

LONG TIME BEHAVIOUR OF THE SOLUTIONS OF NONLINEAR WAVE EQUATION

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ABSTRACT. In this paper, we consider the nonlinear wave equation

$$u_{tt} - \Delta u + mu + f(x, u) = 0, \quad x \in \mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d,$$

where $m > 0$ and f is an analytic function of order at least two in u . The long time behaviour of its solutions is proved by Birkhoff normal form.

Keywords: long time behaviour, nonlinear wave equation, normal form

1. INTRODUCTION AND RESULT

Consider nonlinear wave equation on d -dimensional torus

$$u_{tt} - \Delta u + mu + f(x, u) = 0, \quad x \in \mathbb{T}^d \tag{1.1}$$

with given initial data $u(0) \equiv u(0, x)$ and $\dot{u}(0) \equiv \partial_t u(0, x)$, where $m > 0$ and $f(x, u)$ is analytic function of order at least two with respect to u at the origin. For (1.1) with nonlinearity of the form $f(u)$, i.e., not containing the spatial variable x explicitly, the long time behaviour of the solutions has been proved by Bernier, Faou and Grébert in [BFG20a]. More precisely, they show that for almost all $m > 0$ and all $r \geq 2$, $s_0 > (d+1)/2$, there exists s_* depending on r, s_0 such that for any $s \geq s_*$, if the initial datum satisfies $\|(u(0), \dot{u}(0))\|_{H^s \times H^{s-1}} \leq \varepsilon$ for small enough $\varepsilon > 0$, then $\|u(t)_{\leq N_\varepsilon}\|_{H^{\frac{s}{2}}} \leq 2\varepsilon$ and $\|u(t)_{> N_\varepsilon}\|_{H^{s_0}} \leq \varepsilon^r$ for any $t \leq \varepsilon^{-\frac{r}{s_0+1}}$, where H^s is the Sobolev space on \mathbb{T}^d , $u(t)_{\leq N_\varepsilon}$ and $u(t)_{> N_\varepsilon}$ denote the low and high modes parts according to the threshold $N_\varepsilon = \varepsilon^{-\frac{2r}{s-2s_0}}$, respectively. In the following theorem, we study more general nonlinearity $f(x, u)$. For convenience, we keep fidelity with the notation and terminology from [BFG20a].

Theorem 1.1. For almost all $m > 0$ and any given $r \geq 1$, there exists $\tau, \varepsilon_* > 0$ such that for any $s > 2s_0 > d + 1$ with $s - 2s_0 \geq 2r(2r\tau + d)$, if the initial data $(u(0), \dot{u}(0)) \in H^s \times H^{s-1}$ satisfies $\varepsilon := \|(u(0), \dot{u}(0))\|_{H^s \times H^{s-1}} \leq \varepsilon_*$, then

$$\|u(t)_{\leq N_\varepsilon}\|_{H^{\frac{s}{2}}} \leq c_0\varepsilon \quad \text{and} \quad \|u(t)_{> N_\varepsilon}\|_{H^{s_0}} \leq \varepsilon^r \tag{1.2}$$

for any $t \leq \varepsilon^{-\frac{r}{2s_0}}$, where $N_\varepsilon = \varepsilon^{-\frac{2r}{s-2s_0}}$ and the positive constant c_0 depends on m, s .

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Birkhoff normal form for long time behavior of solutions of Hamiltonian partial differential equations has been widely investigated by many authors. For nonlinear wave equations, see [Bou96, Bam03, BG06, BDGS07, Bam08, Del09, Zha10, BFG20a] for example; for nonlinear Schrödinger equations, see [BG06, GIP09, FG13, CLY16, BMP20, CCMW22] for example; for equations with unbounded nonlinear vector field, see [YZ14, Del15, BD18, CLW20, CMW20, Zha20, CMWZ22] for example; for equations without external parameters, see [Bou00, BFG20b, BG21] for example.

For (1.1), the frequencies of linear equation are $\omega_a := \sqrt{|a|^2 + m}$, $a \in \mathbb{Z}^d$. If one wishes to get the usual long time stability, it is necessary to meet the following non-resonant conditions: any given integer $l \geq 3$, there exists $\gamma, \tau > 0$ such that for any $\mathbf{k} = (k_1, \dots, k_p) \in (\mathbb{Z}^d)^p$, $\mathbf{h} = (h_1, \dots, h_q) \in (\mathbb{Z}^d)^q$ with $p + q \leq l$, one has

$$|\omega_{k_1} + \dots + \omega_{k_p} - \omega_{h_1} - \dots - \omega_{h_q}| \geq \frac{\gamma}{\mu_3(\mathbf{k}, \mathbf{h})^\tau}, \quad (1.3)$$

expect that $(|k_1|, \dots, |k_p|)$ and $(|h_1|, \dots, |h_q|)$ are equal up to a permutation, where $\mu_3(\mathbf{k}, \mathbf{h})$ denotes the third largest number among $\{|k_1|, \dots, |k_p|, |h_1|, \dots, |h_q|\}$. But when $d \geq 2$, (1.3) is not satisfied due to the combinations with two high frequencies in opposite signs, i.e., without loss of generality $|k_p|, |h_q| > N \geq \mu_3(\mathbf{k}, \mathbf{h})$ for some N large enough. In [BFG20a], the non-resonant conditions of this type are removed, and as a result, the corresponding monomials $z_{k_1} \dots z_{k_p} \bar{z}_{h_1} \dots \bar{z}_{h_q}$ are remained in the normal form. Of course, these terms essentially affect the long time stability. Novelty observing that these terms preserve the L^2 norm of high modes, the authors derive the result of long time behavior from the normal form. During the proof of energy estimate of high modes in higher Sobolev space H^s , the conservation of momentum, i.e.,

$$k_1 + \dots + k_p - h_1 - \dots - h_q = 0, \quad (1.4)$$

is crucially used to bound $|k_p|^{2s} - |h_q|^{2s}$.

