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UNIFORMLY EXPONENTIALLY STABLE APPROXIMATION FOR THE TRANSMISSION LINE WITH VARIABLE COEFFICIENTS AND ITS APPLICATION*

Bingfeng Zhang[1], Fu Zheng[2,†], and Yong He[2,†]

Abstract We analyze an ideal transmission line, which is defined by the telegraph equation with variable coefficients, from the perspectives of numerical analysis and control theory in this note. Because the spatially semi-discrete scheme of the original system is insufficient for discussing uniform exponential stability, we apply a similar transform to the continuous system and produce an intermediate system that may be easily analyzed. To begin, we discuss uniform exponential stability for the intermediate system using an so called average central-difference semi-discrete scheme and the direct Lyapunov function approach. The proof is the same as in the continuous case. The Trotter-Kato Theorem is used to demonstrate the stability and consistency of numerical approximation scheme. Finally, we propose a semi-discrete strategy for the original system through an inverse transform. All results on intermediate system are then translated into the original system. The numerical state reconstruction problem is addressed as an essential application of the main results. Furthermore, several numerical simulations are used to validate the effectiveness of the numerical approximating algorithms.

Keywords Transmission line, Exponential stability, State reconstruction; Semi-discretization, Average central-difference

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1. Introduction

Space semi-discretization, which converts PDEs to ODEs, is the first natural step in the numerical discretization of PDEs. In the past decades, many researchers have approached this issue from various perspectives. Mathematicians focused primarily on the stability and consistency of the approximating algorithm and con-

†The corresponding author. Email: fuzheng@hainanu.edu.cn(Fu Zheng); heyong1980@tju.edu.cn(Yong He)
[1]School of Mathematical Science, Bohai University, Jinzhou, Liaoning, 121013, China
[2]School of Science, Hainan University, Haikou, Hainan, 570228, China
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structed numerous profound discrete schemes [5, 14, 16, 32]. The goal of physicists was to preserve structural invariants and geometric features of a continuous system. Several recent papers have addressed these problems for infinite-dimensional port-Hamiltonian systems. Dirac structure is kept by using finite element or finite difference approaches [4, 27, 28].

However, when retaining control properties such as passivity, exponential stability, and observability are taken into account, a complete and elegant analysis is required [9, 10, 18–20]. Banks, Ito, and Wang originally pointed out in [1] that the wave equation with boundary damping does not inherit the exponential stability of continuous system when discretized by the conventional finite difference and finite element schemes. At the same time, Glowinski, Li, and Lions demonstrated in [6] that the exact controllability property was not preserved for several discretization processes. The issue of boundary observability of the wave equation, or the question of whether the total energy of solutions can be uniformly calculated in terms of the energy focused on the boundary as the net-spacing approaches zero, has recently been examined by Infante and Zuazua. Due to the presence of high frequency spurious solutions for both finite-difference and finite-element semi-discretizations, they came to a negative conclusion. In a subspace of solutions produced by the low frequencies of the discrete system, a uniform bound was found [12]. In [33], Zuazua provided a thorough analysis of the observation and control of waves that were approximated by finite difference.

Numerous ideas were put up to get around the problem which is caused by high frequency spurious modes. One proposal is to use the mixed finite element method to get the uniform controllability of the conserved wave equation [4, 27, 28]. Tychonoff regularization [6], two-grid algorithms [25], and non-uniform numerical meshes [2] are some other methods for damping out high frequencies. Another common method is to introduce a vanishing viscosity term over the entire domain of the spatial variables [23, 24, 30]. Very recently, Liu and Guo introduced an average operator for the time derivative of classical finite difference for the wave equation with boundary damping and showed that the scheme uniformly preserves the exponential decay of the continuous system [17]. Using the same idea, Xu, Guo and Zheng proposed two finite-difference schemes for uniform exponential approximations of the wave equation with local viscosity damping [7, 34]. Guo and Zheng *et al.* generalized the results of [7, 17, 34] to the coupled heat-wave system [31] and the Schrödinger equation on $L^2(0, 1)$ space [8], respectively.

In this note, we are going to study the transmission line with variable capacity and inductance from the perspectives of numerical analysis and control theory. This transmission line is described by the telegraph equation with variable coefficients. The electrical transmission line, the flexible string, and the compressible fluid are three commonly used physical systems that carry waves in engineering applications ([4], [22], [13, Section 7.1]). Since an electrical transmission line is fundamentally a wave, it may be numerically studied using the techniques described in cites [7, 17, 31, 34]. But constructing an appropriate numerical approximating approach to maintain exponential stability now presents some new challenges due to the variable coefficients that arise in the space derivatives of PDEs. To the best of our knowledge, the passivity, not even the exponential stability, of the continuous system is rarely preserved by the standard finite-difference method for the spatial semi-discretization of the telegraph equation.

To cope with these problems, we first apply a transform that is comparable to the original system and produce an intermediate system that is simple to study. For the intermediate system, an average central-difference semi-discretization method is suggested (see section 4). Then, using a strategy similar to that used in the continuous example, we study the uniform exponential stability of discrete systems. We also show how the discrete system is uniformly exponentially stabilized under the boundary feedback control (see Remark 3.1). Second, the Trotter-Kato Theorem illustrates the consistency and stability of numerical approximating algorithms. Finally, a discretization scheme for the original system is proposed through another similar transform. The final step is to convert every result from the intermediate system into the primary system. The numerical approximations of the state reconstruction problem are also presented as an application of the main results. We also perform a number of numerical simulations to demonstrate the effectiveness of the numerical approximating strategies.

Thus, the contribution of this work is four-fold:

- Propose a novel approach to study the numerical solution of PDE with variable coefficients.
- Extend the results of uniform exponential stability of [30] and [7, 17, 34] from simple models to more complex systems.
- Provide demonstration to study uniform exponential stability of other complex models described by PDEs such as wave-wave coupled equations, beam equations and so on.
- The results of uniform exponential stability have potential applications in uniform controllability, the approximation of control problem and the state reconstruction etc.

The structure of this paper is as follows. In section 2, the transmission line system is introduced and some results about exponential stability and exact observability of the intermediate system are presented. In section 3, the uniform exponential stability and uniform observability of discrete systems are obtained. In section 4, the stability and consistence of the numerical approximating algorithm are derived from the Trotter-Kato Theorem. In section 5, a semi-discrete scheme of the original system is proposed through a similar transformation. The uniform exponential stability of the original system is proved and convergence analysis of the discretization scheme is made based on the result of section 4. In section 6, several numerical simulations are provided to support the theoretical analysis. In section 7, the state reconstruction problems, both in the continuous case and discrete cases, are solved. In section 8, a number of numerical simulations are performed to demonstrate the effectiveness of the numerical approximations of the state reconstruction problem. In section 9, we give some concluding remarks.

2. Results of continuous model

In this section, we introduce the model discussed in this paper and present some known results about the exponential stability of the continuous system. Consider the transmission line on the spatial interval $[0,1]$:

$$\begin{cases} \frac{\partial}{\partial t}Q(t,x) = -\frac{\partial}{\partial x}\left(\frac{\phi(t,x)}{L(x)}\right),\ x\in(0,1), \\ \frac{\partial}{\partial t}\phi(t,x) = -\frac{\partial}{\partial x}\left(\frac{Q(t,x)}{C(x)}\right),\ t>0, \\ V(t,0)=0, \quad V(t,1)=RI(t,1),\ R>0 \\ Q(0,x)=Q^0(x), \quad \phi(0,x)=\phi^0(x). \end{cases} \tag{2.1}$$

Here $Q(t,x)$ is the charge at position $x\in[0,1]$ and time $t>0$, and $\phi(t,x)$ is the magnetic flux at position x and time t. $C(x)$ is the distributed capacity and $L(x)$ is the distributed inductance. The voltage and the current are given by $V=Q/C$ and $I=\phi/L$, respectively. $V(t,1)=RI(t,1)$ is the boundary feedback and R is the feedback gain constant.$(Q^0(\cdot),\phi^0(\cdot))\in[L^2(0,1)]^2$ is initial configuration of the transmission line model. The energy of this system is given by

$$E(t)=\frac{1}{2}\int_0^1 \frac{|\phi(t,x)|^2}{L(x)}+\frac{|Q(t,x)|^2}{C(x)}\mathrm{d}x. \tag{2.2}$$

