

Solvability of Hilfer fractional differential equations with integral boundary conditions at resonance in \mathbb{R}^m *

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Abstract: In this paper, the solvability of a class of nonlinear Hilfer fractional differential equations boundary value problems is considered at resonance in \mathbb{R}^m . The interesting point is that Hilfer is a more general differential operator that contains both the Riemann-Liouville and the Caputo derivative, and the dimension of the kernel of the fractional differential operator with Rimman-stieltjes integral boundary condition can take any value in $\{1, 2, \dots, m\}$. By applying Mawhin's coincidence degree theory, the existence result of solutions is obtained.

Keywords: Hilfer fractional differential equation; Mawhin's coincidence degree theory; resonance; Rimman-stieltjes integral.

MSC: 34A08; 34B15.

1 Introduction

In this paper, we consider the following Hilfer fractional differential equations boundary value problems at resonance in \mathbb{R}^m :

$$\begin{cases} D_{0+}^{\alpha,\beta} u(t) = f(t, u(t), D_{0+}^{\alpha-1,\beta} u(t)), & t \in [0, 1], \\ u(0) = D_{0+}^{\gamma-2} u(0) = \dots = D_{0+}^{\gamma-n+1} u(0) = \theta, u(1) = A \int_0^1 u(t) dh(t), \end{cases} \quad (1.1)$$

where $1 \leq n-1 < \alpha \leq n$, $0 \leq \beta \leq 1$, $\gamma = \alpha + n\beta - \alpha\beta$, θ is the zero vector in \mathbb{R}^m , A is m -order nonzero square matrices, $h(t)$ is a function of bounded variation, $h'(t)$ is bounded almost everywhere on $[0, 1]$, $D_{0+}^{\alpha,\beta}$ is Hilfer fractional derivative of order α and type β , and $f : [0, 1] \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ satisfies Carathéodory, that is,

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- (i) $f(\cdot, u, v)$ is measurable on $[0,1]$ for all $(u, v) \in \mathbb{R}^m \times \mathbb{R}^m$,
- (ii) $f(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^m \times \mathbb{R}^m$, for almost every $t \in [0, 1]$,
- (iii) The function $m_R(t) = \sup\{|f(t, u, v)| : (u, v) \in R\}$ is Lebesgue integrable on $0 \leq t \leq 1$ for all compact set $R \subset \mathbb{R}^m \times \mathbb{R}^m$, where $|f| = \max\{|f_i|, i = 1, 2, \dots, m\}$.

Fractional differential equations are increasingly used in various fields to solve practical problems, such as physics, chemistry, engineering and so on [1–6]. A large number of results are obtained on the existence of solutions to boundary value problems of Hilfer fractional differential equations [7–15]. It is well known that problem (1.1) is a generalization of elliptic differential equations on smooth surfaces [16]. M. Benchohra et al. [17] considered the existence and uniqueness of the solution to the problem (1.1) by using Banach contraction principle and Krasnoselskii's fixed point theorem. Furthermore, Z. Bouazza et al. [18] also considered problem (1.1) when $\beta = 1$ and established the existence result. A. Hasanen et al. [19] considered the following three-dimensional system of multi-point boundary value problem:

$$\begin{cases} D^\iota \varpi(z) = f_1(z, \varpi(z), \rho(z), \varrho(z)), \\ D^\gamma \varpi(z) = f_2(z, \varpi(z), \rho(z), \varrho(z)), \\ D^\chi \varpi(z) = f_3(z, \varpi(z), \rho(z), \varrho(z)), \\ \varpi(0) = \nu_1(\varpi), \varpi(1) = \eta_1 \varpi(\xi_1), \\ \rho(0) = \nu_2(\rho), \rho(1) = \eta_2 \rho(\xi_2), \\ \varrho(0) = \nu_3(\varrho), \varrho(1) = \eta_3 \varrho(\xi_3), \end{cases}$$

where $\iota, \gamma, \chi \in (1, 2]$, $z \in [0, 1]$, $\eta_1, \eta_2, \eta_3 \in (0, 1)$. Moreover, K. O. Ezekiel et al. [20] established the following multipoint boundary value problem with two-dimensional kernel at resonance:

$$\begin{cases} D_{0+}^\alpha u(t) = f(t, u(t), D_{0+}^{\alpha-3}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-1}u(t), \\ u(0) = D_{0+}^{\alpha-3}u(0) = 0, D_{0+}^{\alpha-2}u(0) = \sum_{i=1}^m \mu_i D_{0+}^{\alpha-2}u(\xi_i), \\ D_{0+}^{\alpha-1}u(+\infty) = \int_0^\eta D_{0+}^{\alpha-2}u(t)dh(t), \end{cases}$$

where $t \in (0, +\infty)$, $h(t)$ is a continuous and bounded variation function on $(0, +\infty)$.

In recent years, there has been some related research on resonance boundary value problems of fractional differential equations in \mathbb{R}^m [21–23]. P. D. Phung et al. [24] studied the following second-order three-point boundary value problems in \mathbb{R}^m :

$$\begin{cases} u'' = f(t, u(t), u'(t)), t \in (0, 1), \\ u'(0) = \theta, u(1) = Au(\eta), \end{cases}$$

where θ is an m -order zero vector, the matrix A satisfies one of the condition: $A^2 = A$ or $A^2 = I$. Ge et al. [25] concerned the following fractional three-point boundary value problems in \mathbb{R}^m :

$$\begin{cases} D_{0+}^{\alpha} x(t) = f(t, x(t), D_{0+}^{\alpha-1} x(t)), & 1 < \alpha \leq 2, t \in (0, 1), \\ x(0) = \theta, D_{0+}^{\alpha-1} x(1) = AD_{0+}^{\alpha-1} x(\xi), \end{cases}$$

where θ is an n -order zero vector, the matrix A satisfies one of the condition: $A^2 = A$ or $A^2 = I$. The author extends the order from integer order to fractional order and obtains the existence result of the solution by using Mawhin's coincidence degree theory. Feng et al. [26] used similar methods to study the following four-point boundary value problems in \mathbb{R}^m :

$$\begin{cases} {}^C D_{0+}^{\alpha} u(t) = f(t, u(t), {}^C D_{0+}^{\alpha-1} u(t)), & t \in (0, 1), \\ u(0) = Bu(\xi), u(1) = Cu(\eta), \end{cases}$$

where $0 < \eta, \xi < 1$, $1 < \alpha \leq 2$, B, C are two n -order nonzero square matrices. In [21–26], the variable u is an n -dimensional vector function, and the kernel dimension can take any value in $\{1, 2, \dots, n\}$.

