colon X_0 \to \mathbb{R}$ is defined as follows:

$$E_{\alpha,\lambda}(u) := \frac{a}{p}\|u\|^p + \frac{b}{p\theta}\|u\|^{p\theta} - \frac{\gamma}{p}\|u^+\|^p_H - \frac{\lambda}{1-\gamma}\int_\Omega f(x)(u^+)^{1-\gamma}$$
$$- \frac{1}{p^*_{\mu,s}}\int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy.$$

Here, for all $u \in X_0$ we denote

$$\|u\|^p_H := \int_\Omega \frac{|u(x)|^p}{|x|^{sp}}\,dx.$$

Using the inequality (1.2), it can be seen that for any $\alpha \in (0, a\mu_0)$, the functional energy $E_{\alpha,\lambda}$ is continuous and well-defined. Also, since $v^+ \le |v|$ and inequality (1.2) yield for any $v \in X_0 \setminus \{0\}$, we have

$$a\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \alpha\int_\Omega \frac{(v^+)^p}{|x|^{sp}}\,dx \ge d_{a,\alpha}\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \tag{2.12}$$

with $d_{a,\alpha} = \left(a - \frac{\alpha}{\mu_0}\right) > 0$ for any $\alpha \in (0, a\mu_0)$. Note that, inequality (2.12) guarantees the positivity of the functional energy.

Now, we recall the following important result.

Lemma 2.2. *([18, Lemma 1.32]) Let $p \in (1,\infty)$, $N \ge 3$, and $\{u_k\}$ be a bounded sequence in $L^p(\mathbb{R}^N)$. If $u_k \to u$ a.e in $\mathbb{R}^N$ as $k \to \infty$. Hence, $u_k \rightharpoonup u$ weakly in $L^p(\mathbb{R}^N)$.*

3 Fibering map analysis

To analyze problem (1.1) and address the fact that the energy functional $E_{\alpha,\lambda}$ is not bounded below on X_0, we can employ a minimization method called the Nehari manifold. This approach enables us to identify critical points of the energy functional $E_{\alpha,\lambda}$.

Let us define the Nehari set for problem (1.1) as follows:

$$\begin{aligned}\mathcal{N}_{\alpha,\lambda} &:= \Big\{u \in X_0 : \langle E_{\alpha,\lambda}(u), u\rangle = 0\Big\} \\ &= \Big\{u \in X_0 : a\|u\|^p + b\|u\|^{p\theta} - \alpha\|u^+\|_H^p - \lambda \int_\Omega f(x)(u^+)^{1-\gamma}\, dx \\ &\quad - \int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy = 0\Big\}.\end{aligned}$$

Now, we define the fibering map $\Phi_u : [0,\infty) \to \mathbb{R}$ as $\Phi_u(t) = E_{\alpha,\lambda}(tu)$. Specifically, for $u \in X_0$, we define

$$\begin{aligned}\phi_u(t) &= \frac{a}{p}t^p\|u\|^p + \frac{b}{p\theta}t^{p\theta}\|u\|^{p\theta} - \frac{\alpha}{p}t^p\|u^+\|_H^p \\ &\quad - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega f(x)(u^+)^{1-\gamma}\, dx - \frac{t^{2p^*_{\mu,s}}}{2p^*_{\mu,s}}\int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy,\end{aligned}$$

so that

$$\phi'_u(t) = at^{p-1}\|u\|^p + bt^{p\theta-1}\|u\|^{p\theta} - \alpha t^{p-1}\|u^+\|_H^p \tag{3.13}$$

$$\qquad - \lambda t^{-\gamma}\int_\Omega f(x)(u^+)^{1-\gamma}\, dx - t^{2p^*_{\mu,s}-1}\int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy, \tag{3.14}$$

and

$$\phi''_u(t) = a\Big(p-1\Big)t^{p-2}\|u\|^p + b\Big(p\theta-1\Big)t^{p\theta-2}\|u\|^{p\theta} \tag{3.15}$$

$$\qquad - \alpha\Big(p-1\Big)t^{p-2}\|u^+\|_H^p + \lambda\gamma t^{-\gamma-1}\int_\Omega f(x)(u^+)^{1-\gamma}\, dx \tag{3.16}$$

$$\qquad - \Big(2p^*_{\mu,s}-1\Big)t^{2p^*_{\mu,s}-2}\int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy. \tag{3.17}$$

It is evident that the Nehari manifold is closely related to the function Φ_u. In particular, $u \in \mathcal{N}_{\alpha,\lambda}$ if and only if $\phi'_u(1) = 0$. Therefore, we decompose the manifold $\mathcal{N}_{\alpha,\lambda}$ into three parts corresponding to local minima, local maxima, and points of inflection as follows:

$$\mathcal{N}^{\pm}_{\alpha,\lambda} := \{u \in \mathcal{N}_{\alpha,\lambda} : \phi''_u(1) \gtrless 0\}, \quad \mathcal{N}^{0}_{\alpha,\lambda} := \{u \in \mathcal{N}_{\alpha,\lambda} : \phi''_u(1) = 0\}.$$

To handle the sign-changing weight g, we introduce the sets:

$$\begin{aligned}\mathfrak{g}^+ &:= \Big\{u \in X_0 : \int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy > 0\Big\}, \\ \mathfrak{g}^- &:= \Big\{u \in X_0 : \int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy \le 0\Big\}.\end{aligned}$$

We now present a crucial lemma that provides important properties of the Nehari manifold and establishes existence and uniqueness results for critical points of the energy functional.

Lemma 3.1. *Let $\alpha \in (0, a\mu_0)$. Then, the following holds:*

(1) Let $u \in \mathfrak{g}^+$. Then, there exist positive constants $\Lambda_1 > 0$ and a unique $t_{\max} := t_{\max}(u) > 0$, $t^+ = t^+(u) > 0$ and $t^- = t^-(u) > 0$ with $t^+ < t_{\max} < t^-$ such that the following conditions are satisfied for any $\lambda \in (0, \Lambda_1)$:

- $t^+u \in \mathcal{N}^+_{\alpha,\lambda}$,
- $t^-u \in \mathcal{N}^-_{\alpha,\lambda}$,
- $E_{\alpha,\lambda}(t^+u) = \min\limits_{0\le t\le t^-} E_{\alpha,\lambda}(tu)$,
- $E_{\alpha,\lambda}(t^-u) = \max\limits_{t\ge t_{\max}} E_{\alpha,\lambda}(tu)$.

(2) Let $u \in \mathfrak{g}^-$, $\alpha \in (0, a\mu_0)$ and $\lambda > 0$. Then, there exists a unique positive constant t^* such that the following conditions are satisfied:

- $t^*u \in \mathcal{N}^+_{\alpha,\lambda}$,
- $E_{\alpha,\lambda}(t^*u) = \inf_{t>0} E_{\alpha,\lambda}(tu)$.

Proof. Fix $u \in X_0$. We define $\Psi_u \colon \mathbb{R}^+ \to \mathbb{R}$ as follows

$$\Psi_u(t) = at^{p-2p^*_{\mu,s}}\|u\|^p + bt^{p\theta-2p^*_{\mu,s}}\|u\|^{p\theta} - \alpha t^{p-2p^*_{\mu,s}}\|u^+\|^p_H - \lambda t^{1-\gamma-2p^*_{\mu,s}}\int_\Omega f(x)(u^+)^{1-\gamma}\,dx. \tag{3.18}$$

We observe that $tu \in \mathcal{N}_{\alpha,\lambda}$ if and only if t satisfies the equation:

$$\Psi_u(t) = \int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy.$$

From Equation (3.18), it is evident that $\lim\limits_{t\to0^+}\Psi_u(t) = -\infty$ and $\lim\limits_{t\to\infty}\Psi_u(t) = 0$. Moreover, by differentiating $\Psi_u(t)$ using Equation (3.18), where $\Psi'_u(t)$ denotes the derivative, we find that:

$$\Psi'_u(t) = \Big(p - 2p^*_{\mu,s}\Big)t^{p-2p^*_{\mu,s}-1}\Big(a\|u\|^p - \alpha\|u^+\|^p_H\Big) + b\Big(p\theta - 2p^*_{\mu,s}\Big)t^{p\theta-2p^*_{\mu,s}-1}\|u\|^{p\theta} \tag{3.19}$$
$$- \lambda\Big(1-\gamma-2p^*_{\mu,s}\Big)t^{-\gamma-2p^*_{\mu,s}}\int_\Omega f(x)(u^+)^{1-\gamma}\,dx. \tag{3.20}$$

Given that $0 < \gamma < 1 < p\theta < 2p^*_{\mu,s}$, we can deduce that

$$\lim_{t\to0^+}\Psi'_u(t) > 0 \ \text{ and } \ \lim_{t\to\infty}\Psi'_u(t) < 0.$$

Hence, there exists a unique $t_{\max} = t_{\max}(u) > 0$ such that $\Psi_u(t)$ is decreasing in $(t_{\max}, \infty)$, increasing in $(0, t_{\max})$, and $\Psi' u(t\max) = 0$. We can estimate $\Psi_u(t_{\max})$ from below as follows

$$\Psi_u(t_{\max}) = \max_{t>0}\Psi_u(t) = \max_{t>0}\Big(bt^{p\theta-2p^*_{\mu,s}}\|u\|^{p\theta} + \overline{\Psi_u}(t)\Big) > \max_{t>0}\overline{\Psi_u}(t),$$

where $\overline{\Psi_u}(t)$ is given by

$$\overline{\Psi_u}(t) = t^{p-2p^*_{\mu,s}}\Big(a\|u\|^p - \alpha\|u^+\|^p_H\Big) - \lambda t^{1-\gamma-2p^*_{\mu,s}} \int_\Omega f(x)(u^+)^{1-\gamma}\, dx.$$

Using the inequality in Equation (1.2), we can infer that for $u \in X_0$, for any $\alpha \in (0, a\mu_0)$, the functional $\overline{\Psi_u}(t)$ is bounded from below by

$$\max_{t>0} \overline{\Psi_u}(t) \geq \overline{\phi_u}(t),$$

with

$$\overline{\phi_u}(t) = d_{a,\alpha}\|u\|^p t^{p-2p^*_{\mu,s}} - \lambda t^{1-\gamma-2p^*_{\mu,s}} \int_\Omega f(x)(u^+)^{1-\gamma}\, dx.$$

Hence, we have

$$\max_{t>0} \overline{\phi_u}(t) = \Big(\frac{p+\gamma-1}{2p^*_{\mu,s}-p}\Big)\Big(\frac{2p^*_{\mu,s}-p}{2p^*_{\mu,s}+\gamma-1}\Big)^{\frac{2p^*_{\mu,s}+\gamma-1}{p+\gamma-1}} \frac{\Big(d_{a,\alpha}\|u\|^p\Big)^{\frac{2p^*_{\mu,s}+\gamma-1}{p+\gamma-1}}}{\Big(\lambda \int_\Omega f(x)(u^+)^{1-\gamma}\, dx\Big)^{\frac{2p^*_{\mu,s}-p}{p+\gamma-1}}}.$$

Hence, using assumption (f) and Eq. (2.8) combine with Hölder inequality, we obtain

$$\begin{aligned} &\Psi_u(t_{\max}) \\ &\geq \Big(\frac{p+\gamma-1}{2p^*_{\mu,s}-p}\Big)\Big(\frac{d_{a,\alpha}(2p^*_{\mu,s}-p)}{2p^*_{\mu,s}+\gamma-1}\Big)^{\frac{2p^*_{\mu,s}+\gamma-1}{p+\gamma-1}} \lambda^{\frac{p-2p^*_{\mu,s}}{p+\gamma-1}} \Big(\|f\|_m\Big)^{\frac{p-2p^*_{\mu,s}}{p+\gamma-1}} S^{\frac{(1-\gamma)(2p^*_{\mu,s}-p)}{p(p+\gamma-1)}} \|u\|^{2p^*_{\mu,s}} > 0. \end{aligned}$$

Now, according to the behavior of g, we split the proof in two cases

Case 1: Let $u \in \mathfrak{g}^+$. Since

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p^*_{\mu,s}}|v(y)|^{p^*_{\mu,s}}}{|x-y|^\mu} dxdy \leq C_g(N,\mu) S^{-\frac{2p^*_{b,s}}{p}} \|u\|^{2p^*_{b,s}} \tag{3.21}$$

where S is a positive constant, we can choose

$$\lambda < \Lambda_1 := \Big(\frac{p+\gamma-1}{2p^*_{\mu,s}-p}\Big)^{\frac{p+\gamma-1}{2p^*_{\mu,s}-p}} \Big(\frac{d_{a,\alpha}(2p^*_{\mu,s}-p)}{2p^*_{\mu,s}+\gamma-1}\Big)^{\frac{(2p^*_{\mu,s}+\gamma-1)}{(2p^*_{\mu,s}-p)}} S^{\frac{(2p^*_{\mu,s}+\gamma-1)}{p^*_{\mu,s}-p}} \frac{1}{\|f\|_m} \Big(\frac{1}{C_g(N,\mu)}\Big)^{\frac{p+\gamma-1}{2p^*_{\mu,s}-p}},$$

to guarantee the existence of a unique $t^+ := t^+(u) < t_{\max}$ and $t^- := t^-(u) > t_{\max}$ satisfying

$$\Psi_u(t^+) = \int_\Omega\int_\Omega \frac{g(x)(u^+(x))^{p^*_{\mu,s}}(u^+(y))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy = \Psi_u(t^-),$$

which implies $t^+u, t^-u \in \mathcal{N}_{\alpha,\lambda}$. Furthermore, from the equation $\phi''_{tu}(1) = t^{2p^*\mu,s+1}\Psi'_u(t)$, we conclude that $\Psi'_u(t^+) > 0$ and $\Psi'_u(t^-) < 0$. Thus, we can deduce that $t^-u \in \mathcal{N}^-_{\alpha,\lambda}$ and $t^+u \in \mathcal{N}^+_{\alpha,\lambda}$. Now, assuming

$$\phi'_u(t) = t^{2p^*_{\mu,s}-1}\Big(\Psi_u(t) - \int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy\Big).$$

We obtain that for all $t \in [0, t^+)$, $\phi'_u(t) < 0$, and for all $t \in (t^+, t^-)$, $\phi'_u(t) > 0$. Consequently,

$$E_{\alpha,\lambda}(t^+ u) = \min_{0 \le t \le t^-} E_{\alpha,\lambda}(tu).$$

Similarly, for all $t \in [t^+, t^-)$, $\phi'_u(t) > 0$, $\phi'_u(t^-) = 0$, and for all $t \in (t^-, \infty)$, $\phi'_u(t) < 0$. Hence,

$$E_{\alpha,\lambda}(t^- u) = \max_{t \ge t_{\max}} E_{\alpha,\lambda}(tu).$$

Case 2: Consider $u \in \mathfrak{g}^-$. By utilizing the fact that $\lim_{t \to 0^+} \Psi_u(t) = -\infty$, we can conclude the existence of a unique $t^* > 0$ satisfying

$$\Psi_u(t^*) = \int_\Omega \int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy \quad \text{for all } \lambda > 0.$$

Since $u \in \mathfrak{g}^-$, it follows that $\Psi'_u(t^) > 0$ and $\Psi_u(t^) < 0$. By repeating the same calculations as in Case 1, we obtain $\phi''_{tu}(1) = t^{2p^*_{\mu,s}+1}\Psi'_u(t)$, where $\Psi'_u(t^) > 0$. Consequently, we have $t^* u \in \mathcal{N}^+_{\alpha,\lambda}$. Thus, we have completed the proof of Lemma 3.1. □

We now demonstrate the uniqueness of the trivial solution in a specific parameter range.

