

CONVERGENCE ANALYSIS ON THE ALTERNATING DIRECTION METHOD OF MULTIPLIERS FOR THE COSPARSE OPTIMIZATION PROBLEM

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Abstract From a dual perspective of the sparse representation model, Nam et al. proposed the cosparse analysis model. In this paper, we aim to investigate the convergence of the alternating direction method of multipliers (ADMM) for the cosparse optimization problem. First, we examine the variational inequality representation of the cosparse optimization problem by introducing auxiliary variables. Second, ADMM is used to solve cosparse optimization problem. Finally, by utilizing a tight frame with a uniform row norm and building upon lemmas and the strict contraction theorem, we establish a worst-case $\mathcal{O}(1/t)$ convergence rate in the ergodic sense. The experimental results verify the practicability and convergence of the ADMM in solving cosparse optimization problems.

Keywords Sparse representation model, cosparse analysis model, alternating direction method of multipliers, variational inequality, convergence analysis

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1. Introduction

Low-dimensional signal recovery takes advantage of the inherent low-dimensionality of many natural signals, despite their high ambient dimension. Utilizing prior information about the low-dimensional space can significantly aid in recovering the signal of interest. Sparsity, a widely recognized form of prior information, serves as the foundation for the burgeoning field of compressive sensing (CS [1–5]). The recovery of sparse inputs has found numerous applications in areas such as imaging, speech, radar signal processing, sub-Nyquist sampling, and more [6–9]. A typical

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sparse recovery problem is associated with the following linear system:

$$y = Mx, \quad (1.1)$$

where $y \in R^m$ is an observed vector, $M \in R^{m \times d}$ is a measurement matrix and $x \in R^d$ is an unknown signal which would be estimated from y . According to the Nyquist-Shannon sampling theorem, if the k -space data is undersampled so much that it fails to meet the Nyquist sampling criterion, then reconstructing the data can be difficult or impossible without prior knowledge of x .

1.1. Sparse synthesis model

Over the past decade, the application of compressed sensing significantly increased the image reconstruction speed and efficiency because of its capability to reconstruct images from highly undersampled signals. Sparse prior is widely used in CS-based reconstruction methods. For the sparse synthesis model, if a vector x is sufficient sparse, under the incoherence assumptions on the measurement matrix M , x can be robustly estimated by the problem

$$\begin{aligned} \min_x \|x\|_p \\ \text{s.t. } y = Mx, \end{aligned} \quad (1.2)$$

where $0 \leq p \leq 1$. The ℓ_p ($0 < p < 1$) (quasi)-norm of a vector x means $\|x\|_p = (\sum_i |x_i|^p)^{\frac{1}{p}}$. In particular, the ℓ_1 norm represents the sum of the absolute values of the elements, and the ℓ_0 norm represents the number of non-zero elements. The advanced ideas and methods have been explored by applications in signals and image processing [10–13]. After years of research, this model is becoming more and more mature and stable.

1.2. Cosparse analysis model

In the recent decade, the cosparse analysis model is an alternative approach has gained popularity [14, 15, 17–20]. Within this framework, a potentially redundant analysis operator $D \in R^{n \times d}$ ($n \geq d$) is employed, and the analyzed vector Dx is expected to be sparse. This implies that a signal $x \in R^d$ belongs to the cosparse analysis model with cosparsity ℓ if $\ell = n - \|Dx\|_0$. In this paper, the quantity ℓ represents the number of rows in D that are orthogonal to the signal. Consequently, x is referred to as ℓ -cosparse or simply cosparse. The specific definitions of cosparse and cosupport can be found in literature [15], for ease of reference, we have listed them below.

Definition 1.1 (Cosparse). A signal $x \in R^d$ is said to be cosparse with respect to an analysis operator $D \in R^{n \times d}$ if the analysis representation vector Dx contains many zero elements. Further, the number of zero elements

$$\ell = n - \|Dx\|_0$$

is called the cosparsity of x , we also say x is ℓ -cosparse.

Definition 1.2 (Cosupport). For a signal $x \in R^d$ and a given analysis operator $D \in R^{n \times d}$ with its rows $D_j \in R^d$ ($1 \leq j \leq n$), the cosupport is defined by

$$\Lambda := \{j | \langle D_j, x \rangle = 0\}.$$

In this paper, D is a tight frame with uniform row norm. We remind the reader that a frame is defined as below.

Definition 1.3 (Frame [21, 22]). Let $\Phi = \{\varphi_i\}_{i=1}^N \subseteq R^n$ be a vector sequence of the Hilbert space with $N \geq n$. If there exist constants $0 < A \leq B < \infty$ such that

$$\forall x \in R^n, \quad A\|x\|^2 \leq \sum_{i=1}^N |\langle x, \varphi_i \rangle|^2 \leq B\|x\|^2, \quad (1.3)$$

then Φ is referred to as a finite frame of R^n .

Remark 1.1. The constants A and B in the above formula are known as the lower and upper bounds of the finite frame Φ , respectively. They are considered to be the optimal bounds, with A being the supremum in the lower bound and B being the infimum in the upper bound. If $A = B$, then the frame Φ is called an A -tight frame. If $A = B = 1$, then Φ is called a Parseval frame. If there exists a constant C such that each meta-norm $\|\varphi_i\| = C$ of the frame Φ , then Φ is called an iso-norm frame. In particular, for a tight frame, if $C = 1$, it is referred to as a uniformly tight frame.