However, we found that there are still some unresolved issues in \mathbb{R}^m . Firstly, the derivative operators in references [21–26] have not been unified. Therefore, it is imperative to mention that the Hilfer fractional differential system considered in the problem (1.1) is a more general form. For instance, the Hilfer fractional differential system in (1.1) corresponds to (i) the Riemann-Liouville fractional differential system for $\beta = 0$; (ii) the Caputo fractional differential system when $\beta = 1$. Secondly, the order of the derivative operator is limited. Therefore, the order was extended from $1 < \alpha \leq 2$ to $n - 1 < \alpha \leq n$ and an interesting new Rimman-stieltjes boundary condition was used. In addition, the use of Moore-Penrose generalized inverse matrix and their properties eliminates the restriction on matrix A .

2 Preliminaries

Definition 2.1. [15] Let X and Y be real Banach spaces. Linear operator $L : \text{dom}L \subset X \rightarrow Y$ to be a Fredholm operator of index zero if

- (A₁) $\text{Im}L$ is a closed subset of Y ;
- (A₂) $\dim \text{Ker}L = \text{codim } \text{Im}L < +\infty$.

If L satisfies (A₁) and (A₂), there exist two continuous projectors $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ such that $\text{Im}P = \text{Ker}L$, $\text{Ker}Q = \text{Im}L$, $X = \text{Ker}L \oplus \text{Ker}P$, $Y = \text{Im}L \oplus \text{Im}Q$. It follows that

$L|_{\text{dom}L \cap \text{Ker}P} : \text{dom}L \cap \text{Ker}P \rightarrow \text{Im}L$ is invertible. We denote the inverse of $L|_{\text{dom}L \cap \text{Ker}P}$ by $K_p : \text{Im}L \rightarrow \text{dom}L \cap \text{Ker}P$.

Definition 2.2. [15] If Ω is an open bounded subset of X , and $\text{dom}L \cap \Omega \neq \emptyset$, the map $N : X \rightarrow Y$ will be called L -compact on Ω if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N(\bar{\Omega})$ is completely continuous.

Lemma 2.3. [15] Let $L : \text{dom}L \subset X \rightarrow Y$ be a Fredholm operator of index zero and $N : X \rightarrow Y$ be L -compact on $\bar{\Omega}$. Suppose the following conditions are satisfied:

- (1) $Lu \neq \lambda Nu$ for every $(u, \lambda) \in [(\text{dom}L \setminus \text{Ker}L) \cap \partial\Omega] \times (0, 1)$;
- (2) $Nu \notin \text{Im}L$ for every $u \in \text{Ker}L \cap \partial\Omega$;
- (3) $\text{deg}(JQN|_{\text{Ker}L}, \Omega \cap \text{Ker}L, 0) \neq 0$, where $Q : Y \rightarrow Y$ is a projection such that $\text{Im}L = \text{Ker}Q$, and $J : \text{Im}Q \rightarrow \text{Ker}L$ is an isomorphism.

Then the equation $Lu = Nu$ has at least one solution in $\text{dom}L \cap \bar{\Omega}$.

Definition 2.4. [13] The Riemann-Liouville fractional integrals of order $\alpha > 0$ of a function $y : (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}y(s)ds,$$

provided the right side is pointwise on $(0, +\infty)$.

Definition 2.5. [13] The Riemann-Liouville fractional derivatives of order $\alpha > 0$ of a function $y : (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$D_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1}y(s)ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α , and this derivative is called the right side is pointwise defined on $(0, +\infty)$.

Lemma 2.6. [13] The left-sided Hilfer fractional derivative of order α and type β for a function $y : (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$D_{a+}^{\alpha, \beta}y(t) = I_{a+}^{\beta(n-\alpha)} \frac{d^n}{dt^n} (I_{a+}^{(1-\beta)(n-\alpha)}y)(t), n-1 < \alpha < n, 0 \leq \beta \leq 1.$$

Lemma 2.7. [13] Let $\alpha > 0$, if $y \in C(0, 1) \cap L(0, 1)$, then the fractional differential equation

$$D_{0+}^{\alpha}y(t) = 0$$

has a unique solution

$$y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}, i = 1, 2, \dots, n, n = [\alpha] + 1$.

Lemma 2.8. [13] If $f \in L(0, 1), \alpha > 0, \beta > 0$, then

$$D_{a\pm}^\alpha I_{a\pm}^\alpha y(t) = y(t).$$

Lemma 2.9. [13] Let $\alpha > 0, n = [\alpha] + 1$, if $y \in L_1(0, 1)$ and $I_{0+}^{n-\alpha} y \in AC^n[0, 1]$, then the following holds

$$I_{0+}^\alpha D_{0+}^\alpha y(t) = y(t) - \sum_{j=1}^n \frac{(I_{0+}^{n-\alpha} y(t))^{(n-j)}|_{t=0}}{\Gamma(\alpha - j + 1)} t^{\alpha-j}.$$

Lemma 2.10. [27] Let T^+ be the Moore-Penrose pseudoinverse matrix of T , then T satisfies

$$(j_1) T^+ T T^+ = T^+,$$

$$(j_2) T T^+ T = T,$$

$$(j_3) \text{Im} T^+ T = \text{Im} T,$$

$$(j_4) \text{Im}(I - T^+ T) = \text{Ker} T.$$

Lemma 2.11. [13] If $\alpha > 0, \nu > -1$, then the following holds

$$D_{0+}^\alpha t^\nu = \frac{\Gamma(\nu + 1)}{\Gamma(n + \nu - \alpha + 1)} \frac{d^n}{dt^n} (t^{n+\nu-\alpha}),$$

where $n = [\alpha] + 1$.

In order to study boundary value problem (1.1). We defined two spaces $X = \left\{ u \mid u, D_{0+}^{\alpha-1, \beta} u \in ([0, 1], \mathbb{R}^m) \right\}$ with the norms $\|u\| = \max \left\{ \|u\|_\infty, \|D_{0+}^{\alpha-1, \beta} u\|_\infty \right\}$, where $\|\cdot\|_\infty = \max_{1 \leq i \leq n} \max_{t \in [0, 1]} |u_i(t)|$ and $Y = L^1([0, 1], \mathbb{R}^m)$ with the norm $\|y\|_1 = \max \int_0^1 |y_i(s)| ds$.

In this paper, let $T = I - A \int_0^1 t^{\gamma-1} dh(t)$ and always assume that:

$$(H_1) \det(I - A \int_0^1 t^{\gamma-1} dh(t)) = 0,$$

$$(H_2) \int_0^1 (t^{\gamma-1} - t^\alpha) dh(t) \neq 0.$$

Define operators $L : \text{dom} L \subset X \rightarrow Y$ and $N : X \rightarrow Y$ as follows

$$Lu = D_{0+}^{\alpha, \beta} u(t), \quad u \in \text{dom} L,$$

$$Nu = f \left(t, u(t), D_{0+}^{\alpha-1, \beta} u(t) \right), \quad u \in X,$$

where

$$\text{dom}L = \left\{ u \mid u \in X, D_{0+}^{\alpha,\beta} u \in Y, u(0) = D_{0+}^{\gamma-2} u(0) = \dots = D_{0+}^{\gamma-n+1} u(0) = \theta, u(1) = A \int_0^1 u(t) dh(t) \right\}.$$

Then the problem(1.1) is equivalent to $Lu = Nu$, $u \in \text{dom}L$.

