The fractional Tikhonov regularization method for simultaneous inversion of the source term and initial value in a space-fractional Allen-Cahn equation *

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Abstract: In this paper, we consider the inverse problem for identifying the source term and initial value simultaneously in a space-fractional Allen-Cahn equation. This problem is ill-posed, i.e., the solution of this problem does not depend continuously on the data. The fractional Tikhonov method is used to solve this problem. Under the a priori and the a posteriori regularization parameter choice rules, the error estimates between the regularization solutions and the exact solutions are obtained, respectively. Different numerical examples are presented to illustrate the validity and effectiveness of our method.

Keywords: Space-fractional Allen-Cahn equation; Fractional Tikhonov regularization method; Ill-posed problem; Identifying the source and initial value

1 Introduction

In recent years, fractional Allen-Cahn equation (ACe) have attracted wide attentions. The ACe was originally proposed by Allen and Cahn [1], which is a phase model that simulates the anti-phase boundary motion of a crystalline solid. And the ACe is widely used in various interface problems, for example, vesicle membranes, the nucleation of solids and the motion by mean curvature and so on [2–4]. Recently, more and more people pay attention to the fractional differential equations [5–8]. In particular, the space fractional Allen-Cahn equation (SFACe) is a class of the fractional differential equations, which can be seen as a fractional analogue of the classic ACe. Currently, most research is focused on numerical solutions to this equation. In [9], Zhang et al. proposed

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energy stability of high-order implicit Runge-Kutta schemes for the SFACe. In [10], Hou et al. constructed a fully discretized Crank-Nicolson scheme for fractional-in-space ACe. And then the nonlinear iteration is required during the solution process. Moreover, He et al. [11] proved that a spatial fourth-order maximum principle preserving operator splitting scheme for the multi-dimensional fractional ACe. For more research on the SFACe, we can refer to the literatures [12, 13].

Compared to the above problems, the study on inverse problem of SFACe is still limited. Especially, the problem of simultaneous inversion of the source term and initial value in a SFACe involve only a few. By reading [16], we know most of the inverse problems are ill-posed, we need to use regularization method to resolve this problem. We can refer to some regularization methods for diffusion equation to solve inverse problem of SFACe. In [14], Yang et al. made use of the Landweber iterative method to identify a space-dependent source for the time-fractional diffusion equation. In [17–19], Yang et al. identified the unknown source for fractional diffusion equation in frequency domain by using different regularization methods. In [15], Yang et al. applied Landweber regularization method to identify the unknown source of the time-fractional inhomogeneous diffusion equation.

In this paper, we consider the following space-fractional Allen-Cahn equation:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \varepsilon^2 L_\alpha u(x,t) + f(x), & x \in \Omega,\ t \in (0,T], \\ u(x,0) = \varphi(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T], \\ u(x,t_0) = g(x), & x \in \Omega,\ t_0 \in (0,T], \\ u(x,T) = h(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega = (a,b)^d$, $d = 1,2$, $0 < \alpha < 1$ is the order of a fractional derivative and the parameter ε is a positive constant. And L_α denotes the Riesz fractional derivative operator. In one-dimension, it is given by

$$L_\alpha u := \frac{\partial^\alpha}{\partial |x|^\alpha} u = -\frac{1}{2}\cos(\frac{\alpha\pi}{2})({}_aD_x^\alpha u +_x D_b^\alpha u) := C_\alpha({}_aD_x^\alpha u +_x D_b^\alpha u), \tag{1.2}$$

here, $C_\alpha = -\frac{1}{2}\cos(\frac{\alpha\pi}{2})$. The left and right-sided Riemann-Liouville fractional derivatives ${}_aD_x^\alpha u$ and ${}_xD_b^\alpha u$ are defined by

$${}_aD_x^\alpha u = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \frac{u(\zeta)}{(x-\zeta)^{\alpha-1}} \mathrm{d}\zeta,$$

and

$${}_xD_b^\alpha u = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^b \frac{u(\zeta)}{(\zeta-x)^{\alpha-1}} \mathrm{d}\zeta,$$

respectively, where $\Gamma(\cdot)$ is Gamma function. Similarly, in two-dimension, the fractional derivative operator L_α can be defined as

$$L_\alpha u := \frac{\partial^\alpha}{\partial |x|^\alpha} u + \frac{\partial^\alpha}{\partial |y|^\alpha} u,$$

where $\frac{\partial^\alpha}{\partial|y|^\alpha}u = C_\alpha({}_aD_y^\alpha u +_y D_b^\alpha u)$.

One will see the relationship between this operator and Laplace is as [21]

$$-(-\Delta)^{\frac{\alpha}{2}}u = C_\alpha({}_aD_x^\alpha u +_x D_b^\alpha u) = L_\alpha u.$$

If the initial value $\varphi(x)$ and source term $f(x)$ are known, we can figure out $u(x,t)$ by solving the initial boundary value problem (1.1), this is the direct problem. But, now $\varphi(x)$ and $f(x)$ are unknown and need to be determined. The inverse problem is: recover the initial value $u(x,0) = \varphi(x)$ and source term $f(x)$ from a pair of measurements (g,h). Because the measurements are error-prone, we remark the measurements with error as $g^\delta(x)$ and $h^\delta(x)$ and satisfy

$$\|g^\delta(x) - g(x)\|_{L^2(\Omega)} \le \delta, \tag{1.3}$$

$$\|h^\delta(x) - h(x)\|_{L^2(\Omega)} \le \delta. \tag{1.4}$$

In our paper, we use the fractional Tikhonov regularized to identify the source term and initial value of problem (1.1). In [22], Klann firstly posed the fractional Tikhonov method in 2008. In [23], Xue et al. used the fractional Tikhonov method to identify the source of a time-fractional diffusion equation. Compared with the standard Tikhonov method, its numerical fitting effect is more better.

The rest of this paper is organized as follows. Section 2 presents some important lemmas used in this paper. The ill-posedness about the simultaneous inversion of the source term and initial value problem is deduced in Section 3. Section 4 constructs the fractional Tikhonov regularization method. In Section 5, error estimations under two regularization parameter choice rules are obtained. Numerical examples are given in Section 6. In the final Section, we give a brief conclusion.

2 Preliminaries

Throughout this paper, we use the following lemmas.

Lemma 2.1 *For $0 \le t_0 \le T$ and $\alpha > 0$, then*

$$e^{-\varepsilon^2\lambda_n^\alpha Tp} \le \left(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{(T-t_0)\varepsilon^2\lambda_n^\alpha}\right)^p. \tag{2.1}$$

Proof: For $0 \le t_0 \le T$, we have

$$\begin{aligned}\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha} &= \frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}(1 - e^{-\varepsilon^2\lambda_n^\alpha (T-t_0)})}{\varepsilon^2\lambda_n^\alpha} \\ &\ge \frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}(T-t_0)\varepsilon^2\lambda_n^\alpha e^{-\varepsilon^2\lambda_n^\alpha (T-t_0)}}{\varepsilon^2\lambda_n^\alpha} = (T-t_0)e^{-\varepsilon^2\lambda_n^\alpha T}.\end{aligned}$$

Thus, we have

$$\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}(1-e^{-\varepsilon^2\lambda_n^\alpha(T-t_0)})}{\varepsilon^2\lambda_n^\alpha(T-t_0)} \geq e^{-\varepsilon^2\lambda_n^\alpha T}.$$

To the power of p on both sides of the above equation, we obtain

$$e^{-\varepsilon^2\lambda_n^\alpha Tp} \leq (\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{(T-t_0)\varepsilon^2\lambda_n^\alpha})^p.$$

Then we complete the proof.

Lemma 2.2 *For* $\lambda_n > \lambda_1 > 0$, $n = 2, 3, \cdots$, $\varepsilon > 0$ *and* $0 \leq t_0 \leq T$, *then*

$$\frac{1-e^{-\varepsilon^2\lambda_n^\alpha t_0}}{\varepsilon^2\lambda_n^\alpha} \leq \frac{1}{\varepsilon^2\lambda_1^\alpha}; \qquad \frac{1-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha} \leq \frac{1}{\varepsilon^2\lambda_1^\alpha}.$$

$$e^{-\varepsilon^2\lambda_n^\alpha t_0} \leq 1; \qquad e^{-\varepsilon^2\lambda_n^\alpha T} \leq 1.$$

Proof: The proof is simple, we omit the proof.

Lemma 2.3 **[20]** *If the constants* $\beta > 0$ *and* $b > a > 0$, *we have the inequality for the variable* $t > 0$,

$$\frac{t^a}{1+\beta t^b} \leq \frac{b-a}{b}(\frac{a}{b-a})^{\frac{a}{b}}\beta^{\frac{-a}{b}}. \tag{2.2}$$

Lemma 2.4 *For* $x \geq 0$, *we have*

$$xe^{-x} \leq 1-e^{-x} \leq x. \tag{2.3}$$

Proof: Let

$$f_1(x) = 1-e^{-x}-xe^{-x}.$$

Taking the derivative of function $f_1(x)$ yields

$$f_1'(x) = e^{-x}-e^{-x}+xe^{-x} \geq 0.$$

Similarly, we have

$$f_2(x) = x-1+e^{-x}.$$

Taking the derivative of function $f_2(x)$ yields

$$f_2'(x) = 1-e^{-x} \geq 0.$$

Thus,

$$xe^{-x} \leq 1-e^{-x} \leq x.$$

Then we complete the proof.

