Solvability for a higher-order Hadamard fractional differential model with a sign-changing nonlinearity dependent on the parameter ϱ

Limin Guo[a], Ying Wang[b,*], Cheng Li[c,d], Jianwen Cai[c], Bo. Zhang[d]

[a]School of Science, Changzhou Institute of Technology, Changzhou 213002, Jiangsu, China

[b] School of Mathematicacs and Statistics, Linyi University, Linyi, 276000, Shandong, China

[c]School of automotive engineering, Changzhou Institute of Technology, Changzhou 213002, Jiangsu,China

[d]Applied Mechanics and Structure Safety Key Laboratory of Sichuan Province, School of Mechanics and Engineering, Southwest Jiaotong University, Chengdu 611756, China

---

Abstract Until now most of the results are obtained in the sense of fractional derivatives such as Caputo and Riemann-Liouville, and there are few models using the Hadamard fractional derivatives. In this paper, based on the properties of the Green's function, the existence of positive solutions are obtained for a Hadamard fractional differential equation with a higher-order sign-changing nonlinearity under some conditions by the fixed point theorem, and the existence of positive solutions is dependent on the parameter ϱ for the Semipositive problem.

Keywords Semipositive problem; Hadamard fractional differential equation; Positive solution; Parameter

MR Subject Classification: 34B16 · 34B18

---

1. Introduction

The empirical formulas of some complex mechanical processes are often expressed in the form of power law functions, and the corresponding mechanical constitutive relations do not conform to the standard "gradient" laws, such as Darcy's law, Fourier heat conduction and Fick diffusion. These mechanical processes have obvious memorability, heritability and path dependence, and when describing mechanical processes with these properties, we usually construct nonlinear differential equations with integer order differentiation, and the constructed nonlinear differential equations need to introduce some artificial empirical parameters and assumptions that are inconsistent with the actual situation, and sometimes due to small changes in materials or external conditions, we need to construct a new model. However, fractional-order differential operators can simply and accurately describe mechanical and physical processes with historical memory and spatial global correlation, and the fractional-order derivative modeling is simple, the physical meaning of parameters is clear and the description is accurate, so fractional differential equation is one of the important tools for mathematical modeling of complex mechanics, physics, medicine and other processes. There has been a significant development in the study of fractional differential equations in recent years, for an extensive collection of such literature, readers can refer to [4–7, 9–12, 14–16, 18–25] and the references therein. There are some periodic achievements for Hadamard fractional differential equation, for example [1, 3]. In [3], the authors consider the following BVP:

*E-mail addresses: lywy1981@163.com.

$$
\begin{aligned}
&{}^{H}D^{\alpha}_{1^+}x(t)+\hbar(t,x(t))=0, \text{for a.e.} t\in J=[1,T],\ 1<\alpha\le 2,\\
&x(1)=0,{}^{H}D^{p}_{1^+}x(T)=\sum_{i=1}^{n}\kappa_i{}^{H}D^{p}_{1^+}x(\mu_i),\ 0<p<1,
\end{aligned}
$$

where $\hbar:[1,T]\times\mathbb{R}\to\mathbb{R}$ is a given function, $\lambda_i\in\mathbb{R}, 1<\mu_i\le T, j=1,2,\dots,n, n\ge 2$, and $(\log T)^{\alpha-p-1}\ne\sum_{i=1}^{n}\kappa_i(\log\mu_i)^{\alpha-p-1}$, ${}^{H}D^{\alpha}_{1^+}$ and ${}^{H}D^{p}_{1^+}$are the Hadamard fractional derivatives of order α and p, respectively. In [1], Algoudi got existence results for the following sequential Hadamard type fractional differential equation:

$$
\begin{aligned}
&({}^{H}D^{p}_{1^+}+\lambda{}^{H}D^{p-1}_{1^+})x(t)=f_1(t,x(t),y(t),{}^{H}D^{r}_{1^+}y(t)),1<p\le 2,0<r<1,\\
&({}^{H}D^{q}_{1^+}+\lambda{}^{H}D^{q-1}_{1^+})y(t)=f_2(t,x(t),{}^{H}D^{v}_{1^+}y(t),y(t)),1<q\le 2,0<v<1,\\
&x(1)=0,x(e)={}^{H}I^{\gamma}_{1^+}y(\eta),\gamma>0,1<\eta<e,\\
&y(1)=0,y(e)={}^{H}I^{\beta}_{1^+}x(\zeta),\beta>0,1<\zeta<e,
\end{aligned}
$$

where $f_1,f_2:[1,e]\times\mathbb{R}^3\to\mathbb{R}$ are continuous functions, ${}^{H}D^{p}_{1^+},{}^{H}I^{\gamma}_{1^+}$ denote the Hadamard fractional derivative and integral p,γ, respectively. In [2], Arul and Karthikeyan considered fractional differential equation of Hadamard type:

$$
{}^{H}D^{\vartheta}_{b^+}x(t)=g(t,x(t),{}^{H}D^{\vartheta}x(t)),t\in(b,\tau),
$$

with an initial value $x(b)=0,x(\tau)=\lambda\int_0^{\sigma}x(s)ds,\sigma\in(b,\tau),\lambda\in\mathbb{R}$, where ${}^{H}D^{\vartheta}_{b^+}$ denotes Hadamard fractional derivative of order ϑ, $1<\vartheta\le 2$, the authors obtained the existence and uniqueness of solutions for this equation.

Motivated by the excellent results above, in this paper, we consider the following Hadamard fractional differential equation:

$$
{}^{H}D^{\nu}_{1^+}u(t)+\varrho\hbar(t,u(t))=0,\ 1<t<e, \tag{1.1}
$$

with nonlocal Hadamard integral boundary conditions

$$
u^{(i)}(1)=0,i \text{ from } 0 \text{ to } n-2,{}^{H}D^{p_0}_{1^+}u(e)=\sum_{i=1}^{m}\chi_i\int_1^{\eta_i}\rho_i(s){}^{H}D^{p_i}_{1^+}u(s)dA_i(s), \tag{1.2}
$$

where $\nu,p_i\in\mathbb{R}^1_+=[0,+\infty)(i=0,1,2,\cdots,m), 0\le p_1\le p_2\le\dots\le p_m\le p_0<\nu-1, p_0\ge 1$, $n\in\mathbb{N}$ (natural number set), $n-1<\nu\le n$, $\rho_i(s)\in L^1(1,e)$, $0<\chi_i\le 1(i=1,2,\dots,m)$ are parameters, $\hbar(t,x)$ may change sign and may be singular at $t=1$ or $t=e$, A_i is a function of bounded variation, $\sum_{i=1}^{m}\chi_i\int_1^{\eta_i}\rho_i(s){}^{H}D^{p_i}_{1^+}u(s)dA_i(s)$ denotes the Riemann-Stieltjes integral with respect to A_i $(i=1,2,\dots,m)$, and ${}^{H}D^{\nu}_{1^+}u$, ${}^{H}D^{p_i}_{1^+}u(i=0,1,2,\dots,m)$ are the standard Hadamard derivatives.