However, in the present paper, the nonlinearity $f(x, u)$ contains the spatial variable x explicitly so that (1.4) is not true. Then we solve this problem by eliminating more terms than [BFG20a]. Precisely, we eliminate the monomials $z_{k_1} \dots z_{k_p} \bar{z}_{h_1} \dots \bar{z}_{h_q}$ with $|k_p|, |h_q| > N \geq \mu_3(\mathbf{k}, \mathbf{h})$ and $||k_p| - |h_q|| \geq C_0 N$ for some positive constants C_0 . See Theorem 2.1 for the normal form and see (3.19) for the energy estimate of high modes in higher Sobolev space.

This paper is organized as follows: in Section 2, we give a normal form theorem, seeing Theorem 2.1, which is a modified version of Theorem 5 in [BFG20a]. In section 3, we apply Theorem 2.1 to the nonlinear wave equation (1.1) and thus prove Theorem 1.1. The main step is to control the high modes, and the key is to estimate the higher Sobolev norm with the help of L^2 norm, seeing (3.19). Besides, instead of Lemma 3 in [BFG20a], we estimate the high modes directly from the vector field in the same way as the low modes, seeing (3.21).

2. NORMAL FORM THEOREM

Define the Hilbert space $l_s^2(\mathbb{Z}^d, \mathbb{C})$ of the complex sequences $\xi = \{\xi_a\}_{a \in \mathbb{Z}^d}$ such that

$$\|\xi\|_s^2 := \sum_{a \in \mathbb{Z}^d} \langle a \rangle^{2s} |\xi_a|^2 < \infty \quad (2.1)$$

with $\langle a \rangle^2 := 1 + |a|^2 = 1 + a_1^2 + \cdots + a_d^2$. Notice that for complex function $u(x) = \sum_{a \in \mathbb{Z}^d} \xi_a e^{ia \cdot x}$ on \mathbb{T}^d with $a \cdot x = a_1 x_1 + \cdots + a_d x_d$, the Sobolev norm $\|u\|_{H^s}$ is equivalent to the norm $\|\xi\|_s$. The scale of phase spaces

$$l_s^2 \oplus l_s^2 \ni (\xi, \bar{\xi}) = (\{\xi_a\}_{a \in \mathbb{Z}^d}, \{\bar{\xi}_a\}_{a \in \mathbb{Z}^d})$$

is endowed by the standard symplectic structure $-i \sum_{a \in \mathbb{Z}^d} d\xi_a \wedge d\bar{\xi}_a$. For a Hamiltonian function $H(\xi, \bar{\xi})$, define its vector field

$$X_H(\xi, \bar{\xi}) = -i \left(\frac{\partial H}{\partial \bar{\xi}}, -\frac{\partial H}{\partial \xi} \right), \quad (2.2)$$

and for two Hamiltonian functions $H(\xi, \bar{\xi})$ and $F(\xi, \bar{\xi})$, define their Poisson bracket

$$\{H, F\} = -i \sum_{a \in \mathbb{Z}^d} \left(\frac{\partial H}{\partial \xi_a} \frac{\partial F}{\partial \bar{\xi}_a} - \frac{\partial H}{\partial \bar{\xi}_a} \frac{\partial F}{\partial \xi_a} \right). \quad (2.3)$$

In a brief statement, we identify $\mathbb{C}^{\mathbb{Z}^d} \times \mathbb{C}^{\mathbb{Z}^d} \simeq \mathbb{C}^{\mathbb{U}_2 \times \mathbb{Z}^d}$ with $\mathbb{U}_2 = \{\pm 1\}$ and use the convenient notation $z = (z_j)_{j=(\delta, a) \in \mathbb{U}_2 \times \mathbb{Z}^d}$, where

$$z_j = \begin{cases} \xi_a, & \text{when } \delta = 1, \\ \bar{\xi}_a, & \text{when } \delta = -1. \end{cases}$$

Set $\langle j \rangle = \langle a \rangle$ and define

$$\|z\|_s^2 := \sum_{j \in \mathbb{U}_2 \times \mathbb{Z}^d} \langle j \rangle^{2s} |z_j|^2 = \|\xi\|_s^2 + \|\bar{\xi}\|_s^2. \quad (2.4)$$

In particular, for any $j \in \mathbb{U}_2 \times \mathbb{Z}^d$, decompose $z = z_{\leq N} + z_{> N}$ with

$$(z_{\leq N})_j = \begin{cases} z_j, & \text{for } |j| \leq N \\ 0, & \text{for } |j| > N \end{cases} \quad \text{and} \quad (z_{> N})_j = \begin{cases} 0, & \text{for } |j| \leq N \\ z_j, & \text{for } |j| > N \end{cases}. \quad (2.5)$$

For $\mathbf{j} = (j_1, \cdots, j_r) = (\delta_k, a_k)_{k=1}^r \in (\mathbb{U}_2 \times \mathbb{Z}^d)^r$, denote the monomial $z_{\mathbf{j}} = z_{j_1} \cdots z_{j_r}$. For a homogeneous polynomial $P(z)$ of order r , namely

$$P(z) = \sum_{\mathbf{j} \in (\mathbb{U}_2 \times \mathbb{Z}^d)^r} P_{\mathbf{j}} z_{\mathbf{j}},$$

define the μ -modulus

$$|P|_{\mu} = \sum_{a \in \mathbb{Z}^d} e^{\mu|a|} \sup_{\delta_1 a_1 + \cdots + \delta_r a_r = a} |P_{\mathbf{j}}|. \quad (2.6)$$

