RESONANCE FOR p-LAPLACIAN AND ASYMMETRIC NONLINEARITIES

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Abstract. In the present paper, we aim to investigate the existence of solutions for the quasilinear boundary value problem involving fractional operators in the ψ-fractional space $\mathcal{H}_p^{\alpha}([0,T],\mathbb{R})$ with asymmetric nonlinearities.

1. Introduction and motivation

The researches on the existence of weak solutions for the resonance problem to p-Laplacian can also be found in the other papers, such as [17, 27] and the references therein. The resonance problems involving p-Laplacian in $\mathbb{R}^N$, have great relevance in the field of partial and ordinary differential equations. What has been noticed during the last two decades is the attention to problems involving resonance, both in the classical approach and in the fractional approach [15–20].

In 1997 Arcoya and Orsina [23], investigated the existence of a solution to the problem

p2
$$\left\{\begin{array}{rcl} -\Delta_p u & = & \lambda_1 |u|^{p-2} + f(x,u) - h,\ in\ \Lambda \\ u \in W_0^{1,p}(\Lambda) & & \end{array}\right. \tag{1.1}$$

where Λ is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, $p > 1$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded Caratheodory function. On the other hand, in 1998, Cuesta et al. [24] carried out a work addressing p-Laplacian on $W_0^{1,p}(\Lambda)$ defined by

p3
$$\left\{\begin{array}{rcl} -\Delta_p u & = & \alpha(u^+)^{p-1} - \beta(u^-)^{p-1},\ in\ \Lambda \\ u & = & 0,\ on\ \partial\Lambda \end{array}\right. \tag{1.2}$$

where $-\Delta_p = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian.

In 2000, Bouchala and Drabek [25], investigated the existence of the weak solutions of the boundary value problem

pp3
$$\left\{\begin{array}{rcl} -\Delta_p u & = & \lambda_1 |u|^{p-2}u + g(u) - h(x),\ in\ \Omega \subset \mathbb{R}^N \\ u & = & 0,\ on\ \partial\Omega \end{array}\right. \tag{1.3}$$

with $N \geq 1$, $p > 1$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, $h \in L^{p'}(\Omega)$ $\left(p' = \frac{p}{p-1}\right)$. See also the interesting work carried out in 2001 by Dancer and Perera [26], where he investigated the existence of positive solutions to the p-Laplacian problem.

On the other hand, the theory of fractional differential equations currently occupies a prominent role in the general theory of differential equations, with intense research, problems of its own, relevant results and a wide range of applications [1, 2]. One front that has recently gained attention are problems involving fractional differential equations and p-Laplacian, in order to discuss

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existence, non-existence and multiplicity of solutions using variational methods [5–12] and the references therein.

Before presenting the main problem to be addressed in this paper, we will make some considerations.

Let $J=[0,T]$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha>0$. The Riemann-Liouville fractional integral (left-sided and right-sided) of a function ϕ on J is defined by [1,3,4,13]

eq111
$$\mathcal{I}^{\alpha}_{0+}\phi\left(\xi\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{\xi}\left(\xi-s\right)^{\alpha-1}\phi\left(s\right)ds \tag{1.4}$$

and

eq112
$$\mathcal{I}^{\alpha}_{0-}\phi\left(\xi\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{\xi}^{0}\left(s-\xi\right)^{\alpha-1}\phi\left(s\right)ds. \tag{1.5}$$

On the other hand, let $\frac{1}{p}<\alpha\leq 1$, with $n\in\mathbb{N}$ and $\phi\in\mathcal{C}^{n}(J,\mathbb{R})$. The Caputo fractional derivative left-sided and right-sided denoted by ${}^{\mathrm{c}}\mathfrak{D}^{\alpha}_{0+}(\cdot)$ (resp. ${}^{\mathrm{c}}\mathfrak{D}^{\alpha}_{0-}(\cdot)$) of a function f of order α, is defined by [1,3,4,13]

eq333
$$ {}^{\mathrm{c}}\mathfrak{D}^{\alpha}_{0+}\phi(\xi)=\mathcal{I}^{1-\alpha}_{0+}\phi'(\xi)\quad\text{and}\quad {}^{\mathrm{c}}\mathfrak{D}^{\alpha}_{0-}\phi(\xi)=\mathcal{I}^{1-\alpha}_{0-}\phi'(\xi) \tag{1.6}$$

where $\mathcal{I}^{\alpha}_{0+}(\cdot)$ and $\mathcal{I}^{\alpha}_{0-}(\cdot)$ there are defined in Eq.(1.4) and Eq.(1.5), respectively. A natural consequence of the definition (1.6), is that in the limit of $\alpha\to 1$, we have the classical derivative (integer order), given by ${}^{\mathrm{c}}\mathfrak{D}^{1}_{0+}\phi(\xi)=\dfrac{d\phi(\xi)}{d\xi}$.