According to the definition of cosparsity, the cosparsity analysis model focuses on the zero elements of the analysis representation vector Dx , rather than the non-zero elements. This perspective contrasts with the sparse synthesis model. If the cosparsity ℓ is significantly large, meaning that the number of zeros ℓ is close to d , we say that x has a cosparsity representation. The cosupport set is identified by iteratively removing rows from D for which $\langle D_j, x \rangle \neq 0$ until the index set Λ remains unchanged, with $|\Lambda| \geq \ell$.

If the analysis representation vector Dx is sparse, similar to the sparse model, the estimation of x from the measurements can be achieved by

$$\begin{aligned} \min_x \|Dx\|_0 \\ \text{s.t. } y = Mx. \end{aligned} \quad (1.4)$$

The minimization problem (1.4) is known to be NP-hard [15], necessitating the use of approximation methods. Similar to the sparse model, one option is to use the greedy analysis pursuit (GAP) approach, which is inspired by the orthogonal matching pursuit (OMP) algorithm [14–16]. Alternatively, the nonconvex ℓ_0 norm can be approximated by the convex ℓ_1 norm, leading to the relaxed problem known as analysis basis pursuit (ABP) [23]. In this case, x can be estimated by solving a modified optimization problem

$$\begin{aligned} \min_x \|Dx\|_1 \\ \text{s.t. } \|y - Mx\|_2 \leq \epsilon, \end{aligned} \quad (1.5)$$

where $\|\cdot\|_1$ is the ℓ_1 norm that sums the absolute values of a vector and ϵ is an upper bound on the noise level $\|v\|_2$.

ABP is equivalent to the unconstrained optimization

$$\min_x \|Dx\|_1 + \frac{\alpha}{2} \|y - Mx\|_2^2, \quad (1.6)$$

which we call analysis LASSO (ALASSO). It can be said that ABP and ALASSO are equivalent in the sense that for any $\epsilon > 0$, there exists an α such that

the optimal solutions of ABP and ALASSO are identical. For the optimization problem (1.6), our previous work presented the modified GAP algorithm and error analysis [18, 24]. The simulations we conducted demonstrated the advantages of the proposed method for the cospase optimization problem. These optimization problems can also be solved using interior point methods [25]. However, as the problem dimension increases, these techniques become time-consuming since they require solutions of linear systems. Other suggested approaches include the alternating direction method of multipliers (ADMM) [26–28, 38] and the accelerated alternating minimization method (AAM) [29]. In this paper, we propose a new way to analyze the convergence theory of the cospase optimization problem based on the variational inequality.

1.3. Organization of the paper

Our focus in this paper is on the cospase optimization problem and its convergence study based on a variational inequality. The paper is structured as follows: In Section 2, we introduce auxiliary variables and investigate the variational inequality characterization of the cospase optimization problem. In Section 3, we present several lemmas that establish the strict contraction of the ADMM for the cospase optimization problem. Using these lemmas and the strict contraction theorem, we establish a worst-case $\mathcal{O}(1/t)$ convergence rate in the ergodic sense. In Section 4, numerical experiments illustrate the validity of the convergence theorem given in this paper. Finally, Section 5 provides a brief conclusion.

2. Preliminaries

To apply the ADMM for solving the cospase optimization problem (1.6), we convert the unconstrained optimization problem mentioned above into a constrained optimization problem as follows

$$\begin{aligned} \min_{x,z} \|z\|_1 + \frac{\alpha}{2} \|y - Mx\|_2^2 \\ \text{s.t. } Dx - z = 0, \end{aligned} \quad (2.1)$$

where an auxiliary variable $z \in R^n$ is introduced in (1.6) to transfer Dx out of the nondifferentiable term $\|\cdot\|_1$ and $\alpha > 0$ is a penalty parameter.

In this section, we summarize the variational inequality (VI) characterization of (2.1). Initially, we present the optimality condition of the constrained optimization problem (2.1), which forms the foundation for our subsequent convergence analysis [30, 33]. We then proceed to express the Lagrangian function of (2.1) as follows

$$L(z, x, \lambda) = \|z\|_1 + \frac{\alpha}{2} \|y - Mx\|_2^2 - \lambda^T (Dx - z). \quad (2.2)$$

In (2.2), we assume that $x \in \mathcal{X}$, $z \in \mathcal{Z}$ and $\lambda \in R^n$ where $\mathcal{X} \subset R^d$ and $\mathcal{Z} \subset R^n$ are closed convex sets, we call $(z^*, x^*, \lambda^*) \in \Omega := \mathcal{Z} \times \mathcal{X} \times R^n$ to be a saddle point of $L(z, x, \lambda)$ if the following inequalities are satisfied

$$L(z^*, x^*, \lambda) \leq L(z^*, x^*, \lambda^*) \leq L(z, x, \lambda^*). \quad (2.3)$$

Obviously, a saddle point (z^*, x^*, λ^*) can be characterized by the system

$$\begin{cases} z^* = \arg \min\{L(z, x^*, \lambda^*) | z \in \mathcal{Z}\}, \\ x^* = \arg \min\{L(z^*, x, \lambda^*) | x \in \mathcal{X}\}, \\ \lambda^* = \arg \max\{L(z^*, x^*, \lambda) | \lambda \in R^n\}, \end{cases} \quad (2.4)$$

which can be rewritten as

$$\begin{cases} z^* \in \mathcal{Z}, L(z, x^*, \lambda^*) - L(z^*, x^*, \lambda^*) \geq 0, \\ x^* \in \mathcal{X}, L(z^*, x, \lambda^*) - L(z^*, x^*, \lambda^*) \geq 0, \\ \lambda^* \in R^n, L(z^*, x^*, \lambda^*) - L(z^*, x^*, \lambda) \geq 0. \end{cases} \quad (2.5)$$

Below, we present a summary of the method for expressing the optimality condition of the cosparse analysis model (2.1) via a variational inequality.