Similarly to the proof of Lemma 5.1 and Lemma 5.2 in [Bam03], one has the following estimate of vector field

$$\|X_P(z)\|_s \leq C|P|_\mu \|z\|_s^{r-1} \quad (2.7)$$

with constant C depending on r, s, μ . For two homogeneous polynomials P, Q of order r_1, r_2 respectively with finite μ -modulus, similarly to Lemma 5.9 of [Bam03], one has the following estimate of Poisson bracket

$$|\{P, Q\}|_\mu \leq r_1 r_2 |P|_\mu |Q|_\mu. \quad (2.8)$$

Theorem 2.1. Fix a positive integer r . Consider the Hamiltonian function

$$H = H_0 + P = \sum_{a \in \mathbb{Z}^d} \omega_a |\xi_a|^2 + P, \quad (2.9)$$

where the frequencies $\omega := \{\omega_a\}_{a \in \mathbb{Z}^d}$ and the higher order perturbation P satisfy the following two assumptions respectively:

- (1) for any positive integer l , there exists $\gamma, \tau, C_0 > 0$ such that for any $N \geq 1$ and $\mathbf{j} = (\delta_k, a_k)_{k=1}^l \in (\mathbb{U}_2 \times \mathbb{Z}^d)^l$, $b_1, b_2 \in \mathbb{Z}^d$ with $|a_k| \leq N$, $|b_1|, |b_2| > N$, we have

$$|\delta_1 \omega_{a_1} + \cdots + \delta_l \omega_{a_l}| \geq \frac{\gamma}{N^\tau}, \quad \text{when } \mathbf{j} \notin \mathcal{A}_l, \quad (2.10)$$

$$|\delta_1 \omega_{a_1} + \cdots + \delta_l \omega_{a_l} + \omega_{b_1}| \geq \frac{\gamma}{N^\tau}, \quad (2.11)$$

$$|\delta_1 \omega_{a_1} + \cdots + \delta_l \omega_{a_l} + \omega_{b_1} + \omega_{b_2}| \geq \frac{\gamma}{N^\tau}, \quad (2.12)$$

$$|\delta_1 \omega_{a_1} + \cdots + \delta_l \omega_{a_l} + \omega_{b_1} - \omega_{b_2}| \geq \frac{\gamma}{N^\tau}, \quad \text{when } ||b_1| - |b_2|| \geq C_0 N, \quad (2.13)$$

where $\mathcal{A}_l := \{\mathbf{j} = (\delta_k, a_k)_{k=1}^l \mid \exists \text{ permutation } \sigma, \text{ s.t. } \forall k, \delta_k = -\delta_{\sigma_k}, |a_k| = |a_{\sigma_k}|\}$ is the set of resonant multi-indices;

- (2) $P = \sum_{l \geq 1} P_l$ with P_l homogenous of order $l+2$, and there exists $\mu, C_1, R_0 > 0$ such that

$$|P_l|_\mu \leq C_1 R_0^{-(l+2)}. \quad (2.14)$$

Then there exists a polynomial Hamiltonian $\chi = \sum_{l=1}^r \chi_l$ of order at most $r+2$ satisfying

$$|\chi_l|_\mu \leq C_2 N^{r\tau} \quad (2.15)$$

with constant $C_2 > 0$ depending on r, C_1 and R_0 such that for any given $s > d/2$, the transformation of time one map Φ_χ^1 generated by χ , whose existence is guaranteed in a neighbourhood of the origin of $l_s^2 \oplus l_s^2$, puts H in normal form:

$$H \circ \Phi_\chi^1 = (H_0 + P) \circ \Phi_\chi^1 = H_0 + \mathcal{Z}^{(0)} + \mathcal{R}^{(ii)} + \mathcal{R}^{(iii)} + \mathcal{R}_{r+3}, \quad (2.16)$$

where

(i) the transformation fulfills the estimate

$$\|z - \Phi_\chi^1(z)\|_s \leq C_3 N^{r\tau} \|z\|_s^2 \quad (2.17)$$

in a neighbourhood of the origin of $l_s^2 \oplus l_s^2$ with constant $C_3 > 0$ depending on s and C_2 . Exactly, the same estimate is fulfilled by the inverse transformation;

(ii) $\mathcal{Z}^{(0)}$ is a polynomial of order at most $r+2$ and contains only resonant monomials, that is to say, for any $a \in \mathbb{Z}^d$,

$$\{J_a, \mathcal{Z}^{(0)}\} = 0, \quad (2.18)$$

where $J_a := \sum_{\langle b \rangle = \langle a \rangle} |z_b|^2$ is the super action;

(iii) $\mathcal{R}^{(ii)} = \sum_{l=1}^r \mathcal{R}_l^{(ii)}$ is a polynomial of order at most $r+2$ with two high modes in opposite signs and having small norm difference: they are (δ, b_1) and $(-\delta, b_2)$ with $||b_1| - |b_2|| < C_0 N$. Moreover, the following estimate holds:

$$|\mathcal{R}_l^{(ii)}|_\mu \leq C_2 N^{(r-1)\tau}; \quad (2.19)$$

(iv) $\mathcal{R}^{(iii)} = \sum_{l=1}^r \mathcal{R}_l^{(iii)}$ is a polynomial of order at most $r+2$ with at least three high modes and

$$|\mathcal{R}_l^{(iii)}|_\mu \leq C_2 N^{(r-1)\tau}; \quad (2.20)$$

(v) \mathcal{R}_{r+3} is a polynomial of order at least $r+3$ and

$$\|X_{\mathcal{R}_{r+3}}(z)\|_s \leq C_3 N^{r\tau} \|z\|_s^{r+2}. \quad (2.21)$$

Proof. The proof is parallel to [BFG20a] except an essential difference: some terms with two high modes of opposite signs will be eliminated, while in [BFG20a], these terms are kept in $\mathcal{R}^{(ii)}$. For convenience, we introduce some notations.