Lemma 3.2. *Consider $\alpha \in (0, a\mu_0)$. Then, there exists a positive constant Λ_2 such that for all $\lambda \in (0, \Lambda_2)$, we have $\mathcal{N}^0_{\alpha,\lambda} = 0$.*

Proof. We proceed by contradiction, assuming the existence of $u \in \mathcal{N}^0_{\alpha,\lambda} \backslash \{0\}$ for all $\lambda \in (0, \Lambda_2)$. We consider two cases.

Case 1: If $\int_\Omega \int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy = 0$, using Eq. (3.13), we have

$$a\|u\|^p + b\|u\|^{p\theta} - \alpha\|u^+\|^p_H - \lambda \int_\Omega f(x)(u^+)^{1-\gamma}\, dx = 0. \tag{3.22}$$

Now, since $p\theta \ge 1 > 1 - \gamma$ and considering $\alpha \in (0, a\mu_0)$ with Eq. (2.12), we obtain

$$\begin{aligned} 0 &= (1+\gamma)[a\|u\|^p - \alpha\|u^+\|^p_H] + b(p\theta + \gamma - 1)\|u\|^{p\theta} \\ &\ge d_{a,\alpha}(1+\gamma)\|u\|^p + b(p\theta + \gamma - 1)\|u\|^{p\theta} > 0, \end{aligned}$$

which leads to a contradiction.

Case 2: If $\int_\Omega \int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy \ne 0$, we can use Eq. (3.13) and Eq. (3.15), with $t = 1$, with $t = 1$ to obtain the following equations

$$\begin{aligned} &(2p^*_{\mu,s} - p)[a\|u\|^p - \alpha\|u^+\|^p_H] + b(2p^*_{\mu,s} - p\theta)\|u\|^{p\theta} \\ &- \lambda(2p^*_{\mu,s} + \gamma - 1)\int_\Omega f(x)(u^+)^{1-\gamma}\, dx = 0, \end{aligned} \tag{3.23}$$

$$\begin{aligned} &(1+\gamma)\left[a\|u\|^p - \alpha\|u^+\|^p_H\right] + b(p\theta + \gamma - 1)\|u\|^{p\theta} \\ &- (2p^*_{\mu,s} + \gamma - 1)\int_\Omega \int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy = 0. \end{aligned} \tag{3.24}$$

We define $J_{\alpha,\lambda}\colon \mathcal{N}_{\alpha,\lambda} \to \mathbb{R}$ as

$$\begin{aligned} J_{\alpha,\lambda} := & \frac{(1+\alpha)[a\|u\|^p - \alpha\|u^+\|_H^p] + b\Big(p\theta + \gamma - 1\Big)\|u\|^{p\theta}}{(2p^*_{\mu,s} + \gamma - 1)} \\ & - \int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy. \end{aligned}$$

Using Eq. (3.24), we find that $J_{\alpha,\lambda} = 0$ for all $u \in \mathcal{N}^0_{\alpha,\lambda}$. However, by assuming condition (g), Eqs. (3.21), (2.12), and the inequality $(c+d) \geq 2\sqrt{cd}$ for any $c, d \geq 0$, we can deduce that

$$\begin{aligned} J_{\alpha,\lambda} &\geq \frac{2\sqrt{(1+\gamma)(p\theta+\gamma-1)d_{a,\alpha}b}\|u\|^{\theta+1}}{(2p^*_{\mu,s}+\gamma-1)} - C_g(N,\mu)S^{-\frac{2p^*_{\mu,s}}{p}}\|u\|^{2p^*_{\mu,s}}, \\ &= \|u\|^{2p^*_{\mu,s}}\Big[\frac{2\sqrt{(1+\gamma)(p\theta+\gamma-1)d_{a,\alpha}b}}{(2p^*_{\mu,s}+\gamma-1)\|u\|^{2p^*_{\mu,s}-\theta-1}} - C_g(N,\mu)S^{-\frac{2p^*_{\mu,s}}{p}}\Big]. \end{aligned}$$

Now using Eq. (2.12), assumption (f) and Hölder inequality in Eq. (3.23), we can obtain the following expression

$$\|u\| \leq \Big[\frac{\lambda(2p^*_{\mu,s}+\gamma-1)\|f\|_m S^{\frac{-(1-\gamma)}{p}}}{2\sqrt{(2p^*_{\mu,s}-p)(2p^*_{\mu,s}-p\theta)d_{a,\alpha}b}}\Big]^{\frac{1}{\theta+\gamma}}.$$

Thus, we obtain

$$\begin{aligned} \lambda < \Lambda_2 := & \frac{S^{\frac{(\theta+1)(2p^*_{\mu,s}+\gamma-1)}{2(2p^*_{\mu,s}-\theta-1)}}}{[C_g(N,\mu)]^{\frac{\theta+\gamma}{2p^*_{\mu,s}-\theta-1}}}\Big[\frac{2\sqrt{(1+\gamma)(p\theta+\gamma-1)d_{a,\alpha}b}}{2p^*_{\mu,s}+\gamma-1}\Big]^{\frac{\theta+\gamma}{2p^*_{\mu,s}-\theta-1}} \\ & \times\Big[\frac{2\sqrt{(2p^*_{\mu,s}-p)(2p^*_{\mu,s}-p\theta)d_{a,\alpha}b}}{(2p^*_{\mu,s}+\gamma-1)\|f\|_m}\Big]. \end{aligned}$$

Hence, for all $u \in \mathcal{N}^0_{\alpha,\lambda} \setminus \{0\}$, $J_{\alpha,\lambda}(u) > 0$, which leads to the desired contradiction. This completes the proof of Lemma 3.2. □

The following lemma establishes the existence of a gap structure in $\mathcal{N}_{\alpha,\lambda}$, demonstrating the presence of distinct magnitudes within the solution space.

Lemma 3.3. *Consider $\alpha \in (0, a\mu_0)$ and $\lambda \in (0, \Lambda_2)$. Then, there exists a gap structure in $\mathcal{N}_{\alpha,\lambda}$ such that*

$$\|U\| > A_0 > A_1 > \|u\|,$$

for every $u \in \mathcal{N}^+_{\alpha,\lambda}$ and $U \in \mathcal{N}^-_{\alpha,\lambda}$, where

$$A_0 := \Big[\frac{2\sqrt{(1+\gamma)(p\theta+\gamma-1)d_{a,\alpha}b}}{(2p^*_{\mu,s}+\gamma-1)C_g(N,\mu)S^{-p^*_{\mu,s}}}\Big]^{\frac{1}{2p^*_{\mu,s}-\theta-1}} \quad \text{and} \quad A_1 := \Big[\frac{\lambda(2p^*_{\mu,s}+\gamma-1)S^{\frac{-(1-\gamma)}{p}}\|f\|_m}{2\sqrt{(2p^*_{\mu,s}-p)(2p^*_{\mu,s}-p\theta)d_{a,\alpha}b}}\Big]^{\frac{1}{\theta+\gamma}}.$$

Proof. If $u \in \mathcal{N}^{+}_{\alpha,\lambda} \subset \mathcal{N}_{\alpha,\lambda}$, we can utilize assumption (f) and combine Equation (2.8) with the Hölder inequality to obtain the following inequality

$$\begin{aligned}
&\Big(2p^*_{\mu,s} - p\Big)\Big[a\|u\|^p - \alpha\|u^+\|^p_H\Big] + b\Big(2p^*_{\mu,s} - p\theta\Big)\|u\|^{p\theta} \\
&< \lambda\Big(2p^*_{\mu,s} + \gamma - 1\Big)\int_\Omega f(x)(u^+)^{1-\gamma}dx \\
&\leq \lambda\Big(2p^*_{\mu,s} + \gamma - 1\Big)\|f\|_m S^{\frac{-(1-\gamma)}{p}}\|u\|^{1-\gamma}.
\end{aligned} \tag{3.25}$$

Combining this inequality with Equation (2.12), we obtain:

$$\|u\| < \Big[\frac{\lambda(2p^*_{\mu,s} + \gamma - 1)S^{\frac{-(1-\gamma)}{p}}\|f\|_m}{2\sqrt{(2p^*_{\mu,s} - p)(2p^*_{\mu,s} - p\theta)d_{a,\alpha}b}}\Big]^{\frac{1}{\theta+\gamma}} := A_1.$$

Now, considering $U \in N^{-}_{\alpha,\lambda}$ and utilizing assumption (g), we have:

$$\begin{aligned}
&(1+\gamma)[a\|U\|^p - \alpha\|U^+\|^p_H] + b\Big(p\theta + \gamma - 1\Big)\|U\|^{p\theta} \\
&< (2p^*_{\mu,s} + \gamma - 1)C_g(N,\mu)S^{-p^*_{\mu,s}}\|U\|^{2p^*_{\mu,s}}.
\end{aligned}$$

By this, and Eq. (2.12), we obtain

$$2\sqrt{(1+\gamma)(p\theta + \gamma - 1)d_{a,\alpha}b}\|U\|^{\theta+1} < (2p^*_{\mu,s} + \gamma - 1)C_g(N,\mu)S^{-p^*_{\mu,s}}\|U\|^{2p^*_{\mu,s}}.$$

This yields

$$\|U\| > \Big[\frac{2\sqrt{(1+\gamma)(p\theta + \gamma - 1)d_{a,\alpha}b}}{(2p^*_{\mu,s} + \gamma - 1)C_g(N,\mu)S^{-p^*_{\mu,s}}}\Big]^{\frac{1}{2p^*_{\mu,s} - \theta - 1}} := A_0.$$

By performing a direct computation, we can verify that $A_0 > A_1$ for all $\lambda \in (0, \Lambda_2)$. Hence, we can conclude that

$$\|U\| > A_0 > A_1 > \|u\| \quad \text{for all } u \in \mathcal{N}^{+}_{\alpha,\lambda},\ U \in \mathcal{N}^{-}_{\alpha,\lambda}.$$

This completes the proof of Lemma 3.3. □

As a direct consequence of the lemma, we can establish the closedness of $\mathcal{N}^{-}_{\alpha,\lambda}$ in the X_0 topology.

Corollary 3.4. *For any $\alpha \in (0, a\mu_0)$, the set $\mathcal{N}^{-}_{\alpha,\lambda}$ is closed in the X_0 topology for all $\lambda \in (0, \Lambda_2)$.*

Proof. Consider a sequence $\{u_k\}_k$ in $\mathcal{N}^{-}_{\alpha,\lambda}$, satisfying $u_k \to u$ in X_0. Therefore, $u \in \mathcal{N}^{-}_{\alpha,\lambda} \cup \{0\}$. By Lemma 3.3, it follows that

$$\|u\| = \lim_{k\to\infty}\|u_k\| \geq A_0 > A_1 > 0. \tag{3.26}$$

Hence, inequality (3.26) implies that u is not identically zero. Therefore, $u \in \mathcal{N}^{-}_{\alpha,\lambda}$. The proof of Corollary 3.4 is now completed. □

The lemma below demonstrates the existence of a continuous function ξ that ensures the preservation of the property $\mathcal{N}^{\pm}_{\alpha,\lambda}$ under small perturbations.

Lemma 3.5. *Let $\alpha \in (0, a\mu_0)$, $\lambda > 0$, and $u \in \mathcal{N}^{\pm}_{\alpha,\lambda}$. Then, there exists $\epsilon > 0$ and a continuous function $\xi\colon B_\epsilon(0) \to \mathbb{R}^+$ such that*

$$\xi(v) > 0,\ \ \xi(0) = 1 \ \ \text{and} \ \ \xi(v)(u+v) \in \mathcal{N}^{\pm}_{\alpha,\lambda} \ \ \text{for all} \quad v \in B_\epsilon(0),$$

where $B_\epsilon(0) = \{v \in X_0 : \|v\| < \epsilon\}$.