Proposition 2.1. *Suppose $\mathcal{X} \subset R^d$ is a closed convex set, and $\theta(x) : R^d \rightarrow R$ is a convex function. Furthermore, let $f(x)$ be differentiable in \mathcal{X} . We assume that the set of solutions for the minimization problem $\min\{\theta(x) + f(x) | x \in \mathcal{X}\}$ is nonempty, then,*

$$x^* = \arg \min\{\theta(x) + f(x) | x \in \mathcal{X}\} \quad (2.6)$$

if and only if

$$x, x^* \in \mathcal{X}, \theta(x) - \theta(x^*) + (x - x^*)^T \nabla f(x^*) \geq 0. \quad (2.7)$$

The proof of Proposition 2.1 is available in [34]. Let $\theta_1(z) = \|z\|_1$ and $\theta_2(x) = \frac{\alpha}{2} \|y - Mx\|_2^2$, according to the above inequality (2.7), a saddle point (z^*, x^*, λ^*) of the Lagrangian function (2.2) can be characterized by a solution point of the following variational inequality

$$\omega, \omega^* \in \Omega, \theta(u) - \theta(u^*) + (\omega - \omega^*)^T F(\omega^*) \geq 0, \quad (2.8)$$

where

$$\theta(u) = \theta_1(z) + \theta_2(x), \quad \Omega = \mathcal{Z} \times \mathcal{X} \times R^n, \quad (2.9)$$

and

$$\omega = \begin{pmatrix} z \\ x \\ \lambda \end{pmatrix}, \quad u = \begin{pmatrix} z \\ x \end{pmatrix}, \quad F(\omega) = \begin{pmatrix} \lambda \\ -D^T \lambda \\ Dx - z \end{pmatrix}, \quad (2.10)$$

Since F is an affine operator, and

$$F(\omega) = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & -D^T \\ -I & D & 0 \end{pmatrix} \begin{pmatrix} z \\ x \\ \lambda \end{pmatrix}, \quad (2.11)$$

According to the antisymmetry of the affine matrix, it follows that

$$(\omega - \bar{\omega})^T [F(\omega) - F(\bar{\omega})] \equiv 0, \quad \forall \omega, \bar{\omega} \in \Omega. \quad (2.12)$$

Using inequality (2.7) and combining (2.2), we derive the following conclusion with $(z^*, x^*, \lambda^*) \in \Omega$,

$$\begin{cases} \theta_1(z) - \theta_1(z^*) + (z - z^*)^T \lambda^* \geq 0, \\ \theta_2(x) - \theta_2(x^*) + (x - x^*)^T (-D^T \lambda^*) \geq 0, \\ (\lambda - \lambda^*)^T (Dx^* - z^*) \geq 0. \end{cases} \quad (2.13)$$

After conducting the aforementioned analysis, the linear constrained cospase optimization problem is reformulated as a variational inequality. Consequently, the task is ultimately simplified to identifying a saddle point of the Lagrangian function. In the subsequent section, the convergence analysis of the ADMM method for addressing the cospase optimization problem, as denoted by equation (2.1), will be discussed.

3. Convergence analysis of the cospase optimization problem

3.1. Variational inequality characterization of ADMM

The augmented Lagrangian function of the problem (2.1) can be formulated as follows

$$\mathcal{L}_\beta(z, x, \lambda) = \|z\|_1 + \frac{\alpha}{2} \|y - Mx\|_2^2 - \lambda^T (Dx - z) + \frac{\beta}{2} \|Dx - z\|_2^2, \quad (3.1)$$

where λ is the Lagrange multiplier and $\beta > 0$ is a penalty parameter for the linear constraints. Thus, applying directly the augmented Lagrangian function (3.1) and starting with an initial iterate $(x^0, \lambda^0) \in \mathcal{X} \times R^n$, the ADMM generates its sequence via following iterative scheme

$$\begin{cases} z^{k+1} = \arg \min \{ \mathcal{L}_\beta(z, x^k, \lambda^k) | z \in \mathcal{Z} \}, \\ x^{k+1} = \arg \min \{ \mathcal{L}_\beta(z^{k+1}, x, \lambda^k) | x \in \mathcal{X} \}, \\ \lambda^{k+1} = \lambda^k - \beta(Dx^{k+1} - z^{k+1}), \quad \lambda \in R^n \end{cases} \quad (3.2)$$

the corresponding variational inequalities of (3.2) can be given as

$$\begin{cases} \theta_1(z) - \theta_1(z^{k+1}) + (z - z^{k+1})^T [\lambda^k - \beta(Dx^k - z^{k+1})] \geq 0, \\ \theta_2(x) - \theta_2(x^{k+1}) + (x - x^{k+1})^T [-D^T \lambda^k + \beta D^T (Dx^{k+1} - z^{k+1})] \geq 0, \\ (\lambda - \lambda^{k+1})^T [(Dx^{k+1} - z^{k+1}) + \frac{1}{\beta}(\lambda^{k+1} - \lambda^k)] \geq 0. \end{cases} \quad (3.3)$$

For some reviews on the classical ADMM, one can refer to literatures [28, 30, 32, 35–37].