For any $l = 1, \dots, r$, decompose homogeneous polynomials P_l of order $l+2$ as follows:

$$P_l = P_l^{(0)} + P_l^{(i)} + P_l^{(ii)} + P_l^{(iii)},$$

where $P_l^{(0)}$ depends only on low modes, namely

$$P_l^{(0)}(z) = \sum_{\substack{j \in (\mathbb{U}_2 \times \mathbb{Z}^d)^{l+2} \\ \mu_1(j) \leq N}} P_{l_j}^{(0)} z_j,$$

$P_l^{(i)}$ contains only one high mode, namely

$$P_l^{(i)}(z) = \sum_{\substack{j \in (\mathbb{U}_2 \times \mathbb{Z}^d)^{l+2} \\ \mu_2(j) \leq N < \mu_1(j)}} P_{l_j}^{(i)} z_j,$$

$P_l^{(ii)}$ contains two high modes, namely

$$P_l^{(ii)}(z) = \sum_{\substack{j \in (\mathbb{U}_2 \times \mathbb{Z}^d)^{l+2} \\ \mu_3(j) \leq N < \mu_2(j)}} P_{l_j}^{(ii)} z_j,$$

and $P_l^{(iii)}$ contains at least three high modes, namely

$$P_l^{(iii)}(z) = \sum_{\substack{j \in (\mathbb{U}_2 \times \mathbb{Z}^d)^{l+2} \\ \mu_3(j) > N}} P_{l_j}^{(iii)} z_j$$

with $\mu_m(\mathbf{j})$ being the m -th largest number amongst the collection $\{|j_k|\}_{k=1}^{l+2}$.

We will not only eliminate the non-resonant terms of $P_l^{(0)}$, all terms of $P_l^{(i)}$, and the terms of $P_l^{(ii)}$ with two high modes in same sign as in [BFG20a], but also eliminate the terms of $P_l^{(ii)}$ with two high modes in opposite signs and having large norm difference, i.e., (δ, b_1) and $(-\delta, b_2)$ with $||b_1| - |b_2|| \geq C_0 N$. For the latter, the corresponding homological equations are solved with the help of non-resonant condition (2.13).

Comparing with [BFG20a], although there is no condition of zero momentum, we still have the estimates of vector field and Poisson bracket, seeing (2.7) and (2.8). Besides, the remaining proof is a standard procedure of non-resonant Birkhoff normal form and thus we omit it. \square

3. PROOF THE MAIN THEOREM

Write the operator $\Lambda := (-\Delta + m)^{1/2}$ and let

$$z = \frac{1}{\sqrt{2}}(\Lambda^{\frac{1}{2}}u + i\Lambda^{-\frac{1}{2}}\dot{u}), \quad (3.1)$$

and then (1.1) is equivalent to

$$\dot{z} = -i\Lambda z - \frac{i}{\sqrt{2}}\Lambda^{-\frac{1}{2}}f(x, \Lambda^{-\frac{1}{2}}(\frac{z + \bar{z}}{\sqrt{2}})). \quad (3.2)$$

Using the Fourier expansion $z(t, x) = \sum_{a \in \mathbb{Z}^d} \xi_a(t) e^{ia \cdot x}$, for any $x \in \mathbb{T}^d$, rewrite (3.2) as

$$\dot{\xi}_a = -i \frac{\partial H}{\partial \xi_a} \quad (3.3)$$

with the Hamiltonian function

$$\begin{aligned} H &= H_0 + P \\ &= \sum_{a \in \mathbb{Z}^d} \omega_a |\xi_a|^2 + \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} F(x, \sum_{a \in \mathbb{Z}^d} (\frac{\xi_a e^{ia \cdot x} + \bar{\xi}_a e^{-ia \cdot x}}{\sqrt{2\omega_a}})) dx, \end{aligned} \quad (3.4)$$

where

$$\omega_a = \sqrt{|a|^2 + m} \quad (3.5)$$

and F is the primitive function of f with respect to the variable u , i.e., $f = \partial_u F$.

In the following, we identify the function z with its sequence of Fourier coefficients

$\{\xi_a\}_{a \in \mathbb{Z}^d}$ (or $\{z_j\}_{j \in \mathbb{U}_2 \times \mathbb{Z}^d}$). In view of (3.1), there exists a constant $c \geq 1$ depending on m and s such that

$$\frac{1}{c} \|(u, \dot{u})\|_{H^s \times H^{s-1}} \leq \|z\|_{s-\frac{1}{2}} \leq c \|(u, \dot{u})\|_{H^s \times H^{s-1}}. \quad (3.6)$$

In the following, we will check that the Hamiltonian H in (3.4) meets two assumptions (1) and (2) in Theorem 2.1.

