DEPENDENCE OF EIGENVALUES ON THE REGULAR FOURTH-ORDER STURM-LIOUVILLE PROBLEM

JIANQING SUO, ZHIJIE SHI, AND ZHEN WEI

Abstract. In this paper, the eigenvalues of a regular fourth-order Sturm-Liouville (SL) problems are studed. The eigenvalues depend not only continuously but smoothly on the problem. An expression for the derivative of the eigenvalues with respect to a given parameter: an endpoint, a boundary condition, a coefficient, or the weight function, are found.

1. Introduction

In the early nineteenth century, Sturm and Liouville published a series of papers on second order linear ordinary differential equations including boundary value problems. The influence of their work was such that this subject became known as Sturm-Liouville theory. Sturm and Liouville were the first to see the need for finding properties of solutions directly from the equation even when no analytic expressions for solutions are available. A large amount of papers have been written since then. Among them, Pöschel and Trubowitz consider the eigenvalue of a regular second-order SL problem

$$-(py')' + qy = \lambda wy \tag{1.1}$$

with Dirichlet boundary conditions

$$y(a) = 0 = y(b)$$

in [1], they show that the n-th eigenvalue $\lambda = \lambda_n(q)$ as a function of q is Frechet differentiable for $q \in L^2(a, b), p = 1 = w$, and give the expression of $d\lambda_q$. Dauge and Helffer in [2] show that the Neumann eigenvalues of a regular second order SL problem on an interval $[a, b]$ are differentiable functions of the right endpoint b and give the expression of $\lambda'(b)$ and they indicate that a similar equation holds for the eigenvalues of other separated BC. In [3] Kong and Zettl give a different proof of the Dauge-Helffer Theorem with substantially weaker hypotheses replace $L^2[a, b]$ by $L^1(a, b)$ and they obtained a similar result for coupled BC. In [4], Kong and Zettl further show that the eigenvalues of regular second-order SL problems are differentiable functions of all the data: the endpoints, the boundary conditions, as well as the coefficients and the weight functions and they give expressions for their derivatives. See Chapter 4 in [5] for an exposition of this theory. In [6], Battle proves continuity and differentiability of the eigenvalues for a more general second order problem. In [7], Kong et al. study that any isolated eigenvalue of a regular self-adjoint or non-self-adjoint ordinary linear n-th order BVP depends

2000 *Mathematics Subject Classification.* Primary 34B20, 34B24; Secondary 47B25.

Key words and phrases. Fourth-order Sturm-Liouville problems, Continuous dependence on parameters, Eigenvalues.

This paper is in final form and no version of it will be submitted for publication elsewhere.

on the problem with the general boundary conditions. In the classical case, these properties play an important role in the Bailey, Everitt, and Zettl code SLEIGN2 [8]. Yet, remarkably, this subject is an intensely active field of research today. Especially in recent years, the research on dependence of the eigenvalues of boundary value problem on the problems has been extended in various aspects. Such as, the dependence of the eigenvalues of Sturm-Liouville problems with interface conditions, with transmission conditions, with eigenparameter-dependent boundary conditions, with distributional potentials[11]-[18], et al. At the same time, such research has also been extended to higher-order situations. For fourth-order case, Suo and Wang study the dependence of eigenvalues on the problem in [9] use the same method as [3,4]. Ge, Wang and Suo study the dependence of eigenvalues on the boundary in [10]. Lv and Ao study the eigenvalues of fourth-order boundary value problems with self-adjoint canonical boundary conditions in [13]. Li and Ao et al. study the dependence of eigenvalues of fourth-order SL problems with discontinuous conditions in [11,12,14]. However, they considered the special fourth-order equation:

$$(p(x)y'')'' + q(x)y = \lambda w(x)y$$

with $q = 1$ or $q \neq 1$.

In this paper, we consider a regular general fourth-order equation with separated conditions and coupled conditions. we show that the eigenvalues of the problems are differentiable functions of all the data: the endpoints, the boundary conditions, as well as the coefficients and the weight functions and we find formulas for their derivatives. It is necessary to study the dependence of eigenvalues of higher-order boundary value problems (BVPs), besides its theoretical importance, the continuous dependence of the eigenvalues and the eigenfunctions on the data is also fundamental from the numerical point of view. The major general purpose codes for the numerical computation of the eigenvalues and eigenfunctions of fourth-order Sturm-Liouville problems are SLEUTH (Sturm-Liouville eigenvalues using theta matrices) [19] and for more general problems see [20]-[25].

2. Notation

Consider the fourth-order symmetric differential equation

(2.1)
$$(p_2(x)y'')''-(p_1(x)y')'+q(x)y = \lambda w(x)y, x \in J' = (a',b'), -\infty \leq a' < b' \leq \infty, \lambda \in \mathrm{R},$$

where

(2.2) $$p_2, p_1, q, w : (a',b') \to \mathrm{R}, 1/p_2, p_1, q, w \in L_{loc}(a',b'), w > 0 \quad a.e. \quad on(a',b').$$

Let

(2.3) $$J = [a,b], \qquad a' < a < b < b'$$

and consider the BC

(2.4) $$A \begin{pmatrix} y(a) \\ y'(a) \\ (p_2y'')(a) \\ (p_2y'')'(a) - (p_1y')(a) \end{pmatrix} + B \begin{pmatrix} y(b) \\ y'(b) \\ (p_2y'')(b) \\ (p_2y'')'(a) - (p_1y')(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where the complex $4*4$ matrices A and B satisfy:

(2.5) $$rank(A \mid B) = 4$$

and

$$AEA^* = BEB^*, E = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.6}$$

A fourth-order SL boundary value problem (BVP) consists of Eq. (2.1) together with BC (2.4). With conditions (2.2), (2.3),(2.5) and (2.6) it is well known that this problem is a regular self-adjoint fourth-order SL problem. In this paper we fix all but one of the parameters that determine the SL problem, i.e., all but one of a; b; A; B; $p_1; 1/p_2$; q; w and study the dependence of the eigenvalues and eigenfunctions on that parameter.

From [26], we know that there are three basic types of self-adjoint boundary conditions (2.4),(2.5),(2.6): separated, coupled and mixed. In the separated case, there are many forms for the fourth-order problems. As other cases are similar , we will only study one form here, and we also study the coupled conditions:

(1) Separated self-adjoint BC .

$$\cos\alpha y(a) - \sin\alpha y'(a) = 0, \tag{2.7}$$

$$\cos\alpha(p_2y'')(a) - \sin\alpha[(p_2y'')' - p_1y'](a) = 0, \qquad 0 \le \alpha < \pi; \tag{2.8}$$

$$\cos\beta y(b) - \sin\beta y'(b) = 0, \tag{2.9}$$

$$\cos\beta(p_2y'')(b) - \sin\beta[(p_2y'')' - p_1y'](b) = 0, \qquad 0 < \beta \le \pi. \tag{2.10}$$

(2) real coupled self-adjoint BC.

$$\begin{pmatrix} y(b) \\ y'(b) \\ (p_2y'')(b) \\ (p_2y'')'(b) - (p_1y')(b) \end{pmatrix} = K \begin{pmatrix} y(a) \\ y'(a) \\ (p_2y'')(a) \\ (p_2y'')'(a) - (p_1y')(a) \end{pmatrix}, \tag{2.11}$$

where K satisfies

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix}, \quad k_{ij} \in R, \quad \det K = 1, \quad KEK^* = E. \tag{2.12}$$

(3) complex coupled self-adjoint BC.

$$\begin{pmatrix} y(b) \\ y'(b) \\ (p_2y'')(b) \\ (p_2y'')'(b) - (p_1y')(b) \end{pmatrix} = e^{i\theta} K \begin{pmatrix} y(a) \\ y'(a) \\ (p_2y'')(a) \\ (p_2y'')'(a) - (p_1y')(a) \end{pmatrix}, \tag{2.13}$$

where K satisfies (2.12), and $-\pi < \theta < 0$ or $0 < \theta < \pi$.