It follows from the exercises 7.1 and 9.1 of [13] that the system (2.1) is exponentially stable with respect to the energy $E(t)$. In order to study the uniform exponential stability in a convenient manner, we begin from the following system on V and I:

$$\begin{cases} C(x)V_t(t,x)=-I_x(t,x),\ x\in(0,1), \\ L(x)I_t(t,x)=-V_x(t,x),\ t>0, \\ V(t,0)=0, \quad V(t,1)=RI(t,1), \\ V(0,x)=V^0(x), \quad I(0,x)=I^0(x). \end{cases} \tag{2.3}$$

The energy of the system (2.3) is also given by

$$E(t)=\frac{1}{2}\int_0^1 C(x)|V(t,x)|^2+L(x)|I(t,x)|^2\mathrm{d}x \tag{2.4}$$

and has different expression in light of $V=Q/C$ and $I=\phi/L$. To give some clues in discrete case, we apply the method of Lyapunov function to obtain the exponential stability of the continuous system (2.3). For this purpose, we assume that the capacity function $C(x)$ and the inductance function $L(x)$ satisfy:

H_1: $C(x)>0$, $L(x)>0$, $\forall x\in[0,1]$;

H_2: $C(x)$, $L(x)\in C^1[0,1]$ and $C(x)\le K$, $L(x)\le K$ for some positive constant K;

H_3: $C'(x)>0$, $L'(x)>0$, $\forall x\in[0,1]$.

It should be pointed out that H_3 is not applied in the proof of the main result, see for instance Theorem 3.1. It is only used in the method of Lyapunov function to verify the exponential stability of the continuous system. If one applies the method of [13, Lemma 9.1.3], H_3 is also useless.

Theorem 2.1. *Under the conditions H_1-H_3, for any $V^0(\cdot)$, $I^0(\cdot)\in L^2[0,1]$, there exist two constants M and ω such that the energy of the solution to the system (2.3) satisfies*

$$E(t)\le Me^{-\omega t}E(0). \tag{2.5}$$

Proof: It is easy to see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) &= -\int_0^1 V(t,x)I_x(t,x) + V_x(t,x)I(t,x)\mathrm{d}x = -[V(t,x)I(t,x)]\,\big|_0^1 \\ &= -R|I(t,1)|^2. \end{aligned} \tag{2.6}$$

Introduce the auxiliary function

$$\rho(t) = -\int_0^1 xC(x)L(x)V(t,x)I(t,x)\mathrm{d}x$$

and Lyapunov function $F(t) = E(t) + \varepsilon\rho(t)$. Here $\varepsilon \in (0, 1/K)$ is a parameter. On one hand, we have $|\rho(t)| \le KE(t)$ by using condition H_2 and Cauchy inequality and $F(t)$ is equivalent to $E(t)$:

$$(1 - K\varepsilon)E(t) \le F(t) \le (1 + K\varepsilon)E(t). \tag{2.7}$$

The parameter $0 < \varepsilon < 1/K$ ensures that $F(t)$ is positive definite. On the other hand, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [L(1)+C(1)R^2]|I(t,1)|^2 - 2E(t) - \int_0^1 xC'(x)|V(t,x)|^2 + xL'(x)|I(t,x)|^2\mathrm{d}x \\ -\frac{\mathrm{d}}{\mathrm{d}t}\rho(t), \end{aligned} \tag{2.8}$$

It follows from the above equality and assumption H_3 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) \le \frac{L(1) + C(1)R^2}{2}|I(t,1)|^2 - E(t). \tag{2.9}$$

By differentiating $F(t)$ and using (2.6), (2.7) and (2.9), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t) = \frac{\mathrm{d}}{\mathrm{d}t}E(t) + \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) \le -\left[R - \frac{\varepsilon(L(1) + C(1)R^2)}{2}\right]|I(t,1)|^2 - \frac{\varepsilon}{1 + K\varepsilon}F(t)$$

Finally, choosing ε to ensure $R - \left(\varepsilon(L(1) + C(1)R^2)\right)/2 \ge 0$ and applying the comparison principle (see Section 3.1 of [15]) and (2.7), we get

$$E(t) \le Me^{-\omega t}E(0)$$

with $M = (1 + K\varepsilon)/(1 - K\varepsilon)$ and $\omega = \varepsilon/(1 - K\varepsilon)$. Therefore we obtain the exponential stability of the system (2.3) and complete the proof of the theorem. □

For convenience in the convergence analysis, we rewrite the exponential stability of the system (2.3) in the language of semigroup of bounded linear operators.

Remark 2.1. Let $H = \left[L^2(0,1)\right]^2$ be the state space and define the inner product on H by: $\langle m, \widetilde{m}\rangle_H = \int_0^1 C(x)V(x)\widetilde{V}(x) + L(x)I(x)\widetilde{I}(x)\mathrm{d}x$, $\forall m = (V, I), \widetilde{m} = (\widetilde{V}, \widetilde{I}) \in H$. Define the system operator A on H through:

$$A\begin{pmatrix} p(x) \\ q(x) \end{pmatrix} = \begin{pmatrix} -\frac{1}{C(x)}\frac{\mathrm{d}}{\mathrm{d}x}q(x) \\ -\frac{1}{L(x)}\frac{\mathrm{d}}{\mathrm{d}x}p(x) \end{pmatrix}, \tag{2.10}$$

$$D(A)=\left\{\begin{pmatrix}p(\cdot)\\ q(\cdot)\end{pmatrix}\in[H^1(0,1)]^2: p(0)=0,\ p(1)=Rq(1)\right\}. \tag{2.11}$$

Thus, the system (2.3) can be transformed into abstract Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}V(t,x)\\ I(t,x)\end{pmatrix}=A\begin{pmatrix}V(t,x)\\ I(t,x)\end{pmatrix},\qquad \begin{pmatrix}V(0,x)\\ I(0,x)\end{pmatrix}=\begin{pmatrix}V^0(x)\\ I^0(x)\end{pmatrix}\in H.$$

A routine method as given in Theorem 3.1.11 of [29] can be applied to show that the operator A generates a contractive semigroup $T(t)$ on H. This means that the system (2.3) has a unique solution for any $V^0(\cdot), I^0(\cdot)\in L^2(0,1)$. Theorem 2.1 further implies that the semigroup $T(t)$ is exponentially stable under the assumptions H_1-H_3, i.e.

$$\|T(t)\|_H\le Me^{-\omega t},$$

here $\|\cdot\|_H$ is the norm induced by the inner product on H.

3. Uniform exponential stability

Firstly, we introduce the spacial semi-discretization scheme for the system (2.3). For this purpose, let $N\in\mathbb{N}$ be a positive integer and $h=1/(N+1)$ mesh size. Insert $N+2$ points and $N+1$ points, denoted by $y_i=ih$ $(i=0,1,\cdots,N+1)$ and $x_j=(j+1/2)h$ $(j=0,1,\cdots,N)$ respectively, in the domain $[0,1]$. If let f_i be the value of any continuous function $f(x)$ at the node $y_i=ih$ $(i=0,1,\cdots,N+1)$, then the notations

$$\delta_x f_j=\frac{f_{j+1}-f_j}{h},\quad \delta_{\frac{1}{2}}f_j=\frac{f_{j+1}+f_j}{2}$$

denote the central difference operator of $f_x(x)$ and the average operator of $f(x)$ at the node x_j, respectively. Inspired by the works of [7,17,23,24,30], we propose the following semi-discretization scheme for (2.3)

$$\begin{cases}C_j\frac{\mathrm{d}}{\mathrm{d}t}\delta_{\frac{1}{2}}V_j(t)=-\delta_x I_j(t),\ L_j\frac{\mathrm{d}}{\mathrm{d}t}\delta_{\frac{1}{2}}I_j(t)=-\delta_x V_j(t),\ j=0,1,\cdots N,\\ V_0(t)=0,\quad V_{N+1}(t)=RI_{N+1}(t),\\ V(0,y_i)\approx V^0(y_i)=V_i^0,\ I(0,y_i)\approx I^0(y_i)=I_i^0,\ i=0,1,\cdots N+1.\end{cases} \tag{3.1}$$

We call this semi-discretization scheme as average central-difference method since the average operator and central-difference operator are applied for the temporal derivative and spatial derivative, respectively.

The energy of (3.1) is

$$E_h(t)=\frac{h}{2}\sum_{j=0}^{N}\left[C_j\left|\delta_{\frac{1}{2}}V_j(t)\right|^2+L_j\left|\delta_{\frac{1}{2}}I_j(t)\right|^2\right],$$

which is discrete counterpart of the energy $E(t)$.