Let $\frac{1}{p}<\alpha\leq 1$ and $1<p<\infty$. The ψ-fractional derivative space $\mathcal{H}^{\alpha}_{p}:=\mathcal{H}^{\alpha}_{p}\left(\left[0,T\right],\mathbb{R}\right)$ is defined by the closure of $C^{\infty}_{0}\left(\left[0,T\right],\mathbb{R}\right)$, and is given by

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$$\mathcal{H}^{\alpha}_{p}([0,T],\mathbb{R})=\left\{\phi(\cdot)\in\mathscr{L}^{p}([0,T],\mathbb{R}):\left|{}^{\mathbf{c}}\mathfrak{D}^{\alpha}_{0+}\phi(\cdot)\right|\in\mathscr{L}^{p}([0,T],\mathbb{R}).\right\} \tag{1.7}$$

The space (1.7), is reflexive, uniform convex Banach and separable space [6,9,10]. In addition, it is equipped with norm

$$\|\phi\|=\|\phi\|_{\mathscr{L}^{p}}+\|{}^{c}\mathfrak{D}^{\alpha}_{0+}\phi\|_{\mathscr{L}^{p}},\ \text{for all}\ \phi\in\mathcal{H}^{\alpha}_{p}([0,T],\mathbb{R}).$$

In this sense, motivated by the problems (1.1)-(1.3) and Caputo fractional derivatives, in this paper, we consider the fractional quasilinear boundary value problem

eq1.1
$${}^{\mathrm{c}}\mathfrak{D}^{\alpha}_{T}\left(|{}^{\mathrm{c}}\mathfrak{D}^{\alpha}_{0+}\phi|^{p-2}\,{}^{\mathrm{c}}\mathfrak{D}^{\alpha}_{0+}\,\phi\right)=\theta_{+}(\xi)(\phi^{+})^{p-1}-\theta_{-}(\xi)(\phi^{-})^{p-1}+f(\xi,\phi),\ \ \phi\in\mathcal{H}^{\alpha}_{p}([0,T],\mathbb{R}) \tag{1.8}$$

where ${}^{\mathrm{c}}\mathfrak{D}^{\alpha}_{T}(\cdot)$ and ${}^{\mathrm{c}}\mathfrak{D}^{\alpha}_{0+}(\cdot)$ are Caputo fractional derivatives of order $\frac{1}{p}<\alpha\leq 1$, $[0,T]$ is a bounded domain in $\mathbb{R}$, $1<p<\infty$, $\phi\pm=\max\{\pm\phi,0\}$, $\theta\pm\in\mathscr{L}^{\infty}([0,T],\mathbb{R})$ and f is a Caratheodory (that is, $f(\xi,\phi)$ is a measurable with respect to ξ in $\Omega=[0,T]$ for every ϕ in $\mathbb{R}$, and continuous with respect to ϕ in $\mathbb{R}$ for almost every $\xi\in\Omega$) function on $[0,T]\times\mathbb{R}$ satisfying a growth condition

eq1.2
$$|f(\xi,\phi)|\leq q\,\mathcal{M}_{1}(\xi)^{p-q}|s|^{q-1}+\mathcal{M}_{2}(\xi)^{p-1} \tag{1.9}$$

with $1\leq q<p$ and $\mathcal{M}_{1},\mathcal{M}_{2}\in\mathscr{L}^{p}([0,T],\mathbb{R})$. Furthermore, we assume that

eq1.3
$$\lambda_{1}\leq\theta\pm(\xi)\leq\lambda_{2}-\varepsilon \tag{1.10}$$

or

eq1.4
$$\lambda_{1}+\varepsilon\leq\theta\pm(\xi)\leq\lambda_{2} \tag{1.11}$$

for two consecutive variational eigenvalues $\lambda_1 < \lambda_2$ of ${}^{\mathrm{c}}\mathfrak{D}_T^{\alpha}\left(|{}^{\mathrm{c}}\mathfrak{D}_{0+}^{\alpha}(\cdot)|^{p-2}\, {}^{\mathrm{c}}\mathfrak{D}_{0+}^{\alpha}(\cdot)\right)$ on $\mathcal{H}_p^{\alpha}([0,T],\mathbb{R})$. During the paper, we condition by Σ the set of solutions of

$$ {}^{\mathrm{c}}\mathfrak{D}_T^{\alpha}\left(|{}^{\mathrm{c}}\mathfrak{D}_{0+}^{\alpha}\phi|^{p-2}\, {}^{\mathrm{c}}\mathfrak{D}_{0+}^{\alpha}\,\phi\right) = \theta_+(\xi)(\phi^+)^{p-1} - \theta_-(\xi)(\phi^-)^{p-1} \tag{1.12} $$

and set

$$ \Theta(\xi,s) : \int_0^s f(\xi,t)dt \text{ and } \Psi(\xi,s) := p\Theta(\xi,s) - s\Theta(\xi,s). \tag{1.13} $$

The main results of this paper is to investigate the following result:

teorema **Theorem 1.1.** *Problem (1.8) has a solution when:*

(1) Eq.(1.10) *holds and* $\displaystyle\int_0^T \Psi(\xi,\phi_j)d\xi \to +\infty$.