3. Continuity of eigenvalues and eigenfunctions

In this section we establish the characterization of the eigenvalues as zeros of an entire function and prove the continuity of the eigenvalues and eigenfunctions for the regular fourth-order SL problems. Let

$$\Omega = \{\omega = (a, b, A, B, p_1, 1/p_2, q, w)\} \tag{3.1}$$

such that (2.2), (2.3), (2.5), (2.6) hold. For the special case of separated BC (2.7)-(2.10) we also use the notation

$$\Omega_s = \{\omega = (a, b, \alpha, \beta, p_1, 1/p_2, q, w)\} \tag{3.2}$$

and for the coupled case (2.11)-(2.13) we let

$$\Omega_c = \{\omega = (a, b, \theta, K, p_1, 1/p_2, q, w)\} \tag{3.3}$$

when $\theta = 0$ we shorten (3.3) to

$$\Omega_{rc} = \{\omega = (a, b, K, p_1, 1/p_2, q, w)\} \tag{3.4}$$

We want to show that a small change of the problem results in only a small change of each eigenvalue and each eigenfunction. This means we have to compare the spectrum of different problems which may be defined on different intervals. Each $\omega \in \Omega$ determines a unique SL problem: a, b the interval, A, B the boundary condition, and the restrictions of $p_1, 1/p_2, q, w$ on $[a; b]$ the equation. Observe that the values of $p_1, 1/p_2, q, w$ outside the interval $[a, b]$, i.e. in $(a', b') \setminus [a, b]$, do not affect the spectrum of the problem determined by ω. To account for this and to facilitate comparisons between eigenvalues of problems defined on different intervals we let

$$\tilde{\Omega} = \{\tilde{\omega} = (a, b, A, B, \widetilde{p_1}, \widetilde{1/p_2}, \tilde{q}, \tilde{w})\} \tag{3.5}$$

where

$$\widetilde{p_1} = \begin{cases} p_1, x \in [a, b] \\ 0, x \in (a', b') \setminus [a, b] \end{cases}, \widetilde{1/p_2} = \begin{cases} 1/p_2, x \in [a, b] \\ 0, x \in (a', b') \setminus [a, b] \end{cases}$$

and $\tilde{q}, \tilde{w}$ are defined similarly. Now we introduce the Banach space

$$X = \mathbf{R} \times \mathbf{R} \times M_{4\times4}(\mathbf{C}) \times M_{4\times4}(\mathbf{C}) \times L'(a', b') \times L'(a', b') \times L'(a', b') \times L'(a', b') \tag{3.6}$$

with its "natural" norm

$$\|\omega\| = \|\tilde{\omega}\| = |a| + |b| + \|A\| + \|B\| + \int_{a'}^{b'} (|\widetilde{p_1}| + \left|\widetilde{1/p_2}\right| + |\tilde{q}| + |\tilde{w}|) \tag{3.7}$$

where $\|A\|$ is any fixed matrix norm. We maintain that this space X is the "natural" setting for the study of regular SL problems. Note that,since $p_1, 1/p_2, q, w$ are only assumed to be in $L_{loc}(a', b')$, ω is not a subset of X but $\tilde{\Omega}$ is since $\widetilde{p_1}, \widetilde{1/p_2}, \tilde{q}, \tilde{w}$ are in $L^1(a', b')$. Now we identify Ω with $\tilde{\Omega}$ as a subset of X. Then Ω inherits the norm from X, and the convergence in Ω is determined by this norm. It is easy to see that every point in Ω is an accumulation point of Ω with respect to the norm in X.

The isolated eigenvalues of a regular fourth-order SL problem depend continuously on the problem. More precisely we will give in Theorem 1. First we give a lemma.

Lemma 1. *A complex* λ_0 *is an eigenvalue of the problem* $(2.1),(2.4)-(2.6)$ *if and only if*

$$\Delta(\omega,\lambda_0)=\det[A+B\Phi(b,a,p_1,1/p_2,q,w,\lambda_0)]=0. \tag{3.8}$$

Proof. Where Φ is defined in Theorem 1, the proof is omitted since it is routine. ∎

Theorem 1. *Let* $\omega_0=(a_0,b_0,A_0,B_0,p_{10},1/p_{20},q_0,w_0)\in\Omega$. *Assume that* $\mu=\lambda(\omega_0)$ *is an isolated eigenvalue of the SL problem* $(2.1),(2.4)-(2.6)$ *determined by* ω_0 *i.e.,* μ *is an isolated eigenvalue of the SL problem*

$$(p_{20}(x)y'')''-(p_{10}(x)y')'+q_0(x)y=\lambda w_0(x)y,\ \ on\ \ [a_0,b_0],\ \ A_0Y_0(a_0)+B_0Y_0(b_0)=0.$$

Then, given any $\varepsilon>0$ *, there exists a* $\delta>0$ *such that for any* $\omega=(a,b,A,B,p_1,1/p_2,q,w)\in\Omega$ *satisfying the inequality*

$$\|\omega-\omega_0\|=|a-a_0|+|b-b_0|+\|A-A_0\|+\|B-B_0\|+$$
$$\int_{a'}^{b'}\left(|\widetilde{p_1}-\widetilde{p_{10}}|+\left|\widetilde{1/p_2}-\widetilde{1/p_{20}}\right|+|\tilde{q}-\tilde{q_0}|+|\tilde{w}-\widetilde{w_0}|\right)<\delta,$$

the SL problem

$$(p_2(x)y'')''-(p_1(x)y')'+q(x)y=\lambda w(x)y,\ \ on\ \ [a,b],\ \ AY(a)+BY(b)=0,$$

has an isolated eigenvalue $\lambda(\omega)$ *satisfying the inequality*

$$|\lambda(\omega)-\lambda(\omega_0)|<\varepsilon.$$

Proof. for $\omega\in\Omega$, and $\lambda\in R$, let $\Phi(b,a,p_1,1/p_2,q,w,\lambda)$ be the matrix solution of the initial value problem

$$Y'=\begin{pmatrix}0&1&0&0\\0&0&1/p_2&0\\0&p_1&0&1\\\lambda w-q&0&0&0\end{pmatrix}Y\qquad on\ \ [a,b] \tag{3.9}$$

with $\Phi(a)=I$ where I is the identity matrix, $Y=\begin{pmatrix}y\\y'\\p_2y''\\(p_2y'')'-(p_1y')\end{pmatrix}$.The characteristic function Δ of problem $(2.1),(2.4)-(2.6)$ is defined as follows

$$\Delta(\omega,\lambda)=\det[A+B\Phi(b,a,p_1,1/p_2,q,w,\lambda)],\quad for\ \ \omega\in\Omega,\lambda\in R \tag{3.10}$$

From Lemma 1, $\lambda(\omega)$ is an eigenvalue of the problem $(2.1),(2.4)-(2.6)$ if and only if $\Delta(\omega,\lambda(\omega))=0$. Furthermore, for any $\omega\in\Omega,\Delta(\omega,\lambda)$ is an entire function of λ and it is continuous in ω see Theorems 2.7, 2.8 of [27], and $\Delta(\omega_0,\mu)=0$ It is obvious that $\Delta(\omega_0,\lambda)$ is not constant in λ since μ is an isolated eigenvalue. Hence there exists $p>0$ such that $\Delta(\omega_0,\lambda)\neq 0$ for $\lambda\in S_p=\{\lambda\in C:|\lambda-\mu|=p\}$. By the well known theorem on continuity of the roots of an equation as a function of parameters, see [28] or [29], the statement of Theorem 1 follows. ∎

Remark 1. *Theorem* 1 *shows that for any fixed eigenvalue* μ *associated with* $\omega = \omega_0$ *there exists a continuous eigenvalue branch* $\lambda(\omega)$ *satisfying* $\lambda(\omega_0) = \mu$. *However, this does not mean that for a fixed* n, *the* n*th eigenvalue* $\lambda_n(\omega)$ *is always continuous in* ω, *see Remark* 3.7 *in* [4].

Below, each eigenvalue $\lambda(\omega)$ of the BVP (2.1), (2.4) − (2.6) as a function of ω for $\omega \in \Omega$, will always be assumed to be embedded in a continuous eigenvalue branch.

Next we state two lemmas needed in the later proofs which are also of independent interest. The first states that the unique solution of any initial value problem of Eq. (2.1) depends continuously on all parameters including the coefficients and the weight function in L^1 norm.