Definition 3.1. If there exist two constants M and ω independent of t and h such that

$$E_h(t) \le M e^{-\omega t} E_h(0),$$

then we call that the system (3.1) is uniform exponentially stable.

The state space of the discrete system (3.1) is $H_N = \left[\mathbb{R}^{N+1}\right]^2$ with the inner product

$$\langle (p_h, q_h), (u_h, v_h) \rangle_N = h \sum_{j=0}^{N} \left[C_j \delta_{\frac{1}{2}} p_j \delta_{\frac{1}{2}} u_j + L_j \delta_{\frac{1}{2}} q_j \delta_{\frac{1}{2}} v_j \right],$$

for all (p_h, q_h), $(u_h, v_h) \in H_N$. Here $p_h = (p_1, \cdots, p_{N+1})$, $q_h = (q_0, \cdots, q_N) \in \mathbb{R}^{N+1}$, (u_h, v_h) and (p_h, q_h) are the same type of vectors. Moreover, set artificially $k = R^{-1}$, $p_0 = u_0 = 0$, $q_{N+1} = kp_{N+1}$, and $v_{N+1} = ku_{N+1}$, to unify the notations of $\delta_{\frac{1}{2}} p_j$ and $\delta_x u_j$ etc.

To verify the uniform exponential stability of the system (3.1), we will follow every step of the proof of Theorem 2.1. However, we need the following lemma additionally.

Lemma 3.1. *Let $\{u_i\}_{i=0}^{N+1}$, $\{v_i\}_{i=0}^{N+1}$ and $\{w_i\}_{i=0}^{N+1}$ be sequences consisting of real numbers, then we have*

$$\begin{aligned}
&\frac{1}{4}\sum_{i=0}^{N}(u_{i+1}-u_i)(v_{i+1}+v_i)(w_{i+1}+w_i)+\frac{1}{4}\sum_{i=0}^{N}(u_{i+1}-u_i)(v_{i+1}-v_i)(w_{i+1}-w_i)\\
&+\frac{1}{4}\sum_{i=0}^{N}(u_{i+1}+u_i)(v_{i+1}-v_i)(w_{i+1}+w_i)+\frac{1}{4}\sum_{i=0}^{N}(u_{i+1}+u_i)(v_{i+1}+v_i)(w_{i+1}-w_i)\\
&= u_{N+1}v_{N+1}w_{N+1} - u_0 v_0 w_0
\end{aligned}$$

Proof: Extracting the factors from the first two terms and the remaining terms respectively and using simple algebraic operations, we obtain

$$\text{Left} = \frac{1}{2}\sum_{i=0}^{N}[(u_{i+1}-u_i)(v_{i+1}w_{i+1}+w_i v_i)+(u_{i+1}+u_i)(v_{i+1}w_{i+1}-w_i v_i)].$$

Breaking the brackets in right hand side of the identity above and eliminating the cross terms, one has

$$\begin{aligned}
&\frac{1}{2}\sum_{i=0}^{N}[(u_{i+1}-u_i)(v_{i+1}w_{i+1}+w_i v_i)+(u_{i+1}+u_i)(v_{i+1}w_{i+1}-w_i v_i)]\\
&=\sum_{i=0}^{N}(u_{i+1}v_{i+1}w_{i+1}-u_i v_i w_i) = u_{N+1}v_{N+1}w_{N+1} - u_0 v_0 w_0.
\end{aligned}$$

We complete the proof of the lemma by the identities above. □

Theorem 3.1. *Under the assumptions H_1 and H_2, the semi-discretized system (3.1) is uniform exponentially stable.*

Proof: Firstly, differentiating the energy $E_h(t)$ with respect to time t along the solution to (3.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_h(t) = -h\sum_{j=0}^{N}\left[\delta_{\frac{1}{2}}V_j(t)\delta_x I_j(t) + \delta_{\frac{1}{2}}I_j(t)\delta_x V_j(t)\right]$$
$$= -\frac{1}{2}\sum_{j=0}^{N}[(V_{j+1}(t) - V_j(t))(I_{j+1}(t) + I_j(t)) + (I_{j+1}(t) - I_j(t))(V_{j+1}(t) + V_j(t))]$$
$$= -\frac{1}{2}\sum_{j=0}^{N}(V_{j+1}(t)I_{j+1}(t) - V_j(t)I_j(t)) = -[V_{N+1}(t)I_{N+1}(t) - V_0(t)I_0(t)]. \quad (3.2)$$

This means that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_h(t) = -R|I_{N+1}(t)|^2. \quad (3.3)$$

Secondly, assume that ε is a parameter and define Lyapunov functions by $F_h(t) = E_h(t)+\varepsilon\rho_h(t)$. Here $\rho_h(t)$ are auxiliary functions given by $\rho_h(t) = -h\sum_{j=0}^{N} C_jL_j\delta_{\frac{1}{2}}y_j$ $\delta_{\frac{1}{2}}V_j(t)\delta_{\frac{1}{2}}I_j(t)$. On one hand, it is easy to see that $|\rho_h(t)| \le KE_h(t)$. This implies that Lyapunov functions $L_h(t)$ are equivalent to the discrete energy $E_h(t)$, i.e.

$$(1 - \varepsilon K)E_h(t) \le F_h(t) \le (1 + \varepsilon K)E_h(t). \quad (3.4)$$

The parameter $0 < \varepsilon < 1/K$ ensures that Lyapunov functions $F_h(t)$ are positive definite.

On the other hand, differentiating the auxiliary function $\rho_h(t)$ and applying (3.1), we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_h(t) = h\sum_{j=0}^{N}C_j\delta_{\frac{1}{2}}y_j\delta_{\frac{1}{2}}I_j(t)\delta_x I_j(t) + h\sum_{j=0}^{N}L_j\delta_{\frac{1}{2}}y_j\delta_{\frac{1}{2}}V_j(t)\delta_x V_j(t)$$
$$= \frac{1}{4}\sum_{j=0}^{N}L_j(y_{j+1} + y_j)(I_{j+1}(t) - I_j(t))(I_{j+1}(t) + I_j(t))$$
$$+\frac{1}{4}\sum_{j=0}^{N}C_j(y_{j+1} + y_j)(V_{j+1}(t) - V_j(t))(V_{j+1}(t) + V_j(t)). \quad (3.5)$$

Using $y_{N+1} = 1$, $y_0 = 0$, $y_{j+1} - y_j = h$ and Lemma 3.1, we obtain

$$\frac{1}{4}\sum_{j=0}^{N}L_j(y_{j+1} + y_j)(I_{j+1}(t) - I_j(t))(I_{j+1}(t) + I_j(t))$$
$$= -\frac{h}{8}\sum_{j=0}^{N}L_j|I_{j+1}(t) - I_j(t)|^2 - \frac{h}{2}\sum_{j=0}^{N}L_j|\delta_{\frac{1}{2}}I_j(t)|^2 + L_N\frac{|I_{N+1}(t)|^2}{2} \quad (3.6)$$

and

$$\frac{1}{4}\sum_{j=0}^{N}C_j(y_{j+1} + y_j)(V_{j+1}(t) - V_j(t))(V_{j+1}(t) + V_j(t))$$

$$= -\frac{h}{8}\sum_{j=0}^{N} C_j|V_{j+1}(t)-V_j(t)|^2 - \frac{h}{2}\sum_{j=0}^{N} C_j\left|\delta_{\frac{1}{2}}V_j(t)\right|^2 + C_N\frac{|V_{N+1}(t)|^2}{2}. \quad (3.7)$$

Combining (3.5) and (3.6)-(3.7), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_h(t) \le \alpha|I_{N+1}(t)|^2 - E_h(t), \quad (3.8)$$

in which $\alpha = (1/2)\left(L_N + C_N R^2\right)$. Note that the inequality (3.8) is a perfect counterpart of inequality (2.8).

Finally, differentiating Lyapunov function $F_h(t)$ and using (3.3)-(3.4) and (3.8), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}F_h(t) = \frac{\mathrm{d}}{\mathrm{d}t}E_h(t) + \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\rho_h(t) \le -\left(R-\varepsilon\alpha\right)|I(t,1)|^2 - \frac{\varepsilon}{1+K\varepsilon}F_h(t).$$

Choosing ε to ensure $R-\varepsilon\alpha \ge 0$ and applying the comparison principle and (3.4), we get

$$E_h(t) \le Me^{-\omega t}E_h(0).$$

M and ω are the same as in the proof of Theorem 2.1. Therefore we obtain uniform exponential stability of the system (3.1) and complete the proof of the theorem. □

At the end of this section, we explain the boundary feedback control mechanism of our numerical scheme.