3 Main results

Lemma 3.1. *Suppose (H_1) holds, then $L : \text{dom}L \subset X \rightarrow Y$ is a Fredholm operator of index zero.*

Proof. It is easy to get that

$$\text{Ker}L = \left\{ u \in \text{dom}L \mid u(t) = ct^{\gamma-1}, c \in \text{Ker}T \right\}.$$

Now, we prove

$$\text{Im}L = \left\{ y \in Y \mid \phi y(t) \in \text{Im}T \right\},$$

where $\phi y(t) : Y \rightarrow \mathbb{R}^m$ is a continuous linear operator defined by

$$\phi(y) = I_{0+}^{\alpha} y(1) - A \int_0^1 I_{0+}^{\alpha} y(t) dh(t). \quad (3.1)$$

Let $|h'(t)| < M$, a.e. $t \in [0, 1]$. For $y_1, y_2 \in Y$, if $\|y_1 - y_2\|_1 < \delta$, then

$$\begin{aligned} & |\phi(y_1) - \phi(y_2)| \\ &= \left| I_{0+}^{\alpha} y_1(1) - I_{0+}^{\alpha} y_2(1) - A \int_0^1 I_{0+}^{\alpha} y_1(t) dh(t) + A \int_0^1 I_{0+}^{\alpha} y_2(t) dh(t) \right| \\ &\leq |I_{0+}^{\alpha} y_1(1) - I_{0+}^{\alpha} y_2(1)| + \|A\|_{\infty} \left| \int_0^1 I_{0+}^{\alpha} [y_2(t) - y_1(t)] dh(t) \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^1 (1-s)^{\alpha-1} y_1(s) - y_2(s) ds \right| + \frac{\|A\|_{\infty}}{\Gamma(\alpha)} \left| \int_0^1 \int_0^t (t-s)^{\alpha-1} y_2(s) - y_1(s) h'(s) ds dt \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \|y_1 - y_2\|_1 + \frac{\|A\|_{\infty} M}{\Gamma(\alpha)} \|y_1 - y_2\|_1, \end{aligned}$$

where $\|A\|_{\infty} = \max_{1 \leq i \leq n} \left(\sum_{j=1}^n |a_{ij}| \right)$. Therefore, the operator ϕ is continuous.

In fact, for any $y \in \text{Im}L$, there exists a function $u \in \text{dom}L$ such that $D_{0+}^{\alpha,\beta} u(t) = y(t)$. By Lemma (2.8) and Lemma (2.9), we obtain $u(t) = I_{0+}^{\alpha} y(t) + c_1 t^{\gamma-1} + c_2 t^{\gamma-2} + \dots + c_n t^{\gamma-n}$. Since $u(0) = D_{0+}^{\gamma-2} u(0) = \dots = D_{0+}^{\gamma-n+1} u(0) = \theta$, we can get

$$u(t) = I_{0+}^{\alpha} y(t) + c_1 t^{\gamma-1}.$$

And then from $u(1) = A \int_0^1 u(t)dh(t)$, we can get

$$I_{0+}^\alpha y(t)|_{t=1} - A \int_0^1 I_{0+}^\alpha y(t)dh(t) = -Tc_1, \quad c_1 \in \mathbb{R}^m,$$

which means that $\phi y(t) \in ImT$. Consequently, $ImL \subseteq \{y \in Y | \phi y(t) \in ImT\}$.

On the other hand, if $y \in Y$ satisfies $\phi y(t) \in ImT$, there exist a constant ξ such that $\phi y(t) = -T\xi$. Let $u(t) = I_{0+}^\alpha y(t) + \xi t^{\gamma-1}$. It is easy to prove that u satisfies the boundary conditions of the problem (1.1), and we have $Lu = y(t)$. Then $ImL \supseteq \{y \in Y | \phi y(t) \in ImT\}$.

In summary, we get

$$ImL = \{y \in Y | \phi y(t) \in ImT\}.$$

Define the operator $Q : Y \rightarrow Y$ by

$$Qy = G(I - TT^+)\phi(y) := C, \quad (3.2)$$

where $G = \frac{\Gamma(\alpha+1) \int_0^1 t^{\gamma-1} dh(t)}{\int_0^1 (t^{\gamma-1} - t^\alpha) dh(t)}$.

For $y \in Y, \forall t \in [0, 1]$

$$\begin{aligned} Q^2 y &= G(I - TT^+)\phi(C) \\ &= G(I - TT^+) \left[\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} C ds - A \int_0^1 \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} C ds dh(t) \right] \\ &= \frac{G}{\Gamma(\alpha+1)} (I - TT^+) (I - A \int_0^1 t^\alpha dh(t)) C \\ &= \frac{G}{\Gamma(\alpha+1)} \left[(I - TT^+) - \frac{1}{\int_0^1 t^{\gamma-1} dh(t)} (I - TT^+) \int_0^1 t^\alpha dh(t) \right] C \\ &= G \frac{\int_0^1 t^{\gamma-1} dh(t) - \int_0^1 t^\alpha dh(t)}{\Gamma(\alpha+1) \int_0^1 t^{\gamma-1} dh(t)} (I - TT^+) C \\ &= (I - TT^+) Qy = Qy. \end{aligned}$$

Actually, $(I - TT^+)A = \frac{1}{\int_0^1 t^{\gamma-1} dh(t)} (I - TT^+)$.

Hence, Q is a linear projection operator. Obviously, $ImL = KerQ$. For $y \in Y$, we can set $y = (y - Qy) + Qy$. By $(y - Qy) \in KerQ = ImL, Qy \in ImQ$, we can get $Y = ImL + ImQ$. It follows from $y \in ImL \cap ImQ$ that $y \in ImL = KerQ$ and $y = Qy$, then $y = \theta$. Hence $Y = ImL \oplus ImQ$. It is obvious that $codim ImL = dim ImQ = dim KerL$. Thus, L is a Fredholm operator of index zero. \square

Define the operator $P : X \rightarrow X$ by

$$Pu(t) = (I - T^+T) \frac{D_{0+}^{\gamma-1} u(0)}{\Gamma(\gamma)} t^{\gamma-1}. \quad (3.3)$$

It is easy to get $P^2u = Pu$ and $ImP = KerL$. Clearly $X = KerL \oplus KerP$. So $P : X \rightarrow X$ is a projector.

Lemma 3.2. Define a linear operator $K_p : ImL \rightarrow domL \cap KerP$,

$$K_P y(t) = I_{0+}^\alpha y(t) - T^+ \phi y(t) t^{\gamma-1}. \quad (3.4)$$

Then $K_P = (L|_{domL \cap KerP})^{-1}$.