On the one hand, we show that for almost all $m > 0$, the family of frequencies $\{\omega_a\}_{a \in \mathbb{Z}^d}$ given in (3.5) is non-resonant, namely satisfies conditions (2.10)–(2.13). It is shown in [Del09] that for almost all $m > 0$ and any positive integer r , there exists $\gamma, \tau > 0$ such that (2.10)–(2.12) hold. Then we only need to check the condition (2.13). For any $N \geq 1$ and $\mathbf{j} = (\delta_k, a_k)_{k=1}^l \in (\mathbb{U}_2 \times \mathbb{Z}^d)^l$ with $l \leq r$ and $\langle a_k \rangle \leq N$, one has

$$\omega_{a_k} = \sqrt{|a_k|^2 + m} \leq \sqrt{N^2 - 1 + m} \leq \sqrt{1 + m} N$$

and thus

$$|\delta_1 \omega_{a_1} + \cdots + \delta_l \omega_{a_l}| \leq \sqrt{1 + m} r N.$$

For any $|b_2| > |b_1| > N$ with $|b_2| - |b_1| \geq 2(1 + m)rN$, one has

$$\begin{aligned} \omega_{b_2} - \omega_{b_1} &= \sqrt{|b_2|^2 + m} - \sqrt{|b_1|^2 + m} \\ &= \frac{|b_2|^2 - |b_1|^2}{\sqrt{|b_2|^2 + m} + \sqrt{|b_1|^2 + m}} \\ &\geq \frac{|b_2|^2 - |b_1|^2}{\sqrt{1 + m}(|b_2| + |b_1|)} \\ &= \frac{|b_2| - |b_1|}{\sqrt{1 + m}} \\ &\geq 2\sqrt{1 + m} r N. \end{aligned}$$

Hence, one has

$$\begin{aligned} |\delta_1 \omega_{a_1} + \cdots + \delta_l \omega_{a_l} + \omega_{b_1} - \omega_{b_2}| &\geq \omega_{b_2} - \omega_{b_1} - |\delta_1 \omega_{a_1} + \cdots + \delta_l \omega_{a_l}| \\ &\geq \sqrt{1 + m} r N. \end{aligned}$$

Taking $C_0 = 2(1 + m)r$, (2.13) holds when $||b_1| - |b_2|| \geq C_0 N$.

On the other hand, since F is analytic with a zero of order at least two, then the assumption (2) in Theorem 2.1 holds.

Applying Theorem 2.1, there exists a normalizing transformation Φ_χ^1 such that $\Phi_\chi^{-1}(z) = z'$. Let $N = N_\varepsilon$ and then $N^{2r(2\tau r + d)} \leq N^{s-2s_0} = \varepsilon^{-2r}$. By (2.17) and (3.6), for any small enough ε , one has

$$\|z'(0)\|_{s-\frac{1}{2}} \leq \|z(0)\|_{s-\frac{1}{2}} + \|z(0) - \Phi_\chi^{-1}(z(0))\|_{s-\frac{1}{2}} \leq 2c\varepsilon. \quad (3.7)$$

Then recall the notation (2.5), we have

$$\|z'(0)_{\leq N}\|_{\frac{s}{2}} \leq \|z'(0)\|_{s-\frac{1}{2}} \leq 2c\varepsilon, \quad (3.8)$$

$$\|z'(0)_{>N}\|_{s_0-\frac{1}{2}} \leq N^{s_0-s} \|z'(0)_{>N}\|_{s-\frac{1}{2}} < 2c\varepsilon^{2r+1}. \quad (3.9)$$

Define

$$\tilde{t} = \inf \left\{ t \geq 0 \mid \|z'(t)_{\leq N}\|_{\frac{s}{2}} = 3c\varepsilon \text{ or } \|z'(t)_{>N}\|_{s_0-\frac{1}{2}} = \varepsilon^{r+1} \right\}.$$

and we will prove that $\tilde{t} \geq \varepsilon^{-\frac{r}{2s_0}}$ in two parts.

(1) Control of the low modes $z'(t)_{\leq N}$.

Define $F_{\leq N}(z) = \|z_{\leq N}\|_{\frac{s}{2}}^2$ and then

$$\begin{aligned} |F_{\leq N}(z'(t)) - F_{\leq N}(z'(0))| &= \left| \int_0^t \{H \circ \Phi_\chi^1, F_{\leq N}\}(z'(t)) dt \right| \\ &\leq \int_0^t |\{\mathcal{R}^{(ii)} + \mathcal{R}^{(iii)} + \mathcal{R}_{r+3}, F_{\leq N}\}(z'(t))| dt \\ &\leq |t| \|X_{\mathcal{R}^{(ii)} + \mathcal{R}^{(iii)} + \mathcal{R}_{r+3}}(z'(t))_{\leq N}\|_{\frac{s}{2}} \|z'(t)_{\leq N}\|_{\frac{s}{2}}. \end{aligned} \quad (3.10)$$

By (2.7), (2.19)–(2.21), there exists constant $C > 0$ such that

$$\begin{aligned} \|X_{\mathcal{R}^{(ii)} + \mathcal{R}^{(iii)}}(z'(t))_{\leq N}\|_{\frac{s}{2}} &\leq N^{\frac{s-2s_0+1}{2}} \|X_{\mathcal{R}^{(ii)} + \mathcal{R}^{(iii)}}(z'(t))_{\leq N}\|_{s_0-\frac{1}{2}} \\ &\leq CN^{\frac{s-2s_0+1}{2}} N^{(r-1)\tau} \|z'(t)_{>N}\|_{s_0-\frac{1}{2}}^2 \\ &\leq C\varepsilon^{r+\frac{3}{2}}, \end{aligned} \quad (3.11)$$

$$\begin{aligned} \|X_{\mathcal{R}_{r+3}}(z'(t))_{\leq N}\|_{\frac{s}{2}} &\leq CN^{r\tau} (\|z'(t)_{\leq N}\|_{\frac{s}{2}}^{r+2} + N^{\frac{s-2s_0+1}{2}} \|z'(t)_{\leq N}\|_{s_0-\frac{1}{2}}^{r+1} \|z'(t)_{>N}\|_{s_0-\frac{1}{2}}) \\ &\leq C\varepsilon^{r+\frac{3}{2}}. \end{aligned} \quad (3.12)$$

When $t \leq \varepsilon^{-\frac{r}{2s_0}}$, by (3.8), (3.10)–(3.12), one has

$$\begin{aligned} \|z'(t)_{\leq N}\|_{\frac{s}{2}}^2 &\leq \|z'(0)_{\leq N}\|_{\frac{s}{2}}^2 + |F_{\leq N}(z'(t)) - F_{\leq N}(z'(0))| \\ &\leq 4c^2\varepsilon^2 + C\varepsilon^{-\frac{r}{2s_0}} \varepsilon^{r+\frac{5}{2}} \\ &\leq 9c^2\varepsilon^2. \end{aligned} \quad (3.13)$$

(2) Control of the high modes $z'(t)_{>N}$.