Compared with [1–3], positive solution of this paper depends on a parameter and therefore the results we obtained is relatively accurate; compared with [24], the derivatives which we used in this paper are Hadamard fractional derivatives and the questions we considered is semipositone problem; compared with [17, 24], the equation we considered in this paper is a very wide type, and [17, 24] is special cases of this paper, moreover, the positive solutions is dependent on parameter ϱ.

2. Preliminaries and lemmas

For the reader's convenience, we first give some basic definitions and lemmas that are useful for the following research, which can be found in recent literature, for example [13].

Definition 2.1([13]). Let $x:(0,\infty)\to\mathbb{R}^1_+$, the Hadamard fractional integral of order $\nu>0$ of a function x is given by

$$^{H}I^{\nu}_{a+}x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(\ln\frac{t}{s})^{\nu-1}\frac{x(s)}{s}ds.$$

Definition 2.2([13]). Let $x:(0,\infty)\to\mathbb{R}^1_+$, the Hadamard fractional derivative of order $\nu>0$ of a continuous function x is given by

$$^{H}D^{\nu}_{a+}x(t)=\frac{1}{\Gamma(n-\nu)}\left(t\frac{d}{dt}\right)^n\int_a^t\frac{x(s)}{s\left(\ln\frac{t}{s}\right)^{\nu-n+1}}ds,$$

where $n=[\nu]+1,\ [\nu]$ denotes the integer part of the number ν, provided that the right-hand side is pointwise defined on $(0,\infty)$.
Lemma 2.1([13]). If $\nu,\mu>0$, then

$$^{H}I^{\nu}_{a}\left(\ln(\frac{t}{a})^{\mu-1}\right)(x)=\frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}\left(\ln\frac{x}{a}\right)^{\mu+\nu-1},$$

$$^{H}D^{\nu}_{a}\left(\ln(\frac{t}{a})^{\mu-1}\right)(x)=\frac{\Gamma(\mu)}{\Gamma(\mu-\nu)}\left(\ln\frac{x}{a}\right)^{\mu-\nu-1}.$$

Lemma 2.2([13]). Suppose that $\nu>0$ and $x\in C[1,\infty)\cap L^1[1,\infty)$, then the solution of Hadamard fractional differential equation $^{H}D^{\nu}_{1+}x(t)=0$ is

$$x(t)=d_1(\ln t)^{\nu-1}+d_2(\ln t)^{\nu-2}+\cdots+d_n(\ln t)^{\nu-n},d_i\in R,i=0,1,\cdots,n,\ n=[\nu]+1.$$

Lemma 2.3([13]). Suppose that $\nu>0,\nu$ is not natural number. If $x\in C[1,\infty)\cap L^1[1,\infty)$, then

$$x(t)={}^{H}I^{\nu}_{1+}{}^{H}D^{\nu}_{1+}x(t)+\sum_{k=1}^{n}d_k(\ln t)^{\nu-k},$$

for $t\in(1,e]$, where $d_k\in R,\ k=1,2,\cdots,n$, and $n=[\nu]+1$.

We consider the linear fractional differential equation

$$^{H}D^{\nu}_{1+}u(t)+g(t)=0,\ 1<t<e,\tag{2.1}$$

with boundary condition (1.2).

Lemma 2.4. Given $g\in L^1(1,e)\cap C(1,e)$, then, BVP (2.1) with boundary conditions (1.2) can be expressed as

$$y(t)=\int_1^e\wp(t,s)\frac{g(s)}{s}ds,\ t\in[1,e],\tag{2.2}$$

where

$$\wp(t,s)=\wp_1(t,s)+\frac{(\ln t)^{\nu-1}}{\Delta}\sum_{i=1}^{m}\chi_i\left(\int_0^{\eta_i}\rho_i(s)\wp_{2i}(\tau,s)dA_i(\tau)\right),$$

$$\wp_1(t,s)=\frac{1}{\Gamma(\nu)}\begin{cases}(\ln t)^{\nu-1}(\ln\frac{e}{s})^{\nu-p_0-1}-(\ln\frac{t}{s})^{\nu-1}, & 1\le s\le t\le e,\\ (\ln t)^{\nu-1}(\ln\frac{e}{s})^{\nu-p_0-1}, & 1\le t\le s\le e,\end{cases}\tag{2.3}$$

$$\wp_{2i}(t,s)=\frac{1}{\Gamma(\nu-p_i)}\begin{cases}(\ln t)^{\nu-p_i-1}(\ln\frac{e}{s})^{\nu-p_0-1}-(\ln\frac{t}{s})^{\nu-p_i-1}, & 1\le s\le t\le e,\\ (\ln t)^{\nu-p_i-1}(\ln\frac{e}{s})^{\nu-p_0-1}, & 1\le t\le s\le e,\end{cases}\tag{2.4}$$

for $i=1,2,\ldots,m$, and $\Delta=\frac{\Gamma(\nu)}{\Gamma(\nu-p_0)}-\sum_{i=1}^m\frac{\chi_i\Gamma(\nu)}{\Gamma(\nu-p_i)}\int_1^{\eta_i}\rho_i(s)(\ln s)^{\nu-p_i-1}dA_i(s)$.

Proof. This proof is similar to Lemma 2.4 of [8] and we omit it here.