Lemma 2. *Let* (2.2) *hold, let* $c \in (a', b')$ *and* $d, k, f, g \in \mathbf{C}$. *Consider the initial value problem*

$$\begin{cases} (p_2(x)y'')'' - (p_1(x)y')' + q(x)y = \lambda w(x)y, \\ y(c) = d, y'(c) = k, (p_2y'')(c) = f, (p_2y'')'(c) - (p_1y')(c) = g. \end{cases} \tag{3.11}$$

Then the unique solution $y = y(\cdot, c, d, k, f, g, p_1, 1/p_2, q, w)$ *is a continuous function of all its variables. More precisely, given* $\varepsilon > 0$ *and any compact subinterval* J *of* (a', b') *there exists a* $\delta > 0$ *such that if*

$$\begin{aligned} &|c - c_0| + |d - d_0| + |k - k_0| + |f - f_0| + |g - g_0| + \\ &\int_a^b (|p_1 - p_{10}| + |1/p_2 - 1/p_{20}| + |q - q_0| + |w - w_0|) < \delta \end{aligned} \tag{3.12}$$

then

$$|y(x, c, d, k, f, g, p_1, 1/p_2, q, w) - y(x, c_0, d_0, k_0, f_0, g_0, p_{10}, 1/p_{20}, q_0, w_0)| < \varepsilon \tag{3.13}$$

$$|y'(x, c, d, k, f, g, p_1, 1/p_2, q, w) - y'(x, c_0, d_0, k_0, f_0, g_0, p_{10}, 1/p_{20}, q_0, w_0)| < \varepsilon \tag{3.14}$$

$$|p_2y''(x, c, d, k, f, g, p_1, 1/p_2, q, w) - p_2y''(x, c_0, d_0, k_0, f_0, g_0, p_{10}, 1/p_{20}, q_0, w_0)| < \varepsilon \tag{3.15}$$

$$\begin{aligned} &|[(p_2y'')' - (p_1y')](x, c, d, k, f, g, p_1, 1/p_2, q, w) - \\ &[(p_2y'')' - (p_1y')](x, c_0, d_0, k_0, f_0, g_0, p_{10}, 1/p_{20}, q_0, w_0)| < \varepsilon \end{aligned} \tag{3.16}$$

for all $x \in J$.

Proof. This follows from Lemma 3.1 in [4]. ∎

As a consequence of Theorem 1 and Lemma 2 we obtain.

Lemma 3. *Let* $\omega_0 = (a_0, b_0, A_0, B_0, p_{10}, 1/p_{20}, q_0, w_0) \in \Omega$. *Let* $\lambda = \lambda(\omega)$ *be an eigenvalue of SL problem* (2.1), (2.4) − (2.6). *If* $\lambda(\omega_0)$ *is simple, then there exists a neighborhood* M *of* ω_0 *in* Ω *such that* $\lambda(\omega)$ *is simple for every* ω *in* M.

Remark 2. *The conclusion of Lemma 3 holds if* ω_0 *is replaced by one of its components and* Ω *by the corresponding subspace of* Ω.

Proof. See [11]. ∎

Definition 1. *A normalized eigenfunction u of an SL problem we mean an eigenfunction u that satisfies*

$$\int_a^b |u|^2 w = 1. \tag{3.17}$$

Theorem 2. *Let the notation and hypotheses of Theorem 1 hold.*

(i) Assume the eigenvalue $\lambda(\omega_0)$ is simple for some $\omega_0 \in \Omega$ and let $u = u(\cdot, \omega_0)$ denote a normalized eigenfunction of $\lambda(\omega_0)$, Then there exist normalized eigenfunctions $u = u(\cdot, \omega)$ of $\lambda(\omega)$ for $\omega \in \Omega$ such that when $\omega \to \omega_0$ in Ω, we have

(3.18)
$$\begin{aligned} u(\cdot,\omega) &\to u(\cdot,\omega_0),\\ u'(\cdot,\omega) &\to u'(\cdot,\omega_0),\\ p_2u''(\cdot,\omega) &\to p_2u''(\cdot,\omega_0),\\ [(p_2u'')' - (p_1u')](\cdot,\omega) &\to [(p_2u'')' - (p_1u')](\cdot,\omega_0), \end{aligned}$$

these uniformly on any compact subinterval J of (a', b').

(ii) Assume that $\lambda(\omega)$ is a eigenvalue of multiplicity $l, (l = 2, 3, 4)$ for all ω in some neighborhood M of ω_0 in Ω . Then there exist l linearly independent normalized eigenfunctions $u_k(\cdot, \omega)$ of $\lambda(\omega)$ such that when $\omega \to \omega_0$. we have

(3.19)
$$\begin{aligned} u_k(\cdot,\omega) &\to u_k(\cdot,\omega_0),\\ {u_k}'(\cdot,\omega) &\to {u_k}'(\cdot,\omega_0),\\ p_2{u_k}''(\cdot,\omega) &\to p_2{u_k}''(\cdot,\omega_0),\\ [(p_2{u_k}'')' - (p_1{u_k}')](\cdot,\omega) &\to [(p_2{u_k}'')' - (p_1{u_k}')](\cdot,\omega_0), \end{aligned}$$

these uniformly on any compact subinterval J of (a', b'). Note that in this case, given l linearly independent normalized eigenfunctions u_k of $\lambda(\omega)$ there exist l linearly independent normalized eigenfunctions of $\lambda(\omega)$ one of which converges to u_1 and the other to u_2 and so on as $\omega \to \omega_0$ in Ω.

Proof. (i) First we show that there exist (not necessarily normalized) eigenfunctions $u(\cdot, \omega)$ such that (3.18) holds uniformly on J. For a solution y of (2.1) and an eigenfunction $u(\cdot, \omega)$ of a SL problem define

$$Y = \begin{pmatrix} y \\ y' \\ p_2y'' \\ (p_2y'')' - (p_1y') \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')' - (p_1u') \end{pmatrix}$$

to be the corresponding vector solution and vector eigenfunction respectively.

Assume the boundary conditions are separated. Suppose $\lambda(\omega_0)$ is simple. Choose eigenfunctions $u = u(\cdot, \omega)$ of $\omega \in \Omega$, ω near ω_0 , all satisfying the same initial condition at $c \in (a, b)$. Then the uniform convergence $U(\cdot, \omega) \to U(\cdot, \omega_0)$ on J follows from Theorem 1 and Lemma 2. Assume the boundary conditions are coupled with $-\pi < \theta \leq \pi$. Suppose $\lambda(\omega_0)$ is simple. Then by Lemma 3 there exists a neighborhood M of ω_0 such that $\lambda(\omega_0)$ is simple for all $\omega \in M$. For all $\omega \in M$

choose an eigenfunction $u = u(\cdot,\omega)$ of $\lambda(\omega)$ satisfying

$$\|U(a_0,\omega)\| = |u(a_0,\omega)| + |u'(a_0,\omega)| + |(p_2u'')(a_0,\omega)| + |[(p_2u'')' - (p_1u')](a_0,\omega)| = 1$$

$$and \quad u(x,\omega) > 0 \quad for \quad x \quad near \quad a_0.$$

It suffices to show that

$$U(a_0,\omega) \to U(a_0,\omega_0), \qquad as \quad \omega \to \omega_0 \quad in \quad \Omega. \tag{3.20}$$

since the uniform convergence on $[a_0, b_0]$ then follows from Lemma 2 and Theorem 1. If (3.20) does not hold, then there exists a sequence $\omega \to \omega_0$ such that

$$U(a_0,\omega_0) - U(a_0,\omega_k) := v_k \to v_0 \neq 0, \qquad as \quad \omega \to \omega_0, \tag{3.21}$$

Let Y_k, Z_k, Y be the vector solutions of (2.1) with the same $\omega = \omega_0, \lambda = \lambda(\omega_0)$ determined by the initial conditions

$$Y_k(a_0) = v_k, Z_k(a_0) = U(a_0,\omega_k), Y(a_0) = v_0, k \in N,$$

respectively. Then by the uniqueness of solutions to initial value problems we have

$$Y_k = U(\cdot,\omega_0) - Z_k, \qquad \text{on the interval } [a_0, b_0].$$

Using the BC (2.13) we get

$$\begin{aligned} Y_k(b_0) &= U(b_0,\omega_0) - Z_k(b_0) = U(b_0,\omega_0) - U(b_k,\omega_k) + U(b_k,\omega_k) - Z_k(b_0) \\ &= \exp(i\theta_0)K_0U(a_0,\omega_0) - \exp(i\theta_k)K_kU(a_k,\omega_k) + U(b_k,\omega_k) - Z_k(b_0) \\ &= \exp(i\theta_0)K_0[U(a_0,\omega_0) - U(a_0,\omega_k)] + \exp(i\theta_0)K_0U(a_0,\omega_k) \\ &\quad - \exp(i\theta_k)K_kU(a_k,\omega_k) + U(b_k,\omega_k) - Z_k(b_0) \\ &= \exp(i\theta_0)K_0Y_k(a_0) + \exp(i\theta_0)K_0U(a_0,\omega_k) - \exp(i\theta_k)K_kU(a_k,\omega_k) \\ &\quad + U(b_k,\omega_k) - Z_k(b_0) \end{aligned} \tag{3.22}$$