3.2. Assertions

To establish that $\{\omega^k\}$ is strictly contractive with respect Ω , we first present several lemmas.

Lemma 3.1. *Let the sequence $\{\omega^k\}$ be generated by (3.2). Then, we have*

$$\begin{aligned} & \theta(u) - \theta(u^{k+1}) + (\omega - \omega^{k+1})^T F(\omega) \\ & \geq (z - z^{k+1})^T \beta(Dx^k - Dx^{k+1}) + \frac{1}{\beta}(\lambda - \lambda^{k+1})^T (\lambda^k - \lambda^{k+1}), \quad \forall \omega \in \Omega. \end{aligned} \quad (3.4)$$

Proof. From (3.3) we know that

$$\theta_1(z) - \theta_1(z^{k+1}) + (z - z^{k+1})^T [\lambda^k - \beta(Dx^k - z^{k+1})] \geq 0, \quad \forall z \in \mathcal{Z} \quad (3.5)$$

and

$$\begin{aligned} & \theta_2(x) - \theta_2(x^{k+1}) \\ & + (x - x^{k+1})^T (-D^T \lambda^k + \beta D^T (Dx^{k+1} - z^{k+1})) \geq 0, \quad \forall x \in \mathcal{X}. \end{aligned} \quad (3.6)$$

Using $\lambda^{k+1} = \lambda^k - \beta(Dx^{k+1} - z^{k+1})$ we can easily deduce

$$\lambda^k = \lambda^{k+1} + \beta(Dx^{k+1} - z^{k+1}) \quad (3.7)$$

and

$$(Dx^{k+1} - z^{k+1}) = \frac{1}{\beta}(\lambda^k - \lambda^{k+1}). \quad (3.8)$$

Putting the formulations (3.7) and (3.8) into (3.5) and (3.6), respectively, then we have the following inequalities

$$\theta_1(z) - \theta_1(z^{k+1}) + (z - z^{k+1})^T [\lambda^{k+1} + \beta(Dx^{k+1} - z^{k+1}) - \beta(Dx^k - z^{k+1})] \geq 0, \quad (3.9)$$

$$\theta_2(x) - \theta_2(x^{k+1}) + (x - x^{k+1})^T (-D^T \lambda^{k+1}) \geq 0, \quad (3.10)$$

and

$$(\lambda - \lambda^{k+1})^T (Dx^{k+1} - z^{k+1}) \geq (\lambda - \lambda^{k+1})^T \frac{1}{\beta}(\lambda^k - \lambda^{k+1}). \quad (3.11)$$

Combining (3.9), (3.10) and (3.11) we have

$$\begin{cases} \theta_1(z) - \theta_1(z^{k+1}) + (z - z^{k+1})^T \lambda^{k+1} \geq (z - z^{k+1})^T \beta(Dx^k - Dx^{k+1}), \\ \theta_2(x) - \theta_2(x^{k+1}) + (x - x^{k+1})^T (-D^T \lambda^{k+1}) \geq 0, \\ (\lambda - \lambda^{k+1})^T (Dx^{k+1} - z^{k+1}) \geq (\lambda - \lambda^{k+1})^T \frac{1}{\beta}(\lambda^k - \lambda^{k+1}), \end{cases} \quad (3.12)$$

which is

$$\begin{aligned} & \theta(u) - \theta(u^{k+1}) + (\omega - \omega^{k+1})^T F(\omega^{k+1}) \\ & \geq (z - z^{k+1})^T \beta(Dx^k - Dx^{k+1}) + (\lambda - \lambda^{k+1})^T \frac{1}{\beta}(\lambda^k - \lambda^{k+1}). \end{aligned}$$

Note that the matrix in the operator F is skew-symmetric, then, using (2.12), we have

$$\begin{aligned} & \theta(u) - \theta(u^{k+1}) + (\omega - \omega^{k+1})^T F(\omega) \\ & \geq (z - z^{k+1})^T \beta (Dx^k - Dx^{k+1}) + (\lambda - \lambda^{k+1})^T \frac{1}{\beta} (\lambda^k - \lambda^{k+1}). \end{aligned} \quad (3.13)$$

The Lemma 3.1 is proved. \square

Lemma 3.2. *Let the sequence $\{\omega^k\}$ be generated by (3.2). Then, we have*

$$\begin{aligned} & \beta(z - z^{k+1})^T (Dx^k - Dx^{k+1}) + \frac{1}{\beta} (\lambda - \lambda^{k+1})^T (\lambda^k - \lambda^{k+1}) \\ & = -\frac{1}{2\beta} \|\lambda^k - \lambda\|_2^2 - \frac{\beta}{2} \|Dx^k - z\|_2^2 + \frac{1}{2\beta} \|\lambda^{k+1} - \lambda\|_2^2 + \frac{\beta}{2} \|Dx^{k+1} - z\|_2^2 \\ & \quad + \frac{\beta}{2} \|Dx^k - z^{k+1}\|_2^2, \quad \forall \omega \in \Omega. \end{aligned} \quad (3.14)$$