Proof. Here, we provide the proof only for the case where $u \in \mathcal{N}^{+}_{\alpha,\lambda}$, while the proof for the case $\mathcal{N}^{-}_{\alpha,\lambda}$ is similar. Let $F\colon X_0 \times \mathbb{R}^+ \to \mathbb{R}$ be a function defined as

$$\begin{aligned} F(v,z) := {}& z^{1+\gamma}\Big(a\|u+v\|^p - \alpha\|(u+v)^+\|^p_H\Big) + z^{p\theta-1+\gamma} b\|u+v\|^{p\theta} \\ & - \lambda \int_\Omega f(x)\left((u+v)^+\right)^{1-\gamma} dx \\ & - z^{2p^*_{\mu,s}+\gamma-1} \int_\Omega\int_\Omega \frac{g(x)((u+v)^+(y))^{p^*_{\mu,s}}((u+v)^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy. \end{aligned}$$

Since $u \in \mathcal{N}^{+}_{\alpha,\lambda} \subset \mathcal{N}_{\alpha,\lambda}$, we obtain

$$\begin{aligned} F(0,1) = {}& a\|u\|^p + b\|u\|^{p\theta} - \alpha\|u^+\|^p_H - \lambda\int_\Omega f(x)(u^+)^{1-\gamma}\, dx \\ & - \int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy = 0, \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} \frac{\partial F}{\partial z}(0,1) = {}& (1+\gamma)\big(a\|u\|^p - \alpha\|u^+\|^p_H\big) + b\Big(p\theta - 1 + \gamma\Big)\|u\|^{p\theta} \\ & - \Big(2p^*_{\mu,s} + \gamma - 1\Big)\int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy > 0. \end{aligned} \tag{3.28}$$

Now, applying the Implicit Function Theorem to the map F at the point $(0,1)$, we obtain the existence of $\bar{\epsilon} > 0$ such that for all $v \in X_0$ where $||v|| < \bar{\epsilon}$, the equation $F(v,z) = 0$ has a unique solution $z = \xi(v) > 0$. Therefore, utilizing Equation (3.27), we find that $\xi(0) = 1$.

Furthermore, since $F(v,\xi(v))=0$ for any $v\in X_0$ with $||v||<\bar{\epsilon}$, we have

$$\begin{aligned}
0 &= \xi(v)^{1+\gamma}\Big(a\|u+v\|^p-\alpha\|(u+v)^+\|_H^p\Big)+\xi(v)^{p\theta-1+\gamma}b\|u+v\|^{p\theta}\\
&\quad-\lambda\int_\Omega f(x)\big((u+v)^+\big)^{1-\gamma}\,dx\\
&\quad-\xi(v)^{2p^*_{\mu,s}+\gamma-1}\int_\Omega\int_\Omega\frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\\
&=\frac{1}{\xi^{1-\gamma}(v)}\Big(a\|\xi(v)(u+v)\|^p-\alpha\|\xi(v)(u+v)^+\|_H^p-\lambda\int_\Omega f(x)\big(\xi(v)(u+v)^+\big)^{1-\gamma}\,dx\\
&\quad+d\|\xi(v)(u+v)\|^{p\theta}\\
&\quad-\int_\Omega\int_\Omega\frac{g(x)\Big[(\xi(v)(u^+(y))\Big]^{p^*_{\mu,s}}\Big[(\xi(v)(u^+(x))\Big]^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\Big).
\end{aligned}$$

This implies that,

$$\xi(v)(u+v)\in\mathcal{N}_{\alpha,\lambda}\ \text{ for every }\ v\in X_0\,;\text{and }\ \|v\|<\bar{\epsilon}.$$

On the other hand, we can calculate the partial derivative of F with respect to z at the point $(v,\xi(v))$

$$\begin{aligned}
&\frac{\partial F}{\partial z}\Big|_{(v,\xi(v))}\\
&=\frac{1}{\xi^{p-\gamma}(v)}\Big[(1+\gamma)\Big(a\|\xi(v)(u+v)\|^p-\alpha\|\xi(v)(u+v)^+\|_H^p\Big)\\
&\quad+\Big(p\theta-1+\gamma\Big)b\|\xi(v)(u+v)\|^{p\theta}\\
&\quad-(2p^*_{\mu,s}+\gamma-1)\int_\Omega\int_\Omega\frac{g(x)\Big[(\xi(v)(u^+(y))\Big]^{p^*_{\mu,s}}\Big[(\xi(v)(u^+(x))\Big]^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\Big].
\end{aligned}$$

Therefore, by Equation (3.28), we can choose $\epsilon>0$ satisfying $\epsilon<\bar{\epsilon}$. For any $v\in X_0$ with $||v||<\epsilon$, we have

$$\begin{aligned}
&(1+\gamma)\Big(a\|\xi(v)(u+v)\|^p-\alpha\|\xi(v)(u+v)^+\|_H^p\Big)+\Big(p\theta-1+\gamma)b\|\xi(v)(u+v)\|^{p\theta}\\
&-\Big(2p^*_{\mu,s}+\gamma-1\gamma\Big)\int_\Omega\int_\Omega\frac{g(x)\Big[(\xi(v)(u^+(y))\Big]^{p^*_{\mu,s}}\Big[(\xi(v)(u^+(x))\Big]^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy>0.
\end{aligned}$$

This imply that

$$\xi(v)(u+v)\in\mathcal{N}^+_{\gamma,\lambda}\ \text{ for all }\ v\in B_\epsilon(0).$$

Thus, the proof of Lemma 3.5 is now completed. □

Now, we show the boundedness from below and coercivity of the functional energy $E_{\alpha,\lambda}$.

Lemma 3.6. *Consider $\alpha \in (0, a\mu_0)$ and $\lambda > 0$. Then, the functional energy $E_{\alpha,\lambda}$ is bounded from below on $\mathcal{N}_{\alpha,\lambda}$ and coercive.*

Proof. Let $u \in \mathcal{N}_{\alpha,\lambda}$. By assumption (f), Eq. (2.8), and Eq. (2.12), we can combine them with Hölder's inequality, noting that $p\theta < 2p^*_{\mu,s}$, to obtain

$$\begin{aligned} E_{\alpha,\lambda}(u) &:= \Big(\frac{1}{p} - \frac{1}{2p^*_{\mu,s}}\Big)\Big(a\|u\|^p - \alpha\|u^+\|_H^p\Big) + \Big(\frac{1}{p\theta} - \frac{1}{2p^*_{\mu,s}}\Big)b\|u\|^{p\theta} \\ &\quad - \lambda\Big(\frac{1}{1-\gamma} - \frac{1}{2p^*_{\mu,s}}\Big)\int_\Omega f(x)(u^+)^{1-\gamma}\,dx \\ &\geq \Big(\frac{1}{p} - \frac{1}{2p^*_{\mu,s}}\Big)d_{a,\alpha}\|u\|^p - \lambda\Big(\frac{1}{1-\gamma} - \frac{1}{2p^*_{\mu,s}}\Big)\|f\|_m S^{\frac{-(1-\gamma)}{p}}\|u\|^{1-\gamma}. \end{aligned}$$

Since $p > 1-\gamma$, it follows that $E_{\alpha,\lambda}$ is coercive on $\mathcal{N}_{\alpha,\lambda}$. Now, let us introduce the function $\mathcal{F}(t)$ defined as

$$\mathcal{F}(t) := \Big(\frac{1}{p} - \frac{1}{2p^*_{\mu,s}}\Big)d_{a,\alpha}t^{\frac{p}{1-\gamma}} - \lambda\Big(\frac{1}{1-\gamma} - \frac{1}{2p^*_{\mu,s}}\Big)\|f\|_m S^{\frac{-(1-\gamma)}{p}}t.$$

We can observe that $\mathcal{F}(t)$ attains its minimum at

$$t_{\min} := \Big(\frac{\lambda(2p^*_{\mu,s} + \gamma - 1)S^{\frac{-(1-\gamma)}{p}}\|f\|_m}{(2p^*_{\mu,s} - p)d_{a,\alpha}}\Big)^{\frac{1-\gamma}{p-1+\gamma}}.$$

Therefore, we have

$$E_{\alpha,\lambda}(u) \geq \frac{d_{a,\alpha}(2p^*_{\mu,s} - p)(1-\gamma-p)}{2pp^*_{\mu,s}(1-\gamma)}\Big[\frac{\Big(\lambda(2p^*_{\mu,s} + \gamma - 1)S^{\frac{-(1-\gamma)}{p}}\|f\|_m\Big)}{\big((2p^*_{\mu,s} - p)d_{a,\alpha}\big)}\Big]^{\frac{p}{(p-1+\gamma)}} > -C,$$

where $C > 0$ is a constant. Hence, $E_{\alpha,\lambda}$ is bounded below on $\mathcal{N}_{\alpha,\lambda}$. This completes the proof of Lemma 3.6. $\square$

4 A compactness result for $E_{\alpha,\lambda}$

In this section, we aim to establish a compactness result for the functional energy $E_{\alpha,\lambda}$. To do this, we start by defining the quantities

$$m^+_{\alpha,\lambda} := \inf_{u \in \mathcal{N}^+_{\alpha,\lambda} \cup \{0\}} E_{\alpha,\lambda}(u) \quad \text{and} \quad m^-_{\alpha,\lambda} := \inf_{u \in \mathcal{N}^-_{\alpha,\lambda}} E_{\alpha,\lambda}(u).$$

Here, $m^+_{\alpha,\lambda}$ represents the infimum of the energy functional $E_{\alpha,\lambda}$ over the set $\mathcal{N}^+_{\alpha,\lambda} \cup \{0\}$, and $m^-_{\alpha,\lambda}$ represents the infimum over $\mathcal{N}^-_{\alpha,\lambda}$.

Using Lemma 3.2 and Corollary 3.4, we establish that both $\mathcal{N}^+_{\alpha,\lambda} \cup \{0\}$ and $\mathcal{N}^-_{\alpha,\lambda}$ are closed sets in X_0 for $\lambda < \Lambda_2$.

By applying Ekeland's variational principle to the functional $E_{\alpha,\lambda}$, we can extract a minimizing sequence u_k from either $\mathcal{N}^+_{\alpha,\lambda} \cup \{0\}$ or $\mathcal{N}^-_{\alpha,\lambda}$. The sequence u_k satisfies the following conditions

$$m^{\pm}_{\alpha,\lambda} < E_{\alpha,\lambda}(u_k) < m^{\pm}_{\alpha,\lambda} + \frac{1}{k}, \quad \text{and} \quad E_{\alpha,\lambda}(u) \geq E_{\alpha,\lambda}(u_k) + \frac{1}{k}\|u - u_k\|. \tag{4.29}$$

Here, $m^{\pm}_{\alpha,\lambda}$ represents the corresponding infimum values defined earlier.

Next, using Lemma 3.6, we can conclude that the sequence $\{u_k\}_k$ is bounded in $\mathcal{N}_{\alpha,\lambda}$. Specifically, we have $||u_k|| \leq C_1$ for all k, where $C_1 > 0$ is a constant.

Therefore, the sequence u_k is bounded in $\mathcal{N}_{\alpha,\lambda}$, and by the weak compactness of X_0, there exists a weakly convergent subsequence u_{k_j} that converges weakly in X_0 to some element u_0, i.e.,

$$u_k \rightharpoonup u_0 \quad \text{weakly in } X_0. \tag{4.30}$$

Now, in order to prove the compactness result for $E_{\alpha,\lambda}$, it is necessary to establish several intermediate lemmas that will aid in the subsequent proof.

Lemma 4.1. *Consider $\alpha \in (0, a\mu_0)$ and $\lambda \in (0, \Lambda_1)$, with Λ_1 is define in Lemma 3.1. Consider $\{u_k\}_k \subset \mathcal{N}^+_{\alpha,\lambda}$ satisfy Eq. (4.30). Then, the following results hold*

(a) If $\{u_k\} \subset \mathcal{N}^+_{\alpha,\lambda}$ for every $k \in \mathbb{N}$, we have

$$\begin{aligned}&(1+\gamma)\Big[a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big] + b(p\theta + \gamma - 1)\|u_k\|^{p\theta} \\ &- \Big(2p^*_{\mu,s} + \gamma - 1\Big)\int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy \geq C_2;\end{aligned}$$

(b) If $\{u_k\} \subset \mathcal{N}^-_{\alpha,\lambda}$ for every $k \in \mathbb{N}$, we have

$$\begin{aligned}&(1+\gamma)\Big[a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big] + b\Big(p\theta + \gamma - 1\Big)\|u_k\|^{p\theta} \\ &- (2p^*_{\mu,s} + \gamma - 1)\int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy \leq -C_2;\end{aligned}$$

where $C_2 > 0$ is a constant.