Lemma 2.5 *For constants $D > 0$, $\mu > 0$, $0 < \beta < 1$, $\varepsilon > 0$, $T > 0$, $s \geq 0$, we have*

$$F(s) = \frac{\mu e^{\varepsilon^2 sT(\beta+1-p)}}{D + \mu e^{\varepsilon^2 sT(\beta+1)}} \leq \begin{cases} \frac{\mu(\frac{(\beta+1-p)D}{\mu p})^{\frac{\beta+1-p}{\beta+1}}}{D+\frac{(\beta+1-p)D}{p}}, & 0 < p < \beta+1, \\ \frac{\mu}{D}, & p \geq \beta+1. \end{cases} \tag{2.4}$$

Proof: For $0 < p < \beta + 1$, let $F'(s_0) = 0$, we obtain $e^{\varepsilon^2 s_0 T(\beta+1)} = \frac{(\beta+1-p)D}{\mu p}$, thus

$$F(s) \leq F(s_0) = \frac{\mu(\frac{(\beta+1-p)D}{\mu p})^{\frac{\beta+1-p}{\beta+1}}}{D + \frac{(\beta+1-p)D}{p}}.$$

For $p \geq \beta + 1$, we have

$$F(s) \leq \frac{\mu}{D}.$$

Then we complete the proof.

3 Ill-posed analysis

In this section, we give some results which are very useful for our main conclusions. Denote the eigenvalues of the operator Δ as λ_n and the corresponding eigenfunctions as X_n, satisfies $(-\Delta)X_n = \lambda_n^2 X_n$. $\{\lambda_n\}_{n=1}^{\infty}$ are the positive zeros and satisfy

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots, \quad \lim_{n\to+\infty} \lambda_n = +\infty. \tag{3.1}$$

Thus, $(-\Delta)^{\frac{\alpha}{2}} X_n = \lambda_n^{\alpha} X_n$.

Using the separation of variables, we obtain the solution for (1.1) as follows:

$$u(x,t) = \sum_{n=1}^{\infty} (\frac{1 - e^{-\varepsilon^2 \lambda_n^{\alpha} t}}{\varepsilon^2 \lambda_n^{\alpha}} f_n + e^{-\varepsilon^2 \lambda_n^{\alpha} t} \varphi_n) X_n(x), \tag{3.2}$$

where $f_n = (f(x), X_n(x))$, $\varphi_n = (\varphi(x), X_n(x))$. $(\cdot,\cdot)$ and $\|\cdot\|$ is the inner product and norm in $L^2(\Omega)$, respectively.

Let $t = t_0$ and $t = T$ in (3.2), we have

$$\sum_{n=1}^{\infty} (\frac{1 - e^{-\varepsilon^2 \lambda_n^{\alpha} t_0}}{\varepsilon^2 \lambda_n^{\alpha}} f_n + e^{-\varepsilon^2 \lambda_n^{\alpha} t_0} \varphi_n) X_n(x) = g(x), \tag{3.3}$$

$$\sum_{n=1}^{\infty} (\frac{1 - e^{-\varepsilon^2 \lambda_n^{\alpha} T}}{\varepsilon^2 \lambda_n^{\alpha}} f_n + e^{-\varepsilon^2 \lambda_n^{\alpha} T} \varphi_n) X_n(x) = h(x). \tag{3.4}$$

For given source term $f(x)$ and initial function $\varphi(x)$, we can define a pair of linear operators K_1 and K_2 to solve problem (1.1):

$$K_1 : (f, \varphi) \mapsto u(x, t_0),$$

$$K_2 : (f, \varphi) \mapsto u(x, T).$$

Similarly, we can define four linear operators respectively

$$K_{11} : f \mapsto u(x, t_0), \quad K_{12} : f \mapsto u(x, T),$$

$$K_{21} : \varphi \mapsto u(x, t_0), \quad K_{22} : \varphi \mapsto u(x, T).$$

By the solution expression (3.3) and (3.4), we can obtain operator equations:

$$(K_{11}(f))(x) = \sum_{n=1}^{\infty} \frac{1 - e^{-\varepsilon^2 \lambda_n^\alpha t_0}}{\varepsilon^2 \lambda_n^\alpha} f_n X_n(x), \quad (K_{12}(f)(x)) = \sum_{n=1}^{\infty} \frac{1 - e^{-\varepsilon^2 \lambda_n^\alpha T}}{\varepsilon^2 \lambda_n^\alpha} f_n X_n(x), \tag{3.5}$$

$$(K_{21}(\varphi))(x) = \sum_{n=1}^{\infty} e^{-\varepsilon^2 \lambda_n^\alpha t_0} \varphi_n X_n(x), \quad (K_{22}(\varphi))(x) = \sum_{n=1}^{\infty} e^{-\varepsilon^2 \lambda_n^\alpha T} \varphi_n X_n(x). \tag{3.6}$$

Then we have the follow expressions for the operator equations

$$u(x, t_0) = (K_{11}(f))(x) + (K_{21}(\varphi))(x) = g(x), \tag{3.7}$$

$$u(x, T) = (K_{12}(f))(x) + (K_{22}(\varphi))(x) = h(x). \tag{3.8}$$

By equations (3.7) and (3.8), the solutions (f, φ) of the system can be obtained:

$$\begin{cases} K_{11} f + K_{21} \varphi = g, \\ K_{12} f + K_{22} \varphi = h. \end{cases} \tag{3.9}$$

Applying operator K_{22} to the first equation in the system (3.9) and operator K_{21} to the second one yield:

$$K_{22} K_{11} f + K_{22} K_{21} \varphi = K_{22} g, \tag{3.10}$$

$$K_{21} K_{12} f + K_{21} K_{22} \varphi = K_{21} h. \tag{3.11}$$

By subtracting (3.10) and (3.11), we have

$$(K_{21} K_{12} - K_{22} K_{11}) f = K_{21} h - K_{22} g.$$

Similarly, we apply operator K_{12} to the first equation in the system (3.9) and K_{11} to the second one,

$$K_{12} K_{11} f + K_{12} K_{21} \varphi = K_{12} g, \tag{3.12}$$

$$K_{11} K_{12} f + K_{11} K_{22} \varphi = K_{11} h. \tag{3.13}$$

We obtain the result as follows

$$(K_{21} K_{12} - K_{22} K_{11}) \varphi = K_{12} g - K_{11} h.$$

So, (3.9) is equivalent to the system

$$\begin{cases} K f = K_{21} h - K_{22} g, \\ K \varphi = K_{12} g - K_{11} h, \end{cases} \tag{3.14}$$

where $K = K_{21}K_{12} - K_{22}K_{11}$.

Using the properties of singular values, the singular values of the operator K_{11}, K_{12}, K_{21}, K_{22} are obtained as follows

$$k_{11} = \frac{1 - e^{-\varepsilon^2\lambda_n^\alpha t_0}}{\varepsilon^2\lambda_n^\alpha}, \quad k_{12} = \frac{1 - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha},$$

$$k_{21} = e^{-\varepsilon^2\lambda_n^\alpha t_0}, \quad k_{22} = e^{-\varepsilon^2\lambda_n^\alpha T}.$$

Thus, it is easy to obtain the singular values of operator K as follows

$$k = \frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha}. \tag{3.15}$$

Now, the problem (1.1) becomes the following operator equation

$$Kf(x) = \eta(x), \tag{3.16}$$

where $\eta(x) = K_{21}h(x) - K_{22}g(x)$. From the operator K, we obtain

$$f = K^{-1}\eta = \sum_{n=1}^{\infty} \frac{1}{k}(\eta, X_n)X_n.$$

So, we can obtain

$$f(x) = \sum_{n=1}^{\infty} \frac{\varepsilon^2\lambda_n^\alpha}{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}(k_{21}h_n - k_{22}g_n)X_n(x), \tag{3.17}$$

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{\varepsilon^2\lambda_n^\alpha}{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}(k_{12}g_n - k_{11}h_n)X_n(x), \tag{3.18}$$

where $g_n = (g(x), X_n(x))$, $h_n = (h(x), X_n(x))$. Since $1/k \to \infty$ as $n \to \infty$, the problem is ill-posed, that is, the solution does not persistently depend on the given data.

Next, we define the priori boundary of $f(x)$ and $\varphi(x)$,

$$\|f(x)\|_{(D(-L_\alpha))^p} = (\sum_{n=1}^{\infty} e^{2\varepsilon^2\lambda_n^\alpha Tp}|(f(x), X_n)|^2)^{\frac{1}{2}} \leqslant E, \quad p > 0, \tag{3.19}$$

$$\|\varphi(x)\|_{(D(-L_\alpha))^p} = (\sum_{n=1}^{\infty} e^{2\varepsilon^2\lambda_n^\alpha Tp}|(\varphi(x), X_n)|^2)^{\frac{1}{2}} \leqslant E, \quad p > 0, \tag{3.20}$$

where $E > 0$ is a constant.

4 The fractional Tikhonov regularization method

In this section, we use the fractional Tikhonov regularization method to solve the problem (1.1) and give the fractional Tikhonov regularization solution.

Since the inverse problem is ill-posed, we use the fractional Tikhonov regularization method to solve it. This kind of idea was proposed by Hochstenbach in [25]. It is a penalized least-squares problem of the following form

$$\min_{f\in L^2(\Omega)}\{\|Kf-\eta\|_Y^2+\mu\|f\|^2\}, \tag{4.1}$$

where $\|\cdot\|_Y$ is a weighted seminorm as $\|z\|_Y=\|Y^{\frac{1}{2}}z\|$, for any z. The problem (4.1) has a unique solution f_μ for all positive values of the regularization parameter μ.