Proof. Applying the identity

$$(a - b)^T (c - d) = \frac{1}{2} \{\|a - d\|_2^2 - \|a - c\|_2^2\} + \frac{1}{2} \{\|c - b\|_2^2 - \|d - b\|_2^2\}$$

to the left-hand side in (3.14) with

$$a = z, \quad b = z^{k+1}, \quad c = Dx^k, \quad d = Dx^{k+1},$$

we obtain

$$\begin{aligned} & \beta(z - z^{k+1})^T (Dx^k - Dx^{k+1}) \\ & = \frac{\beta}{2} \{\|z - Dx^{k+1}\|_2^2 - \|z - Dx^k\|_2^2\} + \frac{\beta}{2} \{\|Dx^k - z^{k+1}\|_2^2 - \|Dx^{k+1} - z^{k+1}\|_2^2\}. \end{aligned} \quad (3.15)$$

Using the identity

$$b^T (b - a) = \frac{1}{2} (\|b\|_2^2 - \|a\|_2^2 + \|b - a\|_2^2),$$

and let

$$a = \lambda - \lambda^k, \quad b = \lambda - \lambda^{k+1},$$

we obtain

$$\frac{1}{\beta} (\lambda - \lambda^{k+1})^T (\lambda^k - \lambda^{k+1}) = \frac{1}{2\beta} \{\|\lambda - \lambda^{k+1}\|_2^2 - \|\lambda - \lambda^k\|_2^2 + \|\lambda^k - \lambda^{k+1}\|_2^2\}. \quad (3.16)$$

Using

$$\beta \|Dx^{k+1} - z^{k+1}\|_2^2 = \frac{1}{\beta} \|\lambda^k - \lambda^{k+1}\|_2^2,$$

and combining (3.15) and (3.16), we complete the proof of this lemma. \square

Lemma 3.3. *Let the sequence $\{x^k\}$, $\{z^k\}$ and $\{\lambda^k\}$ be generated by (3.2), then,*

$$\beta \|Dx^k - z^{k+1}\|_2^2 \geq \beta \|Dx^k - Dx^{k+1}\|_2^2 + \frac{1}{\beta} \|\lambda^k - \lambda^{k+1}\|_2^2. \quad (3.17)$$

Proof. Based on the second inequality of inequality (3.12), we can derive the following result

$$\begin{cases} \theta_2(x) - \theta_2(x^{k+1}) + (x - x^{k+1})^T(-D^T \lambda^{k+1}) \geq 0, \\ \theta_2(x) - \theta_2(x^k) + (x - x^k)^T(-D^T \lambda^k) \geq 0. \end{cases} \quad (3.18)$$

Let $x = x^k$ and $x = x^{k+1}$ in (3.18), respectively, then

$$\begin{cases} \theta_2(x^k) - \theta_2(x^{k+1}) + (x^k - x^{k+1})^T(-D^T \lambda^{k+1}) \geq 0, \\ \theta_2(x^{k+1}) - \theta_2(x^k) + (x^{k+1} - x^k)^T(-D^T \lambda^k) \geq 0. \end{cases}$$

From above inequalities, we have

$$(\lambda^k - \lambda^{k+1})^T(Dx^k - Dx^{k+1}) \geq 0. \quad (3.19)$$

Using

$$(Dx^{k+1} - z^{k+1}) = \frac{1}{\beta}(\lambda^k - \lambda^{k+1}),$$

then we obtain

$$\begin{aligned} & \beta \|Dx^k - z^{k+1}\|_2^2 \\ &= \beta \|Dx^k - Dx^{k+1} + Dx^{k+1} - z^{k+1}\|_2^2 \\ &= \beta \|Dx^k - Dx^{k+1} + \frac{1}{\beta}(\lambda^k - \lambda^{k+1})\|_2^2 \\ &\geq \beta \|Dx^k - Dx^{k+1}\|_2^2 + \frac{1}{\beta} \|\lambda^k - \lambda^{k+1}\|_2^2. \end{aligned} \quad (3.20)$$

The proof of this lemma is completed. \square

3.3. Strict contraction

To present the main result of the paper, it is necessary to establish the strict contractility of the iterative sequence. The following subsection provides a proof of the strong contractility of the iterative sequence $\{\omega^k\}$, which relies on Lemma 3.1, Lemma 3.2, and Lemma 3.3.

Theorem 3.1. *Assuming that the sequence $\{\omega^k\}$ is generated by iteration scheme (3.2), we have*

$$\|v^{k+1} - v^*\|_H^2 \leq \|v^k - v^*\|_H^2 - \|v^k - v^{k+1}\|_H^2 \quad (3.21)$$

where

$$v = \begin{pmatrix} \lambda \\ x \end{pmatrix}, \quad H = \begin{pmatrix} \frac{1}{\beta}I_m & 0 \\ 0 & \beta I_d \end{pmatrix} \text{ and } \mathcal{V}^* = \{(\lambda^*, x^*) | (z^*, x^*, \lambda^*) \in \Omega\}. \quad (3.22)$$

Proof. We can deduce from Lemma 3.1 and Lemma 3.2 that

$$\begin{aligned} & \theta(u^{k+1}) - \theta(u) + (\omega^{k+1} - \omega)^T F(\omega) \\ &\leq \frac{1}{2\beta} \|\lambda^k - \lambda\|_2^2 + \frac{\beta}{2} \|Dx^k - z\|_2^2 - \frac{1}{2\beta} \|\lambda^{k+1} - \lambda\|_2^2 - \frac{\beta}{2} \|Dx^{k+1} - z\|_2^2 \\ &\quad - \frac{\beta}{2} \|Dx^k - z^{k+1}\|_2^2. \end{aligned} \quad (3.23)$$