Letting $k \to \infty$ in (3.22) and using Lemma 2 we get

$$Y(b_0) = \exp(i\theta_0)K_0Y(a_0) \tag{3.23}$$

Since $Y(a_0) = v_0 \neq 0$, Y is a nontrivial vector eigenfunction corresponding to the eigenvalue $\lambda(\omega_0)$. Since $\lambda(\omega_0)$ is simple, there is a constant $h \neq 0$ such that $Y = hU(\cdot,\omega_0)$. In particular $v_0 = Y(a_0) = hU(a_0,\omega_0)$. Letting $k \to \infty$ in (3.21) we obtain that

$$U(a_0,\omega_0) - \lim_{k\to\infty} U(a_0,\omega_k) = v_0 = hU(a_0,\omega_0) \tag{3.24}$$

i.e.

$$\lim_{k\to\infty} U(a_0,\omega_k) = (1-h)U(a_0,\omega_0) \tag{3.25}$$

Noting that $u(x,\omega_k)$ and $u(x,\omega_0)$ have the same sign for x near a_0 , we have that $1 - h > 0$ and hence

$$\lim_{k\to\infty} \|U(a_0,\omega_k)\| = (1-h)\,\|U(a_0,\omega_0)\| \tag{3.26}$$

which contradicts

$$\|U(a_0,\omega_k)\| = \|U(a_0,\omega_0)\| = 1. \tag{3.27}$$

(ii) Suppose $\lambda(\omega)$ is a eigenvalue of multiplicity $l, (l = 2, 3, 4)$ for all ω in some neighborhood M of ω_0. Then we can argue as before by choosing eigenfunctions of $\lambda(\omega)$ all of which satisfy the same initial condition at c for some $c \in (a, b)$ since a linear combination of l linearly independent eigenfunctions can be chosen to satisfy arbitrary initial conditions.

The above discussion shows that for every self-adjoint boundary problem and every eigenvalue $\lambda(\omega)$, the eigenfunction $u(\cdot,\omega)$ and its derivative $u'(\cdot,\omega)$ and quasi-derivative $(p_2u'')(\cdot,\omega)$, $[(p_2u'')'-(p_1u')](\cdot,\omega)$ are uniformly convergent in ω on every compact subinterval of (a',b') . By normalizing the eigenfunctions we complete the proof. ∎

4. Differentiability properties of eigenvalues

In this section we show that the isolated eigenvalues depend continuously on all the data, here we show that this dependence is in fact differentiable. Recall the definition of the Frechet derivative:

Definition 2. *A map T from a Banach space X into a Banach space Y is Frechet differentiable at a point $x\epsilon X$ if there exists a bounded linear operator $dT_x : X \longrightarrow Y$ such that for $h\in X$*

$$|T(x+h)-T(x)-dT_x(h)| = o(h)(h\to 0).$$

Lemma 4. *Assume u and v are solutions of* (2.1) *with $\lambda=\mu$ and $\lambda=\nu$ respectively. Then*

$$\begin{aligned}
&[u(p_2\bar{v}'')'-\bar{v}(p_2u'')'-u(p_1\bar{v}')+\bar{v}(p_1u')-u'(p_2\bar{v}'')+\bar{v}'(p_2u'')]_a^b\\
&:=[u(p_2\bar{v}'')'-\bar{v}(p_2u'')'-u(p_1\bar{v}')+\bar{v}(p_1u')-u'(p_2\bar{v}'')+\bar{v}'(p_2u'')](b)\\
&-[u(p_2\bar{v}'')'-\bar{v}(p_2u'')'-u(p_1\bar{v}')+\bar{v}(p_1u')-u'(p_2\bar{v}'')+\bar{v}'(p_2u'')](a)\\
&=(\nu-\mu)\int_a^b u\bar{v}w.
\end{aligned}\tag{4.1}$$

Lemma 5. *Assume a real valued function $f\epsilon L_{loc}(a',b')$. Then*

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f = f(x) \quad a.e.(A,B)\tag{4.2}$$

Proof. See Lemma 3.2 of [3]. ∎

Theorem 3. *(Eigenvalue-eigenfunction differential equation for special case of separated BVPs). Let* (2.2) *hold. Consider the BVP* (2.1), (2.7)−(2.10) *with $0\le\alpha<\pi$ and $\beta=\pi$. Fix all the components of ω except b and let $\lambda=\lambda(b)$ and $u=u(\cdot,b)$. Then λ is differentiable a.e. and*

$$\lambda'(b) = 2u'(b,b)(p_2u'')'(b,b)-u'(b,b)(p_1u')(b,b), \quad a.e \ \ in \ \ [a,b').\tag{4.3}$$

Proof. For small h, in (4.1) choose $\mu=\lambda(b)$,$\nu=\lambda(b+h)$, and $u=u(\cdot,b)$,$v=u(\cdot,b+h)$. Noting that

$$[u(p_2\bar{v}'')'-\bar{v}(p_2u'')'-u(p_1\bar{v}')+\bar{v}(p_1u')-u'(p_2\bar{v}'')+\bar{v}'(p_2u'')](a)=0$$

$$u(b,b)=0, \quad (p_2u'')(b,b)=0,$$

we have

$$\begin{aligned}
&[\lambda(b+h)-\lambda(b)]\int_a^b u(s,b)u(s,b+h)w(s)ds\\
&=-u(b,b+h)(p_2u'')'(b,b)-u'(b,b)(p_2u'')(b,b+h)+u(b,b+h)(p_1u')(b,b)
\end{aligned}\tag{4.4}$$

and

$$
\begin{aligned}
(p_2u'')(b,b+h) &= (p_2u'')(b,b+h) - (p_2u'')(b+h,b+h) \\
&= -\int_b^{b+h} (p_2u'')'(s,b+h)ds \\
&= -\int_b^{b+h} (p_2u'')'(s,b)ds + \int_b^{b+h} [(p_2u'')'(s,b) - (p_2u'')'(s,b+h)]ds
\end{aligned}
$$

$(p_2u'')'(s,b) - (p_2u'')'(s,b+h) \to 0$ as $h \to 0$. By lemma 5,we have

$$
\lim_{h\to 0} \frac{(p_2u'')(b,b+h)}{h} = -(p_2u'')'(b,b)
$$

$$
\lim_{h\to 0} \frac{u(b,b+h)}{h} = -u'(b,b)
$$

Dividing (4.4) by h and taking the limit as $h \to 0$, we get (4.3). ∎

Theorem 4. *(Eigenvalue-eigenfunction differential equation for special case of separated BVPs). Let* (2.2) *hold.Consider the BVP* (2.1), (2.7) − (2.10) *with* $0 \le \alpha < \pi$ *and* $\beta = \frac{\pi}{2}$*. Fix all the components of* ω *except* b *and let* $\lambda = \lambda(b)$ *and* $u = u(\cdot, b)$*. Then* λ *is differentiable a.e. and*

$$
\lambda'(b) = -u^2(b,b)[\lambda(b)\omega(b) - q(b)] - u''(b,b)(p_2u'')(b,b) \text{ a.e in } [a,b'). \tag{4.5}
$$