Remark 3.1. Let $V_h(t) = (V_1(t), \cdots, V_{N+1}(t))^\top$ and $I_h(t) = (I_0(t), \cdots, I_N(t))^\top$ be the unknown variables of (3.1). We solve $V_1'(t)$ from (3.1) by letting $j=0$ in the first equation of (3.1) since $V_0(t)=0$. Using $V_1'(t)$, we can solve $V_2'(t)$ by letting $j=1$ in the first equation of (3.1). Repeating this process, we can separate all components of $V_h'(t)$. But we obtain $I_h'(t)$ by the converse process in view of $I_{N+1}(t) = kV_{N+1}(t)$. That is the last component $I_N'(t)$ is the starting point of solving $I_h'(t)$ from the second equation of (3.1). Thus we obtain the equivalent form of the discrete system (3.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}(V_h^\top(t), I_h^\top(t))^\top = (B_1I_h(t), \cdots,\ B_{N+1}I_h(t),\ D_0V_h(t), \cdots,\ D_NV_h(t))^\top, \quad (3.9)$$

in which

$$\begin{aligned}
B_1I_h(t) &= -2\frac{\delta_x I_0(t)}{C_0},\\
B_{j+1}I_h(t) &= -2\frac{\delta_x I_j(t)}{C_j} - B_jI_h(t),\ j=1,\cdots,N-1,\\
B_{N+1}I_h(t) &= -2\frac{kV_{N+1}(t)-I_N(t)}{hC_N} - B_NI_h(t),\\
D_NV_h(t) &= -2\frac{\delta_x V_N(t)}{L_N} - kB_{N+1}I_h(t),\\
D_iV_h(t) &= -2\frac{\delta_x V_i(t)}{L_i} - D_{i+1}V_h(t),\ i=0,1,\cdots,N-1
\end{aligned}$$

satisfy the recursion relations and are all mappings from $\mathbb{R}^{N+1}$ to $\mathbb{R}$.

If you track the feedback control $kV_{N+1}(t)$, which corresponds to $kB_{N+1}I_h(t)$, in the dynamical system (3.9), you can find out that it firstly enters into the channel $I'_N(t)$ and then every channel of the system (3.9) one by one. This is caused by the average operator for the time derivative. Without the average operator, the semi-discretization scheme (3.1) degenerates into the classical central difference scheme and the feedback control only appears in one channel. Furthermore, it has been pointed out that there is no uniform exponential stability for this approximating scheme in [30]. More information is given in section 6. This is the main mechanism of boundary feedback control uniformly exponentially stabilizes the discrete system. This has the same effect with the mixed finite element method in [21].

4. Convergence analysis

We will show that the solution to the system (3.1) converges to the corresponding solution of the system (2.2) in the sense of Trotter-Kato.

For every $n = 1, 2, \cdots$, there exist bounded linear operators $P_n : X \to X_n$ and $E_n : X_n \to X$ satisfying

(A_1) There exist two positive constants M_1 and M_2 such that $\|E_n\| \le M_1$ and $\|P_n\| \le M_2$,

(A_2) $\|E_nP_nx - x\| \to 0$ as $n \to \infty$ for all $x \in X$,

(A_3) $P_nE_n = I_n$, where I_n is the identity operator on X_n.

The notation $B \in G(M, \omega, X)$ with $M > 1$ and $\omega \in \mathbb{R}$, means that B is the infinitesimal generator of a C_0-semigroup $S(t)$, $t > 0$, satisfying $\|S(t)\| \le Me^{\omega t}$. The Trotter-Kato Theorem for approximating a linear C_0-semigroup $S(t)$ on a Banach space X is as follows.

Theorem 4.1. *(**Trotter-Kato** [11]). Assume that $(A1)$ and $(A3)$ are satisfied. Let B resp. B_n be in $G(M, \omega, X)$ resp. in $G(M, \omega, X_n)$ and let $S(t)$ and $S_n(t)$ be the semigroups generated by B and B_n on Banach spaces X and X_n, respectively. Then the following statements are equivalent*

(a) There exists a $\lambda_0 \in \rho(B) \cap \bigcap_{n=1}^{\infty} \rho(B_n)$ such that, for all $x \in X$,

$$\|E_n(\lambda_0 - B_n)^{-1}P_nx - (\lambda_0 - B)^{-1}x\| \to 0, \quad \text{as } n \to \infty. \tag{4.1}$$

(b) For every $x \in X$ and $t \ge 0$,

$$\|E_nS_n(t)P_nx - S(t)x\| \to 0, \quad \text{as } n \to \infty \tag{4.2}$$

uniformly on bounded t-intervals.

Note that the assumption $B_n \in G(M, \omega, X_n)$, or equivalently $\|S_n(t)\|_n \le Me^{\omega t}$, $n = 1, 2, \cdots$, usually is called the stability property of the approximations, whereas statement (a) is called the consistency property of the approximations. However, one may face some major difficulties when one applies Theorem 4.1 to perform convergence analysis. The most difficult one is how to verify the consistency property (a). The following property, which can replace (a) by a condition involving convergence of the operators B_n to B in some sense, is useful in this part [11].

Proposition 4.1. *Assume that the assumptions of Theorem 4.1 are satisfied. Then statement (a) of Theorem 4.1 is equivalent to $(A2)$ and the following two statements:*

$(C1)$ There exists a subset $D \subseteq D(B)$ such that $\overline{D} = X$ and $(\lambda_0 I - B)^{-1}D = X$ for a $\lambda_0 > \omega$.

(C2) For all $u \in D$ there exists a sequence $(\overline{u}_n)_{n\in\mathbb{N}}$ with $\overline{u}_n \in D(B_n)$ such that

$$\lim_{n\to\infty} E_n\overline{u} = u, \quad \lim_{n\to\infty} E_nB_n\overline{u} = Bu. \tag{4.3}$$

Now we give the convergence analysis of our systems. In light of (3.9), the approximating operators A_N are obviously defined by

$$A_N\begin{pmatrix}p_h\\ q_h\end{pmatrix} = \begin{pmatrix}Bq_h\\ Dp_h\end{pmatrix}, \text{with} Bq_h = \begin{pmatrix}B_1q_h\\ \vdots\\ B_{N+1}q_h\end{pmatrix}, Dp_h = \begin{pmatrix}D_0p_h\\ \vdots\\ D_Np_h\end{pmatrix}, \forall \begin{pmatrix}p_h\\ q_h\end{pmatrix} \in H_N. \tag{4.4}$$

By the same operations as in (3.2), one has

$$\left\langle A_N\begin{pmatrix}p_h\\ q_h\end{pmatrix}, \begin{pmatrix}p_h\\ q_h\end{pmatrix}\right\rangle_N = -h\sum_{j=0}^{N}\left[\delta_{\frac{1}{2}}p_j\delta_xq_j + \delta_xp_j\delta_{\frac{1}{2}}q_j\right] = -k|p_{N+1}|^2 \le 0.$$

This means that $A_N \in G(1,0,H_N)$. However it follows from Remark 2.1 that A generates some contractive semigroup, i.e. $A \in G(1,0,H)$. This shows that the discrete scheme of (3.1) is stable.