By utilizing Lemma 3.3, we can rewrite equation (3.23) as follows

$$\begin{aligned}
0 &\leq \theta(u^{k+1}) - \theta(u^*) + (\omega^{k+1} - \omega^*)^T F(\omega^*) \\
&\leq \frac{1}{2\beta} \|\lambda^k - \lambda^*\|_2^2 + \frac{\beta}{2} \|Dx^k - z^*\|_2^2 - \frac{1}{2\beta} \|\lambda^{k+1} - \lambda^*\|_2^2 - \frac{\beta}{2} \|Dx^{k+1} - z^*\|_2^2 \\
&\quad - \frac{1}{2\beta} \|\lambda^k - \lambda^{k+1}\|_2^2 - \frac{\beta}{2} \|Dx^k - Dx^{k+1}\|_2^2.
\end{aligned} \tag{3.24}$$

That is

$$\begin{aligned}
&\frac{1}{\beta} \|\lambda^{k+1} - \lambda^*\|_2^2 + \beta \|Dx^{k+1} - z^*\|_2^2 \\
&\leq \frac{1}{\beta} \|\lambda^k - \lambda^*\|_2^2 + \beta \|Dx^k - z^*\|_2^2 - \left(\frac{1}{\beta} \|\lambda^k - \lambda^{k+1}\|_2^2 + \beta \|Dx^k - Dx^{k+1}\|_2^2 \right).
\end{aligned} \tag{3.25}$$

Let

$$Dx^* = z^*, \quad v = \begin{pmatrix} \lambda \\ x \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} \frac{1}{\beta} I_m & 0 \\ 0 & \beta D^T D \end{pmatrix},$$

therefore, the left-hand side of inequality (3.25) becomes

$$\begin{aligned}
&\frac{1}{\beta} \|\lambda^{k+1} - \lambda^*\|_2^2 + \beta \|Dx^{k+1} - z^*\|_2^2 \\
&= \begin{pmatrix} \lambda^{k+1} - \lambda^* \\ x^{k+1} - x^* \end{pmatrix}^T \begin{pmatrix} \frac{1}{\beta} I_m & 0 \\ 0 & \beta D^T D \end{pmatrix} \begin{pmatrix} \lambda^{k+1} - \lambda^* \\ x^{k+1} - x^* \end{pmatrix} \\
&= (v^{k+1} - v^*)^T \begin{pmatrix} \frac{1}{\beta} I_m & 0 \\ 0 & \beta D^T D \end{pmatrix} (v^{k+1} - v^*) \\
&= \|v^{k+1} - v^*\|_H^2.
\end{aligned} \tag{3.26}$$

Likewise, the sum of the first two terms on the right-hand side of inequality (3.25) is

$$\begin{aligned}
&\frac{1}{\beta} \|\lambda^k - \lambda^*\|_2^2 + \beta \|Dx^k - Dx^*\|_2^2 \\
&= \begin{pmatrix} \lambda^k - \lambda^* \\ x^k - x^* \end{pmatrix}^T \begin{pmatrix} \frac{1}{\beta} I_m & 0 \\ 0 & \beta D^T D \end{pmatrix} \begin{pmatrix} \lambda^k - \lambda^* \\ x^k - x^* \end{pmatrix} \\
&= (v^k - v^*)^T \begin{pmatrix} \frac{1}{\beta} I_m & 0 \\ 0 & \beta D^T D \end{pmatrix} (v^k - v^*) \\
&= \|v^k - v^*\|_H^2
\end{aligned} \tag{3.27}$$

and the sum of the last two terms on the right-hand side of inequality (3.25) is

$$\begin{aligned}
& \frac{1}{\beta} \|\lambda^k - \lambda^{k+1}\|_2^2 + \beta \|Dx^k - Dx^{k+1}\|_2^2 \\
&= \begin{pmatrix} \lambda^k - \lambda^{k+1} \\ x^k - x^{k+1} \end{pmatrix}^T \begin{pmatrix} \frac{1}{\beta} I_m & 0 \\ 0 & \beta D^T D \end{pmatrix} \begin{pmatrix} \lambda^k - \lambda^{k+1} \\ x^k - x^{k+1} \end{pmatrix} \\
&= (v^k - v^{k+1})^T \begin{pmatrix} \frac{1}{\beta} I_m & 0 \\ 0 & \beta D^T D \end{pmatrix} (v^k - v^{k+1}) \\
&= \|v^k - v^{k+1}\|_H^2.
\end{aligned} \tag{3.28}$$

Since $D \in R^{n \times d}$ is a unit tight frame, we have that $D^T D = I_d$. By combining formulas (3.26), (3.27), and (3.28), we complete the proof of the Theorem 3.1. \square

Theorem 3.2. *Let H be a positive definite matrix, and let $\{v^k\}$ be a sequence with initial vector $v_0 = (\lambda_0, x_0)^T$. Given inequality (3.21), we can derive the sequence $\{v^k\}$ is convergent to \mathcal{V}^* .*