Proof. Since the proof is similar to that of Theorem 3. For small h, in (4.1) choose $\mu = \lambda(b)$, $\nu = \lambda(b+h)$ and $u = u(\cdot, b)$, $v = u(\cdot, b+h)$. And from the boundary conditions, noting that

$$
[u(p_2\bar{v}'')' - \bar{v}(p_2u'')' - u(p_1\bar{v}') + \bar{v}(p_1u') - u'(p_2\bar{v}'') + \bar{v}'(p_2u'')](a) = 0
$$

$$
u'(b,b) = 0, \quad (p_2u'')'(b,b) = 0
$$

we have

$$
\begin{aligned}
&[\lambda(b+h) - \lambda(b)] \int_a^b u(s,b)u(s,b+h)w(s)ds \\
&= u(b,b)(p_2u'')'(b,b+h) - u(b,b)(p_1u')(b,b+h) + u'(b,b+h)(p_2u'')(b,b)
\end{aligned} \tag{4.6}
$$

and

$$
\begin{aligned}
&(p_2u'')'(b,b+h) = -\int_b^{b+h} (p_2u'')''(s,b+h)ds \\
&= -\int_b^{b+h} [(p_1u')' - qu + \lambda wu](s,b+h)ds \\
&= -\int_b^{b+h} (p_1u')'(s,b+h)ds + \int_b^{b+h} q(s)u(s,b)ds + q(s)\int_b^{b+h} [u(s,b+h) - u(s,b)]ds \\
&- \lambda(b+h)\int_b^{b+h} u(s,b)w(s)ds - \lambda(b+h)\int_b^{b+h} [u(s,b+h) - u(s,b)]w(s)ds
\end{aligned}
$$

as $h \to 0$,$u(s,b+h) - u(s,b) \to 0$. By Lemma 5 and the continuity of λ at b, we have

$$
\lim_{h\to 0} \frac{(p_2u'')'(b,b+h)}{h} = -(p_2u'')''(b,b) = -(p_1u')'(b,b) - [\lambda(b)w(b) - q(b)]u(b,b)
$$

Similarly

$$
\lim_{h\to 0} \frac{(p_1u')(b,b+h)}{h} = -(p_1u')'(b,b)
$$

and

$$\lim_{h\to 0}\frac{u'(b,b+h)}{h} = -u''(b,b)$$

Dividing (4.6) by h and taking the limit as $h \to 0$, we get (4.5). ∎

Theorem 5. *(Eigenvalue-eigenfunction differential equation for separated BVPs). Let* (2.2) *hold. Consider the BVP* (2.1), (2.7)−(2.10) *with* $0 \le \alpha < \pi$ *and* $0 < \beta \le \pi$.

(1) *Fix all the components of* ω *except* a *and let* $\lambda = \lambda(a)$,$u = u(\cdot, a)$.*Then* λ *is differentiable a.e. and*

(4.7)

$$\lambda'(a) = u^2(a,a)[\lambda(a)w(a)-q(a)]-2u'(a,a)(p_2u'')'(a,a)+u'(a,a)(p_1u')(a,a)+\frac{1}{p_2(a)}(p_2u'')^2(a,a)$$

a.e in $(a', b]$.

(2) *Fix all the components of* ω *except* b *and let* $\lambda = \lambda(b)$,$u = u(\cdot, b)$.*Then* λ *is differentiable a.e. and*

(4.8)

$$\lambda'(b) = 2u'(b,b)(p_2u'')'(b,b)-u^2(b,b)[\lambda(b)w(b)-q(b)]-u'(b,b)(p_1u')(b,b)-\frac{1}{p_2(b)}(p_2u'')^2(b,b)$$

a.e in $[a, b')$.

Proof. Since the proof of (4.7), (4.8) are similar we just prove (4.8). The proof is similar to that of Theorem 3 and 4. For small h, in (4.1) choose $\mu = \lambda(b)$, $\nu = \lambda(b + h)$ and $u = u(\cdot, b)$, $v = u(\cdot, b + h)$. And from the boundary conditions, noting that

$$[u(p_2\bar{v}'')' - \bar{v}(p_2u'')' - u(p_1\bar{v}') + \bar{v}(p_1u') - u'(p_2\bar{v}'') + \bar{v}'(p_2u'')](a) = 0$$

we have

$$\begin{aligned}&[\lambda(b+h)-\lambda(b)]\int_a^b u(s,b)u(s,b+h)w(s)ds\\ &= u(b,b)(p_2u'')'(b,b+h) - u(b,b+h)(p_2u'')'(b,b) - u(b,b)(p_1u')(b,b+h)\\ &+ u(b,b+h)(p_1u')(b,b) - u'(b,b)(p_2u'')(b,b+h) + u'(b,b+h)(p_2u'')(b,b)\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}(p_2u'')'(b,b+h) &= -\int_b^{b+h}(p_2u'')''(s,b+h)ds\\ &= -\int_b^{b+h}[(p_1u')' - qu + \lambda wu](s,b+h)ds\end{aligned}$$

$$= -\int_b^{b+h}(p_1u')'(s,b+h)ds + \int_b^{b+h}q(s)u(s,b)ds + \int_b^{b+h}q(s)[u(s,b+h)-u(s,b)]ds$$

$$- \lambda(b+h)\int_b^{b+h}u(s,b)w(s)ds - \lambda(b+h)\int_b^{b+h}[u(s,b+h)-u(s,b)]w(s)ds$$

as $h \to 0$,$u(s, b + h) - u(s, b) \to 0$. by Lemma 5, we have

$$\lim_{h\to 0}\frac{(p_2u'')'(b,b+h)}{h} = -(p_2u'')''(b,b) = -(p_1u')'(b,b) - [\lambda(b)w(b)-q(b)]u(b,b)$$

$$\lim_{h\to 0}\frac{(p_1u')(b,b+h)}{h} = -(p_1u')'(b,b)$$

$$\lim_{h\to 0}\frac{(p_2u'')(b,b+h)}{h}=-(p_2u'')'(b,b)$$

$$\lim_{h\to 0}\frac{u'(b,b+h)}{h}=-u''(b,b)$$

$$\lim_{h\to 0}\frac{u(b,b+h)}{h}=-u'(b,b) \tag{4.10}$$

Dividing (4.9) by h and taking the limit as $h\to 0$, we get (4.8). ∎

Theorem 6. *(Eigenvalue-eigenfunction differential equation for coupled BVPs). Let* (2.2) *hold. Consider the coupled BVP* (2.1) *with* (2.13), (2.12) *where* $-\pi<\theta\leq\pi$.

(1) *Fix all the components of* ω *except* a *and let* $\lambda=\lambda(a)$ *and* $u=u(\cdot,a)$. *Then* λ *is differentiable a.e. and*

(4.11)

$$\lambda'(a)=-2\text{Re}[u'(a)(p_2\bar{u}'')'(a)]+p_1(a)\left|(u')(a)\right|^2+\frac{1}{p_2(a)}\left|(p_2u'')(a)\right|^2-\left|u(a)\right|^2[q(a)-\lambda(a)w(a)]$$

(2) *Fix all the components of* ω *except* b *and let* $\lambda=\lambda(b)$ *and* $u=u(\cdot,b)$. *Then* λ *is differentiable a.e. and*

(4.12)

$$\lambda'(b)=2\text{Re}[u'(b)(p_2\bar{u}'')'(b)]-p_1(b)\left|(u')(b)\right|^2-\frac{1}{p_2(b)}\left|(p_2u'')(b)\right|^2+\left|u(b)\right|^2[q(b)-\lambda(b)w(b)]$$