(2) Eq.(1.11) *holds and* $\displaystyle\int_0^T \Psi(\xi,\phi_j)d\xi \to -\infty$ *for every sequence* (ϕ_j) *in* $\mathcal{H}_p^{\alpha}([0,T],\mathbb{R})$ *such that* $||\phi_j|| \to \infty$ *and* $\dfrac{\phi_j}{|\phi_j|}$ *converges to some element of* Σ*. In particular,* (1.8) *is solvable when* (1.10) *or* (1.11) *holds and* Σ *is empty.*

Below we highlight some comments on the difficulties of working with problems of the type (1.8) and consequences of the results obtained in the paper:

(1) In general, when dealing with resonance problems, a difficulty and often problematic is the lack of compactness of the variational functional associated with the problem. However, it is possible to overcome this difficulty by constructing an approximation sequence of non-resonance problems, to obtain approximate solutions via min-max arguments and passing to the limit.

(2) A natural consequence of the result investigated in this work is that in the limit of $\alpha \to 1$, the integer case is obtained, i.e., the problem (1.8), becomes the following problem

inteiro
$$ \left(|\phi'|^{p-2}\,\phi'\right)' = \theta_+(\xi)(\phi^+)^{p-1} - \theta_-(\xi)(\phi^-)^{p-1} + f(\xi,\phi),\ \phi \in \mathcal{H}_p^1([0,T],\mathbb{R}). \tag{1.14} $$

(3) Consequently, the result Theorem 1.1, is valid for the problem (1.14). On the other hand, taking $p = 2$ in the problem (1.8), we have

$$ \phi'' = \theta_+(\xi)\phi^+ - \theta_-(\xi)\phi^- + f(\xi,\phi),\ \phi \in \mathcal{H}_2^1([0,T],\mathbb{R}). $$

(4) Note that when $\alpha = 1$, $\theta_+(\xi) = \theta_-(\xi) = \lambda_1$ and $q = 1$ we have some special cases that can be found in the literature, namely: Arcoya and Orsina [23], Bouchala and Drabek [25], and Drabek and Robinson [27].

In the rest, the paper is divided into two sections, namely: Section 2, we present some corollaries. In section 3, we investigate the main result of this paper, i.e., the Theorem 1.1.

2. Some Results

This section is intended to discuss the existence of a solution to the Problem (1.8) via some corollaries.

Corollary 2.1. *Problem (1.8) has a solution when:*

(1) Eq.(1.10) *holds,* $\Psi(\xi,s) \to +\infty$*, a.e. as* $|s| \to \infty$*, and* $\Psi(\xi,s) \geq -\epsilon(\xi)$.

(2) *Eq.(1.11) holds, $\Psi(\xi,s)\to-\infty$, a.e., as $|s|\to\infty$ and $\Psi(\xi,s)\le\epsilon(\xi)$ for some $\epsilon\in\mathscr{L}^1([0,T],\mathbb{R})$.*

Proof. If (1) holds, then $\Psi(\xi,\phi_{j(\xi)})=\Psi(\xi,\rho_j\nu_j(\xi))\to+\infty$ for some a.e. ξ such that $\mathcal{M}_1(\xi)\neq 0$ and $\Psi(\xi,\phi_j(\xi))\ge-\epsilon(\xi)$. Using the Fatou's lemma, follows that

eq1.7
$$\int_0^T\Psi(\xi,\phi_j(\xi))d\xi\ge\int_{v\neq 0}\Psi(\xi,\phi_j(\xi))d\xi-\int_{v=0}\epsilon(\xi)d\xi\to+\infty. \tag{2.1}$$

Similarly, follows that $\int_0^T\Psi(\xi,\phi_j(\xi))d\xi\to-\infty$ if (2) holds. □

Since (1) and (2) hold on subsets of $\{\xi\in\Lambda;\mathcal{M}_1(\xi)\neq 0\}$ with positive measure, the ideas discussed above are maintained. Consider $w=\nu^{\pm}$ in

eq1.8
$$\int_0^T|{}^{c}\mathfrak{D}^{\alpha}_{0+}\nu|^{p-2}\ {}^{c}\mathfrak{D}^{\alpha}_{0+}\nu\ {}^{c}\mathfrak{D}^{\alpha}_{0+}w\ d\xi=\int_0^T(\theta_+(\xi)(\nu^+)^{p-1}-\theta_-(\xi)(\nu^-)^{p-1})w\,d\xi \tag{2.2}$$

gives

eq1.9
$$||\nu^{\pm}||^p=\int_0^T\theta_{\pm}(\xi)(\nu^{\pm})^p d\xi\le||\theta_{\pm}||_{\infty}||\nu^{\pm}||^p_{p^+_{\alpha}}\mu(\Lambda_{\pm})^p\le||\theta_{\pm}||_{\infty}S^{-1}||\nu^{\pm}||^p\mu(\Lambda_{\pm})^p \tag{2.3}$$