Proof. According to Theorem 3.1, we know that H is a positive definite matrix, and inequality (3.21) implies that the sequence $\{v^k\}$ is bounded. Assuming that the initial vector is $v_0 = (\lambda_0, x_0)^T$, we can obtain the following expression by summing both sides of inequality (3.21)

$$\sum_{k=0}^{\infty} \|v^k - v^{k+1}\|_H^2 \leq \|v^0 - v^*\|_H^2. \tag{3.29}$$

The above equation indicates that $\lim_{k \rightarrow \infty} \|v^k - v^{k+1}\|_H^2 = 0$. Therefore, any subsequence v^{k_j} of v^k also has $\lim_{j \rightarrow \infty} \|v^{k_j} - v^{k_j+1}\|_H^2 = 0$. Suppose there exists a subsequence that converges to \bar{v} , then formula (3.4) implies that \bar{v} is the solution of formula (3.2). This shows that any accumulation point of the sequence v^k is a solution of (3.2). According to formula (3.21), v^k cannot have more than one accumulation point, and hence v^k converges to $\bar{v} \in \mathcal{V}^*$. \square

3.4. Convergence rate in ergodic sense

Combining with Theorem 3.1, we prove a worst-case $\mathcal{O}(1/t)$ convergence rate in an ergodic sense of the ADMM scheme (3.2) for cospase signal reconstruction problem.

Theorem 3.3. *Let the sequence $\{\omega^k\}$ be generated by (3.2). Then, for any positive integer t , we have*

$$\begin{aligned}
& \theta(u^t) - \theta(u) + (\omega^t - \omega)^T F(\omega) \\
& \leq \frac{1}{2(t+1)} \left[\frac{1}{\beta} \|\lambda^0 - \lambda\|_2^2 + \beta \|Dx^0 - z\|_2^2 \right], \quad \forall \omega \in \Omega
\end{aligned} \tag{3.30}$$

where

$$\omega^t = \frac{1}{t+1} \left(\sum_{k=0}^t \omega^{k+1} \right). \tag{3.31}$$

Proof. For any integer k , by (3.23) we obtain

$$\begin{aligned} & \theta(u^{k+1}) - \theta(u) + (\omega^{k+1} - \omega)^T F(\omega) \\ & \leq \frac{1}{2\beta} \|\lambda^k - \lambda\|_2^2 + \frac{\beta}{2} \|Dx^k - z\|_2^2 - \frac{1}{2\beta} \|\lambda^{k+1} - \lambda\|_2^2 - \frac{\beta}{2} \|Dx^{k+1} - z\|_2^2. \end{aligned} \quad (3.32)$$

Suppose $k = 0, 1, 2, \dots, t$ are non-negative integers. By summing the left and right ends of the inequality (3.32), we deduce that

$$\begin{aligned} & \sum_{k=0}^t \theta(u^{k+1}) - (t+1)\theta(u) + \left[\sum_{k=0}^t \omega^{k+1} - (t+1)\omega \right]^T F(\omega) \\ & \leq \frac{1}{2\beta} \|\lambda^0 - \lambda\|_2^2 + \frac{\beta}{2} \|Dx^0 - z\|_2^2, \quad \forall \omega \in \Omega. \end{aligned} \quad (3.33)$$

The left and right ends of the inequality (3.33) are multiplied by $\frac{1}{t+1}$ at the same time, and let

$$\omega^t = \frac{1}{t+1} \sum_{k=0}^t \omega^{k+1}, \quad (3.34)$$

then, the inequality (3.33) is equivalent to

$$\begin{aligned} & \frac{1}{t+1} \sum_{k=0}^t \theta(u^{k+1}) - \theta(u) + (\omega^t - \omega)^T F(\omega) \\ & \leq \frac{1}{2(t+1)} \left[\frac{1}{\beta} \|\lambda^0 - \lambda\|_2^2 + \beta \|Dx^0 - z\|_2^2 \right]. \end{aligned} \quad (3.35)$$

Given that the function $\theta(u)$ is convex, let

$$u^t = \frac{1}{t+1} \sum_{k=0}^t u^{k+1} = \frac{1}{t+1} (u^1 + u^2 + \dots + u^t),$$

we can derive the following expression

$$\begin{aligned} \theta(u^t) &= \theta \left[\frac{1}{t+1} (u^1 + u^2 + \dots + u^t) \right] \\ &\leq \frac{1}{t+1} [\theta(u^1) + \theta(u^2) + \dots + \theta(u^t)] \\ &= \frac{1}{t+1} \sum_{k=0}^t \theta(u^{k+1}). \end{aligned} \quad (3.36)$$

By utilizing equations (3.35) and (3.36), we complete the proof of the Theorem 3.3. \square

After t -th iterations, then ω^t defined by (3.34) satisfies

$$\tilde{\omega} \in \Omega \quad \text{and} \quad \sup_{\omega \in \mathcal{D}_{\tilde{\omega}}} \{ \theta(\tilde{u}) - \theta(u) + (\tilde{\omega} - \omega)^T F(\omega) \} \leq \frac{d}{2t} = \mathcal{O}\left(\frac{1}{t}\right),$$

where

$$D_{\tilde{\omega}} = \{ \omega \in \Omega \mid \|\omega - \tilde{\omega}\| \leq 1 \}, \quad d := \sup \left\{ \frac{1}{\beta} \|\lambda^0 - \lambda\|_2^2 + \beta \|Dx^0 - z\|_2^2 \mid \omega \in \mathcal{D}_{\tilde{\omega}} \right\},$$

and $v^0 = (\lambda^0, x^0)$ is the initial iteration point. That means ω^t is an $\mathcal{O}\left(\frac{1}{t}\right)$ solution of (3.3).