We propose to let

$$Y=(K^*K)^{\frac{\beta-1}{2}} \tag{4.2}$$

for a suitable value of $0<\beta<1$, and if $\beta<1$, we define Y with the aid of the Moore-Penrose pseudo-inverse of K^*K. The seminorm $\|\cdot\|_Y$ allows the parameter β to be chosen to improve the quality of the computed solution of (4.1). We refer to (4.1) with Y given by (4.2) as the fractional Tikhonov regularization. When $\beta=1$ it is the standard Tikhonov regularization, then Y is the identifying matrix.

The normal equation associate with the Tikhonov minimization problem (4.1) with Y defined by (4.2) is given by

$$((K^*K)^{\frac{(\beta+1)}{2}}+\mu I)f_\mu=(K^*K)^{\frac{(\beta-1)}{2}}K^*\eta, \tag{4.3}$$

the solution of (4.3) is uniquely determined for any $\mu>0$ and $\beta>0$. By the singular values of the linear operator K, we can obtain

$$f_\mu(x)=\sum_{n=1}^{\infty}\frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^\beta}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}(k_{21}h_n-k_{22}g_n)X_n(x), \tag{4.4}$$

$$\varphi_\mu(x)=\sum_{n=1}^{\infty}\frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^\beta}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}(k_{12}g_n-k_{11}h_n)X_n(x), \tag{4.5}$$

where $0<\beta<1$ is the order of the fractional Tikhonov regularization solution. For the noisy data, we have the fractional Tikhonov regularized solution

$$f_\mu^\delta(x)=\sum_{n=1}^{\infty}\frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^\beta}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}(k_{21}h_n^\delta-k_{22}g_n^\delta)X_n(x), \tag{4.6}$$

$$\varphi_\mu^\delta(x)=\sum_{n=1}^{\infty}\frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^\beta}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}(k_{12}g_n^\delta-k_{11}h_n^\delta)X_n(x). \tag{4.7}$$

5 Error estimation

5.1 The priori regularization parameter choice rule of the source term

Theorem 5.1 *Let* $f(x)$ *be given by* (3.17) *and* $f_\mu^\delta(x)$ *be given by* (4.6). *Suppose that* $f(x)$ *satisfies a priori bound condition* (3.19) *and the assumptions* (1.3), (1.4) *hold. Choosing the regularization parameter:*

$$\mu = \begin{cases} (\frac{\delta}{E})^{\frac{\beta+1}{p+1}}, & 0 < p < \beta+1, \\ (\frac{\delta}{E})^{\frac{\beta+1}{\beta+2}}, & p \geq \beta+1, \end{cases} \tag{5.1}$$

then we obtain the following error estimate:

$$\|f_\mu^\delta(x) - f(x)\| \leq \begin{cases} (c+c_1)E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & 0 < p < \beta+1, \\ (c+c_2)E^{\frac{1}{\beta+2}}\delta^{\frac{\beta+1}{\beta+2}}, & p \geq \beta+1, \end{cases} \tag{5.2}$$

where $c := 2\frac{1}{\beta+1}\beta^{\frac{\beta}{\beta+1}}$, $c_1 := \frac{(\frac{(\beta+1-p)(T-t_0)^{\beta+1}}{p})^{\frac{\beta+1-p}{\beta+1}}}{(T-t_0)^{\beta+1}+\frac{(\beta+1-p)(T-t_0)^{\beta+1}}{p}}$, $c_2 := \frac{1}{(T-t_0)^{\beta+1}}$.

Proof: By the triangle inequality, we have

$$\|f_\mu^\delta(x) - f(x)\| \leq \|f_\mu^\delta(x) - f_\mu(x)\| + \|f_\mu(x) - f(x)\|.$$

Firstly, we give an estimate for the first term. From (4.4), (4.6) and lemma 2.2, we have

$$\begin{aligned}
&\|f_\mu^\delta(x) - f_\mu(x)\| \\
&= \|\sum_{n=1}^{\infty}[\frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^\beta}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}((k_{21}h_n^\delta - k_{22}g_n^\delta) - (k_{21}h_n - k_{22}g_n))]X_n(x)\| \\
&\leq \sup_{n\geq 1}|\frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^\beta}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}|(\|\sum_{n=1}^{\infty}k_{21}(h_n^\delta - h_n)X_n(x)\| + \|\sum_{n=1}^{\infty}k_{22}(g_n^\delta - g_n)X_n(x)\|) \\
&\leq \sup_{n\geq 1}|A(n)|(\|\sum_{n=1}^{\infty}(h_n^\delta - h_n)X_n(x)\| + \|\sum_{n=1}^{\infty}(g_n^\delta - g_n)X_n(x)\|) \leq \sup_{n\geq 1}|A(n)|2\delta,
\end{aligned} \tag{5.3}$$

where $A(n) = \frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^\beta}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}$.

Applying lemma 2.3, we obtain

$$A(n) = \frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^\beta}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}$$

$$= \frac{1}{\mu}\frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^\beta}{\frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}}{\mu}+1}$$

$$\leq \frac{1}{\beta+1}\beta^{\frac{\beta}{\beta+1}}(\frac{1}{\mu})^{\frac{1}{\beta+1}}. \tag{5.4}$$

Then, we obtain

$$\|f_\mu^\delta(x)-f_\mu(x)\| \leq c(\frac{1}{\mu})^{\frac{1}{\beta+1}}\delta, \tag{5.5}$$

where $c = 2\frac{1}{\beta+1}\beta^{\frac{\beta}{\beta+1}}$.

Then we estimate the second term by (4.4), (3.17),

$$\|f_\mu(x)-f(x)\| \tag{5.6}$$

$$= \|\sum_{n=1}^{\infty}[(\frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^\beta}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu} - \frac{\varepsilon^2\lambda_n^\alpha}{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}})(k_{21}h_n-k_{22}g_n)]X_n(x)\|$$

$$= \|\sum_{n=1}^{\infty}[\frac{\mu}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu](\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})}(k_{21}h_n-k_{22}g_n)]X_n(x)\|$$

$$\leq \sup_{n\geq 1}|\frac{\mu e^{-\varepsilon^2\lambda_n^\alpha Tp}}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}|E$$

$$= \sup_{n\geq 1}|B(n)|E, \tag{5.7}$$

where $B(n) = \frac{\mu e^{-\varepsilon^2\lambda_n^\alpha Tp}}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}$.

Applying lemma 2.4, we can infer

$$B(n) = \frac{\mu e^{-\varepsilon^2\lambda_n^\alpha Tp}}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}(1-e^{-\varepsilon^2\lambda_n^\alpha(T-t_0)})}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}$$

$$\leq \frac{\mu e^{-\varepsilon^2\lambda_n^\alpha Tp}}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}\varepsilon^2\lambda_n^\alpha(T-t_0)e^{-\varepsilon^2\lambda_n^\alpha(T-t_0)}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}$$

$$= \frac{\mu e^{\varepsilon^2\lambda_n^\alpha T(\beta+1-p)}}{(T-t_0)^{\beta+1}+\mu e^{\varepsilon^2\lambda_n^\alpha T(\beta+1)}}.$$

Let $\lambda_n^\alpha = s$, we obtain

$$B(s) = \frac{\mu e^{\varepsilon^2 sT(\beta+1-p)}}{(T-t_0)^{\beta+1}+\mu e^{\varepsilon^2 sT(\beta+1)}}.$$

By lemma 2.5, we deduce that

$$B(s) \leq \begin{cases} c_1\mu^{\frac{p}{\beta+1}}, & 0<p<\beta+1, \\ c_2\mu, & p\geq \beta+1, \end{cases} \tag{5.8}$$

where $c_1 = \frac{(\frac{(\beta+1-p)(T-t_0)^{\beta+1}}{p})^{\frac{\beta+1-p}{\beta+1}}}{(T-t_0)^{\beta+1}+\frac{(\beta+1-p)(T-t_0)^{\beta+1}}{p}}$, $c_2 = \frac{1}{(T-t_0)^{\beta+1}}$. Then, we have

$$\|f_\mu(x) - f(x)\| \le \begin{cases} c_1 \mu^{\frac{p}{\beta+1}} E, & 0 < p < \beta+1, \\ c_2 \mu E, & p \ge \beta+1. \end{cases} \tag{5.9}$$

Combining (5.5) with (5.9), we choose the regularized parameter μ by

$$\mu = \begin{cases} (\frac{\delta}{E})^{\frac{\beta+1}{p+1}}, & 0 < p < \beta+1, \\ (\frac{\delta}{E})^{\frac{\beta+1}{\beta+2}}, & p \ge \beta+1. \end{cases}$$

We have

$$\|f_\mu^\delta(x) - f(x)\| \le \begin{cases} (c+c_1) E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}, & 0 < p < \beta+1, \\ (c+c_2) E^{\frac{1}{\beta+2}} \delta^{\frac{\beta+1}{\beta+2}}, & p \ge \beta+1. \end{cases}$$

The proof is completed.

5.2 The posteriori regularization choice rule of the source term

Now we consider a posteriori regularization parameter choice rule called the Morozov discrepancy principle. We can obtain a convergence rate for the fractional Tikhonov regularization solution (4.4).