where $\Lambda_{\pm}=\{\xi\in\Lambda=[0,T];\mathcal{M}_1(\xi)\ge 0\}$, S is the best constant for the trace embedding $\mathcal{H}^{\alpha}_p([0,T],\mathbb{R})\hookrightarrow\mathscr{L}^{\infty}([0,T],\mathbb{R})$ and, μ is the Lebesgue measure in $\mathbb{R}$. Thus

eq1.10
$$\mu\left(\Lambda_{\pm}\right)\ge\left(S||\theta_{\pm}||^{-1}_{\infty}\right)^{1/p} \tag{2.4}$$

and so

eq1.11
$$\mu\left(\{\xi\in[0,T];\mathcal{M}_1(\xi)=0\}\right)\le\mu([0,T],\mathbb{R})-S^{1/p}\left(||\theta_+||^{-1/p}_{\infty}+||\theta_-||^{-1/p}_{\infty}\right). \tag{2.5}$$

In this sense, we have the following result.

Corollary 2.2. *Problem (1.8) has a solution when:*

(1) *Eq.(1.10) holds, $\Psi(\xi,s)\to\infty$ in $\Lambda'=[0,T]$ as $|s|\to\infty$, and $\Psi(\xi,s)\ge-\epsilon(\xi)$*

(2) *Eq.(1.11) holds, $\Psi(\xi,s)\to-\infty$ in Λ' as $|s|\to\infty$ and $\Psi(\xi,s)\le\epsilon(\xi)$ for some $\Lambda'\subset\Lambda$ with $\mu(\Lambda')>\mu([0,T],\mathbb{R})-S^{1/p}\left(||\theta_+||^{-1/p}_{\infty}+||\theta_-||^{-1/p}_{\infty}\right)$ and $\epsilon\in\mathscr{L}^1([0,T],\mathbb{R})$.*

Next, note that

eq1.12
$$\begin{aligned}\underline{\Psi}_+(\xi)(\nu^+(\xi))^q+\underline{\Psi}_-(\xi)(\nu^-(\xi))^q\le&\quad \liminf\frac{\Psi(\xi,\phi_j(\xi))}{\rho_j^q}\\ \le&\quad \limsup\frac{\Psi(\xi,\phi_j(\xi))}{\rho_j^q}\\ \le&\quad \overline{\Psi}_+(\xi)(\nu^+(\xi))^q+\overline{\Psi}_-(\xi)(\nu^-(\xi))^q.\end{aligned} \tag{2.6}$$

Moreover,

eq1.14
$$\frac{\Psi(\xi,\phi_j(\xi))}{\rho_j^q}\le(p+q)\mathcal{M}_1(\xi)^{p-q}|\nu_j(\xi)|^q+(p+1)\frac{\mathcal{M}_2(\xi)^{p-q}|\nu_j(\xi)|}{\rho_j^{q-1}} \tag{2.7}$$

by (1.9), so it follows that

$$\begin{aligned}\int_0^T (\underline{\Psi_+}(\nu^+)^q + \underline{\Psi_-}(\nu^-)^q) d\xi \leq & \quad \liminf \frac{\int_0^T \Psi(\xi, \phi_j) d\xi}{\rho_j^q} \\ \leq & \quad \limsup \frac{\int_0^T \Psi(\xi, \phi_j) d\xi}{\rho_j^q} \\ \leq & \quad \int_0^T (\overline{\Psi_+}(\nu^+)^q + \overline{\Psi_-}(\nu^-)^q) d\xi. \end{aligned} \tag{2.8}$$

eq1.15

Thus, we have the following corollary.

Corollary 2.3. *Problem (1.8) has a solution when:*

(1) Eq.(1.10) *holds and* $\int_0^T \underline{\Psi_+}(\nu^+)^q + \underline{\Psi_-}(\nu^-)^q > 0$ *for all* $\nu \in \Sigma$.

(2) Eq.(1.11) *holds and* $\int_0^T \overline{\Psi_+}(\nu^+)^q + \overline{\Psi_-}(\nu^-)^q < 0$ *for all* $\nu \in \Sigma$.