Proof. Since the proof of (4.11) and (4.12) are similar we just prove (4.12).The proof is similar to that of Theorem 3,we have

$$\begin{aligned}
&[\lambda(b+h)-\lambda(b)]\int_a^b u\bar{v}w\\
&=[u(p_2\bar{v}'')'-\bar{v}(p_2u'')'-u(p_1\bar{v}')+\bar{v}(p_1u')-u'(p_2\bar{v}'')+\bar{v}'(p_2u'')]_a^b\\
&=((p_2\bar{v}'')'-(p_1\bar{v}'),-(p_2\bar{v}''),\bar{v}',-\bar{v})(b)\begin{pmatrix}u\\u'\\p_2u''\\(p_2u'')'-p_1u'\end{pmatrix}(b)\\
&-((p_2\bar{v}'')'-(p_1\bar{v}'),-(p_2\bar{v}''),\bar{v}',-\bar{v})(a)\begin{pmatrix}u\\u'\\p_2u''\\(p_2u'')'-p_1u'\end{pmatrix}(a)
\end{aligned}$$

$$
\begin{aligned}
&= ((p_2\bar{v}'')' - (p_1\bar{v}'), -(p_2\bar{v}''), \bar{v}', -\bar{v})(b) \begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')' - p_1u' \end{pmatrix}(b) \\
&- ((p_2\bar{v}'')' - (p_1\bar{v}'), -(p_2\bar{v}''), \bar{v}', -\bar{v})(b+h) \begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')' - p_1u' \end{pmatrix}(b) \\
&= -[((p_2\bar{v}'')' - (p_1\bar{v}'), -(p_2\bar{v}''), \bar{v}', -\bar{v})(b+h) \\
&- ((p_2\bar{v}'')' - (p_1\bar{v}'), -(p_2\bar{v}''), \bar{v}', -\bar{v})(b)] \begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')' - p_1u' \end{pmatrix}(b)
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
&\lim_{h\to 0} \frac{[(p_2\bar{v}'')' - (p_1\bar{v}')](b+h) - [(p_2\bar{v}'')' - (p_1\bar{v}')](b)}{h} \\
&= [(p_2\bar{u}'')' - (p_1\bar{u}')]'(b) \\
&= \bar{u}(b)[\lambda(b)\omega(b) - q(b)]
\end{aligned}
$$

$$
\begin{aligned}
\lim_{h\to 0} \frac{-(p_2\bar{v}'')(b+h) + (p_2\bar{v}'')(b)}{h} &= -(p_2\bar{u}'')'(b) \\
\lim_{h\to 0} \frac{\bar{v}'(b+h) - \bar{v}'(b)}{h} &= \bar{u}''(b) \\
\lim_{h\to 0} \frac{\bar{v}(b) - \bar{v}(b+h)}{h} &= -\bar{u}'(b),
\end{aligned} \tag{4.14}
$$

Then we get (4.12). ∎

Theorem 7. *Let* $\omega = (a, b, A, B, p_1, 1/p_2, q, w) \in \Omega$, *Let* $\lambda = \lambda(b)$ *and let* $u = u(\cdot, b)$ *be a normalized eigenfunction of* λ *for the BVP* (2.1), (2.4) − (2.6). *Assume that either (i)*$\lambda(\omega)$ *is a simple eigenvalue or (ii) that* $\lambda(p_1), \lambda(1/p_2)$ *is a eigenvalue of multiplicity* l *for each* p_1 *and* $1/p_2$ *in some neighborhood* $M \subset \Omega$ *of* ω. *Then* λ *is continuously differentiable with respect to each variable* α, β *for the separated BC* (2.7) − (2.10); *continuously differentiable with respect to each variable* θ, k *for the coupled BC* (2.11) − (2.13); *continuously differentiable with respect to each variable* $p_1, 1/p_2, q, w$ *for the general BC* (2.4) − (2.6) *in the appropriate sense. Then derivatives are given by:*

1.Fix all components of ω *except* α *and* $\lambda = \lambda(\alpha)$ *and* $u = u(\cdot, \alpha)$. *Then* λ *is differentiable and*

$$\lambda'(\alpha) = 2\left\{u(a)(p_2u'')(a) + u'(a)[(p_2u'')'(a) - (p_1u')(a)]\right\}. \tag{4.15}$$

2.Fix all components of ω *except* β *and* $\lambda = \lambda(\beta)$ *and* $u = u(\cdot, \beta)$. *Then* λ *is differentiable and*

$$\lambda'(\beta) = -2\left\{u(b)(p_2u'')(b) + u'(b)[(p_2u'')'(b) - (p_1u')(b)]\right\}. \tag{4.16}$$

3.Fix all components of ω *except* θ *and* $\lambda = \lambda(\theta)$ *and* $u = u(\cdot, \theta)$. *Then* λ *is differentiable at* θ *for any* θ *satisfying* $-\pi < \theta < 0$ *or* $0 < \theta < \pi$ *and*

$$\lambda'(\theta) = 2\mathrm{Im}[u(b)(p_2\bar{u}'')'(b) + \bar{u}(b)(p_1u')(b) + \bar{u}'(b)(p_2\bar{u}'')(b)]. \tag{4.17}$$

4.Fix all components of ω except λ and let $\lambda = \lambda(K)$ and $u = u(\cdot, K)$. Assume K satisfies (2.12). Then λ is differentiable and its Frechet derivative is given by:

(4.18)

$$d\lambda_K(H) = -((p_2\overline{u}'')' - (p_1\overline{u}'), -p_2\overline{u}'', \overline{u}', -\overline{u})(b)HK^{-1}\begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')' - p_1u' \end{pmatrix}(b).$$

5.Fix all components of ω except $1/p_2$ and consider λ as a function of $1/p_2 \in L^1(a,b)$. Then λ is Frechet differentiable and its Frechet derivative is given by:

$$d\lambda_{1/p_2}(h) = \int_a^b |p_2u''|^2 h, h \in L^1(a,b). \tag{4.19}$$

6.Fix all components of ω except p_1 and consider λ as a function of $p_1 \in L^1(a,b)$. Then λ is Frechet differentiable and its Frechet derivative is given by:

$$d\lambda_{p_1}(h) = -\int_a^b |u'|^2 h, h \in L^1(a,b). \tag{4.20}$$

7.Fix all components of ω except q and consider λ as a function of $q\epsilon L^1(a,b)$. Then λ is Frechet differentiable and its Frechet derivative is given by:

$$d\lambda_q(h) = \int_a^b |u|^2 h, h \in L^1(a,b). \tag{4.21}$$

8.Fix all components of ω except w and consider λ as a function of $w\epsilon L^1(a,b)$. Then λ is Frechet differentiable and its Frechet derivative is given by:

$$d\lambda_w(h) = -\lambda\int_a^b |u|^2 h, h \in L^1(a,b). \tag{4.22}$$

Proof. Since the proofs of (4.15),(4.16) are similar we just prove (4.16). From the BVP we have

$$[u(p_2v'')' - v(p_2u'')' - u(p_1v') + v(p_1u') - u'(p_2v'') + v'(p_2u'')](a) = 0$$

Hence

$$\begin{aligned}
&[\lambda(\beta+h) - \lambda(\beta)]\int_a^b uvw \\
&= [u(p_2v'')' - v(p_2u'')' - u(p_1v') + v(p_1u') - u'(p_2v'') + v'(p_2u'')]_a^b \\
&= [u(p_2v'')' - v(p_2u'')' - u(p_1v') + v(p_1u') - u'(p_2v'') + v'(p_2u'')](b) \\
&= \tan\beta u'(b)[(p_2v'')'(b) - (p_1v')(b)] - \tan(\beta+h)u'(b)[(p_2v'')'(b) - (p_1v')(b)] \\
&- \tan(\beta+h)v'(b)[(p_2u'')'(b) - (p_1u')(b)] + \tan\beta v'(b)[(p_2u'')'(b) - (p_1u')(b)] \\
&= -[\tan(\beta+h) - \tan\beta]u'(b)[(p_2v'')'(b) - (p_1v')(b)] - [\tan(\beta+h) - \tan\beta]v'(b)[(p_2u'')'(b) - (p_1u')(b)]
\end{aligned}$$

$$\begin{aligned}
\lambda'(\beta) &= -\sec^2\beta u'(b)[(p_2u'')'(b) - (p_1u')(b)] - \sec^2\beta u'(b)[(p_2u'')'(b) - (p_1u')(b)]\} \\
&= -\tan^2\beta u'(b)[(p_2u'')'(b) - (p_1u')(b)] - u'(b)[(p_2u'')'(b) - (p_1u')(b)] \\
&- \tan^2\beta u'(b)[(p_2u'')'(b) - (p_1u')(b)] - u'(b)[(p_2u'')'(b) - (p_1u')(b)] \\
&= -2\left\{u(b)(p_2u'')(b) + u'(b)[(p_2u'')'(b) - (p_1u')(b)]\right\},
\end{aligned}$$