The Morozov discrepancy principle here is to find μ such that

$$\|Kf_\mu^\delta(x) - (K_{21}h^\delta(x) - K_{22}g^\delta(x))\| = \tau_1 \delta, \tag{5.10}$$

where $\tau_1 > 2$ is a constant. According to the following lemma, we know there exists a unique solution for (5.10) if $\|K_{21}h^\delta(x) - K_{22}g^\delta(x)\| > \tau_1 \delta$.

Lemma 5.2.1 $\rho_1(\mu) = \|Kf_\mu^\delta(x) - (K_{21}h^\delta(x) - K_{22}g^\delta(x))\|$, *the following results hold*

(a) $\rho_1(\mu)$ *is a continuous function;*

(b) $\lim\limits_{\mu\to 0} \rho_1(\mu) = 0$;

(c) $\lim\limits_{\mu\to\infty} \rho_1(\mu) = \|K_{21}h^\delta(x) - K_{22}g^\delta(x)\|$;

(d) $\rho_1(\mu)$ *is a strictly increasing function over* $\mu \in (0, \infty)$.

Proof: By (5.10), we have

$$\rho_1(\mu) = \|Kf_\mu^\delta(x) - (K_{21}h^\delta(x) - K_{22}g^\delta(x))\|$$

$$= \| \sum_{n=1}^{\infty} [\frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu}(k_{21}h_n^\delta - k_{22}g_n^\delta) - (k_{21}h_n^\delta - k_{22}g_n^\delta)]X_n(x)\|$$

$$= \| \sum_{n=1}^{\infty} \frac{\mu(k_{21}h_n^\delta - k_{22}g_n^\delta)}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu} X_n(x)\|. \tag{5.11}$$

Obviously, the conclusions $(a), (b), (c), (d)$ hold.

Lemma 5.2.2 *Suppose* (1.3), (1.4) *and the priori bound condition* (3.19) *hold, we obtain*

$$\mu^{-1} \leq \begin{cases} (\frac{c_4}{\tau_1 - 2})^{\frac{\beta+1}{p+1}} (\frac{E}{\delta})^{\frac{\beta+1}{p+1}}, & 0 < p < \beta, \\ (\frac{c_5}{\tau_1 - 2}) \frac{E}{\delta}, & p \geq \beta, \end{cases} \tag{5.12}$$

where $c_4 := (\frac{1}{T-t_0})^p \frac{\beta-p}{\beta+1} (\frac{p+1}{\beta-p})^{\frac{p+1}{\beta+1}}$, $c_5 := \frac{1}{(T-t_0)^p (\varepsilon^2\lambda_1^\alpha)^{p-\beta}}$.

Proof:

$$\tau_1\delta = \| \sum_{n=1}^{\infty} \frac{\mu(k_{21}h_n^\delta - k_{22}g_n^\delta)}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu} X_n(x)\|$$

$$\leq \| \sum_{n=1}^{\infty} \frac{\mu[k_{21}(h_n^\delta - h_n) - k_{22}(g_n^\delta - g_n)]}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu} X_n(x)\| + \| \sum_{n=1}^{\infty} \frac{\mu(k_{21}h_n - k_{22}g_n)}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu} X_n(x)\|$$

$$\leq 2\delta + \sup_{n\geq 1} |\frac{\mu \frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha} e^{-\varepsilon^2\lambda_n^\alpha T p}}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu}| E$$

$$= 2\delta + \sup_{n\geq 1} |C(n)| E. \tag{5.13}$$

Using lemma 2.1, we obtain

$$C(n) \leq (\frac{1}{T-t_0})^p \frac{\mu(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{p+1}}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu}$$

$$= (\frac{1}{T-t_0})^p \frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{p+1}}{\frac{1}{\mu}(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + 1}.$$

By lemma 2.3 and $0 < p < \beta$, we have

$$C(n) \leq c_4 \mu^{\frac{p+1}{\beta+1}},$$

where $c_4 = (\frac{1}{T-t_0})^p \frac{\beta-p}{\beta+1} (\frac{p+1}{\beta-p})^{\frac{p+1}{\beta+1}}$.
For $p \geq \beta$, we have

$$C(n) \leq (\frac{1}{T-t_0})^p \frac{\mu(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{p+1}}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu}$$

$$\leq \mu \frac{1}{(T-t_0)^p}(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{p-\beta}$$
$$\leq \frac{\mu}{(T-t_0)^p(\varepsilon^2\lambda_1^\alpha)^{p-\beta}}$$
$$= c_5\mu.$$

Thus we obtain

$$\mu^{-1} \leq \begin{cases} (\frac{c_4}{\tau_1-2})^{\frac{\beta+1}{p+1}}(\frac{E}{\delta})^{\frac{\beta+1}{p+1}}, & 0<p<\beta, \\ (\frac{c_5}{\tau_1-2})\frac{E}{\delta}, & p\geq\beta. \end{cases}$$

This proof is completed.

Lemma 5.2.3 *Let $f_\mu(x)$ be given by* (4.4) *and $f_\mu^\delta(x)$ be given by* (4.6)*, then we have*

$$\|f_\mu^\delta(x)-f_\mu(x)\| \leq \begin{cases} c(\frac{c_4}{\tau_1-2})^{\frac{1}{p+1}}E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & 0<p<\beta, \\ c(\frac{c_5}{\tau_1-2})^{\frac{1}{\beta+1}}E^{\frac{1}{\beta+1}}\delta^{\frac{\beta}{\beta+1}}, & p\geq\beta. \end{cases} \tag{5.14}$$

Proof: Using lemma 5.2.2 and (5.5), we have

$$\|f_\mu^\delta(x)-f_\mu(x)\| \leq c(\frac{1}{\mu})^{\frac{1}{\beta+1}}\delta$$
$$\leq \begin{cases} c(\frac{c_4}{\tau_1-2})^{\frac{1}{p+1}}E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & 0<p<\beta, \\ c(\frac{c_5}{\tau_1-2})^{\frac{1}{\beta+1}}E^{\frac{1}{\beta+1}}\delta^{\frac{\beta}{\beta+1}}, & p\geq\beta. \end{cases}$$

Theorem 5.2 *Let $f(x)$ be given by* (3.17) *and $f_\mu^\delta(x)$ be given by* (4.6)*. Suppose that $f(x)$ satisfies a priori bound condition* (3.19) *and the assumptions* (1.3) *and* (1.4) *hold. The regularization parameter $\mu>0$ is chosen by the Morozov discrepancy principle* (5.10)*. Then*

$$\|f_\mu^\delta(x)-f(x)\| \leq \begin{cases} (c(\frac{c_4}{\tau_1-2})^{\frac{1}{p+1}}+c_6)E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & 0<p<\beta, \\ c(\frac{c_5}{\tau_1-2})^{\frac{1}{\beta+1}}E^{\frac{1}{\beta+1}}\delta^{\frac{\beta}{\beta+1}}+c_6E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & p\geq\beta, \end{cases} \tag{5.15}$$

where $c_6 := (\frac{1}{T-t_0})^{\frac{p}{p+1}}(8+2\tau_1^2)^{\frac{p}{2p+2}}$.

Proof: By the triangle inequality, we have

$$\|f_\mu^\delta(x)-f(x)\| \leq \|f_\mu^\delta(x)-f_\mu(x)\|+\|f_\mu(x)-f(x)\|.$$

Firstly, we give an estimate for the second term.

$$\|f_\mu(x)-f(x)\|^2 = \|\sum_{n=1}^{\infty}\frac{\mu(k_{21}h_n-k_{22}g_n)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu](\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})}X_n(x)\|^2$$

$$
\begin{aligned}
&= \| \sum_{n=1}^{\infty} (\frac{\mu(k_{21}h_n - k_{22}g_n)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu](\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})})^{\frac{1}{p+1}} \\
&\cdot (\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\frac{-p}{p+1}} (\frac{\mu(k_{21}h_n - k_{22}g_n)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu]})^{\frac{p}{p+1}} X_n(x) \|^2 \\
&\leq (\sum_{n=1}^{\infty} ((\frac{\mu(k_{21}h_n - k_{22}g_n)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu](\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})})^{\frac{2}{p+1}})^{p+1})^{\frac{1}{p+1}} \\
&\cdot (\sum_{n=1}^{\infty} (((\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\frac{-p}{p+1}})^2 (\frac{\mu(k_{21}h_n - k_{22}g_n)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu]})^{\frac{2p}{p+1}})^{\frac{p+1}{p}})^{\frac{p}{p+1}} \\
&\leq \sup_{n\geq 1} ((\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\frac{-p}{p+1}} e^{\frac{-\varepsilon^2\lambda_n^\alpha T p}{p+1}})^2 E^{\frac{2}{p+1}} \\
&\cdot (\sum_{n=1}^{\infty} ((\frac{\mu(k_{21}h_n - k_{22}g_n)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu]})^{\frac{2p}{p+1}})^{\frac{p+1}{p}})^{\frac{p}{p+1}} \\
&\leq \sup_{n\geq 1} ((\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\frac{-p}{p+1}} e^{\frac{-\varepsilon^2\lambda_n^\alpha T p}{p+1}})^2 E^{\frac{2}{p+1}} \\
&\cdot (\sum_{n=1}^{\infty} (\frac{\mu(k_{21}h_n - k_{22}g_n)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu]})^2)^{\frac{p}{p+1}} \\
&\leq \sup_{n\geq 1} ((\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\frac{-p}{p+1}} e^{\frac{-\varepsilon^2\lambda_n^\alpha T p}{p+1}})^2 E^{\frac{2}{p+1}} \\
&\cdot (2 \sum_{n=1}^{\infty} (\frac{\mu(k_{21}(h_n - h_n^\delta) + k_{22}(g_n^\delta - g_n))}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu]})^2 \\
&+ 2 \sum_{n=1}^{\infty} (\frac{\mu(k_{21}h_n^\delta - k_{22}g_n^\delta)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu]})^2)^{\frac{p}{p+1}} \\
&\leq \sup_{n\geq 1} (D(n))^2 E^{\frac{2}{p+1}} (8 + 2\tau_1^2)^{\frac{p}{p+1}} \delta^{\frac{2p}{p+1}},
\end{aligned}
$$

where $D(n) = (\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\frac{-p}{p+1}} e^{\frac{-\varepsilon^2\lambda_n^\alpha T p}{p+1}}$.
By lemma 2.1, we have

$$
D(n) \leq (\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\frac{-p}{p+1}} (\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha (T - t_0)})^{\frac{p}{p+1}} = (\frac{1}{T - t_0})^{\frac{p}{p+1}}.
$$