Lemma 2.5. The functions $\wp_1$ and $\wp$ given by (2.2) have the following properties:

(1) $\wp_1(t,s)\geq\frac{1}{\Gamma(\nu)}(\ln t)^{\nu-1}\ln s(\ln\frac{e}{s})^{\nu-p_0-1}$, $\forall t,s\in[1,e]$;
(2)$\wp_1(t,s)\leq\frac{1}{\Gamma(\nu-1)}(\ln s)(\ln\frac{e}{s})^{\nu-p_0-1}$,$\forall t,s\in[1,e]$;
(3) $\wp_1(t,s)\leq\frac{1}{\Gamma(\nu-1)}(\ln t)^{\nu-1}(\ln\frac{e}{s})^{\nu-p_0-1}$;
(4) $\wp(t,s)\leq J(s), J(s)=\frac{1}{\Gamma(\nu-1)}(\ln s)(\ln\frac{e}{s})^{\nu-p_0-1}+\frac{1}{\Delta}\sum_{i=1}^m\chi_i\left(\int_1^{\eta_i}\rho_i(s)\wp_{2i}(\tau,s)dA_i(\tau)\right)$ for all $t,s\in[1,e],i=1,2,\ldots,m$;
(5) $\frac{1}{\nu-1}(\ln t)^{\nu-1}J(s)\leq\wp(t,s)\leq\sigma(\ln t)^{\nu-1}$, where

$$\sigma=\frac{1}{\Gamma(\nu)}(\ln\frac{e}{s})^{\nu-p_0-1}\left[\nu-1+\frac{1}{\Delta}\sum_{i=1}^m\chi_i(\int_0^{\eta_i}\rho_i(s)(\ln s)^{\nu-p_0-1}dA_i(\tau)\right],$$

for $\forall t,s\in[1,e]$.
Proof. The proof is similar to that of Lemma 2.4 of [8] and we omit it here.

Lemma 2.6. Assume that $\chi_i\geq 0$ for all $i=1,2,\ldots,m$, $\Delta>0$, $g\in L^1(1,e)\cap C(1,e)$ and $g(t)>0$ for all $t\in(1,e)$. Then the solution u of problem (2.1) and (1.2) satisfies the inequality $u(t)\geq\frac{1}{\nu-1}(\ln t)^{\nu-1}u(t')$ for all $t,t'\in[1,e]$.

Proof. By means of Lemma 2.5, for all $t,t'\in[1,e]$, we get

$$\begin{aligned}u(t)=&\int_1^e\wp(t,s)\frac{g(s)}{s}ds\geq\int_1^e\frac{1}{\nu-1}(\ln t)^{\nu-1}J(s)\frac{g(s)}{s}ds\\ \geq&\frac{1}{\nu-1}(\ln t)^{\nu-1}\int_1^e\wp(t',s)\frac{g(s)}{s}ds=\frac{1}{\nu-1}(\ln t)^{\nu-1}u(t').\end{aligned}$$

In order to get our main results, the following Guo-Krasnosel'skii fixed point theorem is presented below.

Theorem 2.1. Let E be a Banach space and let $R\subset E$ be a cone in E. Assume Λ_1 and Λ_2 are bounded open subsets of E with $\theta\in\Lambda_1\subset\overline{\Lambda}_1\subset\Lambda_2$, and let $A:R\cap(\overline{\Lambda}_1\subset\Lambda_2)\to R$ be a completely continuous operator such that

(i) $\|Av\|\leq\|v\|,v\in R\cap\partial\Lambda_1$ and $\|Av\|\geq\|v\|,v\in R\cap\partial\Lambda_2$,
or
(ii)$\|Av\|\geq\|v\|,v\in R\cap\partial\Lambda_1$ and $\|Av\|\leq\|v\|,v\in R\cap\partial\Lambda_2$. Then A has a fixed point in $R\cap(\overline{\Lambda}_1\subset\Lambda_2)$.

3 Main results

In this section, we investigate the solvability for higher-order Hadamard fractional differential model with a sign-changing nonlinearity. First, we propose the assumptions which we will use.

(H_1) $0\leq p_1<p_2<\ldots<p_m\leq p_0<\nu-1$ and

$$\Delta=\frac{\Gamma(\nu)}{\Gamma(\nu-p_0)}e^{\nu-p_0-1}-\sum_{i=1}^m\frac{\chi_i\Gamma(\nu)}{\Gamma(\nu-p_i)}\int_1^{\eta_i}\rho_i(s)(\ln s)^{\nu-\beta_i-1}dA_i(s)>0.$$

(H_2) The function $\hbar\in C((1,e)\times[0,\infty),R)$ may be singular at $t=1$ and (or) $t=e$, and there exist the functions $\theta,\nu\in C((1,e),[0,\infty)$, $\vartheta\in C([1,e]\times[0,\infty),[0,\infty))$ such that $-\theta(t)\leq\hbar(t,x)\leq\nu(t)\vartheta(t,x)$ for all $t\in(1,e)$ and $x\in[0,\infty)$ with $0<\int_1^e\theta(t)\frac{dt}{t}<\infty,0<\int_1^e\omega(t)\frac{dt}{t}<\infty$.

(H_3) There exist $a \in (1, \frac{e-1}{2})$ such that $\hbar_\infty = \lim_{u\to\infty} \min_{t\in[a,e-a]} \frac{\hbar(t,u)}{u} = \infty$.

(H_4) There exist $a \in (1, \frac{e-1}{2})$ such that $\liminf_{u\to\infty} \min_{t\in[a,e-a]} \hbar(t,u) > M_0$, with $M_0 = (2\sigma \int_1^e \theta(s)\frac{ds}{s})$ $/(\ln a)^{\alpha-1} \int_a^{e-a} J(s)\frac{ds}{s})$, and $\vartheta_\infty = \lim_{u\to\infty} \max_{t\in[1,e]} \frac{\vartheta(t,u)}{u} = 0$, where J and σ are given in Section 2.

Next, let's consider the following fractional differential equation:

$$^{H}D^{\nu}_{1+}x(t) + \varrho\left(\hbar(t, [x(t) - \varrho\varpi(t)]^\star) + \theta(t)\right) = 0,\ 1 < t < e, \tag{3.1}$$

with nonlocal Hadamard integral conditions

$$x^{(i)}(1) = 0, i \text{ is from } 0 \text{ to } n-2, ^{H}D^{p_0}_{1+}x(e) = \sum_{i=1}^{m} \chi_i \int_1^{\eta_i} \rho_i(s)^H D^{p_i}_{1+}x(s)dA_i(s), \tag{3.2}$$

where $\varrho > 0$ and $\varsigma^\star(t) = \varsigma(t)$ if $\varsigma(t) \geq 0$, and $\varsigma^\star(t) = 0$ if $\varsigma(t) < 0$. Here, $\varpi(t) = \int_1^e \wp(t,s)\theta(s)\frac{ds}{s}, t \in [1,e]$ is the solution of problem

$$\begin{aligned}
&^{H}D^{\nu}_{1+}\varpi(t) + \theta(t) = 0,\ 1 < t < e,\\
&\varpi^{(i)}(1) = 0, i \text{ is from } 0 \text{ to } n-2, ^{H}D^{p_0}_{1+}\varpi(e) = \sum_{i=1}^{m} \chi_i \int_1^{\eta_i} \rho_i(s)^H D^{p_i}_{1+}\varpi(s)dA_i(s).
\end{aligned}$$

By (H_1)-(H_2), we get $\varpi(t) \geq 0$ for all $t \in [1,e]$. We will show that there exists a solution x for problem (3.1)-(3.2) with $x(t) \geq \varrho\varpi(t)$ on $[1,e]$ and $x(t) > \lambda\varpi(t)$ on $(1,e)$. Under the circumstances, $u = x - \varrho\varpi$ represents the positive solution to the problem (1.1-1.2). Hence, in the following we shall investigate problem (3.1-3.2).