By Theorem 2.1, $\mathcal{R}^{(ii)}$ can be written as

$$\mathcal{R}^{(ii)} = \sum_{\substack{b_1, b_2 \in \mathbb{Z}^d \\ |b_1|, |b_2| > N \\ \||b_1| - |b_2|\| < 2(1+m)rN}} B_{b_1 b_2}(z'_{\leq N}) z'_{b_1} \bar{z}'_{b_2} \quad (3.14)$$

with the estimate

$$|B_{b_1 b_2}(z'_{\leq N})| \leq CN^{(r-1)\tau} \|z'_{\leq N}\|_{\frac{s}{2}} \quad (3.15)$$

for some positive constants C . Besides, as the Hamiltonian is real, we have $\overline{B_{b_1 b_2}(z'_{\leq N})} = B_{b_2 b_1}(z'_{\leq N})$, i.e., the operator $(B_{b_1 b_2}(z'_{\leq N}))_{b_1, b_2}$ is Hermitian so that

$$\{\mathcal{R}^{(ii)}(z'(t)), \|z'(t)_{>N}\|_0^2\} = 0.$$

Thus one has

$$\begin{aligned}
\left| \|z'(t)_{>N}\|_0^2 - \|z'(0)_{>N}\|_0^2 \right| &= \left| \int_0^t \{H \circ \Phi_\chi^1(z'(t)), \|z'(t)_{>N}\|_0^2\} dt \right| \\
&\leq \int_0^t |\{\mathcal{R}^{(iii)} + \mathcal{R}_{r+3}(z'(t)), \|z'(t)_{>N}\|_0^2\}| dt \\
&\leq t \|X_{\mathcal{R}^{(iii)} + \mathcal{R}_{r+3}}(z'(t))_{>N}\|_0 \|z'(t)_{>N}\|_0. \tag{3.16}
\end{aligned}$$

Notice that by (2.7), (2.20) and (2.21), there exists constant $C > 0$ such that

$$\|X_{\mathcal{R}^{(iii)}}(z'(t))_{>N}\|_{s_0 - \frac{1}{2}} \leq CN^{(r-1)\tau} \|z'(t)_{>N}\|_{s_0 - \frac{1}{2}}^2 \leq C\varepsilon^{2r + \frac{3}{2}}, \tag{3.17}$$

$$\begin{aligned}
\|X_{\mathcal{R}_{r+3}}(z'(t))_{>N}\|_{s_0 - \frac{1}{2}} &\leq CN^{r\tau} (N^{\frac{2s_0 - s - 1}{2}} \|z'(t)_{\leq N}\|_{\frac{s}{2}}^{r+2} + \|z'(t)_{\leq N}\|_{s_0 - \frac{1}{2}}^{r+1} \|z'(t)_{>N}\|_{s_0 - \frac{1}{2}}) \\
&\leq C\varepsilon^{2r + \frac{3}{2}}. \tag{3.18}
\end{aligned}$$

By (3.9), (3.16)–(3.18) and the fact $s_0 > 1/2$, one has

$$\begin{aligned}
\|z'(t)_{>N}\|_0^2 &\leq \|z'(0)_{>N}\|_0^2 + t \|X_{\mathcal{R}^{(iii)} + \mathcal{R}_{r+3}}(z'(t))_{>N}\|_{s_0 - \frac{1}{2}} \|z'(t)_{>N}\|_{s_0 - \frac{1}{2}} \\
&\leq C(1+t)\varepsilon^{3r + \frac{5}{2}}. \tag{3.19}
\end{aligned}$$

Define $F_{>N}(z) = \|z_{>N}\|_{s_0 - \frac{1}{2}}^2$ and then

$$\begin{aligned}
|F_{>N}(z'(t)) - F_{>N}(z'(0))| &= \left| \int_0^t \{H \circ \Phi_\chi^1, F_{>N}\}(z'(t)) dt \right| \\
&\leq \int_0^t |\{\mathcal{R}^{(ii)} + \mathcal{R}^{(iii)} + \mathcal{R}_{r+3}, F_{>N}\}(z'(t))| dt. \tag{3.20}
\end{aligned}$$