Thus

$$
\|f_\mu(x) - f(x)\| \leq (\frac{1}{T - t_0})^{\frac{p}{p+1}} (8 + 2\tau_1^2)^{\frac{p}{2p+2}} E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}} = c_6 E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}. \qquad (5.16)
$$

Combining lemma 5.2.3, we have

$$\|f_\mu^\delta(x)-f(x)\| \le \begin{cases} (c(\frac{c_4}{\tau_1-2})^{\frac{1}{p+1}}+c_6)E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & 0<p<\beta, \\ c(\frac{c_5}{\tau_1-2})^{\frac{1}{\beta+1}}E^{\frac{1}{\beta+1}}\delta^{\frac{\beta}{\beta+1}}+c_6E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & p\ge\beta. \end{cases}$$

5.3 The priori regularization parameter choice rule of the initial value

Theorem 5.3 *Let $\varphi(x)$ be given by* (3.18) *and $\varphi_\mu^\delta(x)$ be given by* (4.7)*. Suppose that $\varphi(x)$ satisfies a priori bounded condition* (3.20) *and the assumptions* (1.3), (1.4) *hold. Choosing the regularization parameter:*

$$\mu = \begin{cases} (\frac{\delta}{E})^{\frac{\beta+1}{p+1}}, & 0<p<\beta+1, \\ (\frac{\delta}{E})^{\frac{\beta+1}{\beta+2}}, & p\ge\beta+1, \end{cases}$$

then we obtain the following error estimate:

$$\|\varphi_\mu^\delta(x)-\varphi(x)\| \le \begin{cases} (c'+c_1)E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & 0<p<\beta+1, \\ (c'+c_2)E^{\frac{1}{\beta+2}}\delta^{\frac{\beta+1}{\beta+2}}, & p\ge\beta+1, \end{cases} \tag{5.17}$$

where $c' := 2\frac{1}{\beta+1}\beta^{\frac{\beta}{\beta+1}}\frac{1}{\varepsilon^2\lambda_1^\alpha}$.

Proof: By the triangle inequality, we have

$$\|\varphi_\mu^\delta(x)-\varphi(x)\| \le \|\varphi_\mu^\delta(x)-\varphi_\mu(x)\| + \|\varphi_\mu(x)-\varphi(x)\|.$$

Firstly, we give an estimate for the first term. From (4.5), (4.7), we have

$$\begin{aligned} \|\varphi_\mu^\delta(x)-\varphi_\mu(x)\| &= \|\sum_{n=1}^{\infty}\frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^\beta[(k_{12}g_n^\delta-k_{11}h_n^\delta)-(k_{12}g_n-k_{11}h_n)]}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}X_n(x)\| \\ &\le \sup_{n\ge1}\left|\frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^\beta}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}\right|(\|\sum_{n=1}^{\infty}k_{12}(g_n^\delta-g_n)X_n(x)\|+\|\sum_{n=1}^{\infty}k_{11}(h_n^\delta-h_n)X_n(x)\|) \\ &\le \sup_{n\ge1}|A(n)|\frac{2}{\varepsilon^2\lambda_1^\alpha}\delta, \end{aligned} \tag{5.18}$$

where $A(n) = \frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^\beta}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu}$.

By (5.4), the following conclusion can be drawn.

$$\|\varphi_\mu^\delta(x)-\varphi_\mu(x)\| \le c'(\frac{1}{\mu})^{\frac{1}{\beta+1}}\delta, \tag{5.19}$$

where $c' = 2\frac{1}{\beta+1}\beta^{\frac{\beta}{\beta+1}}\frac{1}{\varepsilon^2\lambda_1^\alpha}$.
Then we estimate the second term by (4.5) and (3.18),

$$\begin{aligned}
&\|\varphi_\mu(x) - \varphi(x)\| \\
&= \|\sum_{n=1}^{\infty}(\frac{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^\beta}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu} - \frac{\varepsilon^2\lambda_n^\alpha}{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}})(k_{12}g_n - k_{11}h_n)X_n(x)\| \\
&= \|\sum_{n=1}^{\infty}\frac{\mu}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu](\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})}(k_{12}g_n - k_{11}h_n)X_n(x)\| \\
&\le \sup_{n\ge 1}|B(n)|E. \qquad (5.20)
\end{aligned}$$

By (5.8), we deduce that

$$\|\varphi_\mu(x) - \varphi(x)\| \le \begin{cases} c_1\mu^{\frac{p}{\beta+1}}E, & 0 < p < \beta+1, \\ c_2\mu E, & p \ge \beta+1. \end{cases} \qquad (5.21)$$

Combining (5.19) with (5.21), we choose the regularized parameter μ by

$$\mu = \begin{cases} (\frac{\delta}{E})^{\frac{\beta+1}{p+1}}, & 0 < p < \beta+1, \\ (\frac{\delta}{E})^{\frac{\beta+1}{\beta+2}}, & p \ge \beta+1. \end{cases}$$

We have

$$\|\varphi_\mu^\delta(x) - \varphi(x)\| \le \begin{cases} (c' + c_1)E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & 0 < p < \beta+1, \\ (c' + c_2)E^{\frac{1}{\beta+2}}\delta^{\frac{\beta+1}{\beta+2}}, & p \ge \beta+1. \end{cases}$$

The proof is completed.

5.4 The posteriori regularization choice rule of the initial value

The Morozov discrepancy principle is used to find μ as follows:

$$\|K\varphi_\mu^\delta(x) - (K_{12}g^\delta(x) - K_{11}h^\delta(x))\| = \tau_2\delta, \qquad (5.22)$$

where $\tau_2 > \frac{2}{\varepsilon^2\lambda_1^\alpha}$ is a constant. According to the following lemma, we know there exists a unique solution for (5.22) if $\|K_{12}g^\delta(x) - K_{11}h^\delta(x)\| > \tau_2\delta$.

Lemma 5.4.1 $\rho_2(\mu) = \|K\varphi_\mu^\delta(x) - (K_{12}g^\delta(x) - K_{11}h^\delta(x))\|$, *the following results hold*

(a) $\rho_2(\mu)$ *is a continuous function;*

(b) $\lim_{\mu\to 0}\rho_2(\mu) = 0$;

(c) $\lim_{\mu\to\infty} \rho_2(\mu) = \|K_{12}g^\delta(x) - K_{11}h^\delta(x)\|$;

(d) $\rho_2(\mu)$ *is a strictly increasing function over* $\mu \in (0,\infty)$.

Lemma 5.4.2 *Suppose* (1.3), (1.4) *and the priori bound condition* (3.20) *hold, we obtain*

$$\mu^{-1} \le \begin{cases} (\frac{c_4}{\tau_2 - \frac{2}{\varepsilon^2\lambda_1^\alpha}})^{\frac{\beta+1}{p+1}} (\frac{E}{\delta})^{\frac{\beta+1}{p+1}}, & 0 < p < \beta, \\ (\frac{c_5}{\tau_2 - \frac{2}{\varepsilon^2\lambda_1^\alpha}}) \frac{E}{\delta}, & p \ge \beta. \end{cases} \tag{5.23}$$

Proof:

$$\begin{aligned} \tau_2\delta &= \| \sum_{n=1}^{\infty} \frac{\mu(k_{12}g_n^\delta - k_{11}h_n^\delta)}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu} X_n(x) \| \\ &\le \| \sum_{n=1}^{\infty} \frac{\mu[k_{12}(g_n^\delta - g_n) - k_{11}(h_n^\delta - h_n)]}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu} X_n(x) \| + \| \sum_{n=1}^{\infty} \frac{\mu(k_{12}g_n - k_{11}h_n)}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu} X_n(x) \| \\ &\le \frac{2}{\varepsilon^2\lambda_1^\alpha}\delta + \sup_{n\ge1} \left| \frac{\mu \frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha} e^{-\varepsilon^2\lambda_n^\alpha pT}}{(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1} + \mu} \right| E \\ &= \frac{2}{\varepsilon^2\lambda_1^\alpha}\delta + \sup_{n\ge1} |C(n)| E. \end{aligned} \tag{5.24}$$

From the previous analysis of $C(n)$, we obtain

$$\mu^{-1} \le \begin{cases} (\frac{c_4}{\tau_2 - \frac{2}{\varepsilon^2\lambda_1^\alpha}})^{\frac{\beta+1}{p+1}} (\frac{E}{\delta})^{\frac{\beta+1}{p+1}}, & 0 < p < \beta, \\ (\frac{c_5}{\tau_2 - \frac{2}{\varepsilon^2\lambda_1^\alpha}}) \frac{E}{\delta}, & p \ge \beta. \end{cases}$$

This proof is completed.