Proof. For $y \in ImL$, we have $\phi(y) \in ImT$, which means that $\phi(y) = T\xi$, then

$$\begin{aligned} K_P y(0) &= D_{0+}^{\gamma-2} K_P y(0) = \dots = D_{0+}^{\gamma-n+1} K_P y(0) = \theta, \\ K_P y(t)|_{t=1} - A \int_0^1 K_P y(t) dh(t) \\ &= I_{0+}^\alpha y(t)|_{t=1} - T^+ \phi(y) - A \int_0^1 [I_{0+}^\alpha y(t) - T^+ \phi(y) t^{\gamma-1}] dh(t) \\ &= I_{0+}^\alpha y(t)|_{t=1} - A \int_0^1 I_{0+}^\alpha y(t) dh(t) - \left[I - A \int_0^1 t^{\gamma-1} dh(t) \right] T^+ \phi(y) \\ &= \phi(y) - TT^+ \phi(y) \\ &= (I - TT^+) T\xi \\ &= \theta. \end{aligned}$$

Thus, $K_P y \in domL$. It follows from (3.2) and (3.4)

$$\begin{aligned} PK_P y &= (I - TT^+) \frac{D_{0+}^{\gamma-1} K_P y(t)|_{t=0}}{\Gamma(\gamma)} t^{\gamma-1} \\ &= (I - TT^+) \frac{D_{0+}^{\gamma-1} [I_{0+}^\alpha y(t)]|_{t=0}}{\Gamma(\gamma)} t^{\gamma-1} \\ &= \theta. \end{aligned}$$

Thus, $K_P y \in KerP$. And we have

$$\begin{aligned} LK_P y(t) &= D_{0+}^{\alpha, \beta} [I_{0+}^\alpha y(t) - T^+ \phi y(t) t^{\gamma-1}] \\ &= I_{0+}^{\beta(n-\alpha)} D_{0+}^\gamma I_{0+}^\alpha y(t) - I_{0+}^{\beta(n-\alpha)} D_{0+}^\gamma T^+ \phi y(t) t^{\gamma-1} \\ &= I_{0+}^{\beta(n-\alpha)} D_{0+}^{\beta(n-\alpha)} y(t) = y(t). \end{aligned}$$

On the other hand, for $u \in \text{dom}L \cap \text{Ker}P$, we have

$$\begin{aligned}
& K_P L u(t) \\
&= I_{0+}^\alpha D_{0+}^{\alpha,\beta} u(t) - t^{\gamma-1} T^+ \left[I_{0+}^\alpha D_{0+}^{\alpha,\beta} u(t) \Big|_{t=1} - A \int_0^1 I_{0+}^\alpha D_{0+}^{\alpha,\beta} u(t) dh(t) \right] \\
&= I_{0+}^\gamma D_{0+}^\gamma u(t) - t^{\gamma-1} T^+ \left[I_{0+}^\gamma D_{0+}^\gamma u(t) \Big|_{t=1} - A \int_0^1 I_{0+}^\gamma D_{0+}^\gamma u(t) dh(t) \right] \\
&= u(t) - \frac{D_{0+}^{\gamma-1} u(0)}{\Gamma(\gamma)} t^{\gamma-1} - \frac{D_{0+}^{\gamma-2} u(0)}{\Gamma(\gamma-1)} t^{\gamma-2} - \dots - \frac{I_{0+}^{n-\gamma} u(0)}{\Gamma(\gamma-n+1)} t^{\gamma-n} \\
&\quad - t^{\gamma-1} T^+ \left[u(1) - \frac{D_{0+}^{\gamma-1} u(0)}{\Gamma(\gamma)} - \frac{D_{0+}^{\gamma-2} u(0)}{\Gamma(\gamma-1)} - \dots - \frac{I_{0+}^{n-\gamma} u(0)}{\Gamma(\gamma-n+1)} \right. \\
&\quad \left. - A \int_0^1 u(t) - \frac{D_{0+}^{\gamma-1} u(0)}{\Gamma(\gamma)} t^{\gamma-1} - \frac{D_{0+}^{\gamma-2} u(0)}{\Gamma(\gamma-1)} t^{\gamma-2} - \dots - \frac{I_{0+}^{n-\gamma} u(0)}{\Gamma(\gamma-n+1)} t^{\gamma-n} dh(t) \right] \\
&= u(t) - \frac{D_{0+}^{\gamma-1} u(0)}{\Gamma(\gamma)} t^{\gamma-1} - t^{\gamma-1} T^+ \left[u(1) - \frac{D_{0+}^{\gamma-1} u(0)}{\Gamma(\gamma)} - A \int_0^1 u(t) dh(t) + A \int_0^1 \frac{D_{0+}^{\gamma-1} u(0)}{\Gamma(\gamma)} t^{\gamma-1} dh(t) \right] \\
&= u(t) - \frac{D_{0+}^{\gamma-1} u(0)}{\Gamma(\gamma)} t^{\gamma-1} + t^{\gamma-1} T^+ \left[I - A \int_0^1 t^{\gamma-1} dh(t) \right] \frac{D_{0+}^{\gamma-1} u(0)}{\Gamma(\gamma)} \\
&= u(t) - (I - T^+ T) \frac{D_{0+}^{\gamma-1} u(0)}{\Gamma(\gamma)} t^{\gamma-1} \\
&= u(t).
\end{aligned}$$

That means $K_p = (L|_{\text{dom}L \cap \text{Ker}P})^{-1}$. □

Lemma 3.3. *Assume $\Omega \subset X$ is an open bounded subset and $\text{dom}L \cap \overline{\Omega} \neq \emptyset$, then N is L -compact on $\overline{\Omega}$.*

Proof. Let $\Omega \subset X$ is an open bounded subset. By the hypothesis (iii) on the function f , there exists a function $m_R(t) \in L^1[0, 1]$ such that for all $u \in \Omega$,

$$|Nu(t)| = |f(t, u(t), D_{0+}^{\alpha-1, \beta} u(t))| \leq m_R(t), t \in [0, 1].$$

And then we shall prove that $K_p(I - Q)Nu$ is completely continuous. It follows from (3.1) definition of ϕ that

$$|\phi(Nu)| = \left| I_{0+}^\alpha - A \int_0^1 I_0^1 Nu(t) dh(t) \right| \leq \frac{1}{\Gamma(\alpha)} \left(1 + \|A\|_\infty \left| \int_0^1 dh(t) \right| \right) \|m_R\|_1.$$