3. Main results

Note that the eigenvalues of ${}^{\mathrm{c}}\mathfrak{D}_T^{\alpha}\left(|{}^{\mathrm{c}}\mathfrak{D}_{0+}^{\alpha}(\cdot)|^{p-2}\, {}^{\mathrm{c}}\mathfrak{D}_{0+}^{\alpha}(\cdot)\right)$ on $\mathcal{H}_p^{\alpha}([0,T],\mathbb{R})$ correspond to the critical values of

eq2.1

$$\mathbf{E}_{\alpha}(\phi) = \int_0^T |{}^{\mathrm{c}}\mathfrak{D}_{0+}^{\alpha}\phi|^p d\xi \tag{3.1}$$

with $\phi \in \mathfrak{M} = \left\{\phi \in \mathcal{H}_p^{\alpha}([0,T],\mathbb{R}); ||\phi||_p = 1\right\}$. Moreover, $\mathbf{E}_{\alpha}$ satisfies the Palais-Smale condition, i.e., if $\{\phi_n\} \subset \mathcal{H}_p^{\alpha}(\Omega)$ such that $\{\mathbb{E}_{\alpha}(\{\phi_n\})\}$ is bounded and $\mathbb{E}_{\alpha}(\{\phi_n\}) \to 0$ in $\left(\mathcal{H}_p^{\alpha}(\Omega)\right)^*$, then $\{\phi_n\}$ has a subsequence that converges in $\mathcal{H}_p^{\alpha}(\Omega)$.

Consider the unbounded sequence of eigenvalues given by

eq2.2

$$\lambda_1 := \inf_{\mathfrak{A} \in \mathcal{H}_1} \max_{\phi \in \mathfrak{A}} \mathbf{E}_{\alpha}(\phi) \tag{3.2}$$

where

eq2.3

$$\mathcal{H}_1 = \{\mathfrak{A} \subset I : \exists \text{ a continuous add subjection } h : I \to \mathfrak{A}\} \tag{3.3}$$

and I is an interval in $\mathbb{R}$.

Lemma 3.1. λ_1 *is an eigenvalue of* ${}^{\mathrm{c}}\mathfrak{D}_T^{\alpha}\left(|{}^{\mathrm{c}}\mathfrak{D}_{0+}^{\alpha}(\cdot)|^{p-2}\, {}^{\mathrm{c}}\mathfrak{D}_{0+}^{\alpha}(\cdot)\right)$ *and* $\lambda_1 \to \infty$.

Proof. Suppose that λ_1 is a regular value of $\mathbf{E}_{\alpha}$, so there exists an $\varepsilon > 0$ and $\eta : \mathfrak{M} \to \mathfrak{M}$ (homeomorphism) such that $\eta(\mathbf{E}_{\alpha}^{\lambda_1+\varepsilon}) \subset \mathbf{E}_{\alpha}^{\lambda_1+\varepsilon}$, see [21]. On the other hand, taking $\mathfrak{A} \in \mathcal{H}_1$ with $\max \mathbf{E}_{\alpha}(\mathfrak{A}) \leq \lambda_1 + \varepsilon$ and setting $\mathfrak{A} = \eta(\mathfrak{A})$, we have a set in $\mathcal{H}_1$ for which $\max\ \mathbf{E}_{\alpha}(\overline{\mathfrak{A}}) \leq \lambda_1 - \varepsilon$, which contradicts λ_1 (see Eq.(3.2)). Finally, let $\mu_1 \to \infty$ eigenvalues, follows that $\lambda_1 \geq \mu_1$ since the genus of each $\mathfrak{A}$ in $\mathcal{H}_1$ is 1, so $\lambda_1 \to \infty$. $\square$

Let us now discuss the main result of this paper.

Proof. (**Proof of Theorem 1.1**) Let's just do the proof of (1), the case (2) is similarly. Then consider

$$\theta_{\pm}^{j}(\xi)=\begin{cases}\theta_{\pm}(\xi),\ if\ \theta_{\pm}(\xi)\geq\lambda_1+\frac{1}{j}\\ \lambda_1+\frac{1}{j},\ if,\ \theta_{\pm}(\xi)<\lambda_1+\frac{1}{j}\end{cases}\tag{3.4}$$

so that

$$\lambda_1+\frac{1}{j}\leq\theta_{\pm}^{j}\leq\lambda_2-\varepsilon,\quad |\theta_{\pm}^{j}(\xi)-\theta_{\pm}(\xi)|\leq\frac{1}{j}\tag{3.5}$$

and let

$$\Phi_j(\phi)=\int_0^T|{}^{\rm c}\mathfrak{D}_{0+}^{\alpha}\phi|^p-\theta_{+}^{j}(\xi)(\phi^{+})^p-\theta_{-}^{j}(\phi^{-})^p-p\Theta(\xi,\phi)d\xi,\quad \phi\in\mathcal{H}_p^{\alpha}([0,T],\mathbb{R}).\tag{3.6}$$

First, we show that there is a $\phi_j\in\mathcal{H}_p^{\alpha}([0,T],\mathbb{R})$, such that

$$||\phi_j||\ ||\Phi_j'(\phi_j)||\to 0,\ \ \inf\Phi_j(\phi_j)>-\infty.\tag{3.7}$$

Using (3.2), there exists an $\mathfrak{A}\in\mathcal{H}_1$, such that

$$\mathbf{E}_{\alpha}(\phi)\leq\lambda_1+\frac{1}{2j},\ \ \phi\in\mathfrak{A}.\tag{3.8}$$