Let χ_S be the characteristic function of the set S and define the extension operators $E_N : H_N \to H$:

$$E_N\begin{pmatrix}p_h\\ q_h\end{pmatrix} = \begin{pmatrix}\sum_{i=0}^{N}(\delta_{\frac{1}{2}}p_i)\chi_{(y_i,y_{i+1}]}\\ \sum_{i=0}^{N}(\delta_{\frac{1}{2}}q_i)\chi_{(y_i,y_{i+1}]}\end{pmatrix},$$

Choose the dense subset $D \triangleq D(A)\bigcap(C^2[0,1])^2$ of H. For any $(u(\cdot),v(\cdot))^\top \in D$, set $\overline{u} = (u(y_1), \cdots, u(y_{N+1}))^\top$, $\overline{v} = (v(y_0), \cdots, v(y_N))^\top$ and $\overline{U} = (\overline{u},\overline{v})^\top$. By $u(y_0) = u(0) = 0$ and $v(1) = v(y_{N+1}) = ku(y_{N+1}) = ku(1)$, it is easy to see

$$E_N\overline{U} = E_N\begin{pmatrix}\overline{u}\\ \overline{v}\end{pmatrix} = \begin{pmatrix}\sum_{i=0}^{N}\left(\frac{u(y_{i+1})+u(y_i)}{2}\right)\chi_{[y_i,y_{i+1}]}\\ \sum_{i=0}^{N}\left(\frac{v(y_{i+1})+v(y_i)}{2}\right)\chi_{[y_i,y_{i+1}]}\end{pmatrix},$$

and

$$E_NA_N\overline{U} = E_N\begin{pmatrix}B\overline{v}\\ C\overline{u}\end{pmatrix} = \begin{pmatrix}\sum_{i=0}^{N}\left(\frac{v(y_{i+1})-v(y_i)}{hC_i}\right)\chi_{[y_i,y_{i+1}]}\\ \sum_{i=0}^{N}\left(\frac{u(y_{i+1})-u(y_i)}{hL_i}\right)\chi_{[y_i,y_{i+1}]}\end{pmatrix}.$$

Furthermore, we have:

$$E_N\begin{pmatrix}\overline{u}\\ \overline{v}\end{pmatrix} - \begin{pmatrix}u(x)\\ v(x)\end{pmatrix}$$

$$= \begin{pmatrix}\sum_{i=0}^{N}\left[\frac{u(y_{i+1})+u(y_i)}{2}\right]\chi_{[y_i,y_{i+1}]} - \sum_{i=0}^{N}u(x)\chi_{[y_i,y_{i+1}]}\\ \sum_{i=0}^{N}\left[\frac{v(y_{i+1})+v(y_i)}{2}\right]\chi_{[y_i,y_{i+1}]} - \sum_{i=0}^{N}v(x)\chi_{[y_i,y_{i+1}]}\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \sum_{i=0}^{N}[u(y_{i+1}) - u(x) + u(y_i) - u(x)]\chi_{[y_i,y_{i+1}]} \\ \sum_{i=0}^{N}[v(y_{i+1}) - v(x) + v(y_i) - v(x)]\chi_{[y_i,y_{i+1}]} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \sum_{i=0}^{N}[u'(\xi_{i+1}^u)(y_{i+1} - x) + u'(\xi_i^u)(x - y_i)]\chi_{[y_i,y_{i+1}]} \\ \sum_{i=0}^{N}[v'(\xi_{i+1}^v)(y_{i+1} - x) + v'(\xi_i^v)(x - y_i)]\chi_{[y_i,y_{i+1}]} \end{pmatrix},$$

in which the mean value Theorem is applied and $u(x),\ v(x) \in C^1[0,1]$ imply that $|u'(\xi_i^u)|,\ |u'(\xi_{i+1}^u)|,\ |v'(\xi_i^v)|$ and $|v'(\xi_{i+1}^v)|$ are uniformly bounded with respect to $i = 0, 1, \cdots, N$. Let Γ be their common upper bound and so we have

$$\left\| E_N \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} - \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_N^2 \leq \frac{\Gamma^2 h^2}{2} \left\| \begin{pmatrix} \sum_{i=0}^{N} \chi_{[y_i,y_{i+1}]} \\ \sum_{i=0}^{N} \chi_{[y_i,y_{i+1}]} \end{pmatrix} \right\|_N^2 \to 0, \quad \text{as } N \to \infty.$$

Similarly, for $E_N A_N - A$ we obtain

$$E_N A_N \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} - A \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=0}^{N} \left[\frac{v(y_{i+1})-v(y_i)}{hC_i} + \frac{v'(x)}{C(x)}\right] \chi_{[y_i,y_{i+1}]} \\ \sum_{i=0}^{N} \left[\frac{u(y_{i+1})-u(y_i)}{hL_i} + \frac{u'(x)}{L(x)}\right] \chi_{[y_i,y_{i+1}]} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=0}^{N} \left[\frac{v'(\eta_{i+1}^v)}{C_i} + \frac{v'(x)}{C(x)}\right] \chi_{[y_i,y_{i+1}]} \\ \sum_{i=0}^{N} \left[\frac{u'(\eta_{i+1}^u)}{L_i} + \frac{u'(x)}{L(x)}\right] \chi_{[y_i,y_{i+1}]} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=0}^{N} \frac{C(x)[v'(\eta_{i+1}^v)-v'(x)]-[C(x)-C_i]v'(x)}{C_iC(x)} \chi_{[y_i,y_{i+1}]} \\ \sum_{i=0}^{N} \frac{L(x)[u'(\eta_{i+1}^u)-u'(x)]-[L(x)-L_i]u'(x)}{L_iL(x)} \chi_{[y_i,y_{i+1}]} \end{pmatrix}.$$

By the same idea as above, we can show that $E_N A_N \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} - A \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$ converges to zero as $N \to \infty$ since $C(x),\ L(x) \in C^1[0,1]$ and $u(x),\ v(x) \in C^2[0,1]$. Therefore, the statement (b) in Proposition 4.1 holds.

Finally, construct the projecting operators $P_N : H \to H_N$ in light of the expressions of the extensions E_N by

$$P_N \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} I^1 u(x) \\ I^2 v(x) \end{pmatrix}, \text{ with } I^1 u(x) = 2 \begin{pmatrix} I_0^1 u(x) \\ \vdots \\ I_N^1 u(x) \end{pmatrix}, \quad I^2 v(x) = 2 \begin{pmatrix} I_0^2 v(x) \\ \vdots \\ I_N^2 v(x) \end{pmatrix},$$

and

$$I_0^1 u(x) = h^{-1} \int_{y_0}^{y_1} u(x) \mathrm{d}x, \quad I_i^1 u(x) = h^{-1} \int_{y_i}^{y_{i+1}} u(x) \mathrm{d}x - I_{i+1}^1 u(x), \ i = 1, 2, \cdots, N$$

$$I_N^2 v(x) = h^{-1}\int_{y_N}^{y_{N+1}} v(x)\mathrm{d}x - kI_N^1 u(x), \quad I_j^2 v(x) = h^{-1}\int_{y_j}^{y_{j+1}} v(x)\mathrm{d}x - I_{j+1}v(x),$$
$$j = 0, \ \cdots, \ N-1.$$

It is easy to show that E_N and P_N satisfy $(A1)$ and $(A3)$.

To prove that E_N and P_N satisfy $(A2)$, we firstly assume $(u(x), v(x)) \in D(A)$. With this assumption we have

$$E_N P_N \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = h^{-1}\begin{pmatrix} \sum_{i=0}^N \int_{y_i}^{y_{i+1}} u(x)\mathrm{d}x\chi_{[y_i,y_{i+1}]} \\ \sum_{i=0}^N \int_{y_i}^{y_{i+1}} v(x)\mathrm{d}x\chi_{[y_i,y_{i+1}]} \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^N u(\theta_i^u)\chi_{[y_i,y_{i+1}]} \\ \sum_{i=0}^N v(\theta_i^v)\chi_{[y_i,y_{i+1}]} \end{pmatrix},$$

and

$$E_N P_N \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} - \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^N [u(\theta_i^u) - u(x)]\chi_{[y_i,y_{i+1}]} \\ \sum_{i=0}^N [v(\theta_i^v) - v(x)]\chi_{[y_i,y_{i+1}]} \end{pmatrix} \to 0, \text{ as } N \to \infty,$$

here θ_i^u and θ_i^v are chosen such that $\int_{y_i}^{y_{i+1}} u(x)\mathrm{d}x = u(\theta_i^u)h$ and $\int_{y_i}^{y_{i+1}} v(x)\mathrm{d}x = v(\theta_i^v)h$ when the mean value theorem is applied, and the continuity of $u(x)$ and $v(x)$ on the internal $[0,1]$ is applied in the last step. Thus combing this result and the density of $D(A)$ in the state space H, we obtain (A_2). Moreover, let $(V^0, I^0) \in H$ be the initial value of (2.3) and set $(V_h(0), I_h(0))^\top = P_N(V^0, I^0)^\top$ be the initial data of (3.1), then (A_2) implies that $(V_h(0), I_h(0))$ convergent to (V^0, I^0) in the sense of

$$E_N(V_h(0), I_h(0))^\top \to (V^0, I^0)^\top, \text{ as } N \to \infty.$$

In a word, we have completed the verification of (A_1)-(A_3) and (C_1)-(C_2) and this means that the solutions to the discrete systems (3.1) strongly converge to the solution of (2.2), i.e., as $N \to \infty$,

$$\left\| E_N S_N(t) P_N \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} - S(t)\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_H \to 0, \quad \forall \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \in H. \tag{4.5}$$