Using (3.14) and (3.15), one has

$$\begin{aligned}
&|\{\mathcal{R}^{(ii)}, F_{>N}\}(z'(t))| \\
&= \left| \sum_{\substack{|b_1|, |b_2| > N \\ ||b_1| - |b_2|| < 2(1+m)rN}} (\langle b_1 \rangle^{2s_0 - 1} - \langle b_2 \rangle^{2s_0 - 1}) B_{b_1 b_2}(z'_{\leq N}) z'_{b_1} \bar{z}'_{b_2} \right| \\
&\leq \sum_{\substack{|b_1|, |b_2| > N \\ ||b_1| - |b_2|| < 2(1+m)rN}} (2s_0 - 1) |\langle b_1 \rangle - \langle b_2 \rangle| (\langle b_1 \rangle^{2s_0 - 2} + \langle b_2 \rangle^{2s_0 - 2}) |B_{b_1 b_2}(z'_{\leq N})| |z'_{b_1} \bar{z}'_{b_2}| \\
&\leq CN^{(r-1)\tau + 1} \|z'_{\leq N}\|_{\frac{s}{2}} \sum_{\substack{|b_1|, |b_2| > N \\ ||b_1| - |b_2|| < 2(1+m)rN}} \langle b_1 \rangle^{s_0 - \frac{3}{2}} \langle b_2 \rangle^{s_0 - \frac{1}{2}} |z'_{b_1} \bar{z}'_{b_2}| \\
&\leq CN^{(r-1)\tau + d + 1} \|z'_{\leq N}\|_{\frac{s}{2}} \|z'_{>N}\|_{s_0 - \frac{3}{2}} \|z'_{>N}\|_{s_0 - \frac{1}{2}} \\
&\leq \|z'_{>N}\|_0^{\frac{2}{2s_0 - 1}} \|z'_{>N}\|_{s_0 - \frac{1}{2}}^{2 - \frac{2}{2s_0 - 1}}, \tag{3.21}
\end{aligned}$$

where the last inequality follows from the estimate $CN^{(r-1)\tau+d+1}\|z'_{\leq N}\|_{\frac{s}{2}} \leq 1$ and the Hölder inequality $\|z'_{>N}\|_{s_0-\frac{3}{2}} \leq \|z'_{>N}\|_0^{\frac{2}{2s_0-1}} \|z'_{>N}\|_{s_0-\frac{1}{2}}^{1-\frac{2}{2s_0-1}}$. By (3.17) and ((3.18), there exists constant $C > 0$ such that

$$\begin{aligned} |\{\mathcal{R}^{(iii)} + \mathcal{R}_{r+3}, F_{>N}\}(z'(t))| &\leq \|X_{\mathcal{R}^{(iii)}+\mathcal{R}_{r+3}}(z'(t))_{>N}\|_{s_0-\frac{1}{2}} \|z'(t)_{>N}\|_{s_0-\frac{1}{2}} \\ &\leq C\varepsilon^{3r+\frac{5}{2}}. \end{aligned} \quad (3.22)$$

By (3.19)–(3.22), it is easy for any $t \leq \varepsilon^{-\frac{r}{2s_0}}$ to get

$$\begin{aligned} |F_{>N}(z'(t)) - F_{>N}(z'(0))| &\leq t(\|z'_{>N}\|_0^{\frac{2}{2s_0-1}} \|z'_{>N}\|_{s_0-\frac{1}{2}}^{2-\frac{2}{2s_0-1}} + C\varepsilon^{3r+\frac{5}{2}}) \\ &\leq t(C(1+t)\varepsilon^{3r+\frac{5}{2}})^{\frac{1}{2s_0-1}} \varepsilon^{(r+1)(2-\frac{2}{2s_0-1})} + Ct\varepsilon^{3r+\frac{5}{2}} \\ &\leq \frac{1}{2}\varepsilon^{2(r+1)}. \end{aligned} \quad (3.23)$$

By (3.9) and (3.23), we obtain

$$\|z'(t)_{>N}\|_{s_0-\frac{1}{2}}^2 \leq \|z'(0)_{>N}\|_{s_0-\frac{1}{2}}^2 + |F_{>N}(z'(t)) - F_{>N}(z'(0))| \leq \varepsilon^{2(r+1)}. \quad (3.24)$$

Combining (3.13) and (3.24), we can conclude that $\tilde{t} \geq \varepsilon^{-\frac{r}{2s_0}}$.

By (2.7) and (2.15), there exists constant $C > 0$ such that

$$\begin{aligned} \|X_\chi(z'(t))_{\leq N}\|_{\frac{s}{2}} &\leq CN^{r\tau}(\|z'(t)_{\leq N}\|_{\frac{s}{2}}^2 + N^{\frac{s-2s_0+1}{2}}\|z'(t)_{\leq N}\|_{s_0-\frac{1}{2}}\|z'(t)_{>N}\|_{s_0-\frac{1}{2}}) \\ &\leq C\varepsilon^{\frac{3}{2}}, \end{aligned} \quad (3.25)$$

$$\begin{aligned} \|X_\chi(z'(t))_{>N}\|_{s_0-\frac{1}{2}} &\leq CN^{r\tau}(N^{\frac{2s_0-s-1}{2}}\|z'(t)_{\leq N}\|_{\frac{s}{2}}^2 + \|z'(t)_{\leq N}\|_{s_0-\frac{1}{2}}\|z'(t)_{>N}\|_{s_0-\frac{1}{2}}) \\ &\leq C\varepsilon^{r+\frac{3}{2}}. \end{aligned} \quad (3.26)$$

So by (3.13) and (3.24)–(3.26), for any $t \leq \varepsilon^{-\frac{r}{2s_0}}$, we have

$$\begin{aligned} \|z(t)_{\leq N}\|_{\frac{s}{2}} &= \|\Phi_\chi^1(z'(t))_{\leq N}\|_{\frac{s}{2}} \leq \|z'(t)_{\leq N}\|_{\frac{s}{2}} + \|X_\chi(z'(t))_{\leq N}\|_{\frac{s}{2}} < 4c\varepsilon, \\ \|z(t)_{>N}\|_{s_0-\frac{1}{2}} &= \|\Phi_\chi^1(z'(t))_{>N}\|_{s_0-\frac{1}{2}} \leq \|z'(t)_{>N}\|_{s_0-\frac{1}{2}} + \|X_\chi(z'(t))_{>N}\|_{s_0-\frac{1}{2}} \leq 2\varepsilon^{r+1}. \end{aligned}$$

Going back to the original variables u , take $c_0 = 4c^2$ and by (3.6), we can obtain

$$\|u(t)_{\leq N}\|_{H^{\frac{s}{2}}} \leq c_0\varepsilon \quad \text{and} \quad \|u(t)_{>N}\|_{H^{s_0}} \leq \varepsilon^r.$$

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REFERENCES

- [Bam03] D. Bambusi. Birkhoff normal form for some nonlinear PDEs. *Comm. Math. Phys.*, 234(2):253–285, 2003. (Cited on section 1, 2, and 2.)