By means of Lemma 2.4, a positive solution of equation

$$x(t) = \varrho \int_1^e \wp(t,s)\left(\hbar(s, [x(s) - \varrho\varpi(s)]^\star) + \theta(s)\right)\frac{ds}{s} = 0,\ 1 < t < e,$$

is a positive solution for problem (3.1-3.2).

So let's think about Banach space $X = C([1,e])$ with the superemun norm $\|\cdot\|$, and let's define a cone

$$K = \{x \in X : x(t) \geq \frac{1}{\nu - 1}(\ln t)^{\nu-1}\|x\|, \forall t \in [1,e]\}.$$

For $\varrho > 0$, we introduce the operator $T : X \to X$, which is defined by

$$Tx(t) = \varrho \int_1^e \wp(t,s)\left(\hbar(s, [x(s) - \varrho\varpi(s)]^\star) + \theta(s)\right)\frac{ds}{s},\ 1 < t < e, x \in X.$$

Obviously, if x is the fixed point of the operator T, then x is the solution to the problem (3.1-3.2).
Lemma 3.1. If $(H_1$-$H_2)$ hold, then the operator $T : K \to K$ is a completely continuous operator.
Proof. Let $x \in K$ be fixed. By using $(H_1$-$H_2)$, we deduce that $Tx(t) < \infty$ for all $t \in [1,e]$. Besides, by means of Lemma 2.5, for all $t, t' \in [1,e]$, we have

$$\begin{aligned}
Tx(t) \leq& \varrho \int_1^e J(s)\left(\hbar(s, [x(s) - \varrho\varpi(t)]^\star) + \theta(s)\right)\frac{ds}{s},\ x \in X,\\
Tx(t) \geq& \frac{\varrho}{\nu - 1}\int_1^e (\ln t)^{\nu-1} J(s)\left(\hbar(s, [x(s) - \varrho\varpi(s)]^\star) + \theta(s)\right)\frac{ds}{s},\\
\geq& \frac{1}{\nu - 1}(\ln t)^{\nu-1} Tx(t').
\end{aligned}$$

Hence, $Tx(t) \geq \frac{1}{v-1}(\ln t)^{v-1}\|Tx\|$ for all $t \in [1,e]$, then we deduce that $Tx \in K$, and therefore, $T(K) \subset K$. By using the standard method, we get the operator $T : K \to K$ is a completely continuous operator.
Theorem 3.1. Suppose that condition (H_1-H_3) hold, there exists $\varrho^\star > 0$, for any $\varrho \in (0, \varrho^\star)$, such that boundary value problem (1.1-1.2) has at least one positive solution.
Proof. Let's pick a positive number, such that $R_1 > \sigma(\nu-1)\int_1^e \theta(s)\frac{ds}{s} > 0$, and we define the set $\Omega_1 = \{x \in K : \|x\| < R_1\}$.

Let's introduce

$$\varrho^\star = \min\left\{1, R_1\left(\Upsilon_1 \int_1^e J(s)(\nu(s)+\theta(s))\right)^{-1}\right\}$$

with $\Upsilon_1 = \max\{\max_{t\in[1,e],u\in[0,R_1]} \vartheta(t,u), 1\}$.

Let $\varrho \in (0, \varrho^\star]$. Since $\varpi(t) \leq \sigma(\ln t)^{\nu-1}\int_1^e \theta(s)\frac{ds}{s}$ for all $t \in [1,e]$, we deduce for any $x \in K \cap \partial\Omega_1$ and $t \in [1,e]$,

$$[x(t) - \varrho\varpi(t)]^\star \leq x(t) \leq \|x\| \leq R_1,$$

then

$$\begin{aligned}
&x(t) - \varrho\varpi(t)\\
\geq&\frac{1}{\nu-1}(\ln t)^{\nu-1}\|x\| - \varrho\sigma(\ln t)^{\nu-1}\int_1^e \theta(s)\frac{ds}{s} = (\ln t)^{\nu-1}\left(\frac{R_1}{\nu-1} - \varrho\sigma\int_1^e \theta(s)\frac{ds}{s}\right)\\
\geq&(\ln t)^{\nu-1}\left(\frac{R_1}{\nu-1} - \varrho^\star\sigma\int_1^e \theta(s)\frac{ds}{s}\right) \geq (\ln t)^{\nu-1}\left(\frac{R_1}{\nu-1} - \sigma\int_1^e \theta(s)\frac{ds}{s}\right) \geq 0.
\end{aligned}$$

So for any $x \in K \cap \partial\Omega_1$ and $t \in [1,e]$, by (H_2), we have

$$\begin{aligned}
Tx(t) \leq&\varrho\int_1^e J(s)\left((\nu(s)\vartheta(s,[x(s)-\varrho\varpi(s)]^\star)) + \theta(s)\right)\frac{ds}{s}\\
\leq&\varrho^\star\Upsilon_1\int_1^e J(s)(\nu(s)+\theta(s))\frac{ds}{s} \leq R_1 = \|x\|.
\end{aligned}$$

Hence, we get

$$\|Tx\| \leq \|x\|, \forall x \in K \cap \partial\Omega_1. \tag{3.3}$$

Furthermore, for a, by (H_3), we pick a positive constant $L > 0$ such that

$$L > 2(\nu-1)^2\left(\varrho(\ln a)^{2(\nu-1)}\int_a^{e-a} J(s)\frac{ds}{s}\right)^{-1}.$$