Proof. We only prove case (a) since case (b) can be proved similarly. Firstly, considering $\{u_k\}_k \subset \mathcal{N}^+_{\alpha,\lambda}$, it is sufficient to prove the following inequality

$$\begin{aligned}&\liminf_{k\to\infty}\Big[(2p^*_{\mu,s} - p)\Big(a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big) + b(2p^*_{\mu,s} - p\theta)\|u_k\|^{p\theta}\Big] \\ &< \lambda(2p^*_{\mu,s} + \gamma - 1)\int_\Omega f(x)(u_0^+)^{1-\gamma}dx.\end{aligned}$$

We proceed by contradiction and assume that

$$\begin{aligned}&\liminf_{k\to\infty}\Big[(2p^*_{\mu,s} - p)\Big(a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big) + b(2p^*_{\mu,s} - p\theta)\|u_k\|^{p\theta}\Big] \\ &= \lambda(2p^*_{\mu,s} + \gamma - 1)\int_\Omega f(x)(u_0^+)^{1-\gamma}\,dx.\end{aligned}$$

Since $\{u_k\}_k \in \mathcal{N}^+_{\alpha,\lambda}$, it follows that

$$\Big(2p^*_{\mu,s} - p\Big)\Big(a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big) + b(2p^*_{\mu,s} - p\theta)\|u_k\|^{p\theta} < \lambda\Big(2p^*_{\mu,s} + \gamma - 1\Big)\int_\Omega f(x)(u_k^+)^{1-\gamma}dx.$$

On the other hand, using condition (f) and applying Vitali's convergence theorem, we obtain

$$\lim_{k\to\infty}\int_\Omega f(x)(u_k^+)^{1-\gamma}dx = \int_\Omega f(x)(u_0^+)^{1-\gamma}\,dx.$$

Hence, we have

$$\begin{aligned}&\liminf_{k\to\infty}\Big[(2p^*_{\mu,s} - p)\Big(a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big) + b(2p^*_{\mu,s} - p\theta)\|u_k\|^{p\theta}\Big]\\ &\le \limsup_{k\to\infty}\Big[(2p^*_{\mu,s} - p)\Big(a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big) + b\Big(2p^*_{\mu,s} - p\theta\Big)\|u_k\|^{p\theta}\Big]\\ &\le \lambda\Big(2p^*_{\mu,s} + \gamma - 1\Big)\int_\Omega f(x)(u_0^+)^{1-\gamma}dx,\end{aligned}$$

which implies

$$\begin{aligned}&\lim_{k\to\infty}\Big[(2p^*_{\mu,s} - p)\Big(a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big) + b(2p^*_{\mu,s} - p\theta)\|u_k\|^{p\theta}\Big]\\ &= \lambda\Big(2p^*_{\mu,s} + \gamma - 1\Big)\int_\Omega f(x)(u_0^+)^{1-\gamma}dx.\end{aligned} \tag{4.31}$$

Using Equation (4.31), we can find positive constants $A > 0$ and $A_\alpha > 0$ such that $d_{a,\alpha} \le A_\alpha \le aA$ for $\alpha \in (0, a\mu_0)$. Equation (2.12) is also referenced, which leads to the following convergence statements

$$a\|u_k\|^p - \alpha\|u_k^+\|_H^p \to A_\alpha, \qquad \|u_k\|^p \to A \quad \text{as} \quad k \to \infty.$$

Using the above results, we can derive the equation

$$(2p^*_{\mu,s} - p)A_\alpha + b(2p^*_{\mu,s} - p\theta)A^\theta = \lambda(2p^*_{\mu,s} + \gamma - 1)\int_\Omega f(x)(u_0^+)^{1-\gamma}dx.$$

Finally, by rearranging terms, we obtain

$$\lambda\int_\Omega f(x)(u_0^+)^{1-\gamma}\,dx = \frac{(2p^*_{\mu,s} - p)A_\alpha}{(2p^*_{\mu,s} + \gamma - 1)} + \frac{b(2p^*_{\mu,s} - p\theta)A^\theta}{(2p^*_{\mu,s} + \gamma - 1)}. \tag{4.32}$$

Now, according to Lemma 3.2, for any $\lambda \in (0, \Lambda_1)$, we have the following inequality

$$\begin{aligned}0 \le &\Big(\frac{1+\gamma}{2p^*_{\mu,s} - p}\Big)\Big(\frac{2p^*_{\mu,s} - p}{2p^*_{\mu,s} + \gamma - 1}\Big)^{\frac{2p^*_{\mu,s}+\gamma-1}{1+\gamma}}(d_{a,\alpha}A)^{\frac{2p^*_{\mu,s}+\gamma-1}{1+\gamma}}\Big(\lambda\int_\Omega f(x)(u_0^+)^{1-\gamma}dx\Big)^{\frac{p-2p^*_{\mu,s}}{1+\gamma}}\\ &- \lim_{k\to\infty}\int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy.\end{aligned} \tag{4.33}$$

Considering that $\{u_k\}_k \subset \mathcal{N}^+_{\alpha,\lambda} \subset \mathcal{N}_{\alpha,\lambda}$ and Eq. (4.32), we obtain

$$\begin{aligned}&\lim_{k\to\infty}\int_\Omega\int_\Omega \frac{g(x)(u^+(y))^{p^*_{\mu,s}}(u^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\\&= A_\alpha\Big(\frac{1+\gamma}{2p^*_{\mu,s}+\gamma-1}\Big)+bA^\theta\Big(\frac{p\theta+\gamma-1}{2p^*_{\mu,s}+\gamma-1}\Big).\end{aligned}$$

Substituting Equation (4.31) into Equation (4.33), and using $d_{a,\alpha}A \le A_\alpha$, we obtain

$$dA^\theta\Big(\frac{p\theta+\gamma-1}{2p^*_{\mu,s}+\gamma-1}\Big)\le 0,$$

which leads to the desired contradiction. This completes the proof of Lemma 4.1. □

Now, we fix $\psi \in X_0$ with $\psi \ge 0$. Referring to the constants $C_1 > 0$ introduced in Lemma 4.1 where $\|u_k\| \le C_1$, and recalling the constant $C_2 > 0$, we can deduce the following inequality for $k \in \mathbb{N}$

$$\frac{(1-\gamma)C_1}{k} < C_2. \tag{4.34}$$

By utilizing Lemma 3.5, we can establish the existence of a sequence of functions $(\xi_k)_k$ that satisfies $\xi_k(0) = 1$ and $\xi_k(t\psi)(u_k + t\psi) \in \mathcal{N}^\pm_{\alpha,\lambda}$ for $t > 0$ small enough. Since $u_k \in \mathcal{N}_{\alpha,\lambda}$ and $\xi_k(t\psi)(u_k + t\psi) \in \mathcal{N}_{\alpha,\lambda}$, we can conclude that

$$\begin{aligned}&a\|u_k\|^p+b\|u_k\|^{p\theta}-\alpha\|u_k^+\|_H^p-\lambda\int_\Omega f(x)(u_k^+)^{1-\gamma}\,dx\\&-\int_\Omega\int_\Omega\frac{g(x)(u_k^+(y))^{p^*_{\mu,s}}(u_k^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\end{aligned}\tag{4.35}$$

and

$$\begin{aligned}&\xi_k^2(t\psi)\Big(a\|u_k+t\psi\|^p-\alpha\|(u_k+t\psi)^+\|_H^p\Big)\\&-\lambda\xi_k^{1-\gamma}(t\psi)\int_\Omega f(x)\Big((u_k+t\psi)^+)\Big)^{1-\gamma}dx+b\xi_k^{p\theta}(t\psi)\|u_k+t\psi\|^p\\&-\xi_k^{1-\gamma}(t\psi)\int_\Omega\int_\Omega\frac{g(x)\Big((u_k+t\psi)^+(y)\Big)^{p^*_{\mu,s}}\Big((u_k+t\psi)^+(x)\Big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy=0.\end{aligned}\tag{4.36}$$

Now, let us define $\xi_k'(0)$ as the derivative of ξ_k at the point 0 such that $\langle\xi_k'(0),\psi\rangle \in [-\infty,\infty]$ for every $\psi \in X_0$. However, if the derivative of the function ξ_k does not exist, we can replace $\xi_k'(0)$ with $q_k(0) = \lim_{k\to\infty}\dfrac{\xi_k(t\psi)-1}{t_k}$ for some sequence $(t_k)_k$ satisfying $t_k \to 0$ as $k \to \infty$ and $t_k > 0$.

In the following lemma, we establish a key property of the sequence $\xi_k'(0)$ that will be crucial for the subsequent analysis.

Lemma 4.2. *Consider $\alpha \in (0, a\mu_0)$, $\lambda \in (0,\Lambda_1)$, and suppose $\{u_k\}_k \subset \mathcal{N}^\pm_{\alpha,\lambda}$ satisfies Eq. (4.29) and Eq. (4.30). Then, $\langle\xi_k'(0),\psi\rangle$ is uniformly bounded for any $\psi \in X_0$ with $\psi \ge 0$.*

Proof. We just prove the case that $\mathcal{N}^+_{\alpha,\lambda}$. The case $\mathcal{N}^-_{\alpha,\lambda}$ can be done similarly. In view of Eq. (4.35) and Eq. (4.36), we obtain the following expression

$$\begin{aligned}
0 = {}& [\xi_k^p(t\psi) - 1]\Big(c\|u_k + t\psi\|^p - \alpha\|(u_k + t\psi)^+\|_H^p\Big) \\
&+ \Big(a\|u_k + t\psi\|^p - \alpha\|(u_k + t\psi)^+\|_H^p\Big) \\
&- \Big(a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big) + b\Big([\xi_k^{p\theta}(t\psi) - 1]\|u_k + t\psi\|^{p\theta} + \|u_k + t\psi\|^{p\theta} - \|u_k\|^{p\theta}\Big) \\
&- \lambda \int_\Omega f(x)\Big[\Big((u_k + t\psi)^+\Big)^{1-\gamma} - (u_k^+)^{1-\gamma}\Big] dx \\
&- \lambda[\xi_k^{1-\gamma}(t\psi) - 1] \int_\Omega f(x)\Big((u_k + t\psi)^+\Big)^{1-\gamma} dx \\
&- [\xi_k^p(t\psi) - 1] \int_\Omega\int_\Omega \frac{g(x)((u_k + t\psi)^+(y))^{p^*_{\mu,s}}((u_k + t\psi)^+(x))^{p^*_{\mu,s}}}{|x-y|^\mu} dx\, dy \\
&- \int_\Omega\int_\Omega g(x)\Big[\frac{((u_k + t\psi)^+(x))^{p^*_{\mu,s}}((u_k + t\psi)^+(y))^{p^*_{\mu,s}}}{|x-y|^\mu} \\
&- \frac{\big(u_k^+(x)\big)^{p^*_{\mu,s}}\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\Big] dx\, dy.
\end{aligned}$$

Dividing the above equation by $t > 0$ and taking the limit as t approaches 0^+, we obtain

$$\begin{aligned}
0 = {}& \langle \xi_k^{'}(0), \psi\rangle \Big[p\left(a\|u_k\|^p - \gamma\|u_k^+\|_H^p\right) + bp\theta\|u_k\|^{p\theta} \\
&- \lambda(1-\gamma)\int_\Omega f(x)\big((u_k)^+\big)^{1-\gamma} \\
&- 2p^*_{\mu,s}\int_\Omega\int_\Omega \frac{g(x)(u_k^+(x))^{p^*_{\mu,s}}(u_k^+(y))^{p^*_{\mu,s}}}{|x-y|^\mu} dx\, dy\Big] \\
&+ \Big(ap + p\theta b\|u_k\|^{p\theta-p}\Big)\int_\Omega\int_\Omega \frac{|u_k(x) - u_k(y)|^{p-2}\left(u_k(x) - u_k(y)\right)\left(\psi(x) - \psi(y)\right)}{|x-y|^{N+sp}} dx\, dy \\
&- p\alpha\int_\Omega \frac{u_k^+\psi}{|x|^{sp}} dx - 2p^*_{\mu,s}\int_\Omega\int_\Omega \frac{g(x)(u_k^+(x))^{p^*_{\mu,s}-1}\psi(x)(u_k^+(y))^{p^*_{\mu,s}}}{|x-y|^\mu} dx\, dy.
\end{aligned}$$

Therefore, using Eq. (4.35), we obtain

$$\begin{aligned}
0 \le {}& \langle \xi_k'(0), \psi\rangle \Big[(1+\gamma)\Big(a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big) + b\Big(p\theta + \gamma - 1\Big)\|u_k\|^{p\theta} \\
&- \Big(2p^*_{\mu,s} - \gamma + 1\Big)\int_\Omega\int_\Omega \frac{g(x)(u_k^+(x))^{p^*_{\mu,s}}(u_k^+(y))^{p^*_{\mu,s}}}{|x-y|^\mu} dx\, dy\Big]
\end{aligned}$$

$$\begin{aligned}
&+ \Big(ap + p\theta b\|u_k\|^{p\theta-p}\Big)\int_\Omega\int_\Omega \frac{|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))(\psi(x) - \psi(y))}{|x-y|^{N+sp}} dx\, dy \\
&- p\alpha\int_\Omega \frac{u_k^+\psi}{|x|^{sp}} dx - 2p^*_{\mu,s}\int_\Omega\int_\Omega \frac{g(x)(u_k^+(x))^{p^*_{\mu,s}-1}\psi(x)(u_k^+(y))^{p^*_{\mu,s}}}{|x-y|^\mu} dx\, dy.
\end{aligned}$$

Using Lemma 4.1 (a) in combination with the fact that the sequence u_k is bounded, it follows that $\langle \xi_k'(0), \psi\rangle$ is bounded from below for every $\psi \in X_0$ with $\psi \geq 0$. Using Lemma 4.1 (a) combine with the fact that the sequence $\{u_k\}$ is bounded, it follows that $\langle \xi_k'(0), \psi\rangle$ is bounded from below for every $\psi \in X_0$ with $\psi \geq 0$.