Lemma 5.4.3 *Let* $\varphi_\mu(x)$ *be given by* (4.5) *and* $\varphi_\mu^\delta(x)$ *be given by* (4.7), *then we have*

$$\|\varphi_\mu^\delta(x) - \varphi_\mu(x)\| \le \begin{cases} c'(\frac{c_4}{\tau_2 - \frac{2}{\varepsilon^2\lambda_1^\alpha}})^{\frac{1}{p+1}} E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}, & 0 < p < \beta, \\ c'(\frac{c_5}{\tau_2 - \frac{2}{\varepsilon^2\lambda_1^\alpha}})^{\frac{1}{\beta+1}} E^{\frac{1}{\beta+1}} \delta^{\frac{\beta}{\beta+1}}, & p \ge \beta. \end{cases} \tag{5.25}$$

Proof: Using lemma 5.4.2 and (5.19), we have

$$\begin{aligned} \|\varphi_\mu^\delta(x) - \varphi_\mu(x)\| &\le c'(\frac{1}{\mu})^{\frac{1}{\beta+1}} \delta \\ &\le \begin{cases} c'(\frac{c_4}{\tau_2 - \frac{2}{\varepsilon^2\lambda_1^\alpha}})^{\frac{1}{p+1}} E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}, & 0 < p < \beta, \\ c'(\frac{c_5}{\tau_2 - \frac{2}{\varepsilon^2\lambda_1^\alpha}})^{\frac{1}{\beta+1}} E^{\frac{1}{\beta+1}} \delta^{\frac{\beta}{\beta+1}}, & p \ge \beta. \end{cases} \end{aligned}$$

Theorem 5.4 *Let $\varphi(x)$ be given by (3.18) and $\varphi_\mu^\delta(x)$ be given by (4.7). Suppose that $\varphi(x)$ satisfies a priori bound condition (3.20) and the assumptions (1.3) and (1.4) hold. The regularization parameter $\mu > 0$ is chosen by the Morozov discrepancy principle (5.22). Then*

$$\|\varphi_\mu^\delta(x)-\varphi(x)\| \le \begin{cases} (c'(\frac{c_4}{\tau_2-\frac{2}{\varepsilon^2\lambda_1^\alpha}})^{\frac{1}{p+1}}+c_7)E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & 0<p<\beta,\\ c'(\frac{c_5}{\tau_2-\frac{2}{\varepsilon^2\lambda_1^\alpha}})^{\frac{1}{\beta+1}}E^{\frac{1}{\beta+1}}\delta^{\frac{\beta}{\beta+1}}+c_7E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & p\ge\beta,\end{cases} \tag{5.26}$$

where $c_7 := (\frac{1}{T-t_0})^{\frac{p}{p+1}}(\frac{8}{(\varepsilon^2\lambda_1^\alpha)^2}+2\tau_2^2)^{\frac{p}{2p+2}}$.

Proof: By the triangle inequality, we have

$$\|\varphi_\mu^\delta(x)-\varphi(x)\| \le \|\varphi_\mu^\delta(x)-\varphi_\mu(x)\|+\|\varphi_\mu(x)-\varphi(x)\|.$$

Firstly, we give an estimate for the second term.

$$\|\varphi_\mu(x)-\varphi(x)\|^2 = \|\sum_{n=1}^\infty \frac{\mu(k_{12}g_n-k_{11}h_n)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu](\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})}X_n(x)\|^2$$

$$= \|\sum_{n=1}^\infty (\frac{\mu(k_{12}g_n-k_{11}h_n)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu](\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})})^{\frac{1}{p+1}}$$

$$\cdot (\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\frac{-p}{p+1}}(\frac{\mu(k_{12}g_n-k_{11}h_n)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu]})^{\frac{p}{p+1}}X_n(x)\|^2$$

$$\le (\sum_{n=1}^\infty ((\frac{\mu(k_{12}g_n-k_{11}h_n)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu](\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})})^{\frac{2}{p+1}})^{p+1})^{\frac{1}{p+1}}$$

$$\cdot (\sum_{n=1}^\infty ((\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\frac{-2p}{p+1}}(\frac{\mu(k_{12}g_n-k_{11}h_n)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu]})^{\frac{2p}{p+1}})^{\frac{p+1}{p}})^{\frac{p}{p+1}}$$

$$\le \sup_{n\ge1}((\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\frac{-p}{p+1}}e^{\frac{-\varepsilon^2\lambda_n^\alpha Tp}{p+1}})^2E^{\frac{2}{p+1}}$$

$$\cdot (\sum_{n=1}^\infty ((\frac{\mu(k_{12}g_n-k_{11}h_n)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu]})^{\frac{2p}{p+1}})^{\frac{p+1}{p}})^{\frac{p}{p+1}}$$

$$\le \sup_{n\ge1}(D(n))^2E^{\frac{2}{p+1}}(\sum_{n=1}^\infty (\frac{\mu(k_{12}g_n-k_{11}h_n)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu]})^2)^{\frac{p}{p+1}}$$

$$\le \sup_{n\ge1}(D(n))^2E^{\frac{2}{p+1}}$$

$$\cdot (2\sum_{n=1}^\infty (\frac{\mu(k_{12}(g_n-g_n^\delta)+k_{11}(h_n^\delta-h_n))}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu]})^2$$

$$+2\sum_{n=1}^{\infty}\Big(\frac{\mu(k_{12}g_n^\delta-k_{11}h_n^\delta)}{[(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\beta+1}+\mu]}\Big)^2\Big)^{\frac{p}{p+1}}$$

$$\le \sup_{n\ge1}(D(n))^2E^{\frac{2}{p+1}}\Big(\frac{8}{(\varepsilon^2\lambda_1^\alpha)^2}+2\tau_2^2\Big)^{\frac{p}{p+1}}\delta^{\frac{2p}{p+1}},$$

where $D(n)=(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0}-e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha})^{\frac{-p}{p+1}}e^{\frac{-\varepsilon^2\lambda_n^\alpha Tp}{p+1}}$.
Thus

$$\|\varphi_\mu(x)-\varphi(x)\|\le\Big(\frac{1}{T-t_0}\Big)^{\frac{p}{p+1}}\Big(\frac{8}{(\varepsilon^2\lambda_1^\alpha)^2}+2\tau_2^2\Big)^{\frac{p}{2p+2}}E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}. \tag{5.27}$$

Combining lemma 5.4.3, we have

$$\|\varphi_\mu^\delta(x)-\varphi(x)\|\le\begin{cases}\Big(c'\big(\frac{c_4}{\tau_2-\frac{2}{\varepsilon^2\lambda_1^\alpha}}\big)^{\frac{1}{p+1}}+c_7\Big)E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & 0<p<\beta,\\ c'\big(\frac{c_5}{\tau_2-\frac{2}{\varepsilon^2\lambda_1^\alpha}}\big)^{\frac{1}{\beta+1}}E^{\frac{1}{\beta+1}}\delta^{\frac{\beta}{\beta+1}}+c_7E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & p\ge\beta,\end{cases}$$

where $c_7=(\frac{1}{T-t_0})^{\frac{p}{p+1}}(\frac{8}{(\varepsilon^2\lambda_1^\alpha)^2}+2\tau_2^2)^{\frac{p}{2p+2}}$.

6 Numerical implementation

In this section, we are going to use numerical examples and software to verify the efficiency of our method. We solve the following direct problem to obtain $g(x)$ and $h(x)$.

$$\begin{cases}\frac{\partial u(x,t)}{\partial t}=\varepsilon^2L_\alpha u(x,t)+f(x), & x\in\Omega,\ t\in(0,T],\\ u(x,0)=\varphi(x), & x\in\Omega,\\ u(x,t)=0, & x\in\partial\Omega,\ t\in(0,T]\\ u(x,t_0)=g(x), & x\in\Omega,\ t_0\in(0,T],\\ u(x,T)=h(x), & x\in\Omega.\end{cases}\tag{6.1}$$

We define

$$x_i=i\Delta x\ (i=0,1,\cdots,M+1),t_j=j\Delta t\ (j=0,1,\cdots,N),$$

where $\Delta x=\frac{1}{M+1}$ is the step size of space and Δt is the step size of time.
Let $g(x)=0$, $\lambda=0$ in [24], we can obtain

$$(-\Delta)^{\frac{\alpha}{2}}U=C_\alpha BU,$$

$$U_i=u(x_i),i=1,2,\cdots,M,$$

$$U=(U_1,U_2,\cdots,U_M)^T,$$

$$B=(h_{i,p})_{i,p=1}^M,$$

$$\varphi=(\varphi(x_1),\varphi(x_2),\cdots,\varphi(x_M))^T,$$

$$F=(f(x_1),f(x_2),\cdots,f(x_M))^T,$$

where $C_\alpha=\frac{4^{\frac{\alpha}{2}}\Gamma(1/2+\frac{\alpha}{2})}{\pi^{1/2}|\Gamma(-\frac{\alpha}{2})|}$, and B is a strictly diagonally dominant symmetric positive-definite matrix.