By (H_3), we deduce that there exists a constant $L_0 > 0$ such that

$$\hbar(t,u) \geq Lu, \forall t \in [a, e-a], u \geq L_0. \tag{3.4}$$

Now we define $R_2 = \max\{2R_1, 2(\nu-1)L_0/(\ln a)^{\nu-1}\}$ and let $\Omega_2 = \{x \in K : \|x\| < R_2\}$. Then for any $x \in K \cap \partial\Omega_2$, we have

$$\begin{aligned}
x(t) - \varrho\varpi(t) \geq&\frac{1}{\nu-1}(\ln t)^{\nu-1}\|x\| - \varrho\sigma(\ln t)^{\nu-1}\int_1^e \theta(s)\frac{ds}{s} = (\ln t)^{\nu-1}\left(\frac{R_2}{\nu-1} - \varrho\sigma\int_1^e \theta(s)\frac{ds}{s}\right)\\
\geq&(\ln t)^{\nu-1}\left(\frac{R_1}{\nu-1} - \varrho^\star\sigma\int_1^e \theta(s)\frac{ds}{s}\right) \geq (\ln t)^{\nu-1}\left(\frac{R_1}{\nu-1} - \sigma\int_1^e \theta(s)\frac{ds}{s}\right) \geq 0.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
[x(t) - \varrho\varpi(t)]^\star =&x(t) - \varrho\varpi(t) \geq \frac{1}{\nu-1}(\ln t)^{\nu-1}\left(\frac{R_2}{\nu-1} - \varrho\sigma\int_1^e \theta(s)\frac{ds}{s}\right)\\
\geq&\frac{(\ln a)^{\nu-1}R_2}{2(\nu-1)} \geq L_0, \forall t \in [a, e-a].
\end{aligned} \tag{3.5}$$

Then for any $x \in K \cap \partial\Omega_2$ and $t \in [a, e-a]$, by (3.4) and (3.5), we have

$$
\begin{aligned}
Tx(t) \geq & \varrho \int_a^{e-a} \wp(t,s)\left(\hbar(s,[x(s)-\varrho\varpi(s)]^\star)+\theta(s)\right)\frac{ds}{s} \\
\geq & \varrho \int_a^{e-a} \wp(t,s) L[x(s)-\varrho\varpi(s)]\frac{ds}{s} \geq \frac{\varrho L}{\nu-1}\int_a^{e-a} (\ln t)^{\nu-1} J(s)\frac{(\ln a)^{\nu-1}R_2}{2(\nu-1)}\frac{ds}{s} \\
\geq & \frac{\varrho L(\ln a^{2(\nu-1)})R_2}{2(\nu-1)^2}\int_a^{e-a} J(s)\frac{ds}{s} \geq R_2 = \|x\|.
\end{aligned}
$$

Hence, we conclude

$$
\|Tx\| \geq \|x\|, \forall x \in K \cap \partial\Omega_2. \tag{3.6}
$$

By (3.3),(3.6) and Theorem 3.1, we get that T has a fixed point $x_1 \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, that is $R_1 \leq \|x_1\| \leq R_2$. Since $\|x_1\| \geq R_1$, we have

$$
\begin{aligned}
x_1(t) - \varrho\varpi(t) \geq & (\ln t)^{\nu-1}\left(\frac{\|x_1\|}{\nu-1} - \sigma\varrho\int_1^e \theta(s)\frac{ds}{s}\right) \geq (\ln t)^{\nu-1}\left(\frac{R_1}{\nu-1} - \sigma\int_1^e \theta(s)\frac{ds}{s}\right) \\
= & \Lambda_1(\ln t)^{\nu-1},
\end{aligned}
$$

and so $x_1(t) \geq \varrho\varpi(t) + \Lambda_1(\ln t)^{\nu-1}$ for all $t \in [1,e]$, where $\Lambda_1 = \frac{R_1}{\nu-1} - \sigma\int_1^e \theta(s)\frac{ds}{s}$.

Let $u_1(t) = x_1(t) - \varrho\varpi(t)$ for all $t \in [1,e]$. Then u_1 is a positive solution of problem (1.1-1.2) with $u_1(t) \geq \Lambda_1(\ln t)^{\nu-1}$for all $t \in [1,e]$. This completes the proof of Theorem 3.1.

Theorem 3.2. Assume that $(H_1), (H_2)$ and (H_4) hold. Then there exists $\varrho_\star > 0$ such that, for any $\varrho \geq \varrho_\star$, the boundary value problem (1.1-1.2) has at least one positive solution.

Proof. By (H_4), there exists $\Upsilon_2 > 0$ such that

$$
\hbar(t,u) \geq 2\sigma\int_1^e \theta(s)ds\left((\ln a)^{\nu-1}\int_a^{e-a} J(s)\frac{ds}{s}\right)^{-1}, \forall t \in [a, e-a], u \geq \Upsilon_2.
$$

Let's define

$$
\varrho_\star = \Upsilon_2\left((\ln a)^{\nu-1}\sigma\int_a^{e-a} J(s)\frac{ds}{s}\right)^{-1}.
$$

Let's assume $\varrho \geq \varrho_\star$. Let $R_3 = 3(\nu-1)\varrho\sigma\int_1^e \theta(s)\frac{ds}{s}$ and $\Omega_3 = \{x \in K : \|x\| < R_3\}$. Then for any $x \in K \cap \partial\Omega_3$, we have

$$
\begin{aligned}
& x(t) - \varrho\varpi(t) \\
\geq & \frac{1}{\nu-1}(\ln t)^{\nu-1}\|x\| - \varrho\sigma(\ln t)^{\nu-1}\int_1^e \theta(s)\frac{ds}{s} = (\ln t)^{\nu-1}\left(\frac{R_3}{\nu-1} - \varrho\sigma\int_1^e \theta(s)\frac{ds}{s}\right) \\
\geq & 2(\ln t)^{\nu-1}\varrho\sigma\int_1^e \theta(s)\frac{ds}{s} \geq 2(\ln t)^{\nu-1}\varrho_\star\sigma\int_1^e \theta(s)\frac{ds}{s} \\
= & \frac{2\Upsilon_2(\ln t)^{\nu-1}}{(\ln a)^{\nu-1}} \geq 0, \forall t \in [1,e].
\end{aligned} \tag{3.7}
$$

Hence, for any $x \in K \cap \partial\Omega_3$ and $t \in [a, e-a]$, we get

$$
[x(t) - \varrho\varpi(t)]^\star = x(t) - \varrho\varpi(t) \geq \frac{2\Upsilon_2(\ln t)^{\nu-1}}{(\ln a)^{\nu-1}} \geq \Upsilon_2.
$$

Therefore, for any $x \in K \cap \partial\Omega_3$ and $t \in [a, e-a]$, we have

$$\begin{aligned}
Tx(t) \geq & \varrho \int_a^{e-a} \wp(t,s)\hbar(s,[x(s)-\varrho\varpi(s)]^\star)\frac{ds}{s} \\
\geq & \varrho \int_a^{e-a} \frac{1}{\nu-1}(\ln t)^{\nu-1} J(s)\hbar(s, x(s)-\varrho\varpi(s))\frac{ds}{s} \\
\geq & \varrho \int_a^{e-a} \frac{1}{\nu-1}(\ln t)^{\nu-1} J(s) \left(2\sigma \int_1^e \theta(s)\frac{ds}{s}\left((\ln a)^{\nu-1}\int_a^{e-a} J(s)\frac{ds}{s}\right)^{-1}\right)\frac{ds}{s} \\
= & \frac{R_3(\ln t)^{\nu-1}}{(\ln a)^{\nu-1}} \geq R_3.
\end{aligned}$$