Combing with (3.3), one has

$$\left| QNu(t) \right| = \left| \frac{\Gamma(\alpha+1) \int_0^1 t^{\gamma-1} dh(t)}{\int_0^1 (t^{\gamma-1} - t^\alpha) dh(t)} (I - TT^+) \phi(Nu) \right|$$

$$\begin{aligned}
&= \left| \frac{\Gamma(\alpha+1) \int_0^1 t^{\gamma-1} dh(t)}{\int_0^1 (t^{\gamma-1} - t^\alpha) dh(t)} (I - TT^+) \left[I_{0+}^\alpha Nu(1) - A \int_0^1 I_{0+}^\alpha Nu(t) dh(t) \right] \right| \\
&\leq \left| \frac{\alpha \int_0^1 t^{\gamma-1} dh(t)}{\int_0^1 (t^{\gamma-1} - t^\alpha) dh(t)} \right| \|I - TT^+\|_\infty \left(\left| \int_0^1 (1-s)^{\alpha-1} Nu(s) ds \right| \right. \\
&\quad \left. + \left| A \int_0^1 \int_0^t (t-s)^{\alpha-1} Nu(s) ds dh(t) \right| \right) \\
&\leq \left| \frac{\alpha \int_0^1 t^{\gamma-1} dh(t)}{\int_0^1 (t^{\gamma-1} - t^\alpha) dh(t)} \right| \|I - TT^+\|_\infty \left(\int_0^1 |Nu(s)| ds + \|A\|_\infty \int_0^1 \int_0^1 |Nu(s)| ds dh(t) \right) \\
&\leq \left| \frac{\alpha \int_0^1 t^{\gamma-1} dh(t)}{\int_0^1 (t^{\gamma-1} - t^\alpha) dh(t)} \right| \|I - TT^+\|_\infty \left(1 + \|A\|_\infty \left| \int_0^1 dh(t) \right| \right) \|m_R\|_1.
\end{aligned}$$

Thus, $QN(\bar{\Omega})$ is bounded.

For $u \in \bar{\Omega}$,

$$\begin{aligned}
&\phi(QNu) - \phi(Nu) \\
&= I_{0+}^\alpha QNu(t)|_{t=1} - A \int_0^1 I_{0+}^\alpha QNu(t) dh(t) - \phi(Nu) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} QNu(s) ds - A \int_0^1 \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} QNu(s) ds dh(t) - \phi(Nu) \\
&= \frac{G(I - TT^+) \phi(Nu)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds - \frac{AG(I - TT^+) \phi(Nu)}{\Gamma(\alpha)} \int_0^1 \int_0^t (t-s)^{\alpha-1} ds dh(t) - \phi(Nu) \\
&= \frac{G(I - TT^+)}{\Gamma(\alpha+1)} \phi(Nu) - \frac{AG(I - TT^+)}{\Gamma(\alpha+1)} \phi(Nu) \int_0^1 t^\alpha dh(t) - \phi(Nu) \\
&= \frac{\int_0^1 t^{\gamma-1} dh(t)}{\int_0^1 (t^{\gamma-1} - t^\alpha) dh(t)} \left(I - A \int_0^1 t^\alpha dh(t) \right) (I - TT^+) \phi(Nu) - \phi(Nu) \\
&= D\phi(Nu),
\end{aligned}$$

where $D = \frac{\int_0^1 t^{\gamma-1} dh(t)}{\int_0^1 (t^{\gamma-1} - t^\alpha) dh(t)} \left(I - A \int_0^1 t^\alpha dh(t) \right) (I - TT^+) - I$.

For every $u \in \Omega$, we have

$$\begin{aligned}
K_p(I - Q)Nu(t) &= I_{0+}^\alpha Nu(t) - t^{\gamma-1} T^+ \phi Nu(t) - I_{0+}^\alpha QNu(t) + t^{\gamma-1} T^+ \phi(QNu(t)) \\
&= I_{0+}^\alpha Nu(t) - I_{0+}^\alpha QNu(t) + t^{\gamma-1} T^+ D\phi(Nu),
\end{aligned} \tag{3.5}$$

and

$$D_{0+}^{\alpha-1} K_p(I - Q)Nu(t) = I_{0+}^1 Nu(t) - I_{0+}^1 QNu(t) + I_{0+}^{\beta(n-\alpha)} \Gamma(\gamma) T^+ D\phi(Nu). \tag{3.6}$$

Combining (3.4) and (3.5) we have

$$\begin{aligned}
& \left| K_p(I - Q)Nu(t) \right| \\
&= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Nu(s) ds - \frac{G(I - TT^+)}{\Gamma(\alpha+1)} \phi(Nu)t^\alpha + t^{\gamma-1} T^+ D\phi(Nu) \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^1 |Nu(s)| ds + \left| \frac{G(I - TT^+)}{\Gamma(\alpha+1)} \phi(Nu) \right| + |T^+ D\phi(Nu)| \\
&\leq \frac{1}{\Gamma(\alpha)} \|m_R\|_1 + \left(\left| \frac{G}{\Gamma(\alpha+1)} \right| \|I - TT^+\|_\infty + \|T^+ D\|_\infty \right) |\phi(Nu)| \\
&= \frac{1}{\Gamma(\alpha)} \left[1 + \left(\left| \frac{\int_0^1 t^{\gamma-1} dh(t)}{\int_0^1 (t^{\gamma-1} - t^\alpha) dh(t)} \right| \|I - TT^+\|_\infty + \|T^+ D\|_\infty \right) \left(1 + \|A\|_\infty \left| \int_0^1 dh(t) \right| \right) \right] \|m_R\|_1
\end{aligned}$$

and

$$\begin{aligned}
& |D_{0+}^{\alpha-1} K_p(I - Q)Nu(t)| \\
&= |I_{0+}^1 Nu(t) - I_{0+}^1 QNu(t) + I_{0+}^{\beta(n-\alpha)} \Gamma(\gamma) T^+ D\phi(Nu)| \\
&\leq \int_0^1 |Nu(s)| ds + \int_0^1 |QNu(s)| ds + |I_{0+}^{\beta(n-\alpha)} \Gamma(\gamma) T^+ D\phi(Nu)| \\
&\leq \|m_R\|_1 + |G(I - TT^+) \phi(Nu)| + \frac{\Gamma(\gamma)}{\Gamma(\beta(n-\alpha) + 1)} \|T^+ D\|_\infty |\phi(Nu)| \\
&= \left[1 + \left(\frac{|\alpha \int_0^1 t^{\gamma-1} dh(t)| \|I - TT^+\|_\infty}{\left| \int_0^1 (t^{\gamma-1} - t^\alpha) dh(t) \right|} + \frac{\Gamma(\gamma) \|T^+ D\|_\infty}{\Gamma(\alpha) \Gamma(\beta(n-\alpha) + 1)} \right) \left(1 + \|A\|_\infty \left| \int_0^1 dh(t) \right| \right) \right] \|m_R\|_1.
\end{aligned}$$

That is, $K_p(I - Q)N(\bar{\Omega})$ is uniformly bounded in X .