$B\triangleq(h_{i,p})_{i,p=1}^M$, where

$$h_{i,p}\triangleq\begin{cases}-(Z_1(i,p+1)+Z_2(i,p))\frac{1}{i-p}, & 1\le p\le i-2,\\ -\frac{h^{-\alpha}}{2-\alpha}-Z_2(i,i-1), & p=i-1,\\ -\frac{h^{-\alpha}}{2-\alpha}-Z_3(i,i+2), & p=i+1,\\ -(Z_3(i,p+1)+Z_4(i,p))\frac{1}{p-i}, & i+2\le p\le M,\end{cases}$$

and $h_{i,i}$ satisfies:

$$h_{i,i}+\sum_{p=1,p\neq i}^{M}h_{i,p}-Y_1(i)-Y_2(i)=\begin{cases}\frac{h^{-\alpha}}{2-\alpha}+\frac{Z_4(i,M+1)}{M+1-i}, & i=1,\\ \frac{Z_1(i,1)}{i}+\frac{Z_4(i,M+1)}{M+1-i}, & 2\le i\le M-1,\\ \frac{h^{-\alpha}}{2-\alpha}+\frac{Z_1(i,1)}{i}, & i=M.\end{cases}$$

By a simple calculation, for $\alpha\in(0,2)$, we have

$$Z_1(i,p+1)+Z_2(i,p)=Z_3(i,p+1)+Z_4(i,p)=\begin{cases}\frac{h^{-\alpha}}{(\alpha-1)(2-\alpha)}[2|i-p|^{2-\alpha}-(|i-p|-1)^{2-\alpha}\\ -(|i-p|+1)^{2-\alpha}], & \alpha\neq1,\\ \frac{1}{h}[-2|i-p|\ln(|i-p|)\\ +(|i-p|+1)\ln(|i-p|+1)\\ +(|i-p|-1)\ln(|i-p|-1)], & \alpha=1,\end{cases}$$

$$Z_2(i,i-1)=Z_3(i,i+2)=\begin{cases}\frac{h^{-\alpha}}{(\alpha-1)(2-\alpha)}(3-\alpha-2^{2-\alpha}), & \alpha\neq1,\\ \frac{1}{h}[2\ln2-1], & \alpha=1,\end{cases}$$

$$Z_1(i,1)=\begin{cases}\frac{h^{-\alpha}}{(\alpha-1)(2-\alpha)}[i^{2-\alpha}-(i-1)^{2-\alpha}-(2-\alpha)i^{1-\alpha}], & \alpha\neq1,\\ \frac{1}{h}[(1-i)\ln(\frac{i}{i-1})+1], & \alpha=1,\end{cases}$$

$$Z_4(i,M+1)=\begin{cases}\frac{h^{-\alpha}}{(\alpha-1)(2-\alpha)}[(M+1-i)^{2-\alpha}-(M-i)^{2-\alpha}\\ -(2-\alpha)(M+1-i)^{1-\alpha}], & \alpha\neq1,\\ \frac{1}{h}[(i-M)\ln(\frac{M+1-i}{M-i})+1], & \alpha=1,\end{cases}$$

$$Y_1(i)=\frac{(x_i)^{-\alpha}}{\alpha},\quad Y_2(i)=\frac{(1-x_i)^{-\alpha}}{\alpha}.$$

According to the numerical differentiation formula, we have

$$\frac{\partial u(x_i,t_j)}{\partial t} \approx \frac{1}{\Delta t}(u_i^j - u_i^{j-1}).$$

Thus, we can obtain

$$\frac{1}{\Delta t}(U^j - U^{j-1}) + C_\alpha B U^j = F, \quad 1 \le j \le N,$$
$$U_0 = \varphi.$$

The following iterative format can be obtained.

$$(I + \Delta t C_\alpha B)U^j = U^{j-1} + \Delta t F, \quad 1 \le j \le N. \tag{6.2}$$

Using the Tikhonov regularization method, we obtain the regularized solutions of measurement data $g^\delta(x)$ and $h^\delta(x)$ with error:

$$f_\mu^\delta(x) = \sum_{n=1}^{m} \frac{\left(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha}\right)^\beta}{\left(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha}\right)^{\beta+1} + \mu}(k_{21}h_n^\delta - k_{22}g_n^\delta)X_n(x),$$

$$\varphi_\mu^\delta(x) = \sum_{n=1}^{m} \frac{\left(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha}\right)^\beta}{\left(\frac{e^{-\varepsilon^2\lambda_n^\alpha t_0} - e^{-\varepsilon^2\lambda_n^\alpha T}}{\varepsilon^2\lambda_n^\alpha}\right)^{\beta+1} + \mu}(k_{12}g_n^\delta - k_{11}h_n^\delta)X_n(x).$$

m is the truncation parameter and $m = 10$.

We generate the noise-contaminated data by adding a random perturbation, i.e.,

$$g^\delta(x) = g(x) + \varepsilon \cdot g(x) rand(size(g)), \tag{6.3}$$

$$h^\delta(x) = h(x) + \varepsilon \cdot h(x) rand(size(h)), \tag{6.4}$$

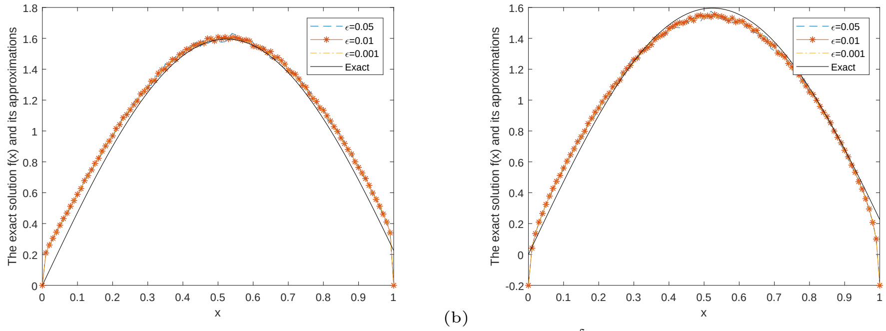
here, $size(g)$ represents the size of g in space, $size(h)$ represents the size of h in space, the function $rand(\cdot)$ generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and the noise level is:

$$\delta_1 = \|g^\delta - g\| = \sqrt{\frac{1}{M+1}\sum_{i=1}^{M+1}(g_i - g_i^\delta)^2}, \tag{6.5}$$

$$\delta_2 = \|h^\delta - h\| = \sqrt{\frac{1}{M+1}\sum_{i=1}^{M+1}(h_i - h_i^\delta)^2}. \tag{6.6}$$

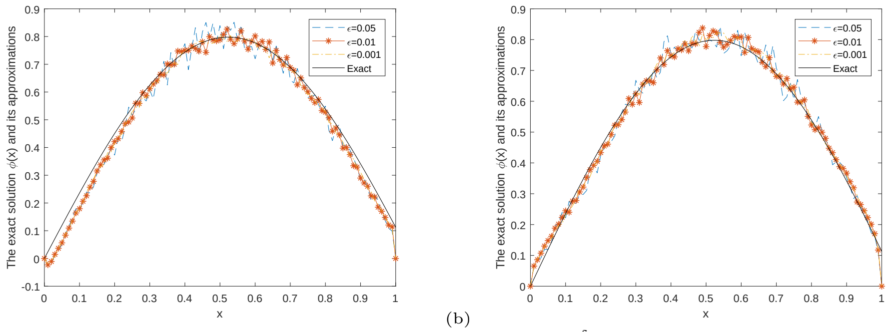
In general, the priori bound E is difficult to obtain, thus we choose the posteriori parameter rule which is independent of E and let takes $\tau_1 = 2.1, \tau_2 = 1.1$. To verify the stability of numerical results, the following Root-mean-square deviation is defined:

$$e_{r_1} = \frac{\sqrt{\sum(f - f_\mu^\delta)^2}}{\sqrt{\sum f^2}}, \tag{6.7}$$



(a) (b)

Figure 1. The exact solution $f(x)$ and regularization solution $f_\mu^\delta(x)$ for (a)$\alpha = 0.3$, (b)$\alpha = 0.7$.



(a) (b)

Figure 2. The exact solution $\varphi(x)$ and regularization solution $\varphi_\mu^\delta(x)$ for (a)$\alpha = 0.3$, (b)$\alpha = 0.7$.

$$e_{r_2} = \frac{\sqrt{\sum(\varphi - \varphi_\mu^\delta)^2}}{\sqrt{\sum \varphi^2}}. \tag{6.8}$$

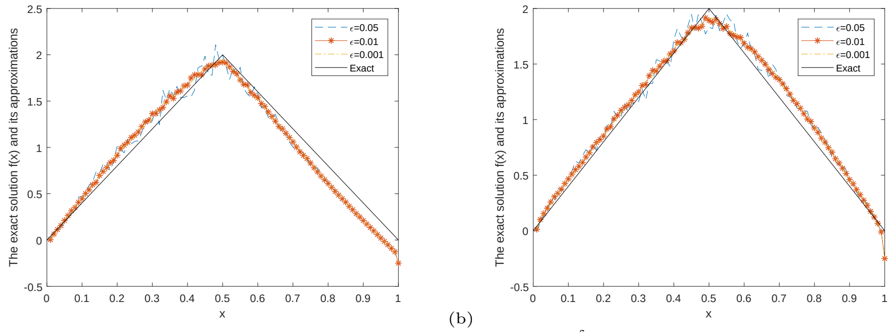
For convenience, we let $M = 100$, $N = 30$, $\varepsilon = 1$, $d = 1$, $\Omega = (0, 1)$, $T_0 = 0.5$ and $T = 1$. $X_n(x)$ and λ_n^α are the characteristic functions and eigenvalues of operator $(-\Delta)^{\frac{\alpha}{2}}$. By calculation, the characteristic function $X_n(x) = \sqrt{2}sin(n\pi x)$ and eigenvalues $\lambda_n = n\pi$ can be obtained, where $n = 1, 2, \cdots$.