Therefore, we conclude

$$\|Tx\| \geq \|x\|, \forall x \in K \cap \partial\Omega_3. \tag{3.8}$$

Furthermore, we consider the positive number

$$\varepsilon = \left(2\varrho \int_1^e J(s)\omega(s)\frac{ds}{s}\right)^{-1},$$

then from (H_4), we obtain that there exists $\Upsilon_3 > 0$ such that $\vartheta(t,u) \leq \varepsilon u$ for all $t \in [1,e], u \geq \Upsilon_3$. Hence, we have $\vartheta(t,u) \leq \Upsilon_4 + \varepsilon u$, for all $t \in [1,e], u \geq 0$, where $\Upsilon_4 = \max_{t\in[1,e],u\in[0,\Upsilon_3]} \vartheta(t,u)$.

Let's define

$$R_4 > \max\left\{R_3, 2\varrho \max\{\Upsilon_4, 1\} \int_1^e J(s)(\nu(s)+\theta(s))\frac{ds}{s}\right\},$$

and let $\Omega_4 = \{x \in K : \|x\| < R_4\}$. Then for any $x \in K \cap \partial\Omega_4$, we have

$$\begin{aligned}
& x(t) - \varrho\varpi(t) \\
\geq & \frac{1}{\nu-1}(\ln t)^{\nu-1}\|x\| - \varrho\sigma(\ln t)^{\nu-1}\int_1^e \theta(s)\frac{ds}{s} = (\ln t)^{\nu-1}\left(\frac{R_4}{\nu-1} - \varrho\sigma\int_1^e \theta(s)\frac{ds}{s}\right) \\
\geq & (\ln t)^{\nu-1}\left(\frac{R_3}{\nu-1} - \varrho^\star\sigma\int_1^e \theta(s)\frac{ds}{s}\right) \\
= & 2(\ln t)^{\nu-1}\varrho\sigma\int_1^e \theta(s)\frac{ds}{s} \\
\geq & 2(\ln t)^{\nu-1}\varrho_\star\sigma\int_1^e \theta(s)\frac{ds}{s} = 2\frac{\Upsilon_2(\ln t)^{\nu-1}}{(\ln a)^{\nu-1}} \geq 0, \forall t \in [1,e].
\end{aligned} \tag{3.9}$$

Then for any $x \in K \cap \partial\Omega_4$, we have

$$\begin{aligned}
Tx(t) \leq & \varrho \int_1^e J(s)\left((\nu(s)\vartheta(s,[x(s)-\varrho\varpi(s)]^\star)) + \theta(s)\right)\frac{ds}{s} \\
\leq & \varrho^\star \int_1^e J(s)[(\nu(s)(\Upsilon_4 + \varepsilon(x(s)-\varrho\varpi(s)))+\theta(s)]\frac{ds}{s} \\
\leq & \varrho\max\{\Upsilon_4,1\}\int_1^e J(s)(\nu(s)+\theta(s))\frac{ds}{s} + \varrho\varepsilon R_4\int_1^e J(s)\nu(s)\frac{ds}{s} \\
\leq & \frac{R_4}{2} + \frac{R_4}{2} = R_4 = \|x\|R_1 = \|x\|, \forall t \in [1,e].
\end{aligned}$$

Hence,

$$\|Tx\| \leq \|x\|, \forall x \in K \cap \partial\Omega_4. \tag{3.10}$$

By (3.8), (3.10) and Theorem 3.1, we conclude that T has a fixed point $x_1 \in K \cap (\overline{\Omega}_4 \ \Omega_3)$, so $R_3 \le \|x_1\| \le R_4$. In addition, we deduce that for all $t \in [1, e]$,

$$\begin{aligned}
&x_1(t) - \varrho\varpi(t)\\
\ge& x_1(t) - \varrho\sigma(\ln t)^{\nu-1}\int_1^e \theta(s)\frac{ds}{s}\\
\ge& \frac{1}{\nu-1}(\ln t)^{\nu-1}\|x_1\| - \varrho\sigma\frac{1}{\nu-1}(\ln t)^{\nu-1}\int_1^e \theta(s)\frac{ds}{s}\\
\ge& \frac{1}{\nu-1}R_3 - \varrho\sigma\frac{1}{\Gamma(\nu-1)}(\ln t)^{\nu-1}\int_1^e \theta(s)\frac{ds}{s} = 2\varrho\sigma(\ln t)^{\nu-1}\int_1^e \theta(s)\frac{ds}{s}\\
\ge& 2\varrho_\star\sigma(\ln t)^{\nu-1}\int_1^e \theta(s)\frac{ds}{s} = \frac{2\Upsilon_2(\ln t)^{\nu-1}}{(\ln a)^{\nu-1}} \ge 0, \forall t \in [1, e].
\end{aligned} \tag{3.11}$$

Let $u_1(t) = x_1(t) - \varrho\varpi(t)$, for all $t \in [1, e]$. Then $u_1(t) \ge \overline{\Lambda}_1 \frac{1}{\nu-1}(\ln t)^{\nu-1}$, for all $t \in [1, e]$, where $\overline{\Lambda}_1 = \Upsilon_2/(\ln a)^{\nu-1}$. Hence, we conclude that u_1 is a positive solution of problem (1.1-1.2), which completes the proof of Theorem 3.2.

Theorem 3.3. Assume that $(H_1), (H_2)$ and

$(\widetilde{H_4})$ There exists $a \in (1, \frac{e-1}{2})$ such that $\widetilde{\hbar}_\infty = \lim_{u\to\infty} \min_{t\in[a,1-c]} \hbar(t, u) = \infty$ and $\vartheta_\infty = \lim_{u\to\infty} \max_{t\in[1,e]} \frac{\vartheta(t,u)}{u} = 0$ hold. Then there exists $\varrho_\star > 0$ such that, for any $\varrho \ge \widetilde{\varrho}_\star$, the boundary value problem $(1.1, 1.2)$ has at least one positive solution.