For $\phi\in\mathfrak{A}$, and $\mathcal{R}>0$, one has

$$\begin{aligned}\Phi_j(\mathcal{R}\phi)&=\int_0^T|{}^{\rm c}\mathfrak{D}_{0+}^{\alpha}\mathcal{R}\phi|^p-\theta_{+}^{j}(\xi)(\mathcal{R}\phi^{+})^p-\theta_{-}^{j}(\xi)(\mathcal{R}\phi^{-})^p-p\Theta(\xi,\mathcal{R}\phi)d\xi\\ &=\mathcal{R}^p\left\{\int_0^T\left(|{}^{\rm c}\mathfrak{D}_{0+}^{\alpha}\phi|^p-\theta_{+}^{j}(\xi)(\phi^{+})^p-\theta_{-}^{j}(\xi)(\phi^{-})^p-p\Theta(\xi,\phi)\right)d\xi\right\}\qquad(3.9)\\ &\leq-\frac{\mathcal{R}^p}{2j}+p\left(||\mathcal{M}_1||_p^{p-q}\mathcal{R}^q+||\mathcal{M}_2||_p^{p-1}\mathcal{R}\right).\end{aligned}$$

Using (1.9), (3.5) and (3.8), so

$$\max_{\phi\in\mathfrak{A}}\Phi_j(\mathcal{R}\phi)\to-\infty,\ as\ \mathcal{R}\to\infty.\tag{3.10}$$

Next let,

$$\mathcal{E}=\left\{\phi\in\mathcal{H}_p^{\alpha}([0,T],\mathbb{R}):\int_0^T|{}^{\rm c}\mathfrak{D}_{0+}^{\alpha}r\phi|^pd\xi\geq\lambda_2\int_0^T|\phi|^pd\xi\right\}.\tag{3.11}$$

For $\phi\in\mathcal{E}$ and (3.6), yields

$$\Phi_j(\phi)\geq\varepsilon||\phi||_p^p-p\left(||\mathcal{M}_1||_p^{p-q}\,||\phi||_p^q+||\mathcal{M}_2||_p^{p-1}\,||\phi||_p\right).\tag{3.12}$$

Hence

$$\inf_{\phi\in\mathcal{E}}\Phi_j(\phi)\geq\epsilon:=\min_{r\geq0}\left[\varepsilon r^p-p\left(||\mathcal{M}_1||_p^{p-q}\,r^q+||\mathcal{M}_2||_p^{p-1}r\right)\right]>-\infty.\tag{3.13}$$

Using (3.10), follows that

$$\max\ \Phi_j(\mathcal{R}\mathfrak{A})<\epsilon\tag{3.14}$$

where $\mathcal{R}\mathfrak{A}=\{\mathcal{R}\phi:\phi\in\mathfrak{A}\}$ for $\mathcal{R}>0$ fix and large. Since $\mathfrak{A}\in\mathcal{H}_1$, there exists a continuous add subjection $h:I\to\mathfrak{A}$. Consider

$$\Gamma=\left\{\varphi\in C(D,\mathcal{H}_p^{\alpha}([0,T],\mathbb{R})):\varphi|_I=\mathcal{R}h\right\}\tag{3.15}$$

where D is the interval in $\mathbb{R}$ with boundary I.

Affirmation 1: $\varphi(D)\cap\mathcal{E}\neq\emptyset,\ \forall\varphi\in\Gamma$.

The proof will be divided into two parts. The first part is to investigate when $0 \in \varphi(D)$. Consider π the radial projection. For $\mathfrak{M}$, $\overline{\mathfrak{A}} := \pi(\varphi(D) \cup -\pi(\varphi(D))) \in \mathcal{H}_1$, yields

eq2.17 (3.16)
$$\max_{\phi \in \pi(\varphi(D))} \mathbf{E}_\alpha(\phi) = \max_{\phi \in \overline{\mathfrak{A}}} \mathbf{E}_\alpha(\phi) \geq \lambda_2 \tag{3.16}$$

then $\pi(\varphi(D)) \cap \mathcal{E} \neq \emptyset$. Consequently, follow that $\varphi(D) \cap \mathcal{E} \neq \emptyset$.

Note that, there exists a ϕ_j such that

eq2.18 (3.17)
$$||\phi_j||\ ||\Phi_j'(\phi_j)|| \to 0, \ \ |\Phi_j'(\phi_j) - c_j| \to 0 \tag{3.17}$$

where

eq2.19 (3.18)
$$\epsilon_j := \inf_{\varphi \in \Gamma} \max_{\phi \in \varphi(D)} \Phi_j(\phi) \geq \epsilon. \tag{3.18}$$

Thus, we get (3.7). Obtaining Eq.(3.17), follows the deformation argument [22]. In this sense, we finish the proof of first part.

Affirmation 2: A subsequence of (ϕ_j) converges to a solution of (1.8).