For $0 \leq t_1 < t_2 \leq 1, u \in \bar{\Omega}$, we have

$$\begin{aligned}
& \left| K_p(I - Q)Nu(t_2) - K_p(I - Q)Nu(t_1) \right| \\
&\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} Nu(s) ds - \frac{G(I - TT^+) \phi(Nu)}{\Gamma(\alpha+1)} t_2^\alpha + t_2^{\gamma-1} T^+ D\phi(Nu) \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} Nu(s) ds - \frac{G(I - TT^+) \phi(Nu)}{\Gamma(\alpha+1)} t_1^\alpha + t_1^{\gamma-1} T^+ D\phi(Nu) \right| \\
&= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_2-s)^{\alpha-1} Nu(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} Nu(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} Nu(s) ds \right. \\
&\quad \left. + \frac{G(I - TT^+) \phi(Nu)}{\Gamma(\alpha+1)} (t_1^\alpha - t_2^\alpha) + (t_2^{\gamma-1} - t_1^{\gamma-1}) T^+ D\phi(Nu) \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] m_R(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} m_R(s) ds \\
&\quad + \left| \frac{G(I - TT^+) \phi(Nu)}{\Gamma(\alpha+1)} \right| |t_1^\alpha - t_2^\alpha| + |t_2^{\gamma-1} - t_1^{\gamma-1}| \|T^+ D\|_\infty |\phi(Nu)|.
\end{aligned}$$

According to the uniform continuity of binary functions, for any $\varepsilon > 0$, there is always a positive integer δ that only depends on ε , so that for all points $(t_1, t_2) \in [0, 1] \times [0, 1]$, as long as $|t_1 - t_2| < \delta$, there is $|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| < \varepsilon$.

$$\begin{aligned}
& \left| D_{0+}^{\alpha-1, \beta} K_p(I - Q)Nu(t_2) - D_{0+}^{\alpha-1, \beta} K_p(I - Q)Nu(t_1) \right| \\
&= \left| \int_0^{t_2} Nu(s)ds - G(I - TT^+)\phi(Nu)t_2 + \frac{\Gamma(\gamma)}{\Gamma(\beta(n - \alpha) + 1)} T^+ D\phi(Nu) \right. \\
&\quad \left. - \int_0^{t_1} Nu(s)ds + G(I - TT^+)\phi(Nu)t_1 - \frac{\Gamma(\gamma)}{\Gamma(\beta(n - \alpha) + 1)} T^+ D\phi(Nu) \right| \\
&= \left| \int_{t_1}^{t_2} Nu(s)ds + G(I - TT^+)\phi(Nu)(t_1 - t_2) \right| \\
&\leq \int_{t_1}^{t_2} m_R(s)ds + \left| \frac{\alpha \int_0^1 t^{\gamma-1} dh(t)}{\int_0^1 (t^{\alpha-1} - t^\alpha) dh(t)} \right| \|I - TT^+\|_\infty \left(1 + \|A\|_\infty \left| \int_0^1 dh(t) \right| \right) |t_1 - t_2| \|m_R\|_1.
\end{aligned}$$

Thus, $K_p(I - Q)N(\bar{\Omega})$ is equicontinuous. By the Ascoli-Arzelà theorem, we can conclude that the operator N is L-compact in $\bar{\Omega}$. \square

Theorem 3.4. *Suppose $(H_1), (H_2)$ and the following conditions hold:*

(H_3) *There exists a constant $\delta_1 > 0$ such that for $u \in \text{dom}L$, if $|t^{1-\gamma}u(t)| > \delta_1$ for all $t \in [0, 1]$, then*

$$\phi f(t, u(t), D_{0+}^{\alpha-1, \beta} u(t)) \notin \text{Im}T.$$

(H_4) *There exist three nonnegative functions $a, b, c \in C^1[0, 1]$ such that*

$$|f(t, u, v)| \leq a(t)|u| + b(t)|v| + c(t), \quad \text{for all } t \in [0, 1], u, v \in \mathbb{R}^m,$$

where $B\Gamma(\alpha + 1) > 2(\|a\|B + \|b\|D)$, $B = \Gamma(\alpha + 1)\Gamma(\beta(n - \alpha) + 1)(1 - \|b\|) - \|b\|\Gamma(\gamma)$, $D = [\Gamma(\alpha + 1)\Gamma(\beta(n - \alpha) + 1) + \Gamma(\gamma)]$.

(H_5) *There exists a constant $\delta_2 > 0$ such that for any $\beta \in \mathbb{R}^m$, satisfying $\beta = \int_0^1 t^{\gamma-1} dh(t)A\beta$ and $|\beta| > \delta_2$, either*

$$\langle \beta, QN\beta \rangle \geq 0 \quad \text{or} \quad \langle \beta, QN\beta \rangle \leq 0,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{R}^m .

Then the problem (1.1) has at least one solution in X .

Before we prove theorem 3.4, we show three Lemmas.

Lemma 3.5. Let $\Omega_1 = \left\{ u | u \in \text{dom}L \setminus \text{Ker}L, Lu = \lambda Nu, \lambda \in (0, 1) \right\}$. Assume $(H_1) - (H_4)$ hold. Then Ω_1 is bounded in X .

Proof. Let $u \in \Omega_1$, we have $Lu = \lambda Nu$, $Nu \in \text{Im}L$, we get $\phi(Nu) \in \text{Im}T$. It follows from (H_3) that there exists a constant $t_0 \in [0, 1]$ such that $|t_0^{1-\gamma}u(t_0)| \leq \delta_1$.

By $Lu = \lambda Nu(t)$ and boundary condition, we have

$$u(t) = \lambda I_{0+}^{\alpha} Nu(t) + \xi t^{\gamma-1}. \quad (3.7)$$

Taking $t = t_0$ into equation (3.6), we have $u(t_0) = \lambda I_{0+}^{\alpha} Nu(t_0) + \xi t_0^{\gamma-1}$. That means

$$\begin{aligned} |\xi| &\leq |t_0^{1-\gamma}u(t_0)| + |\lambda t_0^{1-\gamma} I_{0+}^{\alpha} Nu(t_0)| \\ &\leq \delta_1 + t_0^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_0^{t_0} (t_0 - s)^{\alpha-1} |Nu(s)| ds \\ &\leq \delta_1 + t_0^{1-\gamma} t_0^{\alpha} \frac{1}{\Gamma(\alpha+1)} (\|a\| \|u\|_{\infty} + \|b\| \|D_{0+}^{\alpha-1, \beta} u\|_{\infty} + \|c\|) \\ &\leq \delta_1 + \frac{1}{\Gamma(\alpha+1)} (\|a\| \|u\|_{\infty} + \|b\| \|D_{0+}^{\alpha-1, \beta} u\|_{\infty} + \|c\|). \end{aligned}$$