4 Example

In this paper, we consider the following Hadamard fractional differential equation

$$ {}^H D_{1^+}^{\frac{5}{2}} u(t) + \varrho\hbar(t, u(t)) = 0, \ 1 < t < e, \tag{4.1}$$

with nonlocal Hadamard integral boundary conditions

$$\begin{aligned}
&u(1) = u'(1) = 0,\\
&{}^H D_{1^+}^{\frac{4}{3}} u(e) = \int_1^e s^{\frac{3}{4}}(1-s)^2 D_{1^+}^{\frac{1}{2}} u(s) dA_1(s) + \frac{1}{2}\int_1^{\frac{2}{3}} s^{\frac{7}{8}}(1+s^2)^2 D_{1^+}^{\frac{1}{2}} u(s) dA_2(s),
\end{aligned} \tag{4.2}$$

$$\wp_1(t,s) = \frac{1}{\Gamma(\frac{5}{2})}\begin{cases} (\ln t)^{\frac{3}{2}}(\ln\frac{e}{s})^{\frac{1}{6}} - (\ln\frac{t}{s})^{\frac{3}{2}}, & 1 \le s \le t \le e,\\ (\ln t)^{\frac{3}{2}}(\ln\frac{e}{s})^{\frac{1}{6}}, & 1 \le t \le s \le e, \end{cases} \tag{2.3}$$

$$\wp_2(t,s) = \frac{1}{\Gamma(2)}\begin{cases} (\ln t)(\ln\frac{e}{s})^{\frac{1}{6}} - (\ln\frac{t}{s}), & 1 \le s \le t \le e,\\ (\ln t)(\ln\frac{e}{s})^{\frac{1}{6}}, & 1 \le t \le s \le e, \end{cases}$$

and

$$\wp(t,s) = \wp_1(t,s) + \frac{(\ln t)^{\frac{3}{2}}}{\Delta}\left(\int_1^e s^{\frac{3}{4}}(1-s)^2\wp_{21}(\tau,s)dA_1(\tau) + \frac{1}{2}\int_1^2 s^{\frac{7}{8}}(1+s^2)\wp_{22}(\tau,s)dA_2(\tau)\right),$$

where

$$\begin{aligned}
\Delta =& \frac{\Gamma(\nu)}{\Gamma(\nu p_0)}e^{\nu-p_0-1} - \sum_{i=1}^m \frac{\alpha_i\Gamma(\nu)}{\Gamma(\nu-p_i)}\int_1^{\eta_i}\rho_i(s)(\ln s)^{\nu-p_i-1}dA_i(s)\\
=& \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{7}{6})}e^{\frac{1}{6}} - \frac{\Gamma(\frac{5}{2})}{\Gamma(2)}\int_1^e s^{\frac{3}{4}}(1-s)^2(\ln s)dA_1(s) - \frac{1}{2}\int_1^2 s^{\frac{7}{8}}(1+s^2)(\ln s)dA_2(s) \approx 1.4563,
\end{aligned}$$

$$J(s)=\frac{1}{\Gamma(\frac{3}{2})}(\ln s)(\ln\frac{e}{s})^{\frac{1}{6}}+\frac{1}{\Delta}\Big(\Big(\int_1^e s^{\frac{3}{4}}(1-s)^2\wp_{21}(\tau,s)dA_1(\tau)+\frac{1}{2}\int_1^2 s^{\frac{7}{8}}(1+s^2)\wp_{22}(\tau,s)dA_2(\tau)\Big)\Big).$$

Example 1. Considering the function

$$\hbar(t,u)=\frac{u^3+2u+1}{t\sqrt[3]{\ln t(1-\ln t)^2}}+\ln t, t\in(1,e), u\geq 0,$$

then, $\theta(t)=-\ln t$, $\nu(t)=\frac{1}{t\sqrt[3]{\ln t(1-\ln t)^2}}$ for all $t\in(1,e)$, $\vartheta(t,u)=u^3+2u+1$ for all $t\in[1,e]$ and $u\geq 0$, $\int_1^e\theta(t)\frac{dt}{t}=1$, $\displaystyle\int_1^e\nu(t)\frac{dt}{t}=\int_1^e\frac{1}{t\sqrt[2]{\ln t(1-\ln t)}}dt=\int_1^e(\ln t)^{-\frac{1}{2}}(1-\ln t)^{-\frac{1}{2}}\frac{dt}{t}=B(\frac{1}{2},\frac{1}{2})=\pi$. Hence, assumption (H_2) is satisfied. Obviously, for fixed $c\in(0,\frac{1}{2})$, the condition (H_3) is also satisfied$(f_\infty=\infty)$.

Moreover, with a few simple calculations, we have

$$\begin{aligned}&\int_1^e J(s)(\nu(s)+\theta(s))\frac{ds}{s}\\ =&\frac{1}{\Gamma(\frac{3}{2})}(\ln s)(\ln\frac{e}{s})^{\frac{1}{6}}+\frac{1}{\Delta}\Big(\Big(\int_1^e s^{\frac{3}{4}}(1-s)^2\wp_1(\tau,s)dA_1(\tau)+\frac{1}{2}\int_1^2(\ln s)^{\frac{7}{8}}(1+(\ln s)^2)\wp_2(\tau,s)dA_2(\tau)\Big)\Big)\\ &\times(\frac{1}{s\sqrt[3]{\ln s(1-\ln s)^2}}-\ln s)\approx 2.9130.\end{aligned}$$

Choosing $R_1=2(R_1>\sigma\int_1^e\theta(s)\frac{ds}{s})$, and then we obtain $M_1=5$ and $\varrho^\star\approx 0.3752$, then we have that problem (4.1-4.2) has at least one positive solution for any $\varrho\in(0,\varrho^\star]$ by Theorem 3.2.

Example 2. Considering the function

$$\hbar(t,u)=\frac{\sqrt{u+1}}{t\sqrt[4]{(\ln t)^3(1-(\ln t))}}-\frac{2}{\sqrt{\ln t}}, t\in(1,e), u\geq 0.$$

then, $\theta(t)=\frac{2}{\sqrt{\ln t}}$ and $z(t)=\frac{1}{t\sqrt[4]{(\ln t)^3(1-(\ln t))}}$ for all $t\in(1,e)$, $g(t,u)=\sqrt{u+1}$ for all $t\in[1,e]$ and $u\geq 0$, $\int_1^e\theta(t)\frac{dt}{t}=1$, $\int_1^e z(t)\frac{dt}{t}=\int_1^e\frac{1}{t\sqrt[4]{(\ln t)^3(1-(\ln t))}}\frac{dt}{t}=B(-\frac{1}{3},\frac{3}{4})$. Hence, for $c\in(1,\frac{e-1}{2})$ fixed, the conditon $(H_2,H_4))$ are satisfied.