5. Return to the original system (2.1)

Now, all results for the systems (2.3) and (3.1) are going to be translated into the system (2.1) and its semi-discretized systems, respectively. Recall that the state space H and the system operator A corresponding to (2.3) have defined in Remark 2.1. Similarly, we introduce the state space H_O and the system operator A_O for the system (2.1). The space H_O is $[L^2(0,1)]^2$ with the inner product given by:

$$\left\langle \begin{pmatrix} p(x) \\ q(x) \end{pmatrix}, \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\rangle_{H_0} = \int_0^1 \frac{p(x)\overline{u(x)}}{C(x)} + \frac{q(x)\overline{v(x)}}{L(x)}\mathrm{d}x,$$

for $\forall(p(\cdot),q(\cdot))^\top,(u(\cdot),v(\cdot))^\top \in [L^2(0,1)]^2$. Define the system operator A_O on H_O through

$$A_O \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{v(x)}{C(x)}\right) \\ -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u(x)}{L(x)}\right) \end{pmatrix}, \tag{5.1}$$

$$D(A_0) = \left\{ \begin{pmatrix} u(\cdot) \\ v(\cdot) \end{pmatrix} \in [L^2(0,1)]^2 : u(0)=0,\ u(1)=Rv(1) \right\}. \tag{5.2}$$

Thus, the system (2.1) can be transformed into abstract Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} Q(t,x) \\ \phi(t,x) \end{pmatrix} = A_O \begin{pmatrix} Q(t,x) \\ \phi(t,x) \end{pmatrix}, \qquad \begin{pmatrix} Q(0,x) \\ \phi(0,x) \end{pmatrix} = \begin{pmatrix} Q^0(x) \\ \phi^0(x) \end{pmatrix} \in H_O.$$

Furthermore, the relations $V = Q/C$ and $I = \phi/L$ determine a mapping $\Psi : H_O \to H$ given by

$$\begin{pmatrix} V(x) \\ I(x) \end{pmatrix} = \Psi \begin{pmatrix} Q(x) \\ \phi(x) \end{pmatrix} := \begin{pmatrix} \frac{1}{C(x)} & 0 \\ 0 & \frac{1}{L(x)} \end{pmatrix} \begin{pmatrix} Q(x) \\ \phi(x) \end{pmatrix},\ \forall \begin{pmatrix} Q(x) \\ \phi(x) \end{pmatrix} \in H_O$$

The operators A, A_O and Ψ have following basic properties.

Proposition 5.1. *The operators A, A_O and Ψ satisfies:*

(1)The operator Ψ is an isometric isomorphism from H_0 to H.

(2)Let Ψ^{-1} be the inverse operator of Ψ, then the operators A and A_O are similar, i.e. $A_O = \Psi^{-1}A\Psi$.

Proof: (1) Obviously, Ψ is a linear operator from H_0 to H and invertible. The inverse of Ψ is $\Psi^{-1} = \begin{pmatrix} C(x) & 0 \\ 0 & L(x) \end{pmatrix}$ and also a linear operator from H to H_0. This implies that the operator Ψ is an isomorphism. Because the identity

$$\left\| \Psi \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_H^2 = \int_0^1 \frac{|u(x)|^2}{C(x)} + \frac{|v(x)|^2}{L(x)} \mathrm{d}x = \left\| \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_{H_0}^2$$

holds, we know that Ψ is isometric.

(2) For any $\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \in D(A) = D(A_O)$, we have $\Psi^{-1} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \in D(A_O)$

$$A_O\Psi^{-1} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = -\begin{pmatrix} u'(x) \\ v'(x) \end{pmatrix} = \Psi^{-1}A \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},$$

which gives $A_O = \Psi^{-1}A\Psi$. □

Now, we can give exponential stability result of the original system (2.1).

Theorem 5.1. *Let the assumptions H_1-H_3 hold and the semigroup on the space H_O generated by the operator A_O be $T_O(t)$, then $T_O(t)$ is exponentially stable.*

Proof: Recall that the semigroup $T(t)$ given in Remark 2.1 satisfies $\|T(t)\|_H \leq Me^{-\omega t}$ for some positive constants M and ω. However, it follows from $A_O = \Psi^{-1}A\Psi$ and similar semigroup theory in [3] that $T_O(t) = \Psi^{-1}T(t)\Psi$. Therefore, from (1) of Proposition 5.1 we have

$$\|T_O(t)\|_{H_O} = \|\Psi^{-1}T(t)\Psi\|_{H_O} = \|T(t)\|_H \leq Me^{-\omega t},$$

which means that $T_O(t)$ is exponentially stable. $\square$

Certainly, we can obtain Theorem 5.1 by the same method in Theorem 2.1. However, this indirect method for obtaining exponential stability of the original system (2.1) is important in studying the uniform exponential stability of its discrete systems. From semi-discretization scheme (3.1) and the relations $V_j(t) = Q_j(t)/C_j$ and $I_j(t) = \phi_j(t)/L_j$, we obtain semi-discretization scheme for (2.1)

$$\begin{cases} C_j \frac{\mathrm{d}}{\mathrm{d}t}\delta_{\frac{1}{2}}\left(\frac{Q_j(t)}{C_j}\right) = -\delta_x\left(\frac{\phi_j(t)}{L_j}\right), \\ L_j \frac{\mathrm{d}}{\mathrm{d}t}\delta_{\frac{1}{2}}\left(\frac{\phi_j(t)}{L_j}\right) = -\delta_x\left(\frac{Q_j(t)}{C_j}\right), \quad j = 0, 1, \cdots N, \\ Q_0(t) = 0, \quad Q_{N+1}(t) = RC_{N+1}L_{N+1}^{-1}\phi_{N+1}(t), \\ Q(0, y_i) \approx Q^0(y_i) = Q_i^0, \ \phi(0, y_i) \approx \phi^0(y_i) = \phi_i^0, i = 0, 1, \cdots N+1. \end{cases} \tag{5.3}$$

The energy of discrete system (5.3) is

$$E_{Oh}(t) = \frac{h}{2}\sum_{j=0}^{N}\left[C_j\left|\delta_{\frac{1}{2}}\left(\frac{Q_j(t)}{C_j}\right)\right|^2 + L_j\left|\delta_{\frac{1}{2}}\left(\frac{\phi_j(t)}{L_j}\right)\right|^2\right],$$

Now we show that the system (5.3) is uniform exponentially stable in the sense of Definition of 3.1. The state space of the discrete system (5.3) is $H_{ON} =: \left[\mathbb{R}^{N+1}\right]^2$ with the inner product

$$\langle(\widetilde{p}_h, \widetilde{q}_h), (\widetilde{u}_h, \widetilde{v}_h)\rangle_{ON} = h\sum_{j=0}^{N}\left[C_j\delta_{\frac{1}{2}}\frac{\widetilde{p}_j}{C_j}\delta_{\frac{1}{2}}\frac{\widetilde{u}_j}{C_j} + L_j\delta_{\frac{1}{2}}\frac{\widetilde{q}_j}{L_j}\delta_{\frac{1}{2}}\frac{\widetilde{v}_j}{L_j}\right], \forall(\widetilde{p}_h, \widetilde{q}_h), (\widetilde{u}_h, \widetilde{v}_h) \in H_{ON},$$

in which $\widetilde{p}_0 = \widetilde{u}_0 = 0$, $\widetilde{q}_{N+1} = k'\widetilde{p}_{N+1}$, $\widetilde{v}_{N+1} = k'\widetilde{u}_{N+1}$, and $k' = R^{-1}C_{N+1}^{-1}L_{N+1}$ are used. Let Ψ_N be matrixes of order $2N+2$ given by

$$\Psi_N = \mathrm{diag}\left\{\frac{1}{C_0}, \cdots, \frac{1}{C_N}, \frac{1}{L_0}, \cdots, \frac{1}{L_N}\right\},$$

which is well-defined since the assumption H_1 is right. Using these matrixes, we define the linear operators from H_{ON} to H_N by

$$\begin{pmatrix} p_h \\ q_h \end{pmatrix} = \Psi_N \begin{pmatrix} \widetilde{p}_h \\ \widetilde{q}_h \end{pmatrix}, \ \forall \begin{pmatrix} \widetilde{p}_h \\ \widetilde{q}_h \end{pmatrix} \in H_{ON}.$$

Thus we can get the results similar to those of Proposition 5.1 and Theorem 5.1, which are main results of this paper.