To proof (4.17),

$$
\begin{aligned}
&[\lambda(\theta+h)-\lambda(\theta)]\int_a^b u\bar{v}w \\
&=[u(p_2\overline{v}'')'-\overline{v}(p_2u'')'-u(p_1\overline{v}')+\overline{v}(p_1u')-u'(p_2\overline{v}'')+\overline{v}'(p_2u'')]_a^b \\
&=((p_2\overline{v}'')'-(p_1\overline{v}'),-p_2\overline{v}'',\overline{v}',-\overline{v})(b)\begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')'-p_1u' \end{pmatrix}(b) \\
&-((p_2\overline{v}'')'-(p_1\overline{v}'),-p_2\overline{v}'',\overline{v}',-\overline{v})(a)\begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')'-p_1u' \end{pmatrix}(a) \\
&=e^{i\theta}((p_2\overline{v}'')'-(p_1\overline{v}'),-p_2\overline{v}'',\overline{v}',-\overline{v})(b)K\begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')'-p_1u' \end{pmatrix}(a) \\
&-e^{i(\theta+h)}((p_2\overline{v}'')'-(p_1\overline{v}'),-p_2\overline{v}'',\overline{v}',-\overline{v})(b)K\begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')'-p_1u' \end{pmatrix}(a) \\
&=-e^{i\theta}((p_2\overline{v}'')'-(p_1\overline{v}'),-p_2\overline{v}'',\overline{v}',-\overline{v})(b)K\begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')'-p_1u' \end{pmatrix}(a)(e^{ih}-1)
\end{aligned}
$$

Dividing both sides of above equation by h and taking the limit as $h\to 0$ we obtain

$$
\begin{aligned}
\lambda'(\theta)&=-ie^{i\theta}((p_2\overline{u}'')'-(p_1\overline{u}'),-p_2\overline{u}'',\overline{u}',-\overline{u})(b)K\begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')'-p_1u' \end{pmatrix}(a) \\
&=-i((p_2\overline{u}'')'-(p_1\overline{u}'),-p_2\overline{u}'',\overline{u}',-\overline{u})(b)\begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')'-p_1u' \end{pmatrix}(b) \\
&=-i[u(p_2\overline{u}'')'-\overline{u}(p_2u'')'-u(p_1\overline{u}')+\overline{u}(p_1u')-u'(p_2\overline{u}'')+\overline{u}'(p_2u'')](b) \\
&=2\mathrm{Im}[u(p_2\overline{u}'')'+\overline{u}(p_1u')+\overline{u}'(p_2u'')](b)
\end{aligned}
$$

To proof (4.18), let $u = u(\cdot, K)$, $v = u(\cdot, K + H)$,

$$[\lambda(K+H) - \lambda(K)]\int_a^b u\overline{v}w$$

$$= [u(p_2\overline{v}'')' - \overline{v}(p_2u'')' - u(p_1\overline{v}') + \overline{v}(p_1u') - u'(p_2\overline{v}'') + \overline{v}'(p_2u'')]_a^b$$

$$= e^{i\theta}((p_2\overline{v}'')' - (p_1\overline{v}'), -p_2\overline{v}'', \overline{v}', -\overline{v})(b)K\begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')' - p_1u' \end{pmatrix}(a)$$

$$- e^{i\theta}((p_2\overline{v}'')' - (p_1\overline{v}'), -p_2\overline{v}'', \overline{v}', -\overline{v})(b)(K+H)\begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')' - p_1u' \end{pmatrix}(a)$$

$$= -e^{i\theta}((p_2\overline{v}'')' - (p_1\overline{v}'), -p_2\overline{v}'', \overline{v}', -\overline{v})(b)H\begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')' - p_1u' \end{pmatrix}(a)$$

$$= -((p_2\overline{u}'')' - (p_1\overline{u}'), -(p_2\overline{u}''), \overline{u}', -\overline{u})(b)HK^{-1}\begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')' - p_1u' \end{pmatrix}(b) + o(H)$$

Hence

$$\lambda(K+H) - \lambda(K)$$

$$= -((p_2\overline{u}'')' - (p_1\overline{u}'), -(p_2\overline{u}''), \overline{u}', -\overline{u})(b)HK^{-1}\begin{pmatrix} u \\ u' \\ p_2u'' \\ (p_2u'')' - p_1u' \end{pmatrix}(b) + o(H).$$

Defined by Frechet differential, we obtain (4.18).
To proof (4.19)

$$[\lambda(1/p_{2h}) - \lambda(1/p_2)]\int_a^b u\overline{v}w$$

$$= \int_a^b \lambda(1/p_{2h})u\overline{v}w - \int_a^b \lambda(1/p_2)u\overline{v}w$$

$$= \int_a^b [(p_{2h}\overline{v}'')'' - (p_1\overline{v}')' + q\overline{v}]u - \int_a^b [(p_2u'')'' - (p_1u')' + qu]\overline{v}$$

$$= \int_a^b ud(p_{2h}\overline{v}'')' - \int_a^b ud(p_1\overline{v}') + \int_a^b q\overline{v}u - \int_a^b \overline{v}d(p_2u'')' + \int_a^b \overline{v}d(p_1u') - \int_a^b qu\overline{v}$$

$$= [u(p_{2h}\overline{v}'')' - \overline{v}(p_2u'')' - u(p_1\overline{v}') + \overline{v}(p_1u')]_a^b - \int_a^b u'd(p_{2h}\overline{v}'') + \int_a^b \overline{v}'d(p_2u'')$$

$$= [u(p_{2h}\overline{v}'')' - \overline{v}(p_2u'')' - u(p_1\overline{v}') + \overline{v}(p_1u') - u'(p_{2h}\overline{v}'') + \overline{v}'(p_2u'')]_a^b + \int_a^b (p_{2h} - p_2)\overline{v}''u'',$$

where

$$[u(p_{2h}\overline{v}'')' - \overline{v}(p_2u'')' - u(p_1\overline{v}') + \overline{v}(p_1u') - u'(p_{2h}\overline{v}'') + \overline{v}'(p_2u'')]_a^b = 0.$$

Then

$$\begin{aligned}[\lambda(1/p_{2h}) - \lambda(1/p_2)](1+o(1)) &= \int_a^b (p_{2h} - p_2)u''\overline{v}'' + o(h) \\ &= \int_a^b p_2 p_{2h} h u''\overline{v}'' + o(h) \\ &= \int_a^b (p_2 u'')(p_{2h}\overline{v}'')h + o(h) \\ &= \int_a^b |p_2 u''|^2 h + o(h).\end{aligned}$$

$$\lambda(1/p_{2h}) - \lambda(1/p_2) = [\int_a^b |p_2 u''|^2 h + o(h)](1+o(1))^{-1} = \int_a^b |p_2 u''|^2 h + o(h)$$

Hence

$$d\lambda_{1/p_2}(h) = \int_a^b |p_2 u''|^2 h, h \in L^1(a,b)$$

To proof (4.20)

$$\begin{aligned}&[\lambda(p_1+h) - \lambda(p_1)]\int_a^b u\overline{v}w \\ &= \int_a^b \lambda(p_1+h)u\overline{v}w - \int_a^b \lambda(p_1)u\overline{v}w \\ &= \int_a^b [(p_2\overline{v}'')'' - ((p_1+h)\overline{v}')' + q\overline{v}]u - \int_a^b [(p_2 u'')'' - (p_1 u')' + qu]\overline{v} \\ &= \int_a^b [(p_2\overline{v}'')'' - (p_1\overline{v}')' + q\overline{v}]u - \int_a^b [(p_2 u'')'' - (p_1 u')' + qu]\overline{v} - \int_a^b (h\overline{v}')'u \\ &= [u(p_2\overline{v}'')' - \overline{v}(p_2 u'')' - u(p_1\overline{v}') + \overline{v}(p_1 u') - u'(p_2\overline{v}'') + \overline{v}'(p_2 u'')]_a^b - \int_a^b (h\overline{v}')'u\end{aligned}$$

where

$$[u(p_2\overline{v}'')' - \overline{v}(p_2 u'')' - u(p_1\overline{v}') + \overline{v}(p_1 u') - u'(p_2\overline{v}'') + \overline{v}'(p_2 u'')]_a^b = 0.$$

Then

$$[\lambda(p_1+h) - \lambda(p_1)](1+o(1)) = -\int_a^b |u'|^2 h + o(h),$$

$$\lambda(p_1+h) - \lambda(p_1) = [-\int_a^b |u'|^2 h + o(h)](1+o(1))^{-1} = -\int_a^b |u'|^2 h + o(h),$$