Example 1 We consider the following equations:

$$f(x) = 2\sqrt{\frac{2}{\pi}}sin(x), \quad \varphi(x) = \sqrt{\frac{2}{\pi}}sin(x), \quad x \in [0, 1].$$

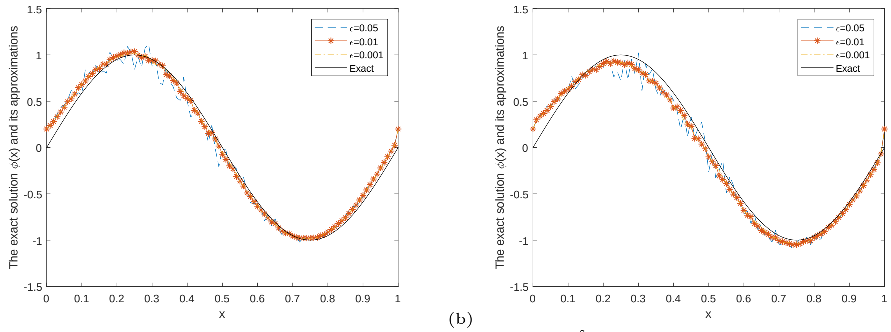
Table 1: Numerical results of Example 1 for different ε

ε			0.05	0.01	0.001
$f(x)$	$\alpha = 0.3$	e_{r1}	0.0040	0.0039	0.0039
	$\alpha = 0.7$	e_{r1}	0.0037	0.0035	0.0035
$\varphi(x)$	$\alpha = 0.3$	e_{r2}	0.0054	0.0033	0.0029
	$\alpha = 0.7$	e_{r2}	0.0029	0.0015	0.0012



(a) (b)

Figure 3. The exact solution $f(x)$ and regularization solution $f_\mu^\delta(x)$ for (a)$\alpha = 0.3$, (b)$\alpha = 0.55$.



(a) (b)

Figure 4. The exact solution $\varphi(x)$ and regularization solution $\varphi_\mu^\delta(x)$ for (a)$\alpha = 0.3$, (b)$\alpha = 0.55$.

In Table 1, we can see when $\alpha = 0.3$ and $\alpha = 0.7$, the larger the noisy level, the larger the relative error level for the exact solutions and the regularization solutions, respectively. It can say that when the space-fractional order α is fixed a constant, as the noise level increases, the numerical effect becomes worse and worse.

Figure 1 shows the exact $f(x)$ and its Tikhonov regularization solution $f_\mu^\delta(x)$ for the relative error levels $\varepsilon = 0.05, 0.01, 0.001$ with various values $\alpha = 0.3, 0.7$. Figure 2 shows the exact $\varphi(x)$ and its Tikhonov regularization solution $\varphi_\mu^\delta(x)$ for the relative error levels $\varepsilon = 0.05, 0.01, 0.001$ with various values $\alpha = 0.3, 0.7$.

It can be seen from Figures 1-2 that the Tikhonov regularization method is very effective for solving the inverse problem of space-fractional Allen-Cahn equation.

Example 2 Consider the following equations:

$$f(x) = \begin{cases} 4x, & x \in [0, \frac{1}{2}), \\ -4(x-1), & x \in [\frac{1}{2}, 1], \end{cases}$$

$$\varphi(x) = sin(2\pi x), \qquad x \in [0, 1].$$

Figure 3 shows the exact $f(x)$ and its Tikhonov regularization solution $f_\mu^\delta(x)$ for the relative error levels $\varepsilon = 0.05, 0.01, 0.001$ with value $\alpha = 0.3, 0.55$. Figure 4 shows the exact $\varphi(x)$ and its Tikhonov regularization solution $\varphi_\mu^\delta(x)$ for the relative error levels $\varepsilon = 0.05, 0.01, 0.001$ with value $\alpha = 0.3, 0.55$. From the images, it can be seen that

Table 2: Numerical results of Example 2 for different ε

ε			0.05	0.01	0.001
$f(x)$	$\alpha = 0.3$	e_{r1}	0.0156	0.0130	0.0127
	$\alpha = 0.55$	e_{r1}	0.0066	0.0051	0.0050
$\varphi(x)$	$\alpha = 0.3$	e_{r2}	0.0212	0.0099	0.0094
	$\alpha = 0.55$	e_{r2}	0.0283	0.0163	0.0169

Tikhonov regularization method has certain limitations in handing inflection points.

Example 3 Consider the following discontinuous equations:

$$f(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}), \\ 1, & x \in [\frac{1}{2}, 1], \end{cases}$$

$$\varphi(x) = \begin{cases} 1, & x \in [0, \frac{3}{10}), \\ 0, & x \in [\frac{3}{10}, \frac{3}{4}), \\ 1, & x \in [\frac{3}{4}, 1]. \end{cases}$$

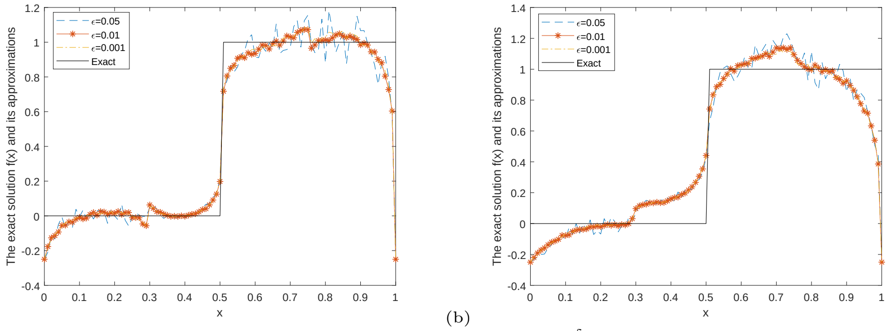
Figure 5 shows the exact $f(x)$ and its Tikhonov regularization solution $f_\mu^\delta(x)$ for

Table 3: Numerical results of Example 3 for different ε

ε			0.05	0.01	0.001
$f(x)$	$\alpha = 0.3$	e_{r1}	0.0517	0.0464	0.0466
	$\alpha = 0.45$	e_{r1}	0.0088	0.0061	0.0060
$\varphi(x)$	$\alpha = 0.3$	e_{r2}	0.0260	0.0158	0.0160
	$\alpha = 0.45$	e_{r2}	0.0860	0.0612	0.0805

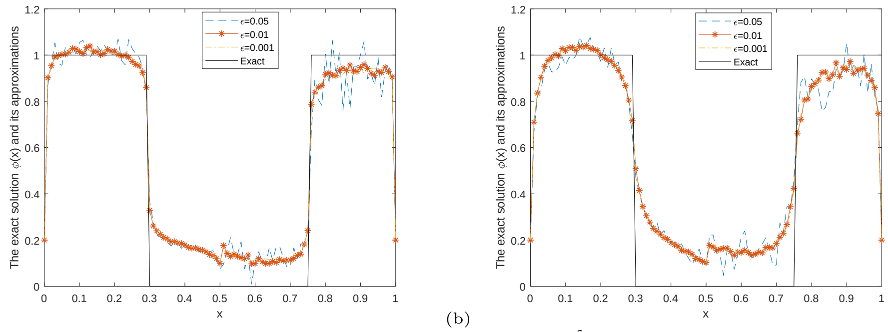
the relative error levels $\epsilon = 0.05, 0.01, 0.001$ with value $\alpha = 0.3, 0.45$. Figure 6 shows the exact $\varphi(x)$ and its Tikhonov regularization solution $\varphi_\mu^\delta(x)$ for the relative error levels $\epsilon = 0.05, 0.01, 0.001$ with value $\alpha = 0.3, 0.45$. Obviously, this method produces large errors when dealing with discontinuous function, but it can still be used to approximate the exact solution.

Through the above examples, we find that from Figures 1-6, it can be seen that the fitting results of different α are not significantly different. From Tables 1-3, it can be seen that the smaller the relative error level, the better the approximation effect. This indicates that regardless of how α changes in $[0, 1]$, image fitting is relatively stable. This also means that the fractional order Tikhonov regularization method is effective.



(a) (b)

Figure 5. The exact solution $f(x)$ and regularization solution $f_{\mu}^{\delta}(x)$ for (a)$\alpha = 0.3$, (b)$\alpha = 0.45$.



(a) (b)

Figure 6. The exact solution $\varphi(x)$ and regularization solution $\varphi_{\mu}^{\delta}(x)$ for (a)$\alpha = 0.3$, (b)$\alpha = 0.45$.

7 Conclusion

In this paper, we consider an inverse problem to identify simultaneously the source term and initial value of space-fractional Allen-Cahn equation. We use the fractional Tikhonov method to overcome the ill-posedness. The error estimations are obtained under a priori regularization parameter choice rule and a posteriori regularization parameter choice rule, respectively. And we compare this method from error estimates and numerical results. The numerical tests are presented to show the validity and the advantage of the proposed schemes.

Disclosure statement

No potential conflict of interest was reported by the author.

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