Proposition 5.2. *For any fixed positive integer N, the operator Ψ_N is an isometric isomorphism from H_{0N} to H_N.*

Proof: Obviously, for a fixed positive integer N, Ψ_N is an isomorphism from H_{0N} to H_N since it is invertible. The inverse of Ψ_N is $\Psi_N^{-1} = \text{diag}\,\{C_0, \cdots, C_N, L_0, \cdots, L_N\}$. Because the identity

$$\left\| \Psi_N \begin{pmatrix} \widetilde{p}_h \\ \widetilde{q}_h \end{pmatrix} \right\|_{H_N}^2 = h\sum_{j=0}^{N} C_j \left| \delta_{\frac{1}{2}} \left(\frac{\widetilde{p}_j}{C_j} \right) \right|^2 + h\sum_{j=0}^{N} L_j \left| \delta_{\frac{1}{2}} \left(\frac{\widetilde{q}_j}{L_j} \right) \right|^2 = \left\| \begin{pmatrix} \widetilde{p}_h \\ \widetilde{q}_h \end{pmatrix} \right\|_{H_{0N}}^2$$

holds, we know that Ψ_N is isometric.

Theorem 5.2. *Define operator A_{ON} by the formulas $A_{ON} = \Psi_N^{-1} A_N \Psi_N$. Then the abstract Cauchy problem*

$$\frac{d}{dt} \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix} = A_{ON} \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix} \tag{5.4}$$

determined by the operator A_{ON} is equivalent to the discrete system (5.3) without initial data. Furthermore, the discrete system (5.3) is uniform exponentially stable with respect to energy $E_{Oh}(t)$.

Proof: It is easy to see that the discrete system (3.1) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} V_h(t) \\ I_h(t) \end{pmatrix} = A_N \begin{pmatrix} V_h(t) \\ I_h(t) \end{pmatrix} \tag{5.5}$$

from (3.9) and (4.4). Let $Q_h(t) = (Q_1(t), \cdots, Q_{N+1}(t))$, $\phi_h(t) = (\phi_0(t), \cdots, \phi_N(t))$ and $(Q_h(t), \phi_h(t))^\top$ be state variables of the discrete system (5.3). Then we have the identity

$$\begin{pmatrix} V_h(t) \\ I_h(t) \end{pmatrix} = \Psi_N \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix}$$

since $V_j(t) = Q_j(t)/C_j$ and $I_j(t) = \phi_j(t)/L_j$ for $j = 0, 1, \cdots, N+1$. Substituting the identity above into (5.5), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \Psi_N \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix} = A_N \Psi_N \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix},$$

which is equivalent to the discrete system (5.3). This means that the abstract Cauchy problem corresponding to the discrete system (5.3) without initial data is

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix} = A_{ON} \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix} = \Psi_N^{-1} A_N \Psi_N \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix}. \tag{5.6}$$

That is to say $A_{ON} = \Psi_N^{-1} A_N \Psi_N$. Let $T_{ON}(t)$ and $T_N(t)$ be semigroups generated by A_{ON} and A_N respectively, we have $T_O(t) = \Psi^{-1}T(t)\Psi$. Therefore, from Proposition 5.2 and Theorem 3.1 we know

$$\|T_{ON}(t)\|_{H_{ON}} = \|\Psi_N^{-1}T_N(t)\Psi_N\|_{H_{ON}} = \|T_N(t)\|_{H_N} \le Me^{-\omega t},$$

which means that $T_{ON}(t)$ is uniform exponentially stable, i.e. the the exponential decay rates of $T_{ON}(t)$ are independent of N. But if you restate this result in the language of energy, you can obtain that the discrete systems (5.3) are uniform exponentially stable with respect to energy $E_{Oh}(t)$. □

Finally, the solution to the discrete system (5.3) is convergent to the one of the continuous system (2.1) in the meaning of Trotter-Kato Theorem. In fact, constructing the extensions E_{ON} from H_{ON} to H_O by $E_{ON} = \Psi^{-1}E_N\Psi_N$ and the projecting operators P_{ON} from H_O to H_{ON} by $P_{ON} = \Psi_N^{-1}E_N\Psi$, we have the following convergence result.

Theorem 5.3. *As $N \to \infty$, we have*

$$\left\| E_{ON}T_{ON}(t)P_{ON}\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} - T_O(t)\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_{H_O} \to 0, \quad \forall \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \in H_O. \tag{5.7}$$

Proof: From the relation $T_O(t) = \Psi^{-1}T(t)\Psi$ of the semigroups, Proposition 5.1 and Proposition 5.2 we have

$$\begin{aligned} \left\| [E_{ON}T_{ON}(t)P_{ON} - T_O(t)]\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_{H_O} &= \left\| \Psi^{-1}[E_NT_N(t)P_N - T(t)]\Psi\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_{H_O} \\ &= \left\| [E_NT_N(t)P_N - T(t)]\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_{H}. \end{aligned}$$

The identities above and (4.5) imply that (5.7) holds and the proof of the theorem is finished. □

6. Numerical simulations for the uniform exponential stability

In this section, we show the effectiveness of our numerical approximating schemes (3.1) and (5.5) through some numerical experiments. Recall that

$$\Psi_N = \mathrm{diag}\left\{\frac{1}{C_0}, \cdots, \frac{1}{C_N}, \frac{1}{L_0}, \cdots, \frac{1}{L_N}\right\}$$

and $A_{ON} = \Psi_N^{-1}A_N\Psi_N$ are given in last section. Because the operators A_N and A_{ON} are similar, we only give eigenvalue distributions of A_N to analyze uniformly exponential stability of (3.1) and (5.5). For this purpose, we should express the operator A_N as matrix. Let $G_h = \mathrm{diag}\{0, \cdots, 0, 1\}$,

$$B_h = \begin{pmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}, \quad M_h = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{pmatrix}$$

belong to $\mathbb{R}^{(N+1)\times(N+1)}$. Set

$$\Phi_N = \frac{1}{2}\begin{pmatrix} B_h^\top & 0 \\ kG_h & B_h \end{pmatrix}, \text{ and } \Omega_N = \frac{1}{h}\begin{pmatrix} -kG_h & M_h^\top \\ -M_h & 0 \end{pmatrix}.$$

Then (3.1) and (5.5) are equivalent to

$$\Phi_N \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} V_h(t) \\ I_h(t) \end{pmatrix} = \Psi_N \Omega_N \begin{pmatrix} V_h(t) \\ I_h(t) \end{pmatrix}, \tag{6.1}$$

in which $A_N = \Phi_N^{-1}\Psi_N\Omega_N$ is used. It is easy to see that Φ_N is corresponding to the average operator of time derivative of (3.1). If one replace B_h by the identity operator, then the classical finite difference scheme of (2.3) is easily restored from (6.1), i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} V_h(t) \\ I_h(t) \end{pmatrix} = \Psi_N \Omega_N \begin{pmatrix} V_h(t) \\ I_h(t) \end{pmatrix}, \tag{6.2}$$

If $C(x) = L(x) = 1$, the numerical approximating scheme (3.1) degenerates to the numerical approximating scheme (2.5) of [17].

Now we explain the significance of the discrete scheme (6.1) or (3.1). We plot two figures in Figure 1 and Figure 2, respectively. Figure 1 depicts the maximal real parts of A_N and $\Psi_N\Omega_N$ for $N = 40 : 5 : 400$. Figure 2 depicts the distributions of the eigenvalues of A_N and $\Psi_N\Omega_N$ with $N = 500$.

We see that the real parts of the eigenvalues of $\Psi_N\Omega_N$ approach to zero and those of A_N approach to a negative number from both figures. In both figures, we take $k = R = 1$, $C(x) = \ln(1 + x)$ and $L(x) = e^x$. Numerical simulation results show that the classical finite difference scheme of (6.2) is not uniformly exponentially stable. This is consistent with earlier research results of [30]. However, Figure 1 and Figure 2 manifest that (3.1) and (5.5) are uniformly exponentially stable and this is in accordance with theory result of section 3.