Next, we will prove the boundedness of the sequence $\langle \xi_k'(0), \psi\rangle$ from above. Assuming the contrary, let us suppose that $\langle \xi_k'(0), \psi\rangle = \infty$. Since this assumption is made, we can consider the following

$$\|\xi_k(t\psi)(u_k + t\psi) - u_k\| \leq \xi_k(t\psi)\|t\psi\| + |\xi_k(t\psi) - 1|\|u_k\| \tag{4.37}$$

and $\xi_k(t\psi) > \xi_k(0) = 1$ for sufficiently large k. From the definition of $\xi_k'(0)$ and Equation (4.29) with $u = \xi_k(t\psi)(u_k + t\psi) \in \mathcal{N}^+_{\alpha,\lambda}$, we obtain

$$\begin{aligned}
&|\xi_k(t\psi) - 1|\frac{\|u_k\|}{k} + \xi_k(t\psi)\frac{\|t\psi\|}{k} \\
&\geq \frac{1}{k}\|\xi_k(t\psi)(u_k + t\psi) - u_k\| \\
&\geq E_{\alpha,\lambda}(u_k) - E_{\alpha,\lambda}\Big(\xi_k(t\psi)(u_k + t\psi)\Big) \\
&= \Big(\frac{1}{1-\gamma} - \frac{1}{p}\Big)\Big[\Big(a\|(u_k + t\psi)\|^p - \alpha\Big\|\Big((u_k + t\psi)^+\Big)\Big\|_H^p\Big) - \Big(a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big)\Big] \\
&\quad + \Big(\frac{1}{1-\gamma} - \frac{1}{p\theta}\Big)b\Big(\|u_k + t\psi\|^{p\theta} - \|u_k\|^{p\theta}\Big) \\
&\quad + \Big(\frac{1}{1-\gamma} - \frac{1}{p\theta}\Big)b[\xi_k^{p\theta}(t\psi) - 1]\|u_k + t\psi\|^{p\theta} \\
&\quad + \Big(\frac{1}{1-\gamma} - \frac{1}{p}\Big)\Big[\xi_k^{p}(t\psi) - 1\Big]\Big(a\|u_k + t\psi\|^p - \alpha\|(u_k + t\psi)^+\|_H^p\Big) \\
&\quad - \Big(\frac{1}{1-\gamma} - \frac{1}{2p^*_{\mu,s}}\Big)\Big[\xi_k^{2p^*_{\mu,s}}(t\psi) - 1\Big]\int_\Omega\int_\Omega \frac{g(x)\Big((u_k)^+(x)\Big)^{p^*_{\mu,s}}\Big((u_k)^+(y)\Big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy \\
&\quad - \Big(\frac{1}{1-\gamma} - \frac{1}{2p^*_{\mu,s}}\Big)\xi_k^{2p^*_{\mu,s}}(t\psi)\int_\Omega\int_\Omega g(x)\Big[\frac{((u_k + t\psi)^+(x))^{p^*_{\mu,s}}\Big((u_k + t\psi)^+(y)\Big)^{p^*_{\mu,s}}}{|x-y|^\mu} \\
&\quad - \frac{\Big((u_k)^+(x)\Big)^{p^*_{\mu,s}}\Big((u_k)^+(y)\Big)^{p^*_{\mu,s}}}{|x-y|^\mu}\Big]\,dx\,dy.
\end{aligned}$$

By dividing both sides of the inequality by $t > 0$ and letting t approach 0, it follows that

$$
\begin{aligned}
&\langle \xi_k'(0), \psi\rangle \frac{\|u_k\|}{k} + \frac{\|\psi\|}{k} \\
&\geq \frac{1+\gamma}{1-\gamma}\Big(a \iint_{R^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy - \alpha \int_\Omega \frac{u_k^+\psi}{|x|^{sp}}\,dx\Big) \\
&\quad + \frac{1+\gamma}{1-\gamma}\langle \xi_k'(0), \psi\rangle \Big(a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big) + \Big(\frac{p\theta+\gamma-1}{1-\gamma}\Big) b \langle \xi_k'(0), \psi\rangle \|u_k\|^{p\theta} \\
&\quad + \Big(\frac{2\theta+q-1}{1-q}\Big) d\|u_k\|^{2\theta-2} \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad - \frac{(2p_{\mu,s}^*+\gamma-1)}{(1-\gamma)} \int_\Omega\int_\Omega \frac{g(x)\big(u_k^+(x)\big)^{p_{\mu,s}^*-1}\psi(x)\big(u_k^+(y)\big)^{p_{\mu,s}^*}}{|x-y|^\mu} \\
&\quad - \frac{(2p_{\mu,s}^*+\gamma-1)}{(1-\gamma)} \langle \xi_k'(0), \psi\rangle \int_\Omega\int_\Omega \frac{g(x)\,((u_k)^+(x))^{p_{\mu,s}^*}\,((u_k)^+(y))^{p_{\mu,s}^*}}{|x-y|^\mu}\,dx\,dy. \\
&\geq \frac{\langle \xi_k'(0), \psi\rangle}{(1-\gamma)}\Big[(1+\gamma)\Big(a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big) + \Big(p\theta+\gamma-1\Big) b\|u_k\|^{p\theta} \\
&\quad - \Big(2p_{\mu,s}^*+\gamma-1\Big) \int_\Omega\int_\Omega \frac{g(x)\,((u_k)^+(x))^{p_{\mu,s}^*}\,((u_k)^+(y))^{p_{\mu,s}^*}}{|x-y|^\mu}\,dx\,dy\Big] \\
&\quad + \Big(\frac{1+\gamma}{1-\gamma}\Big)\Big(a \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy - \gamma \int_\Omega \frac{u_k^+\psi}{|x|^{sp}}\,dx\Big) \\
&\quad + \Big(\frac{p\theta+\gamma-1}{1-\gamma}\Big) b\|u_k\|^{p\theta-p} \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad - \frac{(2p_{\mu,s}^*+\gamma-1)}{(1-\gamma)} \int_\Omega\int_\Omega \frac{g(x)\big(u_k^+(x)\big)^{p_{\mu,s}^*-1}\psi(x)\big(u_k^+(y)\big)^{p_{\mu,s}^*}}{|x-y|^\mu}.
\end{aligned}
$$

That is,

$$
\begin{aligned}
\frac{\|\psi\|}{k} \geq\ & \frac{\langle \xi_k'(0), \psi\rangle}{(1-\gamma)}\Big[(1+\gamma)\big(a\|u_k\|^p - \mu\|u_k^+\|_H^p\big) + \Big(p\theta+\gamma-1\Big) b\|u_k\|^{p\theta} \\
& - \Big(2p_{\mu,s}^*+\gamma-1\Big) \int_\Omega\int_\Omega \frac{g(x)\,((u_k)^+(x))^{p_{\mu,s}^*}\,((u_k)^+(y))^{p_{\mu,s}^*}}{|x-y|^\mu}\,dx\,dy - (1-\gamma)\frac{\|u_k\|}{k}\Big] \\
& + \Big(\frac{1+\gamma}{1-\gamma}\Big)\Big(a \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy - \alpha \int_\Omega \frac{u_k^+\psi}{|x|^{sp}}\,dx\Big) \\
& + \Big(\frac{p\theta+\gamma-1}{1-\gamma}\Big) b\|u_k\|^{p\theta-p} \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\
& - \frac{(2p_{\mu,s}^*+\gamma-1)}{(1-\gamma)} \int_\Omega\int_\Omega \frac{g(x)\big(u_k^+(x)\big)^{p_{\mu,s}^*-1}\psi(x)\big(u_k^+(y)\big)^{p_{\mu,s}^*}}{|x-y|^\mu}.
\end{aligned}
$$

We arrive at a contradiction with our assumption that $\langle \xi_k'(0), \psi\rangle = \infty$. By applying Lemma

3.6 (1) and considering the fact that $\{u_k\}_k$ is a bounded sequence, we can conclude that

$$\Big[(1+\gamma)\Big(a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big) + \Big(p\theta+\gamma-1\Big)b\|u_k\|^{p\theta}$$
$$-\Big(2p^*_{\mu,s}+\gamma-1\Big)\int_\Omega\int_\Omega \frac{g(x)\left(u_k^+(x)\right)^{p^*_{\mu,s}}\left((u_k)^+(y)\right)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy - (1-\gamma)\frac{\|u_k\|}{k}\Big]$$
$$\geq C_2 - (1-\gamma)\frac{C_1}{k} > 0.$$

By utilizing Equation (4.34), we can deduce that $\langle \xi_k'(0), \psi\rangle$ is uniformly bounded for sufficiently large k for any $\psi \in X_0$ with $\psi \geq 0$. This completes the proof of Lemma 4.2. $\square$

The following lemma provides important result.

Lemma 4.3. *Consider a sequence $\{u_k\}_k \subset \mathcal{N}^{\pm}_{\alpha,\lambda}$ that satisfies Equation (4.29) and Equation (4.30). Let $\alpha \in (0, a\mu_0)$ and $\lambda \in (0, \Lambda_1)$. For every $\psi \in X_0$, as $k \to \infty$, the following hold*

$$f(x)(u_k^+)^{-\gamma}\psi \in L^1(\Omega),$$

and

$$\Big(a+b\|u_k\|^{p\theta-p}\Big)\iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\left(u_k(x)-u_k(y)\right)\left(\psi(x)-\psi(y)\right)}{|x-y|^{N+sp}}\,dx\,dy$$
$$-\alpha\int_\Omega \frac{(u_k^+)^{p-1}\psi}{|x|^{sp}}dx - \lambda\int_\Omega f(x)(u_k^+)^{-\gamma}\psi dx - \int_\Omega\int_\Omega \frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}\psi(x)\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy = o_k(1). \tag{4.38}$$

Proof. Consider $\psi \in X_0$ with $\psi \geq 0$. Therefore, utilizing Equation (4.29) and Equation (4.37), we have

$$[\xi_k(t\psi)-1]\frac{\|u_k\|}{k} + \xi_k(t\psi)\frac{\|t\psi\|}{k}$$
$$\geq E_{\alpha,\lambda}(u_k) - E_{\alpha,\lambda}\left(\xi_k(t\psi)(u_k+t\psi)\right)$$
$$= -\frac{(\xi_k^p(t\psi)-1)}{p}\Big(a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big) - \frac{\Big(\xi_k^p(t\psi)-1\Big)}{p\theta}b\|u_k\|^{p\theta}$$
$$-\frac{\xi_k^p(t\psi)}{p}\Big[\Big(a\|u_k+t\psi\|^p - \alpha\|(u_k+t\psi)^+\|_H^p\Big) - \Big(a\|u_k\|^p - \alpha\|u_k^+\|_H^p\Big)\Big]$$
$$-\frac{\xi_k^{p\theta}(t\psi)}{p\theta}b\Big[\|u_k+t\psi\|^{p\theta} - \|u_k\|^{p\theta}\Big]$$
$$+\frac{\lambda\Big(\xi_k^{1-\gamma}(t\psi)-1\Big)}{1-\gamma}\int_\Omega f(x)\left((u_k+t\psi)^+\right)^{1-\gamma}dx$$
$$+\frac{\lambda}{1-\gamma}\int_\Omega f(x)\Big[\left((u_k+t\psi)^+\right)^{1-\gamma} - \left((u_k)^+\right)^{1-\gamma}\Big]dx$$
$$+\frac{\xi_k^{2p^*_{\mu,s}}(t\psi)-1}{2p^*_{\mu,s}}\int_\Omega\int_\Omega \frac{g(x)\left((u_k+t\psi)^+(x)\right)^{p^*_{\mu,s}}\left((u_k+t\psi)^+(y)\right)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy$$

$$+\frac{1}{2p^*_{\mu,s}}\int_\Omega\int_\Omega \frac{g(x)}{|x-y|^\mu}\Big[\big((u_k+t\psi)^+(x)\big)^{p^*_{\mu,s}}\big((u_k+t\psi)^+(y)\big)^{p^*_{\mu,s}}$$
$$-\big((u_k)^+(x)\big)^{p^*_{\mu,s}}\big((u_k)^+(y)\big)^{p^*_{\mu,s}}\Big]\,dx\,dy.$$

By dividing the above equation by $t>0$ and letting t approach 0^+, it follows that

$$\begin{aligned}
&|\langle \xi_k'(0),\psi\rangle|\frac{\|u_k\|}{k}+\frac{\|\psi\|}{k}\\
&\geq -\langle \xi_k'(0),\psi\rangle\Big[\big(a\|u_k\|^p-\alpha\|u_k^+\|_H^p\big)-\lambda\int_\Omega f(x)(u_k^+)^{1-\gamma}dx+b\|u_k\|^{p\theta}\\
&\quad-\int_\Omega\int_\Omega\frac{((u_k)^+(x))^{p^*_{\mu,s}}((u_k)^+(y))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\Big]\\
&\quad-\Big(a+b\|u_k\|^{p\theta-p}\Big)\iint_{\mathbb{R}^{2N}}\frac{|u_k(x)-u_k(y)|p-2\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy+\alpha\int_\Omega\frac{(u_k^+)^{p-1}\psi}{|x|^{sp}}\,dx\\
&\quad+\int_\Omega\int_\Omega\frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}\psi(x)\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x|^\alpha|x-y|^\mu}\,dx\,dy\\
&\quad+\lim_{t\to 0^+}\inf\frac{\lambda}{1-\gamma}\int_\Omega\frac{f(x)\Big[\big((u_k+t\psi)^+\big)^{1-\gamma}-(u_k^+)^{1-\gamma}\Big]}{t}\,dx\\
&=-\Big(a+b\|u_k\|^{p\theta-p}\Big)\iint_{\mathbb{R}^{2N}}\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy+\alpha\int_\Omega\frac{(u_k^+)^{p-1}\psi}{|x|^{sp}}\,dx\\
&\quad+\int_\Omega\int_\Omega\frac{\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}\psi(x)\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\\
&\quad+\lim_{t\to 0^+}\inf\frac{\lambda}{1-\gamma}\int_\Omega\frac{f(x)\Big[\big((u_k+t\psi)^+\big)^{1-\gamma}-(u_k^+)^{1-\gamma}\Big]}{t}\,dx. \qquad (4.39)
\end{aligned}$$

Applying Equation (4.39), we have

$$\liminf_{t\to 0^+}\int_\Omega\frac{f(x)\Big[\big((u_k+t\psi)^+\big)^{1-\gamma}-(u_k^+)^{1-\gamma}\Big]}{t}\,dx<\infty.$$