- [Bam08] D. Bambusi. A Birkhoff normal form theorem for some semilinear PDEs. In *Hamiltonian dynamical systems and applications*, NATO Sci. Peace Secur. Ser. B Phys. Biophys., pages 213–247. Springer, Dordrecht, 2008. (Cited on section 1.)
- [BDGS07] D. Bambusi, J.-M. Delort, B. Grébert, and J. Szeftel. Almost global existence for Hamiltonian semilinear Klein-Gordon equations with small Cauchy data on Zoll manifolds. *Comm. Pure Appl. Math.*, 60(11):1665–1690, 2007. (Cited on section 1.)
- [BG06] D. Bambusi and B. Grébert. Birkhoff normal form for partial differential equations with tame modulus. *Duke Math. J.*, 135(3):507–567, 2006. (Cited on section 1.)
- [BFG20a] J. Bernier, E. Faou, and B. Grébert. Long time behavior of the solutions of NLW on the d -dimensional torus. *Forum Math. Sigma*, 8:12–26, 2020. (Cited on section 1, 1, 1, 1, and 2.)
- [BFG20b] J. Bernier, E. Faou, and B. Grébert. Rational Normal Forms and Stability of Small Solutions to Nonlinear Schrödinger Equations. *Ann. PDE*, 6(2):14, 2020. (Cited on section 1.)
- [BG21] J. Bernier and B. Grébert. Long time dynamics for generalized Korteweg-de Vries and Benjamin-Ono equations. *Arch. Ration. Mech. Anal.*, 241(3):1139–1241, 2021. (Cited on section 1.)
- [BD18] M. Berti and J. M. Delort. *Almost global solutions of capillary-gravity water waves equations on the circle*, volume 24 of *Lecture Notes of the Unione Matematica Italiana*. Springer, Cham; Unione Matematica Italiana, [Bologna], 2018. (Cited on section 1.)
- [BMP20] L. Biasco, J.-E. Massetti, and M. Procesi. An abstract Birkhoff normal form theorem and exponential type stability of the 1d NLS. *Comm. Math. Phys.*, 375(3):2089–2153, 2020. (Cited on section 1.)
- [Bou96] J. Bourgain. Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations. *Geom. Funct. Anal.*, 6(2):201–230, 1996. (Cited on section 1.)
- [Bou00] J. Bourgain. On diffusion in high-dimensional Hamiltonian systems and PDE. *J. Anal. Math.*, 80:1–35, 2000. (Cited on section 1.)
- [CCMW22] Q. Chen, H. Cong, L. Meng, and X. Wu. Long time stability result for 1-dimensional nonlinear Schrödinger equation. *J. Differential Equations*, 315:90–121, 2022. (Cited on section 1.)
- [CLW20] H. Cong, C. Liu, and P. Wang. A Nekhoroshev type theorem for the nonlinear wave equation. *J. Differential Equations*, 269(4):3853–3889, 2020. (Cited on section 1.)
- [CLY16] H. Cong, J. Liu, and X. Yuan. Stability of KAM tori for nonlinear Schrödinger equation. *Mem. Amer. Math. Soc.*, 239(1134):vii+85, 2016. (Cited on section 1.)
- [CMW20] H. Cong, L. Mi, and P. Wang. A Nekhoroshev type theorem for the derivative nonlinear Schrödinger equation. *J. Differential Equations*, 268(9):5207–5256, 2020. (Cited on section 1.)
- [CMWZ22] H. Cong, L. Mi, X. Wu, and Q. Zhang. Exponential stability estimate for the derivative nonlinear Schrödinger equation. *Nonlinearity*, 35(5):2385–2423, 2022. (Cited on section 1.)
- [Del09] J. M. Delort. On long time existence for small solutions of semi-linear Klein-Gordon equations on the torus. *J. Anal. Math.*, 107:161–194, 2009. (Cited on section 1 and 3.)
- [Del15] J. M. Delort. Quasi-linear perturbations of Hamiltonian Klein-Gordon equations on spheres. *Mem. Amer. Math. Soc.*, 234(1103):vi+80, 2015. (Cited on section 1.)
- [FG13] E. Faou and B. Grébert. A Nekhoroshev-type theorem for the nonlinear Schrödinger equation on the torus. *Anal. PDE*, 6(6):1243–1262, 2013. (Cited on section 1.)
- [GIP09] B. Grébert, R. Imekraz, and É. Paturel. Normal forms for semilinear quantum harmonic oscillators. *Comm. Math. Phys.*, 291(3):763–798, 2009. (Cited on section 1.)
- [YZ14] X. Yuan and J. Zhang. Long time stability of Hamiltonian partial differential equations. *SIAM J. Math. Anal.*, 46(5):3176–3222, 2014. (Cited on section 1.)

- [Zha10] Q. Zhang. Long-time existence for semi-linear Klein-Gordon equations with quadratic potential. *Comm. Partial Differential Equations*, 35(4):630–668, 2010. (Cited on section 1.)
- [Zha20] J. Zhang. Almost global solutions to Hamiltonian derivative nonlinear Schrödinger equations on the circle. *J. Dynam. Differential Equations*, 32(3):1401–1455, 2020. (Cited on section 1.)

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