Hence

$$d\lambda_{p_1}(h) = -\int_a^b |u'|^2 h, h \in L^1(a,b),$$

To proof (4.21)

$$
\begin{aligned}
&[\lambda(q+h)-\lambda(q)]\int_a^b u\overline{v}w \\
&= \int_a^b \lambda(q+h)u\overline{v}w - \int_a^b \lambda(q)u\overline{v}w \\
&= \int_a^b [(p_2\overline{v}'')'' - (p_1\overline{v}')' + (q+h)\overline{v}]u - \int_a^b [(p_2u'')'' - (p_1u')' + qu]\overline{v} \\
&= \int_a^b [(p_2\overline{v}'')'' - (p_1\overline{v}')' + q\overline{v}]u - \int_a^b [(p_2u'')'' - (p_1u')' + qu]\overline{v} + \int_a^b h\overline{v}u \\
&= [u(p_2\overline{v}'')' - \overline{v}(p_2u'')' - u(p_1\overline{v}') + \overline{v}(p_1u') - u'(p_2\overline{v}'') + \overline{v}'(p_2u'')]_a^b + \int_a^b h\overline{v}u
\end{aligned}
$$

where

$$[u(p_2\overline{v}'')' - \overline{v}(p_2u'')' - u(p_1\overline{v}') + \overline{v}(p_1u') - u'(p_2\overline{v}'') + \overline{v}'(p_2u'')]_a^b = 0$$

Then

$$[\lambda(q+h)-\lambda(q)](1+o(1)) = \int_a^b |u|^2 h + o(h)$$

$$\lambda(q+h)-\lambda(q) = [\int_a^b |u|^2 h + o(h)](1+o(1))^{-1} = \int_a^b |u|^2 h + o(h)$$

Hence

$$d\lambda_q(h) = \int_a^b |u|^2 h + o(h), h \in L^1(a,b)$$

The proof of (4.22) is similar to that of (4.21) and hence omitted. ∎

Acknowledgements

The work of the first author is supported by the Inner Mongolia Autonomous Region Natural Science Foundation (No.2017MS0119), and Specialized Research Fund for the Doctoral Program of Higher Education (No.20131501120006).

References

[1] J. Pöschel, E. Trubowitz. Inverse Spectral Theory, Academic Press, New York, London, Sydney, 1987.

[2] M. Dauge, B. Helffer. Eigenvalues variation. I. Neumann problem for Sturm-Liouville operators. J. Differ. Equa, 1993, 104: 243-262.

[3] Q. Kong, A. Zettl. Dependence of eigenvalues of Sturm-Liouville problems on the boundary. J. Differ. Equat, 1996, 126: 389-407.

[4] Q. Kong, A. Zettl. Eigenvalues of regular Sturm-Liouville problems. J. Differ. Equa, 1996, 131: 1-19.

[5] A. Zettl. Sturm-Liouville Theory, Mathematical Surveys and Monographs, American Mathematical Society, vol. 121, 2005.

[6] L.E. Battle. Solution dependence on problem parameters for initial value problems associated with the Stieltjes Sturm-Liouville equations, Electron. J. Differ. Equa, 2005, 2: 1-18.

[7] Q. Kong, H. Wu, A. Zettl. Dependence of eigenvalues on the problems. Math. Nachr, 1997, 188: 173-201.

[8] P.B. Bailey , W.N. Everitt , A. Zettl. The SLEIGN2 Sturm-Liouville code. ACM Transactions on Mathematical Software 2001, 21:143-192.

[9] J. Suo, W. Wang. Eigenvalues of a class of regular fourth-order Sturm-Liouville problems. Applied Mathematics and Computation, 2012, 218: 9716-9729.

[10] S. GE, W. Wang, J. Suo. Dependence of eigenvalues of a class of Fourth-order Sturm-liouville problems on the boundary. Applied Mathematics and Computation, 2013, 220: 268-276.

[11] K. Li, J. Sun, X. Hao. Eigenvalues of regular fourth-order Sturm-Liouville problems with transmission conditions. Math. Methods Appl. Sci.2017, 40: 3538-3551.

[12] X. Lv, J. Ao, A. Zettl. Dependence of eigenvalues of fourth-order differential equations with discontinuous boundary conditions on the problem. J. Math. Anal. Appl. 2017, 456: 671-685.

[13] X. Lv, J. Ao. Eigenvalues of Fourth-Order Boundary Value Problems with Self-Adjoint Canonical Boundary Conditions. Bull. Malays. Math. Sci. Soc. 2020, 43: 833-846.

[14] Zhang, H.Y., Ao, J.J., Bo, F.Z. Eigenvalues of fourth-order boundary value problems with distributional potentials. AIMS Mathematics, 2022, 7(5): 7294-7317.

[15] Zhang, H.Y., Ao, J.J., Li, M.L. Dependence of Eigenvalues of Sturm-Liouville problems with eigenparameter-dependent boundary conditions and interface conditions. Mediterr. J. Math.,2022, 19: 90.

[16] Zhang, H.Y., Ao, J.J., Mu, D. Eigenvalues of discontinuous third-order boundary value problems with eigenparameter-dependent boundary conditions. J. Math. Anal. Appl., 2022, 506: 125680.

[17] Zhang, H.Y., Ao, J.J. Eigenvalue Properties of Third-Order Boundary Value Problems with Distributional Potentials. Acta Mathematicae Applicatae Sinica, English Series,2022,38(4):861-881.

[18] Zhang, M.Z., Li, K. Dependence of eigenvalues of Sturm-Liouville problems with eigenparameter dependent boundary conditions. Appl. Math. Comput.,2020, 378: 125214.

[19] L. Greenberg, M. Marletta. The code SLEUTH for solving fourth-order Sturm-Liouville problems. ACM Trans. Math. Softw. 1997, 23: 453-493.

[20] M.E. Lord, M.R. Scott, H.A. Watts. Computation of eigenvalue/eigenfunctions for two-point boundary-value problems. Applied Non-Linear Analysis, V.Lakshmikantham (Ed.), Academic Press, 1979, 635-656.

[21] L. Greenberg, An oscillation method for fourth order self-adjoint two-point boundary value problems with nonlinear eigenvalues, SIAM J. Math. Anal.22 (1991) 1021-1042.

[22] L. Greenberg, M. Marletta. Oscillation theory and numerical solution of fourth order Sturm-Liouville problems. IMA J. Numer. Anal, 1995, 15: 319-356.

[23] B. Chanane. Eigenvalues of fourth order Sturm-Liouville problems using Fliess series. J. Comput. Appl. Math, 1998, 96: 91-97.

[24] L. Greenberg, M. Marletta. Numerical methods for higher order Sturm-Liouville problems. J. Comput. Appl. Math, 2000, 125: 367-383.

[25] B. Chanane. Fliess series approach to the computation of the eigenvalues of fourth-order Sturm-Liouville problems. Appl. Math. Lett, 2002, 15: 459-702.

[26] A.P. Wang, J. Sun, A. Zettl. The classification of self-adjoint boundary conditions: separated coupled and mixed. J. Funct. Anal, 2008, 255: 1554-1573.

[27] Q. Kong, A. Zettl. Linear ordinary Differential Equations, in Inequalities and Applications. R.P. Agarwal, (Ed.), WSSIAA, 1994, 3: 381-397.

[28] R. Wang, Z. Wu, Lecture of Ordinary Differential Equations, Beijing People's Education Press, 1963, 198-207, (in chinese).

[29] J. Dieudonn. "Foundations of Modern Analysis." Academic Press, New York, London, 1969.

[30] Y. Liu, Q. Li. Existence and Uniqueness of Solutions and Lyapunov-type Inequality for a Mixed Fractional Boundary Value Problem. Journal of Nonlinear Modeling and Analysis,2022,4(2): 207-219.

[31] S.Fu, Z.Qiao, Z. Wang. Eigenvalues and Eigenfunctions of a Schrödinger Operator Associated with a Finite Combination of Dirac-Delta Functions and CH Peakons. Journal of Nonlinear Modeling and Analysis,2021,3(1): 131-144. , London, 1969.

Jianqing Suo, Math. Dept., Inner Mongolia University, Hohhot, 010021, China
E-mail address: sjq.hello@163.com

Zhijie Shi, Math. Dept., Inner Mongolia University, Hohhot, 010021, China
E-mail address: SZJshizhijie@163.com

Zhen Wei, Math. Dept., Inner Mongolia University, Hohhot, 010021, China
E-mail address: 15044974134@163.com