On the other hand, since $f(x)\Big[\big((u_k+t\psi)^+\big)^{1-\gamma}-(u_k^+)^{1-\gamma}\Big]\geq 0$ and considering the boundedness of the sequence $\{u_k\}_k$ in X_0, combined with Fatou's lemma and Lemma 4.1, it follows that

$$\begin{aligned}
&\lambda\int_\Omega f(x)(u_k^+)^{-\gamma}\psi dx\\
&\leq \lim_{t\to 0^+}\inf\frac{\lambda}{1-\gamma}\int_\Omega\frac{f(x)[((u_k+t\psi)^+)^{1-\gamma}-(u_k^+)^{1-\gamma}]}{t}\,dx\\
&\leq\frac{\langle\xi_k'(0),\psi\rangle\|u_k\|+\|\psi\|}{k}-\int_\Omega\int_\Omega\frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}\psi(x)\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\\
&\quad+\big(a+b\|u_k\|^{p\theta-p}\Big)\iint_{\mathbb{R}^{2N}}\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy-\alpha\int_\Omega\frac{(u_k^+)^{p-1}\psi}{|x|^{sp}}\,dx
\end{aligned}$$

$$\leq \frac{C_1C_3 + \|\psi\|}{k} + \Big(a + b\|u_k\|^{p\theta-p}\Big) \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$-\alpha \int_\Omega \frac{(u_k^+)^{p-1}\psi}{|x|^{sp}}\,dx - \int_\Omega\int_\Omega \frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}\psi(x)\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy,$$

where $C_3 > 0$ is determined by the boundedness of $\langle \xi_k'(0), \psi\rangle$ and $\|u_k\| \leq C_1$. This implies that as $k \to \infty$,

$$\begin{aligned}
&\Big(a + b\|u_k\|^{p\theta-p}\Big) \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&-\alpha \int_\Omega \frac{(u_k^+)^{p-1}\psi}{|x|^{sp}}\,dx - \int_\Omega f(x)(u_k^+)^{-\gamma}\psi dx \\
&-\int_\Omega\int_\Omega \frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}\psi(x)\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy \\
&\geq o(1).
\end{aligned} \tag{4.40}$$

In the following, we aim to prove that Eq. (4.40) holds for arbitrary $\psi \in X_0$. Let $\Psi_\epsilon = u_k^+ + \epsilon\psi$ with $\epsilon > 0$. By choosing $\psi = \Psi_\epsilon^+$ as a test function in Eq. (4.40), we obtain the following as $k \to \infty$

$$\begin{aligned}
o(1) \leq{}& \Big(a + b\|u_k\|^{p\theta-p}\Big) \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\Psi_\epsilon^+(x)-\Psi_\epsilon^+(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&-\alpha \int_\Omega \frac{(u_k^+)^{p-1}\Psi_\epsilon^+}{|x|^{sp}}\,dx - \lambda\int_\Omega f(x)(u_k^+)^{-\gamma}\Psi_\epsilon^+ dx \\
&-\int_\Omega\int_\Omega \frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}\Psi_\epsilon^+\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy \\
={}& \Big(a + b\|u_k\|^{p\theta-p}\Big) \\
&\times \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,((\Psi_\epsilon+\Psi_\epsilon^-)(x)-(\Psi_\epsilon+\Psi_\epsilon^-)(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&-\alpha \int_\Omega \frac{(u_k^+)^{p-1}(\Psi_\epsilon+\Psi_\epsilon^-)}{|x|^{sp}}\,dx - \lambda\int_\Omega f(x)(u_k^+)^{-\gamma}(\Psi_\epsilon+\Psi_\epsilon^-) dx \\
&-\int_\Omega\int_\Omega \frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}\big(u_k^+(y)\big)^{p^*_{\mu,s}}(\Psi_\epsilon+\Psi_\epsilon^-)}{|x-y|^\mu}\,dx\,dy.
\end{aligned} \tag{4.41}$$

We notice that by using the following inequality $(a-b)(a^- - b^-) \leq -(a^- - b^-)^2$, we obtain, for almost every $x, y \in \mathbb{R}^N$, the following inequality

$$\begin{aligned}
0 &\leq \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\
&\leq -\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u^-(x)-u^-(y))^2}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{4.42}$$

It follows that,

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\left(u_k^+(x)-u_k^+(y)\right)}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \leq \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p}}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y. \tag{4.43}$$

Applying Eq. (4.43), we have

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,((\Psi_\epsilon+\Psi_\epsilon^-)(x)-(\Psi_\epsilon+\Psi_\epsilon^-)(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\left(u_k^+(x)-u_k^+(y)\right)}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad + \epsilon \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\Psi_\epsilon^-(x)-\Psi_\epsilon^-(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&\leq \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p}}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad + \epsilon \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\Psi_\epsilon^-(x)-\Psi_\epsilon^-(y))}{|x-y|^{N+sp}}\,dx\,dy.
\end{aligned} \tag{4.44}$$

Furthermore, we obtain

$$\begin{aligned}
\int_\Omega \frac{(u_k^+)^{p-1}(\Psi_\epsilon+\Psi_\epsilon^-)}{|x|^{sp}}\,dx &= \int_\Omega \frac{|u_k^+|^p}{|x|^{sp}}\,dx + \epsilon\int_\Omega \frac{(u_k^+)^{p-1}\psi}{|x|^{sp}}\,dx + \int_\Omega \frac{(u_k^+)^{p-1}\Psi_\epsilon^-}{|x|^{sp}}\,dx \\
&\geq \int_\Omega \frac{|u_k^+|^p}{|x|^{sp}}\,dx + \epsilon\int_\Omega \frac{(u_k^+)^{p-1}\psi}{|x|^{sp}}\,dx + \epsilon\int_{\Omega_\epsilon} \frac{(u_k^+)^{p-1}\psi}{|x|^{sp}}\,dx,
\end{aligned} \tag{4.45}$$

with $\Omega_\epsilon = \{x \in \mathbb{R}^N : \Psi_\epsilon \leq 0\}$. Now, by combining Eq. (4.45), Eq. (4.44), and Eq. (4.41), it follows that as $k \to \infty$

$$\begin{aligned}
o(1) \leq \Bigg[&\left(a+b\|u_k\|^{p\theta-p}\right)\|u_k\|^p - \alpha\|u_k^+\|_H^p - \lambda\int_\Omega f(x)(u_k^+)^{1-\gamma}\,dx \\
&- \int_\Omega\int_\Omega \frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}}\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^{\mu}}\,dx\,dy\Bigg] \\
&+ \epsilon\Bigg[\left(a+b\|u_k\|^{p\theta-p}\right)\iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&- \alpha\int_\Omega \frac{(u_k^+)^{p-1}\psi}{|x|^{sp}}\,dx - \lambda\int_\Omega f(x)(u_k^+)^{-\gamma}\psi dx \\
&- \int_\Omega\int_\Omega \frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}\psi(x)\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^{\mu}}\,dx\,dy\Bigg]
\end{aligned}$$

$$+\Big(a+b\|u_k\|^{p\theta-p}\Big)\iint_{\mathbb{R}^{2N}}\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\Psi_\epsilon^-(x)-\Psi_\epsilon^-(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$-\epsilon\alpha\int_{\Omega_\epsilon}\frac{(u_k^+)^{p-1}\psi}{|x|^{sp}}\,dx+\lambda\int_{\Omega_\epsilon}f(x)(u_k^+)^{-\gamma}(u_k^++\epsilon\psi)dx$$
$$+\int_\Omega\int_{\Omega_\epsilon}\frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}(u_k^++\epsilon\psi)(x)\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy.$$

Since $\{u_k\}_k\in\mathcal{N}_{\alpha,\lambda}$ and $f(x)>0$, as $k\to\infty$, it follows that

$$\begin{aligned}
o(1)\le\epsilon\Big[&\Big(a+b\|u_k\|^{p\theta-p}\Big)\iint_{\mathbb{R}^{2N}}\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&-\alpha\int_\Omega\frac{(u_k^+)^{p-1}\psi}{|x|^{sp}}\,dx-\lambda\int_\Omega f(x)(u_k^+)^{-\gamma}\psi dx\\
&-\int_\Omega\int_\Omega\frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}\psi(x)\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\Big]\\
&+\big(a+b\|u_k\|^{p\theta-p}\big)\iint_{R^{2N}}\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\Psi_\epsilon^-(x)-\Psi_\epsilon^-(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&-\epsilon\alpha\int_{\Omega_\epsilon}\frac{(u_k^+)^{p-1}\psi}{|x|^{sp}}\,dx\\
&+\int_\Omega\int_{\Omega_\epsilon}\frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}(u_k^++\epsilon\psi)(x)\,\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy.
\end{aligned}\tag{4.46}$$

Now, utilizing the symmetry of the fractional kernel and employing a similar argument as in Eq. (4.43), we have

$$|u_k^+(x)-u_k^+(y)|^p\le|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,\big(u_k^+(x)-u_k^+(y)\big)\,,$$

it follows that

$$\begin{aligned}
&\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\Psi_\epsilon^-(x)-\Psi_\epsilon^-(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&=\iint_{\Omega_\epsilon\times\Omega_\epsilon}\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\Psi_\epsilon^-(x)-\Psi_\epsilon^-(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&\quad+2\iint_{\Omega_\epsilon\times(\mathbb{R}^N\setminus\Omega_\epsilon)}\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\Psi_\epsilon^-(x)-\Psi_\epsilon^-(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&\le-\epsilon\Big(\iint_{\Omega_\epsilon\times\Omega_\epsilon}\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&\quad+2\iint_{\Omega_\epsilon\times(\mathbb{R}^N\setminus\Omega_\epsilon)}\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy\Big)\\
&\le2\epsilon\iint_{\Omega_\epsilon\times\mathbb{R}^N}\Big|\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\Big|\,dx\,dy
\end{aligned}$$

So, by employing the Hölder inequality and considering the fact that the sequence $\{u_k\}_k$ is bounded in X_0, we obtain

$$\begin{aligned}&\iint_{\Omega_\epsilon\times\mathbb{R}^N}\Big|\frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\Big|\,dx\,dy\\&\leq C\Big(\iint_{\Omega_\epsilon\times\mathbb{R}^N}\big|\frac{(\psi(x)-\psi(y))}{|x-y|^{(N+sp)/p}}\big|^p\,dx\,dy\Big)^{1/p}.\end{aligned}\tag{4.47}$$

Clearly $\dfrac{(\psi(x)-\psi(y))}{|x-y|^{(N+sp)/p}}\in L^p(\mathbb{R}^{2N})$. Therefore, for every $\sigma>0$, there exists R_σ sufficiently large such that

$$\iint_{(\operatorname{supp}\psi)\times[\mathbb{R}^N\setminus B_{R_\sigma}]}\Big|\frac{(\psi(x)-\psi(y))}{|x-y|^{(N+sp)/p}}\Big|^p\,dx\,dy<\frac{\sigma}{p}.$$

So, utilizing the definition of Ω_ϵ, it follows that $\Omega_\epsilon\subset\operatorname{supp}\psi$ and we have $|\Omega_\epsilon\times B_{R_\sigma}|\to 0$ as $\epsilon\to 0^+$. Now, since $\dfrac{(\psi(x)-\psi(y))}{|x-y|^{(N+sp)/p}}\in L^p(\mathbb{R}^{2N})$, we can establish the existence of $\epsilon_\sigma>0$ and $\delta_\sigma>0$ such that for every $\epsilon\in(0,\epsilon_\sigma]$, we have

$$|\Omega_\epsilon\times B_{R_\sigma}|<\delta_\nu\quad\text{and}\iint_{\Omega_\epsilon\times B_{R_\sigma}}\big|\frac{(\psi(x)-\psi(y))}{|x-y|^{(N+sp)/p}}\big|^p\,dx\,dy<\frac{\sigma}{p}.$$

Consequently, for every $\epsilon\in(0,\epsilon_\sigma]$, it follows that

$$\lim_{\epsilon\to0^+}\iint_{\Omega_\epsilon\times\mathbb{R}^N}\Big|\frac{(\psi(x)-\psi(y))}{|x-y|^{(N+sp)/p}}\Big|^p\,dx\,dy=0.\tag{4.48}$$

Hence, by Eq. (4.47), we can conclude that

$$\lim_{\epsilon\to0^+}\iint_{\Omega_\epsilon\times\mathbb{R}^N}\Big|\frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\Big|\,dx\,dy=0.$$

Next, we proceed to demonstrate that

$$\lim_{\epsilon\to0^+}\frac{1}{\epsilon}\int_\Omega\int_{\Omega_\epsilon}\frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}(u_k^++\epsilon\psi)(x)\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy=0.\tag{4.49}$$