3.5. Comparison with existing work

Existing literature, such as He and Yuan [30] has shown a convergence rate of $\mathcal{O}(1/t)$ for the Douglas–Rachford alternating direction methods scheme. He [31] has given a proof of the convergence properties of ADMM, which includes the strong contractility and $\mathcal{O}(1/t)$ ergodic (where t is a non-negative integer, indicating the number of iterations). Although in the ergodic sense, our convergence order is the same as that of the existing literature, in our work, we provided a new perspective and specific convergence proof for the cospase optimization problem, offering a more general convergence analysis under the framework of strict contraction and variational inequalities. This extends the applicability of ADMM to cospase optimization problems. On the other hand, it must be assumed in the literature that H is a symmetric positive semidefinite matrix, but the results we give do not require this assumption, because for the particularity of the cospase optimization problem, the redundant analysis operator D is a uniform tight frame, therefore, the H matrix is a symmetric positive definite matrix.

4. Numerical experiments

This experiment aims to demonstrate how to use the ADMM to solve the cospase optimization problem. Through experiments, we verified the effectiveness and convergence of the ADMM in solving the cospase signal reconstruction problem. To conduct the experiments, we first generated the problem instance, including the signal dimension d , the number of rows of the measurement matrix m , and the observation vector y . We use a randomly generated measurement matrix M . The redundant analysis operator D is generated as a tight frame with uniform row norm and $D^T D = I$. In addition, we also introduce Gaussian noise to simulate the noise situation in actual observations. And the ϵ is a upper bound on the noise level $\|v\|_2$. All experiments are performed under Windows 10 and MATLAB R2021a running on a Lenovo desktop with an Intel Core(TM)i7-9700, CPU at 3.6 GHz and 8 GB of memory.

Algorithm 1 ADMM

Task: Approximate the solution of (2.1).

Input: Initial point $x_0 = 0, z_0 = 0, \lambda_0 = 0$ and the parameters α, β . Number of iterations $k = 1, 2, \dots, max_iter$. Generate the redundant analysis operator D [17].

while the termination criterion is not met, **do**

- $z^{k+1} = \arg \min \mathcal{L}_\beta(z, x^k, \lambda^k),$
- $x^{k+1} = \arg \min \mathcal{L}_\beta(z^{k+1}, x, \lambda^k),$
- $\lambda^{k+1} = \lambda^k - \beta(Dx^{k+1} - z^{k+1}).$

$k \leftarrow k + 1.$

end while

Check convergence conditions.

Output: $z \leftarrow z^{k+1}, x \leftarrow x^{k+1}, \lambda \leftarrow \lambda^{k+1}.$

The maximum number of iterations is set to $max_iter = 1000$, the penalty parameters are set to $\alpha = 1 \times 10^{-4}$ and $\beta = 1 \times 10^{-2}$, the convergence tolerance is set to $tol = 1 \times 10^{-6}$, the noise level is set to $\epsilon = 1 \times 10^{-2}$. By running the

ADMM algorithm, we record the residual $\|r^{k+1}\|_2 = \|Dx^{k+1} - z^{k+1}\|_2$ and error $\|x^{k+1} - x^k\|_2$ for each iteration, and plot their convergence curves. For figure 1, we set $d = 128, n = 140, m = 256, \ell = 118$. For figure 2, we set $d = 256, n = 512, m = 512, \ell = 230$. Experimental results show that as the number of iterations increases, the residuals and errors gradually decrease. This illustrates that the numerical results are consistent with the theoretical results.

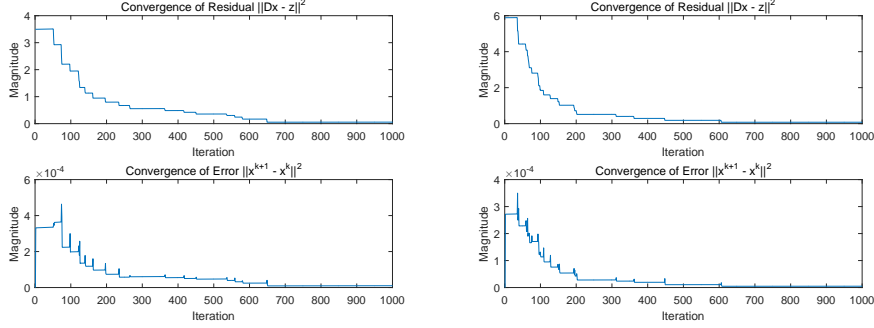


Figure 1. The residuals and errors curves with $d = 128, n = 140, m = 256, \ell = 118$. **Figure 2.** The residuals and errors curves with $d = 256, n = 512, m = 512, \ell = 230$.

5. Conclusions

This paper studies the convergence of the ADMM on cospase optimization problems from the dual perspective of sparse representation models. The paper first introduces auxiliary variables and examines the cospase optimization problem through the expression of variational inequalities. Then, the ADMM is applied to solve the cospase optimization problem, and the tight frame with unified row norm are used, combined with the lemma and the strict contraction theorem, to establish the worst-case convergence rate of $\mathcal{O}(1/t)$ in the ergodic sense. In addition, the paper also verifies the effectiveness and convergence of ADMM in the cospase signal reconstruction problem through numerical experiments. Experimental results show that as the number of iterations increases, the residuals and errors gradually decrease, proving the practicability and convergence of the ADMM method in solving cospase optimization problems.

Researchers currently rely on a range of methods to solve separable convex optimization problems. Two popular approaches are the generalized symmetric ADMM and parameterizable proximal point algorithms [38–40]. These methods have demonstrated their effectiveness and superiority in various experiments. In our future work, we plan to explore the potential of combining these methods to solve the cospase signal reconstruction problem.

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