According to some simple computations, we have

$$\begin{aligned}&\int_1^e J(s)(z(s)+\theta(s))\frac{ds}{s}\\ =&\frac{1}{\Gamma(\frac{3}{2})}(\ln s)(\ln\frac{e}{s})^{\frac{1}{6}}+\frac{1}{\Delta}\Big(\Big(\int_1^e s^{\frac{3}{4}}(1-s)^2\wp_{21}(\tau,s)dA_1(\tau)+\frac{1}{2}\int_1^2 s^{\frac{7}{8}}(1+s^2)\wp_{22}(\tau,s)dA_2(\tau)\Big)\Big)\\ &\times(\frac{1}{t\sqrt[4]{(\ln t)^3(1-(\ln t))}}-\frac{2}{\sqrt{t}})\approx 5.6870.\end{aligned}$$

For $c=\frac{1}{4}$, we have $\int_{\frac{5}{4}}^{e-\frac{1}{4}}J(s)\frac{ds}{s}\approx 0.6830$ and $\Upsilon_0\approx 80.5420$. Through by the proof process of theorem 3.2, we have that $\Upsilon_2\approx 8478.3$ and $\varrho_\star\approx 560367$. Then we have that the problem (4.1-4.2) has at least one positive solution for any $\varrho\in(0,\varrho^\star]$ by Theorem 3.2.

Acknowledgements

This research was supported by the National Natural Science Foundation of China (12271232,12101086), the major project of Basic science (Natural science) research in colleges and universities of Jiangsu Province(22KJA460001, 23KJA580001), Changzhou Science and Technology Plan Project (CE20235049), and Open Project of Applied Mechanics and Structure Safety Key Laboratory of Sichuan Province (SZDKF-202102).

References

[1] Shorog Aljoudi, Bashir Ahmad, Juan J. Nieto, and Ahmed Alsaedi. A coupled system of hadamard type sequential fractional differential equations with coupled strip conditions. *Chaos, Solitons Fractals*, 91:39–46, 2016.

[2] R. Arul and P. Karthikeyan. Integral boundary value problems for implicit fractional differential equations involving hadamard and caputo-hadamard fractional derivatives. *Kragujevac Journal of Mathematics*, 45(3 (2021)):331–341, 2019.

[3] Wafaa Benhamida, John Graef, and Samira Hamani. Boundary value problems for hadamard fractional differential equations with nonlocal multi-point boundary conditions. *Fractional Differential Calculus*, pages 165–176, 01 2018.

[4] A. Cabada and Z. Hamdi. Nonlinear fractional differential equations with integral boundary value conditions. *Appl.Math. Comput.*, 228(2012):251–257, 2014.

[5] L. Guo, C. Li, and J. Zhao. Existence of monotone positive solutions for caputochadamard nonlinear fractional differential equation with infinite-point boundary value conditions. *Symmetry*, 15:970, 2023.

[6] L. Guo, H. Liu, C. Li, J. Zhao, and J. Xu. Existence of positive solutions for singular p-laplacian hadamard fractional differential equations with the derivative term contained in the nonlinear term. *Nonlinear Anal:Model. Control.*, 4:1–25, 2023.

[7] L. Guo and L. Liu. Maximal and minimal iterative positive solutions for singular infinite-point p-laplacian fractional differential equations. *Nonlinear Anal:Model. Control.*, 23(6):851–865, 2018.

[8] L. Guo, L. Liu, and Y. Wang. Maximal and minimal iterative positive solutions for p-laplacian hadamard fractional differential equations with the derivative term contained in the nonlinear term. *AIMS.Math.*, 6(11):12583–12598, 2021.

[9] L. Guo, L. Liu, and Y. Wu. Existence of positive solutions for singular fractional differential equations with infinite-point boundary conditions. *Nonlinear Anal:Model. Control.*, 21(5):635–650, 2016.

[10] L. Guo, L. Liu, and Y. Wu. Existence of positive solutions for singular higher-order fractional differential equations with infinite-point boundary conditions. *Bound. Value Probl.*, 2016(1):1–22, 2016.

[11] L. Guo, L. Liu, and Y. Wu. Iterative unique positive solutions for singular p-laplacian fractional differential equation system with several parameters. *Nonlinear Anal:Model. Control.*, 23(2):182–203, 2018.

[12] M. Jleli and B. Samet. Existence of positive solutions to an arbitrary order fractional differential equation via a mixed monotone operator method. *Nonlinear Anal:Model. Control.*, 20(3):367–376, 2015.

[13] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations.* Elsevier Science BV, Amsterdam, 2006.

[14] V. Lakshmikantham and A. Vatsala. Basic theory of fractional differential equations. *Nonlinear Anal.*, 204:2677–2682, 2008.

[15] H. Lu, Z. Han, C. Zhang, and Y. Zhao. Positive solutions for boundary value problem of nonlinear fractional differential equation with p-laplacian operator. *International Conference on Finite Difference Methods*, 9045:274–281, 2014.

[16] D. Ma. Positive solutions of multi-point boundary value problem of fractional differential equation. *Arab. J. Math. Sci.*, 21(2):225–236, 2015.

[17] G. Wang and T. Wang. On a nonlinear hadamard type fractional differential equation with p-laplacian operator and strip condition. *Journal of Nonlinear Sciences and Applications*, 09:5073–5081, 2016.

[18] X. Zhang. Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions. *Appl. Math. Lett.*, 39:22–27, 2015.

[19] X. Zhang, M. Feng, and W. Ge. Symmetric positive solutions for p-laplacian fourth-order differential equations with integral boundary conditions. *J.Comput.Appl.Math.*, 222:561–573, 2008.

[20] X. Zhang, L. Liu, B. Wiwatanapataphee, and Y. Wu. The eigenvalue for a class of singular p-laplacian fractional differential equations involving the riemanncstieltjes integral boundary condition. *Appl. Math. Comput.*, 235(4):412–422, 2014.

[21] X. Zhang, L. Liu, Y. Wu, and B. Wiwatanapataphee. The spectral analysis for a singular fractional differential equation with a signed measure. *Appl. Math. Comput.*, 257:252–263, 2015.

[22] X. Zhang, Z. Shao, and Q. Zhong. Positive solutions for semipositone $(k, n-k)$ conjugate boundary value roblems with singularities on space variables. *Appl. Math. Lett.*, 217(16):50–57, 2017.

[23] X. Zhang, L. Wang, and Q. Sun. Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter. *Appl. Math. Comput.*, 226:708–718, 2014.

[24] X. Zhang and Q. Zhong. Uniqueness of solution for higher-order fractional differential equations with conjugate type integral conditions. *Fract. Calc. Appl.Anal.*, 20(6):1471–1484, 2017.

[25] X. Zhang and Q. Zhong. Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables. *Appl. Math. Lett.*, 80:12–19, 2018.