To accomplish this, let's consider

$$\begin{aligned}&\int_\Omega\int_{\Omega_\epsilon}\frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}(u_k^++\epsilon\psi)(x)\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\\&\leq\int_\Omega\int_{\Omega_\epsilon}\frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}}\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\\&\quad+\epsilon\int_\Omega\int_{\Omega_\epsilon}\frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}\psi(x)\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\\&\leq\int_\Omega\int_{\Omega_\epsilon}\frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}}\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\end{aligned}$$

$$
\begin{aligned}
&+\epsilon\Big(\int_{\Omega}\int_{\Omega_\epsilon}\frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}\psi(x)\big(u_k^+(y)\big)^{p^*_{\mu,s}-1}\psi(y)}{|x-y|^{\mu}}\,dx\,dy\Big)^{1/p}\\
&\times\Big(\int_{\Omega}\int_{\Omega}\frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}}\big(u_k^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^{\mu}}\,dx\,dy\Big)^{1/p}\\
&\le CC_g(N,\mu)\Big(\int_{\Omega_\epsilon}\big(u_k^+(x)\big)^{p^*_s}\,dx\Big)^{p^*_{\mu,s}/p^*_s}\\
&+C\epsilon C_g(N,\mu)\Big(\int_{\Omega_\epsilon}\big([u_k^+(x)]^{p^*_{\mu,s}-1}\psi(x)\big)^{p^*_s/p^*_{\mu,s}}\,dx\Big)^{p^*_{\mu,s}/p^*_s}\\
&\le CC_g(N,\mu)\Big(\int_{\Omega_\epsilon}\big(u_k^+(x)\big)^{p^*_s}\,dx\Big)^{p^*_{\mu,s}/p^*_s}\\
&+C\epsilon C_g(N,\mu)\Big(\int_{\Omega_\epsilon}\big(u_k^+(x)\big)^{p^*_s}\,dx\Big)^{(p^*_{\mu,s}-1)/p^*_s}\Big(\int_{\Omega_\epsilon}|\psi(x)|^{p^*_s}\,dx\Big)^{1/p^*_s}\\
&\le CC_g(N,\mu)\epsilon^{p^*_{\mu,s}}\Big(\int_{\Omega_\epsilon}|\psi(x)|^{p^*_s}\,dx\Big)^{p^*_{\mu,s}/p^*_s}+\tilde{C}\epsilon C_g(N,\mu)\epsilon^{p^*_{\mu,s}}\Big(\int_{\Omega_\epsilon}|\psi(x)|^{p^*_s}\,dx\Big)^{p^*_{\mu,s}/p^*_s}. \quad (4.50)
\end{aligned}
$$

Hence, dividing Eq. (4.50) by ϵ and taking into account the fact that $|\Omega_\epsilon|\to 0$ as $\epsilon\to 0^+$, we can establish the validity of Eq. (4.49). Furthermore, we claim that

$$
\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\frac{|u_k^+|^{p-2}u_k^+\psi}{|x|^{sp}}\,dx=0. \quad (4.51)
$$

So, for $x\in\Omega_\epsilon$, we have $u_k^+ + \epsilon\psi\le 0$ and $\psi(x)\le 0$. Consequently, utilizing Eq. (1.2), we can conclude that

$$
0\le\Big|\int_{\Omega_\epsilon}\frac{|u_k^+|^{p-2}u_k^+\psi}{|x|^{sp}}\,dx\Big|\le\int_{\Omega_\epsilon}\frac{|u_k^+|^{p-2}u_k^+|\psi|}{|x|^{sp}}\,dx\le\epsilon\int_{\Omega_\epsilon}\frac{|\psi|^p}{|x|^{sp}}\,dx\le\epsilon\|\psi\|_H^p\le\epsilon\frac{\|\psi\|^p}{\mu_0}
$$

from which we establish the validity of Eq. (4.51) as $\epsilon\to 0$.

Then, by dividing Eq. (4.46) by ϵ and utilizing Eq. (4.48), Eq. (4.51), Eq. (4.49), and the fact that $|\Omega_\epsilon|\to 0$ as $\epsilon\to 0^+$, we obtain

$$
\begin{aligned}
o(1)\le&\Big(a+b\|u_k\|^{p\theta-p}\Big)\iint_{\mathbb{R}^{2N}}\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&-\alpha\int_{\Omega}\frac{|u_k^+|^{p-2}u_k^+\psi}{|x|^{sp}}\,dx-\lambda\int_{\Omega}f(x)(u_k^+)^{-\gamma}\psi dx\\
&-\int_{\Omega}\int_{\Omega}\frac{g(x)\big(u_k^+(x)\big)^{p^*_{\mu,s}-1}\big(u_k^+(y)\big)^{p^*_{\mu,s}}\psi(x)}{|x-y|^{\mu}}\,dx\,dy.
\end{aligned}
$$

This proves Eq. (4.40). Since ψ is arbitrary, we can conclude that Eq. (4.38) holds for any $\psi\in X_0$. The proof of Lemma 4.3 is now complete. □

To demonstrate the compactness property of the functional energy $E_{\alpha,\lambda}$, we define

$$
\begin{aligned}
c_{\alpha,\lambda}: &= \Big(\frac{1}{p}-\frac{1}{2p^*_{\mu,s}}\Big)a^{\frac{p^*_{\mu,s}}{p^*_{\mu,s}-p}}\mathbb{S}_{C,\mu}^{-\frac{p}{p^*_{\mu,s}-p}}||g||_r^{-\frac{p}{p^*_{\mu,s}-p}}\\
&- \lambda^{\frac{p\theta}{p\theta-1+\gamma}}\Big(\frac{p\theta-1+\gamma}{p^*_{\mu,s}(1-\gamma)p\theta}\Big)\frac{\Big[(p^*_{\mu,s}+\gamma-1)||f||_m S^{-\frac{\gamma-1}{p^*_{\mu,s}}}\Big]^{\frac{p\theta}{p\theta-1+\gamma}}}{\Big[b(p^*_{\mu,s}-p\theta)\Big]^{\frac{1-\gamma}{p\theta-1+\gamma}}}.
\end{aligned}
\tag{4.52}
$$

The following lemma provides conditions under which a subsequence of the sequence u_{kk} converges strongly to a limit in the function space X_0.

Lemma 4.4. *Consider a sequence $\{u_k\}_k \subset \mathcal{N}^{\pm}_{\alpha,\lambda}$ with $E_{\alpha,\lambda}(u_k) \to c < c_{\alpha,\lambda}$ ans let $\lambda \in (0,\Lambda_1)$, $\alpha \in (0, a\gamma_0)$. Then, the sequence $\{u_k\}_k$ has a subsequence that converges strongly to u_0 in X_0.*

Proof. Considering Eq. (4.30), we can deduce the boundedness of the sequence u_{kk} in X_0. Furthermore, we can establish the boundedness of the sequence $\{u_k^-\}_k$ in X_0. By substituting $\psi = u_k^-$ into Eq. (4.38) as $k \to \infty$, we obtain

$$
\lim_{k\to\infty}\Big(a+b\|u_k\|^{p\theta-p}\Big)\iint_{\mathbb{R}^{2N}}\frac{|u_k(x)-u_k(y)|^{p-2}\left(u_k(x)-u_k(y)\right)\left(u_k^-(x)-u_k^-(y)\right)}{|x-y|^{N+sp}}\,dx\,dy = 0.
$$

Therefore, using Eq. (4.42), we have $\|u_k^-\| \to 0$ as $k \to \infty$. Hence, $\{u_k\}_k$ is a positive sequence. By applying Lemma 2.2 in conjunction with Eq. (1.2), we can establish the existence of a subsequence, which we still denote as $\{u_k\}_k$, satisfying

$$
\begin{aligned}
&u_k \rightharpoonup u_0 \quad \text{weakly in } L^{p_s^*}(\Omega), \quad \|u_k\| \to \nu\\
&u_k \to u_0 \quad \text{in } L^p(\Omega) \quad \text{for any } p \in (1, p_s^*)\\
&u_k \rightharpoonup u_0 \quad \text{in } L^{p_s^*}(\Omega, |x|^{-sp}), \quad \|u_k\|_H \to l\\
&u_k \to u_0 \quad \text{a.e. in } \Omega \quad u_k \le h \text{ a. e. in } \Omega,
\end{aligned}
\tag{4.53}
$$

as $k \to \infty$, where $h \in L^p(\Omega)$ and $p \in [1, p_s^*)$. Therefore, since the sequence $\{u_k\}_k$ is positive, we have $u_0 \ge 0$. Now, let's consider the case where $\nu = 0$ in Eq. (4.53). In this case, we can deduce that $\lim\limits_{k\to\infty} u_k = 0$ in X_0.

Now, let's suppose that $\nu > 0$. Then, by utilizing Lemma 2.3 in [9], Lemma 2.4 in [14], and Lemma 3.2 in [17], we can conclude that

$$
\|u_k\|^p = \|u_k-u_0\|^p + \|u_0\|^p + o(1), \tag{4.54}
$$

$$
\|u_k\|_H^p = \|u_k-u_0\|_H^p + \|u_0\|_H^p + o(1), \tag{4.55}
$$

$$
\begin{aligned}
&\int_\Omega\int_\Omega \frac{g(x)(u_k(x))^{p^*_{\mu,s}}(u_k(y))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\\
&= \int_\Omega\int_\Omega \frac{g(x)((u_k-u_0)(x))^{p^*_{\mu,s}}(u_k-u_0)(y))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\\
&\quad + \int_\Omega\int_\Omega \frac{g(x)(u_0(x))^{p^*_{\mu,s}}(u_0(y))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy + o(1).
\end{aligned}
\tag{4.56}
$$

It follows, from Eq. (4.54), Eq. (4.55) and Eq. (4.56), that

$$
\begin{aligned}
o(1) = & \left(a + b\|u_k\|^{p\theta-p}\right) \iint_{R^{2N}} \frac{|u_k(x)-u_k(y)|^{p-2}\left(u_k(x)-u_k(y)\right)\left((u_k-u_0)(x)-(u_k-u_0)(y)\right)}{|x-y|^{N+sp}}\, dx\, dy \\
& - \alpha \int_\Omega \frac{u_k^{p-1}(u_k-u_0)}{|x|^{sp}}\, dx - \lambda \int_\Omega f(x)(u_k)^{-\gamma}(u_k-u_0) dx \\
& - \int_\Omega \int_\Omega \frac{g(x)(u_k(x))^{p^*_{\mu,s}-1}(u_k(y))^{p^*_{\mu,s}}}{|x-y|^\mu}(u_k-u_0)\, dx\, dy \\
= & \left(a + b\nu^{p\theta-p}\right)\left(\nu^p - \|u_0\|^p\right) - \alpha\left(\|u_k\|_H^p - \|u_0\|_H^p\right) \\
& - \lambda \int_\Omega f(x)(u_k)^{-\gamma}(u_k-u_0) dx - \int_\Omega \int_\Omega \frac{g(x)(u_k(x))^{p^*_{\mu,s}}((u_k(y))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy \\
& + \int_\Omega \int_\Omega \frac{g(x)(u_0(x))^{p^*_{\mu,s}}(u_0(y))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy + o(1)
\end{aligned}
$$

$$
\begin{aligned}
= & \left(a + b\nu^{p\theta-p}\right)\|u_k-u_0\|^p - \alpha\|u_k-u_0\|_H^p - \lambda \int_\Omega f(x)(u_k)^{-\gamma}(u_k-u_0) dx \\
& - \int_\Omega \int_\Omega \frac{g(x)((u_k(x)-u_0(x)))^{p^*_{\mu,s}}((u_k(y)-u_0(y)))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy + o(1).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
& \left(a + b\nu^{p\theta-p}\right)\lim_{k\to\infty}\|u_k-u_0\|^p - \alpha \lim_{k\to\infty}\|u_k-u_0\|_H^p \\
& = \lambda \lim_{k\to\infty} \int_\Omega f(x)(u_k)^{-\gamma}(u_k-u_0) dx \\
& \quad + \lim_{k\to\infty} \int_\Omega \int_\Omega \frac{g(x)((u_k(x)-u_0(x)))^{p^*_{\mu,s}}((u_k(y)-u_0(y)))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy.
\end{aligned}
\tag{4.57}
$$

By Eq. (4.53), we have $u_k^{1-\gamma} \leq h^{1-\gamma}$. Then, by applying the Lebesgue dominated convergence theorem, we can conclude that

$$
\lim_{k\to\infty} \int_\Omega f(x)(u_k^+)^{1-\gamma}\, dx = \int_\Omega f(x)(u_0^+)^{1-\gamma}\, dx.
$$

Therefore, utilizing Lemma 4.3, we have $f(x)u_k^{-\gamma}u_0 \in L^1(\Omega)$ for every $k \in \mathbb{N}$. Now, considering Fatou's lemma, we can deduce that

$$
\int_\Omega f(x)u_0^{1-\gamma}\, dx \leq \liminf_{k\to\infty} \int_\Omega f(x)u_k^{-\gamma}u_0 dx.
$$

Now, let us denote

$$
\lim_{k\to\infty} \int_\Omega \int_\Omega \frac{g(x)((u_k(x)-u_0(x)))^{p^*_{\mu,s}}((u_k(y)-u_0(y)))^{p^*_{\mu,s}}}{|x-y|^\mu}\, dx\, dy = l^{2p^*_{\mu,s}}. \tag{4.58}
$$

Hence, by employing Eqs. (4.57)-(4.58), we obtain

$$\Big(a+b\nu^{p\theta-p}\Big)\lim_{k\to\infty}\|u_k-u_0\|^p\le l^{2p^*_{\mu,s}}. \tag{4.59}$$

Hence, from Eq. (4.59), we obtain $l\ge 0$. If $l=0$, considering the fact that $\nu>0$ and combining Eq. (4.53) with Eq. (4.59), we have $\lim_{k\to\infty}u_k=u_0$ in X_0, which completes the proof of the theorem. Therefore, let us assume that $l>0$. By utilizing Eq. (2.8), we obtain

$$\|u_k-u_0\|^p+o(1)\ge ||g||_r^{-\frac{p}{p^*_{\mu,s}}}l^p\mathbb{S}_{C,\mu}. \tag{4.60}$$

Now, considering Eqs. (4.57)-(4.59) and Eq. (4.60), we have

$$\Big(a+b\nu^{p\theta-p}\Big)||g||_r^{-\frac{p}{p^*_{\mu,s}}}\mathbb{S}_{C,\mu}\le l^{2p^*_{\mu,s}-p}. \tag{4.61}$$

By substituting Eq. (4.61) into Eq. (4.60), we obtain

$$\nu^p\ge\mathbb{S}_{C,\mu}^{\frac{2p^*_{\mu,s}}{2p^*_{\mu,s}-p}}\Big(\frac{a}{||g||_r}\Big)^{\frac{p}{2p^*_{\mu,s}-p}}. \tag{4.62}$$