Based on

$$\begin{aligned} |D_{0+}^{\alpha-1, \beta} u(t)| &= |\lambda I_{0+}^1 Nu(t) + I_{0+}^{\beta(n-\alpha)} \xi \Gamma(\gamma)| \\ &\leq \|a\| \|u\|_{\infty} + \|b\| \|D_{0+}^{\alpha-1, \beta} u\|_{\infty} + \|c\| + \left(\delta_1 + \frac{\|a\| \|u\|_{\infty} + \|b\| \|D_{0+}^{\alpha-1, \beta} u\|_{\infty} + \|c\|}{\Gamma(\alpha+1)} \right) \frac{\Gamma(\gamma)}{\Gamma(\beta(n-\alpha)+1)} \\ &\leq \frac{\|a\| \Gamma(\gamma)}{\Gamma(\alpha+1) \Gamma(\beta(n-\alpha)+1)} \|u\|_{\infty} + \frac{\|b\| \Gamma(\gamma)}{\Gamma(\alpha+1) \Gamma(\beta(n-\alpha)+1)} \|D_{0+}^{\alpha-1, \beta} u\|_{\infty} + \|a\| \|u\|_{\infty} \\ &\quad + \|b\| \|D_{0+}^{\alpha-1, \beta} u\|_{\infty} + \|c\| + \left(\delta_1 + \frac{\|c\|}{\Gamma(\alpha+1)} \right) \frac{\Gamma(\gamma)}{\Gamma(\beta(n-\alpha)+1)}, \end{aligned}$$

we obtain

$$\begin{aligned} \|D_{0+}^{\alpha-1, \beta} u\|_{\infty} &\leq \frac{[\Gamma(\alpha+1) \Gamma(\beta(n-\alpha)+1) + \Gamma(\gamma)] \|a\|}{\Gamma(\alpha+1) \Gamma(\beta(n-\alpha)+1) (1 - \|b\|) - \|b\| \Gamma(\gamma)} \|u\|_{\infty} \\ &\quad + \frac{[\Gamma(\alpha+1) \Gamma(\beta(n-\alpha)+1) + \Gamma(\gamma)] \|c\| + \delta_1 \Gamma(\alpha+1) \Gamma(\gamma)}{\Gamma(\alpha+1) \Gamma(\beta(n-\alpha)+1) (1 - \|b\|) - \|b\| \Gamma(\gamma)}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} |\xi| &\leq \delta_1 + \frac{\|a\|}{\Gamma(\alpha+1)} \|u\|_{\infty} + \frac{\|b\|}{\Gamma(\alpha+1)} \left\{ \frac{[\Gamma(\alpha+1) \Gamma(\beta(n-\alpha)+1) + \Gamma(\gamma)] \|a\|}{\Gamma(\alpha+1) \Gamma(\beta(n-\alpha)+1) (1 - \|b\|) - \|b\| \Gamma(\gamma)} \|u\|_{\infty} \right. \\ &\quad \left. + \frac{[\Gamma(\alpha+1) \Gamma(\beta(n-\alpha)+1) + \Gamma(\gamma)] \|c\| + \delta_1 \Gamma(\alpha+1) \Gamma(\gamma)}{\Gamma(\alpha+1) \Gamma(\beta(n-\alpha)+1) (1 - \|b\|) - \|b\| \Gamma(\gamma)} \right\} + \frac{\|c\|}{\Gamma(\alpha+1)} \end{aligned}$$

$$\leq \delta_1 + \frac{\|a\|(B + \|b\|D)}{\Gamma(\alpha + 1)B} \|u\|_\infty + \frac{\|b\|E + \|c\|B}{\Gamma(\alpha + 1)B},$$

where $E = [\Gamma(\alpha + 1)\Gamma(\beta(n - \alpha) + 1) + \Gamma(\gamma)]\|c\| + \delta_1\Gamma(\alpha + 1)\Gamma(\gamma)$.

By simple calculation, we can get

$$\begin{aligned} \|u\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} |Nu(s)| ds + |\xi| \\ &\leq \frac{1}{\Gamma(\alpha + 1)} (\|a\|\|u\|_\infty + \|b\|\|D_{0+}^{\alpha-1, \beta}\|_\infty + \|c\|) + \delta_1 + \frac{\|a\|(B + \|b\|D)}{\Gamma(\alpha + 1)B} \|u\|_\infty + \frac{\|b\|E + \|c\|B}{\Gamma(\alpha + 1)B} \\ &\leq \delta_1 + 2 \frac{\|a\|(B + \|b\|D)}{\Gamma(\alpha + 1)B} \|u\|_\infty + 2 \frac{\|b\|E + \|c\|B}{\Gamma(\alpha + 1)B}. \end{aligned}$$

Therefore,

$$\|u\|_\infty \leq \frac{\delta_1 B \Gamma(\alpha + 1) + 2(\|b\|E + \|c\|B)}{B \Gamma(\alpha + 1) - 2(\|a\|B + \|b\|D)}.$$

We can conclude that Ω_1 is bounded in X . □

Lemma 3.6. Let $\Omega_2 = \{u | u \in \text{Ker}L, Nu \in \text{Im}L\}$. Suppose $(H_1) - (H_3)$ hold. Then Ω_2 is bounded in X .

Proof. Let $u \in \Omega_2$, for any $t \in [0, 1]$, we have $u(t) = \xi t^{\gamma-1}$, $\xi \in \text{Ker}T$. Since $Nu \in \text{Im}L$, then $\phi(Nu) \in \text{Im}T$. According to (H_3) , there exists $t_0 \in [0, 1]$ such that $|t_0^{1-\gamma}u(t_0)| \leq \delta_1$. Thus we get that $|\xi| = |t_0^{1-\gamma}u(t_0)| \leq \delta_1$.

Therefore, Ω_2 is bounded in X . □

Lemma 3.7. Let $\Omega_3 = \{u \in \text{Ker}L | \rho\lambda u + (1 - \lambda)QNu = \theta, \lambda \in [0, 1]\}$.

$$\rho = \begin{cases} 1, & \text{if } \langle \beta, QN\beta \rangle \geq 0 \text{ holds,} \\ -1, & \text{if } \langle \beta, QN\beta \rangle \leq 0 \text{ holds.} \end{cases}$$

Then Ω_3 is bounded in X .

Proof. Let $u \in \Omega_3$, we know that $u(t) = \beta t^{\gamma-1}$ with $\beta \in \text{Ker}T$ and $(1 - \lambda)QNu = -\rho\lambda u$.

If $\lambda = 0$, then $QNu = \theta$, $Nu \in \text{Ker}L = \text{Im}L$. Thus we have $u \in \Omega_2$, so $\|u\| \leq \delta_1$.

If $\lambda \in (0, 1]$ and $\rho = 1$, suppose $|\beta| > \delta_2$. Then, from (H_5) , we get a contradiction

$$0 > -\lambda|\beta|^2 = -\lambda\langle \beta, \beta \rangle = (1 - \lambda)\langle \beta, QN\beta \rangle \geq 0.$$

Thus, we have $\|u\| \leq \delta_2$.

If $\lambda \in (0, 1]$ and $\rho = -1$, suppose $|\beta| > \delta_2$. Similarly,

$$0 < \lambda|\beta|^2 = \lambda\langle\beta, \beta\rangle = (1 - \lambda)\langle\beta, QN\beta\rangle \leq 0.$$

Thus $\|u\| \leq \delta_2$. In conclusion, Ω_3 is bounded in X . \square

The following is the proof of Theorem 3.4.