Note that, if (ϕ_j) so bounded, hence suppose that $\rho := ||\phi_j|| \to \infty$. Consider $\nu_j := \dfrac{\phi_j}{\rho_j}$. Without loss of generality , we may assume that $\nu_j \to \nu$ weakly in $\mathcal{H}_p^\alpha([0,T],\mathbb{R})$, strongly in $\mathscr{L}^p([0,T],\mathbb{R})$, and a.e in Λ. So, we obtain

$$\begin{aligned}
&\int_0^T |{}^{\mathrm{c}}\mathfrak{D}^\alpha_{0+}\phi|^{p-2}\ {}^{\mathrm{c}}\mathfrak{D}^\alpha_{0+}\nu_j\ {}^{\mathrm{c}}\mathfrak{D}^\alpha_{0+}(\nu_j - \nu) d\xi \\
&= \frac{(\Phi'(\phi_j)\nu_j - \nu)}{p\rho_j^{p-1}} + \int_0^T \left((\theta^j_+)(\xi)(\nu_j^+)^{p-1} - \theta^j_-(\xi)(\nu_j^-)^{p-1} + \frac{f(\xi,\phi_j)}{\rho_j^{p-1}} \right)(\nu_j - \nu) d\xi \to 0.
\end{aligned} \tag{3.19}$$
eq2.21 (3.19)

In this sense, follows that $\nu_j \to \nu$ strongly in $\mathcal{H}_p^\alpha([0,T],\mathbb{R})$. In particular, $||\mathcal{M}_1|| = 1$, then $\nu \neq 0$. Moreover, for each $w \in \mathcal{H}_p^\alpha([0,T],\mathbb{R})$, yields

$$\begin{aligned}
\frac{(\Phi'(\phi_j)\nu_j - \nu)}{p\rho_j^{p-1}} &= \int_0^T |{}^{\mathrm{c}}\mathfrak{D}^\alpha_{0+}\phi|^{p-2}\ {}^{\mathrm{c}}\mathfrak{D}^\alpha_{0+}\nu_j\ {}^{\mathrm{c}}\mathfrak{D}^\alpha_{0+} w d\xi \\
&\quad - \left[\theta^j_+(\xi)(\nu_j^+)^{p-1} - \theta^j_-(\xi)(\nu_j^-)^{p-1} + \frac{f(\xi,\phi_j)}{\rho_j^{p-1}} \right] w
\end{aligned} \tag{3.20}$$

gives that

$$\int_0^T |{}^{\mathrm{c}}\mathfrak{D}^\alpha_{0+}\phi|^{p-2}\ {}^{\mathrm{c}}\mathfrak{D}^\alpha_{0+}\nu\ {}^{\mathrm{c}}\mathfrak{D}^\alpha_{0+} w\ d\xi - \left[\theta_+(\xi)(\nu^+)^{p-1} - \theta_-(\xi)(\nu^-)^{p-1}\right] w = 0$$

so $\nu \in \Sigma$. Thus,

$$\frac{(\Phi_j'(\phi_j), \phi_j)}{p} - \Phi_j(\phi_j) = \int_0^T \Psi(\xi, \phi_j) d\xi \to \infty$$

contradicting (3.7). Thus, we complete the proof of the theorem. □

4. Conclusion and future work

The present paper is a natural motivation for the works [23–25] as discussed in the introduction. The objective of investigating the existence of a solution for a new class of fractional quasilinear boundary value problem was only possible once we circumvented the compactness problem as presented in the course of the work via variational methods. Although we were able to achieve the objective, we were left with some questions that arose during the discussion of the results, and are better described as follows:

- Would it be possible to investigate the nonresonance of the problem (1.8)? What conditions would be necessary? And discuss the Fucık spectrum for the problem (1.8)?
- Finally, would it be possible to extend the results here to a double-phase p-Laplacian? What conditions must be imposed for the results to still be valid?

In this sense, we believe that the present work allowed to contribute with new results for the theory of fractional differential equations and raise some interesting questions about the problem (1.8), allowing a natural continuation of this present paper.

Declarations

Conflict of interest The author have not disclosed any competing interests.

References

frac2 [1] Kilbas, A. A., Hari M. Srivastava, and Juan J. Trujillo. Theory and applications of fractional differential equations. Vol. 204. elsevier, 2006. 1, 2

novo1 [2] Milici, C., G. Daganescu, and J. A. Tenreiro Machado. Introduction to fractional differential equations. Vol. 25. Springer, 2018. 1

novo2 [3] Ortigueira, M. D., and J. A. Tenreiro Machado. What is a fractional derivative?. J. Comput. Phys. 293 (2015): 4-13. 2

novo3 [4] Teodoro, G. Sales, J. A. Tenreiro Machado, and E. Capelas De Oliveira. A review of definitions of fractional derivatives and other operators. J. Comput. Phys. 388 (2019): 195-208. 2