Now, let us define, for any $k\in\mathbb{N}$ and $\phi\in X_0$

$$\begin{aligned}H(u_k,\phi):=&\Big(a+b||u_k||^{p\theta-p}\Big)\iint_{\mathbb{R}^{2N}}\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\,dx\,dy\\&-\alpha\int_\Omega\frac{u_k^{p-1}\phi(x)}{|x|^{sp}}\,dx-\lambda\int_\Omega f(x)(u_k)^{-\gamma}\phi(x)dx-\int_\Omega\int_\Omega\frac{g(x)(u_k(x))^{p^*_{\mu,s}-1}u_k(y)^{p^*_{\mu,s}}\phi(x)}{|x-y|^\mu}\,dx\,dy.\end{aligned} \tag{4.63}$$

Therefore, for every $k\in\mathbb{N}$, as $k\to\infty$, we obtain

$$\begin{aligned}c&=E_{\alpha,\lambda}(u_k)-\frac{1}{2p^*_{\mu,s}}H(u_k,\phi)\\&=\Big(\frac1p-\frac{1}{2p^*_{\mu,s}}\Big)\Big(a||u_k||^p-\alpha||u_k||_H^p\Big)+\Big(\frac{1}{p\theta}-\frac{1}{2p^*_{\mu,s}}\Big)b||u_k||^{p\theta}-\lambda\Big(\frac1\gamma-\frac{1}{2p^*_{\mu,s}}\Big)\int_\Omega f(x)(u_k^+)^{1-\gamma}\,dx+o(1)\\&=\Big(\frac1p-\frac{1}{2p^*_{\mu,s}}\Big)\Big(a\nu^p-\alpha l^p\Big)+\Big(\frac{1}{p\theta}-\frac{1}{2p^*_{\mu,s}}\Big)b||u_k||^{p\theta}-\lambda\Big(\frac1\gamma-\frac{1}{2p^*_{\mu,s}}\Big)\int_\Omega f(x)(u_k^+)^{1-\gamma}\,dx+o(1)\\&\ge\Big(\frac1p-\frac{1}{2p^*_{\mu,s}}\Big)a^{\frac{p^*_{\mu,s}}{p^*_{\mu,s}-p}}\mathbb{S}_{C,\mu}^{-\frac{p}{p^*_{\mu,s}-p}}||g||_r^{-\frac{p}{p^*_{\mu,s}-p}}\\&+b\Big(\frac{1}{p\theta}-\frac{1}{2p^*_{\mu,s}}\Big)||u_k||^{p\theta}-\lambda\Big(\frac1\gamma-\frac{1}{2p^*_{\mu,s}}\Big)S^{-\frac{\gamma-1}{p^*_{\mu,s}}}||f||_m||u_k^+||^{1-\gamma}+o(1).\end{aligned} \tag{4.64}$$

Let us denote

$$F_b(t)=b\Big(\frac{1}{p\theta}-\frac{1}{2p^*_{\mu,s}}\Big)t^{p\theta}-\lambda\Big(\frac1\gamma-\frac{1}{2p^*_{\mu,s}}\Big)S^{-\frac{\gamma-1}{p^*_{\mu,s}}}||f||_qt^{1-\gamma}.$$

Then, by performing a direct calculation, we can establish the existence of a lower bound and global minima for the function F_b. Specifically, we have

$$F_b(t) \geq -\lambda^{\frac{p\theta}{p\theta-1+\gamma}} \Big(\frac{p\theta-1+\gamma}{p^*_{\mu,s}(1-\gamma)p\theta}\Big) \frac{\Big[(p^*_{\mu,s}+\gamma-1)||f||_m S^{-\frac{\gamma-1}{p^*_{\mu,s}}}\Big]^{\frac{p\theta}{p\theta-1+\gamma}}}{\Big[b(p^*_{\mu,s}-p\theta)\Big]^{\frac{1-\gamma}{p\theta-1+\gamma}}}.$$

By letting $k \to \infty$, we obtain

$$\begin{aligned} c &\geq \Big(\frac{1}{p}-\frac{1}{2p^*_{\mu,s}}\Big) a^{\frac{p^*_{\mu,s}}{p^*_{\mu,s}-p}} \mathbb{S}_{C,\mu}^{-\frac{p}{p^*_{\mu,s}-p}} ||g||_r^{-\frac{p}{p^*_{\mu,s}-p}} \\ &- \lambda^{\frac{p\theta}{p\theta-1+\gamma}} \Big(\frac{p\theta-1+\gamma}{p^*_{\mu,s}(1-\gamma)p\theta}\Big) \frac{\Big[(p^*_{\mu,s}+\gamma-1)||f||_m S^{-\frac{\gamma-1}{p^*_{\mu,s}}}\Big]^{\frac{p\theta}{p\theta-1+\gamma}}}{\Big[b(p^*_{\mu,s}-p\theta)\Big]^{\frac{1-\gamma}{p\theta-1+\gamma}}}. \end{aligned}$$

This contradicts the assumption $c < c_{\alpha,\lambda}$. Therefore, we conclude that $\nu = 0$. As a result, $\lim_{k\to\infty} u_k = u_0$ in X_0. The proof of Lemma 4.4 is now complete. □

5 The first solution of the problem (1.1) in $\mathcal{N}^+_{\alpha,\lambda}$

In this section, our goal is to establish the existence of a solution to the problem (1.1) by employing a minimization method on the function space $\mathcal{N}^+_{\alpha,\lambda}$. We now present a lemma that demonstrates the existence of a negative minimizer for the functional energy $E_{\alpha,\lambda}$ in the function space $\mathcal{N}^+_{\alpha,\lambda}$.

Lemma 5.1. *Let λ be a positive parameter and let $\alpha \in (0, a\mu 0)$. Then, we have*

$$m^+_{\alpha,\lambda} = \inf_{u\in\mathcal{N}^+_{\alpha,\lambda}} E_{\alpha,\lambda}(u) < 0.$$

Proof. For $u \in \mathcal{N}^+_{\alpha,\lambda} \subset \mathcal{N}_{\alpha,\lambda}$, we have

$$\begin{aligned} E_{\alpha,\lambda}(u) &= \Big(\frac{1}{p}-\frac{1}{1-\gamma}\Big)[a\|u\|^p - \alpha\|u^+\|^p_H] + \Big(\frac{1}{p\theta}-\frac{1}{1-\gamma}\Big)b\|u\|^{p\theta} \\ &\quad - \Big(\frac{1}{2p^*_{\mu,s}}-\frac{1}{1-\gamma}\Big)\int_\Omega\int_\Omega \frac{g(x)(u^+(x))^{p^*_{\mu,s}}(u^+(y))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy \\ &= -\frac{1}{2p^*_{\mu,s}(1-\gamma)}\Big[(1+\gamma)[a\|u\|^p - \alpha\|u^+\|^p_H] + \Big(p\theta+q-1\Big)b\|u\|^{p\theta} \\ &\quad - \Big(2p^*_{\mu,s}+\gamma-1\Big)\int_\Omega\int_\Omega \frac{g(x)(u^+(x))^{p^*_{\mu,s}}(u^+(y))^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy\Big] < 0, \end{aligned}$$

since $u \in \mathcal{N}^+_{\alpha,\lambda}$ and $p^*_{\mu,s} > p\theta$. Therefore, $m^+_{\alpha,\lambda} < 0$. The proof of Lemma 5.1 is now completes. □

The following theorem guarantees the existence of a non-negative solution in $\mathcal{N}^+_{\alpha,\lambda}$ for problem (1.1), given that the assumptions (f) and (g) hold.

Theorem 5.2. *Suppose that the assumptions* (f) *and* (g) *are fulfilled. Then, problem* (1.1) *has a non-negative solution in* $\mathcal{N}^+\alpha, \lambda$ *for every* $0 < \Lambda * = \min(\Lambda_1, \Lambda_2)$.

Proof. Let us fix $0 < \lambda < \Lambda_* = \min(\Lambda_1, \Lambda_2)$. According to the variational principle of Ekeland combined with Lemma 3.2, we obtain the existence of a minimizing sequence $\{u_k\}_k \subset \mathcal{N}^+_{\alpha,\lambda} \cup \{0\}$, that satisfies Eq. (4.29) and Eq. (4.30). Consequently, we have

$$E_{\alpha,\lambda}(u_k) \to m^+_{\alpha,\lambda} < 0 \quad \text{as} \;\; k \to \infty,$$

which implies $\{u_k\}_k \subset \mathcal{N}^+_{\alpha,\lambda}$. Therefore, using Lemma 4.4 with the fact $c = m^+_{\alpha,\lambda}$, it follows that $u_k \to u_0$ in X_0, up to a subsequence. Furthermore, using Eq. (3.28) combined with Lemma 4.1, we obtain

$$\begin{aligned}&(1+\gamma)\Big[a\|u_0\|^p - \alpha\|u_0\|^p_H\Big] + b\Big(p\theta + \gamma - 1\Big)\|u_0\|^{p\theta} \\ &- \Big(2p^*_{\mu,s} + \gamma - 1\Big)\int_\Omega\int_\Omega \frac{g(x)\big(u_0^+(x)\big)^{p^*_{\mu,s}}\big(u_0^+(y)\big)^{p^*_{\mu,s}}}{|x-y|^\mu}\,dx\,dy > 0,\end{aligned}$$

which implies $u_0 \in \mathcal{N}^+_{\alpha,\lambda}$, and $m^+_{\alpha,\lambda}$ is achieved at u_0 by $E_{\alpha,\lambda}$ is continuous on X_0. Taking $k \to \infty$, together with Fatou's Lemma in Eq. (4.38), we deduce that $H(u_0, \psi) \geq 0$ [where H is defined in Eq. (4.63)] for $\psi \in X_0$ with $\psi \geq 0$.

Next, we take $\psi = \Psi^+_\epsilon$ as a test function, where $\Psi_\epsilon = u_0^+ + \epsilon\psi$ and $\psi \in X_0$. By repeating the steps from Eq. (4.38) to Eq. (4.49) with u_0 in the place of u_k, we obtain $H(u_0, \psi) \geq 0$ for arbitrary $\psi \in X_0$. Thus, we have

$$\lambda f(x)(u_0^+)^{-\gamma}\psi \in L^1(\Omega) \quad \text{and} \;\; u_0 \in \mathcal{N}^+_{\alpha,\lambda}.$$

Since $0 \notin \mathcal{N}^+_{\alpha,\lambda}$ by Lemma 3.2, we have $u_0 \not\equiv 0$. Moreover, by Eq. (2.10) with $\psi = u_0^-$ together with Eq. (4.42), we obtain $\|u_0^-\| = 0$. Hence u_0 is positive. By applying the maximum principle, we can conclude that u_0 is a non-negative solution of (1.1). This completes the proof of Theorem 5.2. □

6 The second solution of the problem (1.1) in $\mathcal{N}^-_{\alpha,\lambda}$

In this section, to prove the existence of a solution in $\mathcal{N}^-_{\alpha,\lambda}$, we can follow a similar approach as in the proof of Theorem 5.2. However, in this case, we will consider the space $\mathcal{N}^-_{\alpha,\lambda}$, which consists of non-negative functions in the Nehari manifold. The following theorem establishes the existence of a non-negative solution in $\mathcal{N}^-_{\alpha,\lambda}$ for problem (1.1), provided that the assumptions (f) and (g) are satisfied.

Theorem 6.1. *Suppose the assumptions* (f) *and* (g) *are satisfied. Then, for* $0 < \Lambda_* = \min(\Lambda_1, \Lambda_2)$*, problem* (1.1) *has a non-negative solution in* $\mathcal{N}^-_{\alpha,\lambda}$.

Proof. We start by observing that $\mathcal{N}^-_{\alpha,\lambda}$ is a closed set in X_0. By the variational principle of Ekeland, we can extract a minimizing sequence $\{v_k\}_k \subset \mathcal{N}^-_{\alpha,\lambda}$ that satisfies the condition for $\inf_{u\in\mathcal{N}^-_{\alpha,\lambda}} E_{\alpha,\lambda}(u)$. Moreover, since the sequence $\{v_k\}_k$ is bounded in X_0, we can choose a subsequence such that $\{v_k\}_k \rightharpoonup v_0$ in X_0. Applying Lemma 4.4, we have that $\{v_k\}_k \to v_0$ in X_0 up to a subsequence. Since, $\mathcal{N}^-_{\gamma,\lambda}$ is closed, we conclude that $v_0 \in \mathcal{N}^-_{\alpha,\lambda}$ with $E_{\alpha,\lambda}(v_0) = m^-_{\alpha,\lambda}$. By repeating the same argument as in Section 5, we have $H(v_0,\psi) \geq 0$, so that $\lambda f(x)(v_0^+)^{-\gamma}\psi \in L^1(\Omega)$ for all $\psi \in X_0$, and v_0 belongs to $\mathcal{N}^-_{\alpha,\lambda}$. Combining this with Lemma 3.2, we deduce that v_0 is a nontrivial solution of problem (1.1).

Finally, by applying the strong maximum principle, we conclude that v_0 is a non-negative solution of problem (1.1). This completes the proof of Theorem 6.1. □

7 Proof of Theorem 1.1

Proof. By applying Theorems 5.2 and 6.1, we conclude that problem (1.1) has two non-negative solutions, denoted as u_0 and v_0, respectively. Since $\mathcal{N}^+_{\alpha,\lambda} \cap \mathcal{N}^-_{\alpha,\lambda} = \emptyset$, the solutions u_0 and v_0 must be distinct. This completes the proof of the Theorem 1.1. □

Conflict of interest:
Both the authors declare that there are no conflicts of interest in any form whatsoever.

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