Proof. Let $\Omega \supset \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3 \cup \{\theta\}$ be a bounded open subset of X . It follows from Lemma 3.3 that N is L -compact on $\overline{\Omega}$. By Lemmas 3.5 and 3.6, we have

$$(1) Lu \neq \lambda Nu, \text{ for every } (u, \lambda) \in [(doml \setminus KerL) \cap \partial\Omega] \times (0, 1);$$

$$(2) Nu \notin ImL \text{ for every } u \in KerL \cap \partial\Omega.$$

We need only to prove $deg(JQN|_{KerL}, \Omega \cap KerL, \theta) \neq 0$.

Take $H(u, \lambda) = \lambda Ju + \rho(1 - \lambda)QNu$. According to Lemma 3.7, we know that $H(u, \lambda) \neq \theta$ for $u \in \partial\Omega \cap KerL$. Therefore, via the homotopy property of degree, we obtain

$$\begin{aligned} deg(JQN|_{KerL}, \Omega \cap KerL, \theta) &= deg(\rho H(\cdot, 0), \Omega \cap KerL, \theta) \\ &= deg(\rho H(\cdot, 1), \Omega \cap KerL, \theta) \\ &= deg(\pm \rho I, \Omega \cap KerL, \theta) = \pm 1 \neq 0. \end{aligned}$$

Applying Lemma 2.3, we conclude that problem (1.1) has at least one solution in X . The proof is completed. \square

4 Conclusions

This paper mainly studied a class of Hilfer fractional differential boundary value problem systems at resonance that state variable $u \in \mathbb{R}^m$ and gave a new theorem on the existence of solutions in kernel spaces by using the Mawhin coincidence degree theorem. We provided an example to illustrate the obtained results. Our results also provide some methods for φ -Hilfer and Hadamard fractional differential. These contributions will advance research in other fields.

5 Example

In this section, we present an example to verify our main results. Let's consider the following boundary value problem at resonance:

$$\left\{ \begin{array}{l} D_{0+}^{\frac{7}{2}, \frac{1}{2}} x(t) = f_1(t, x(t), y(t), D_{0+}^{\frac{5}{2}, \frac{1}{2}} x(t), D_{0+}^{\frac{5}{2}, \frac{1}{2}} y(t)), \quad t \in [0, 1], \\ D_{0+}^{\frac{7}{2}, \frac{1}{2}} y(t) = f_2(t, x(t), y(t), D_{0+}^{\frac{5}{2}, \frac{1}{2}} x(t), D_{0+}^{\frac{5}{2}, \frac{1}{2}} y(t)), \quad t \in [0, 1], \\ x(0) = y(0) = 0, \\ D_{0+}^{\frac{7}{4}} x(0) = D_{0+}^{\frac{7}{4}} y(0) = 0, \\ x(1) = \frac{357}{52} \int_0^1 y(t) d(t^2 - t), \\ y(1) = \frac{357}{52} \int_0^1 y(t) d(t^2 - t). \end{array} \right. \quad (5.1)$$

The problem (5.1) has a solution if and only if problem (1.1) has a solution, where $\alpha = \frac{7}{2}$, $\beta = \frac{1}{2}$, $h(t) = t^2 - t$ and $A = \begin{bmatrix} 0 & \frac{285}{44} \\ 0 & \frac{285}{44} \end{bmatrix}$. Define the function $f_i : [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}^2$, $i = 1, 2$ by

$$\begin{aligned} f(t, u, v) &= (f_1(t, x_1, y_1, x_2, y_2), f_2(t, x_1, y_1, x_2, y_2))^T \\ &= \left(\frac{t^2}{16}(x_1 + y_1) + \frac{t}{3}(|x_2| + |y_2|), \frac{t^2}{16}(x_1 + y_1) + \frac{t}{3}(\sin^2 x_2 + \sin^2 y_2) \right)^T, \end{aligned} \quad (5.2)$$

for all $t \in [0, 1]$ and $u = (x_1, y_1)$, $v = (x_2, y_2) \in \mathbb{R}^2$. Let $T = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $T^+ = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \end{bmatrix}$, and

$$\text{you get } I - TT^+ = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then problem (5.1) has one solution if and only if the problem (1.1), with A and f defined as above, has one solution. So we only need prove that the conditions of Theorem 3.4 are satisfied. By (5.2) f satisfies carathéodory conditions.

First of all, prove the first condition of theorem 3.4. Let $a(t) = \frac{t^2}{8}$, $b(t) = \frac{2}{3}t$, $c(t) = 1$. It follows from (5.2) that $|f(t, u, v)| \leq a(t)|u| + b(t)|v| + c(t)$ for all $t \in [0, 1]$ and $u, v \in \mathbb{R}^2$. By simple calculation we have $B = \Gamma(\alpha + 1)\Gamma(\beta(n - \alpha) + 1)(1 - \|b\|) - \|b\|\Gamma(\gamma) \approx 5.5533$, $D = [\Gamma(\alpha + 1)\Gamma(\beta(n - \alpha) + 1) + \Gamma(\gamma)] \approx 14.9644$ and $B\Gamma(\alpha + 1) - 2(\|a\|B + \|b\|D) \approx 58.9026 > 0$. Hence, (H3) is satisfied.

Next, let's check (H_4) , we note that

$$|f_2(t, x_1, y_1, x_2, y_2)| > 0,$$

for all $u = (x_1, y_1), v = (x_2, y_2) \in \mathbb{R}^2$. We calculated that

$$|\phi(f_2)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f_2(s) ds - \frac{285}{44} \int_0^1 \int_0^1 f_2(s) ds d(t^2 - t) \right| \neq 0$$

therefore

$$\phi f(t, x_1, y_1, x_2, y_2) = (\phi f_1(t, x_1, y_1, x_2, y_2), \phi f_2(t, x_1, y_1, x_2, y_2)) \notin \text{Im}T,$$

due to $\text{Im}T = \{(\eta, 0)^\top; \eta \in \mathbb{R}\}$. Hence, (H_4) is satisfied.

And finally, let's prove the condition (H_5) . For any $\beta \in \mathbb{R}^2$, satisfying $\beta = \frac{285}{44}A\beta$, and $|\beta| > 0$. β can be written as $\beta = (\beta_0, \beta_0)^\top$ for $\beta_0 \in \mathbb{R}$. By (3.1) and (4.2) we have

$$N\beta = (f_1(t, \beta, 0), f_2(t, \beta, 0))^\top = \left(\frac{t^2}{8}\beta_0, \frac{t^2}{8}\beta_0\right)^\top,$$

and we get $G = 138.4249$, $\phi(N\beta) = (0.0376\beta_0, 0.0376\beta_0)^\top$. It follows from (3.3) that

$$Q(N\beta) = 138.4249(0, 0.0376\beta_0)^\top,$$

and

$$\langle \beta, Q(N\beta) \rangle = 0.1957\beta_0^2 > 0.$$

Then, the condition (H_5) holds. Therefore, by an application of Theorem 3.4, we obtain that problem (4.1) has at least one solution.

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