novo4 [5] Ma, Y., F. Zhang, and C. Li. Existence and uniqueness of the solutions to the fractional differential equations. Recent Advances in Applied Nonlinear Dynamics with Numerical Analysis: Fractional Dynamics, Network Dynamics, Classical Dynamics and Fractal Dynamics with Their Numerical Simulations. 2013. 23-48. 2

novo5 [6] Sousa, J. Vanterler da C., N. Nyamoradi, and M. Lamine. Nehari manifold and fractional Dirichlet boundary value problem. Anal. Math. Phys. 12.6 (2022): 143. 2

novo6 [7] Ezati, R., and N. Nyamoradi. Existence and multiplicity of solutions to a ψ-Hilfer fractional p-Laplacian equations. Asian-European J. Math. (2022): 2350045. 2

novo7 [8] Ezati, R., and N. Nyamoradi. Existence of solutions to a Kirchhoff ψ-Hilfer fractional p-Laplacian equations. Math. Meth. Appl. Sci. 44.17 (2021): 12909-12920. 2

novo8 [9] Sousa, J. Vanterler da C., César T. Ledesma, Mariane Pigossi and Jiabin Zuo Nehari manifold for weighted singular fractional p-Laplace equations. Bull. Braz. Math. Soc., New Series 53.4 (2022): 1245-1275. 2

Sousa1 [10] Sousa, J. Vanterler da C., Jiabin Zuo, and Donal O'Regan. The Nehari manifold for a ψ-Hilfer fractional p-Laplacian. Applicable Anal. (2021), 1-31. 2

Sousa80 [11] Sousa, J. Vanterler da C. Existence and uniqueness of solutions for the fractional differential equations with p-Laplacian in $\mathcal{H}_p^{\nu,\eta;\psi}$. J. Appli. Anal. Comput. 12(2) (2022), 622-661. 2

Ezati [12] Ezati, R., and N. Nyamoradi. Existence and multiplicity of solutions to a ψ-Hilfer fractional p-Laplacian equations. Asian-European J. Math. (2022), 2350045. 2

J1 [13] Sousa, J. Vanterler da C., and E. Capelas de Oliveira. On the ψ-Hilfer fractional derivative. Commun. Nonlinear Sci. Numer. Simul. 60 (2018), 72-91. 2

Landesman [14] Landesman, E. M., and A. C. Lazer. Nonlinear perturbations of linear elliptic boundary value problems at resonance. J. Math. Mechanics 19.7 (1970): 609-623.

Liu [15] Liu, S. Multiple solutions for elliptic resonant problems. Proc. Royal Soc. Edinburgh Sec. A: Mathematics 138.6 (2008): 1281-1289. 1

Song [16] Song, S.-Z., and C.-L. Tang. Resonance problems for the p-Laplacian with a nonlinear boundary condition. Nonlinear Analysis: Theory, Methods & Appl. 64.9 (2006): 2007-2021. 1

Drabek [17] Drabek, P., and S. B. Robinson. Resonance problems for the p-Laplacian. J. Funct. Anal. 169.1 (1999): 189-200. 1

Arcoya12 [18] Arcoya, D. A. V. I. D., M. C. M. Rezende, and E. A. B. Silva. Quasilinear problems under local Landesman–Lazer condition. Calc. Var. Partial Diff. Equ. 58.6 (2019): 1-27. 1

Hung [19] Hung, B. Q., and H. Q. Toan. On fractional p-Laplacian equations at resonance. Bull. Malaysian Math. Sci. Soc. 43.2 (2020): 1273-1288. 1

Chen [20] Chen, Y., and J. Su. Bounded resonant problems driven by fractional Laplacian. Topol. Meth. Nonlinear Anal. 57.2 (2021): 635-661. 1

bonet [21] Bonnet, A. A deformation lemma on a C^1 manifold. Manuscripta Math. 81(3) (1993): 339–359. 5

cerani [22] Cerami, G. An existence criterion for the critical points on unbounded manifolds Istit. Lombardo Accad. Sci. Lett. Rend. A 112(2) (1978): 332–336. 7

1 [23] Arcoya, D. and L. Orsina. Landesman-Lazer conditions and quasilinear elliptic equations. Nonlinear Anal. 28 (1997), 1623–1632. 1, 3, 8

6 [24] Cuesta, M., D. de Figueiredo, and J.-P. Gossez. The beginning of the Fucik spectrum for the p-Laplacian. J. Diff. Eq. 159 (1999), 212–238. 1, 8

3 [25] Bouchala, J. and P. Drabek. Strong resonance for some quasilinear elliptic equations. J. Math. Anal. Appl. 245 (2000), 7–19. 1, 3, 8

7 [26] Dancer, E. N. and K. Perera. Some remarks on the Fucik spectrum of the p-Laplacian and critical groups. J. Math. Anal. Appl. 254 (2001), 164–177. 1

8 [27] Drabek, P. and S. B. Robinson. Resonance problems for the p-Laplacian. J. Funct.Anal. 169 (1999), 189–200. 1, 3

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