7. State reconstruction problems

This section studies the state reconstruction of the system (2.3). Assume that the initial value $(V_0(x), I_0(x)) \in H$, as well as its orbit, is unknown. Assuming a given time T, the output $O(t) = V(t, 1)$ is known within the time period of $[0, T]$ at endpoint $x = 1$, and it has control input $I(t, 1) = u(t)$ at the same position. The

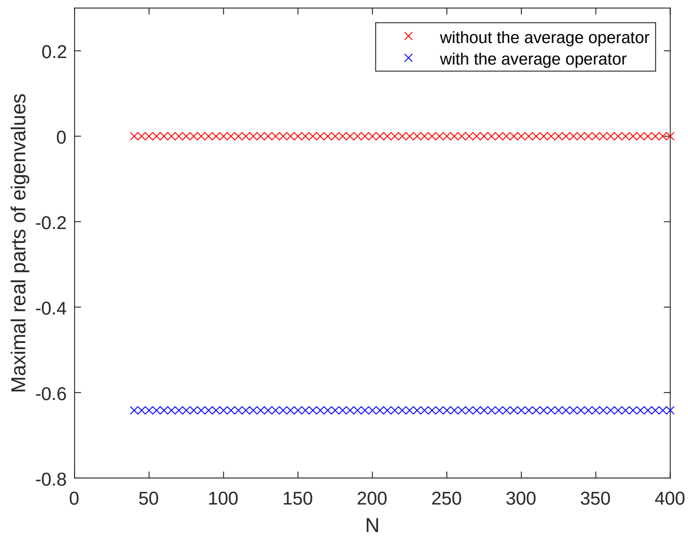


Figure 1. Maximal real parts of eigenvalues of the semi-discrete schemes

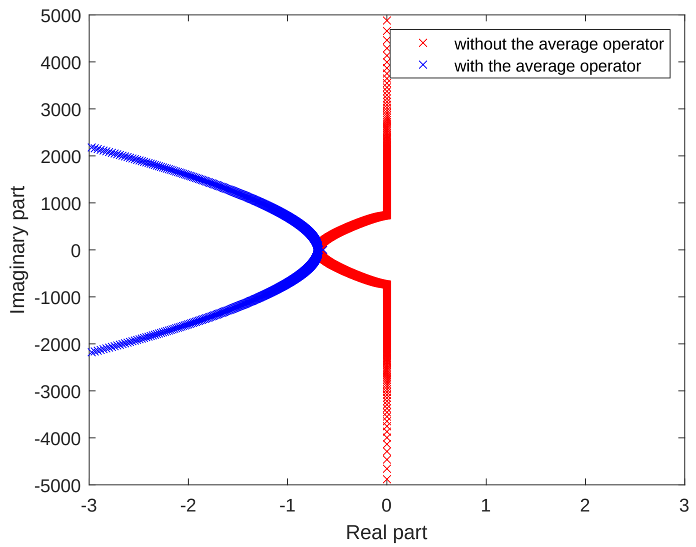


Figure 2. Maximal real parts of eigenvalues of the semi-discrete scheme with $N = 500$.

forward-backward observers-based algorithm, which was introduced in [34] and [26], is utilized to calculate the unknown initial value and those of its semi-discretization systems. To sum up, we consider the following wave equation with collocated boundary observation and control

$$\begin{cases} C(x)V_t(t,x) + I_x(t,x) = 0, \\ L(x)I_t(t,x) + V_x(t,x) = 0, \\ V(t,0) = 0, \\ I(t,1) = u(t), \quad O(t) = V(t,1), \\ V(0,x) = V_0(x), \ I(0,x) = I_0(x). \end{cases} \tag{7.1}$$

7.1. State reconstruction for the continuous system

To estimate the initial value of the system (7.1), a forward Luenberger observer is first designed

$$\begin{cases} C(x)V_t^+(t,x) + I_x^+(t,x) = 0, \\ L(x)I_t^+(t,x) + V_x^+(t,x) = 0, \\ V^+(t,0) = 0, \\ I^+(t,1) = u(t) + k(V^+(t,1) - O(t)), \\ V^+(0,x) = I^+(0,x) = 0. \end{cases} \tag{7.2}$$

Set

$$(u^+(t,x), v^+(t,x)) = (V^+(t,x) - V(t,x), I^+(t,x) - I(t,x)).$$

The difference between system (7.1) and (7.2) yields a forward error system

$$\begin{cases} C(x)u_t^+(t,x) + v_x^+(t,x) = 0, \\ L(x)v_t^+(t,x) + u_x^+(t,x) = 0, \\ u^+(t,0) = 0, \\ v^+(t,1) = ku^+(t,1), \\ u^+(0,x) = -V_0(x), \ v^+(0,x) = -I_0(x). \end{cases} \tag{7.3}$$

Secondly, build a backward Luenberger observer

$$\begin{cases} C(x)V_t^-(t,x) + I_x^-(t,x) = 0, \\ L(x)I_t^-(t,x) + V_x^-(t,x) = 0, \\ V^-(t,0) = 0, \\ I^-(t,1) = u(t) - k(V^-(t,1) - O(t)), \\ V^-(T,x) = V^+(T,x), \ I^-(T,x) = I^+(T,x). \end{cases} \tag{7.4}$$

Set

$$(u^-(t,x), v^-(t,x)) = (V^-(T-t,x) - V(T-t,x), I^-(T-t,x) - I(T-t,x)).$$

The difference between system (7.1) and (7.4) yields a backward error system

$$\begin{cases} C(x)u_t^-(t,x) - v_x^-(t,x) = 0, \\ L(x)v_t^-(t,x) - u_x^-(t,x) = 0, \\ u^-(t,0) = 0, \\ v^-(t,1) = -ku^-(t,1), \\ u^-(T,x) = V^-(0,x) - V_0(x), v^-(T,x) = I^-(0,x) - I_0(x). \end{cases} \tag{7.5}$$

Finally, the following results can be obtained.

Lemma 7.1. *Let H be the state space defined in the Section 2. Define the operator B on H through:*

$$B \begin{pmatrix} p(x) \\ q(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{C(x)} \frac{d}{dx} q(x) \\ \frac{1}{L(x)} \frac{d}{dx} p(x) \end{pmatrix}, \tag{7.6}$$

$$D(A) = \left\{ \begin{pmatrix} p(\cdot) \\ q(\cdot) \end{pmatrix} \in [H^1(0,1)]^2 : p(0) = 0, \ p(1) = -Rq(1) \right\}. \tag{7.7}$$

Thus, the system (7.5) can be transformed into abstract Cauchy problem

$$\frac{d}{dt} \begin{pmatrix} u^-(t,x) \\ v^-(t,x) \end{pmatrix} = B \begin{pmatrix} u^-(t,x) \\ v^-(t,x) \end{pmatrix}, \qquad \begin{pmatrix} u^-(0,x) \\ v^-(0,x) \end{pmatrix} \in H.$$

Then, the semigroup $S(t)$ generated by B is exponentially stable under the assumptions H_1-H_3, i.e.

$$\|S(t)\|_H \le Me^{-\omega t},$$

here $\|\cdot\|_H$ is the norm induced by the inner on H.

We delete the proof of this lemma since the method of Theorem 2.1 can be applied to prove it.

Theorem 7.1. *Let $\mathbb{T}(t)$ be the semigroup given in Remark 2.1, $\mathbb{S}(t)$ be the semigroup corresponding to (7.5) and $K_t = \mathbb{S}(t)\mathbb{T}(t)$. Chose T sufficiently large such that $\delta := \|K_T\| < 1$, then it holds*

$$\begin{pmatrix} V_0(x) \\ I_0(x) \end{pmatrix} = (I - K_T)^{-1} \begin{pmatrix} V^-(0,x) \\ I^-(0,x) \end{pmatrix}. \tag{7.8}$$

Where $V^-(0,x)$ and $I^-(0,x)$ are initial values of (7.4).

If the Neumann series is used to expand $(I-K_T)^{-1}$, then (7.8) can provide an accurate calculation formula for the initial values $(V_0(x), I_0(x))$.

$$\begin{pmatrix} V_0(x) \\ I_0(x) \end{pmatrix} = \sum_{i=0}^{\infty} [K_T]^i \begin{pmatrix} V^-(0,x) \\ I^-(0,x) \end{pmatrix}. \tag{7.9}$$

Theorem 7.2. *Moreover, we can build an approximating sequence from any guessed value $(a^{(0)}(x), b^{(0)}(x)) \in H$ of the initial values $(V_0(x), I_0(x))$. In fact, construct the iterative sequence as follows:*

$$(a^{(n)}(x), b^{(n)}(x))^\top = K_T(a^{(n-1)}(x), b^{(n-1)}(x))^\top,\ n \in \mathbb{N}. \tag{7.10}$$

If T satisfies the condition in Theorem7.1, then when $n \to \infty$, $(a^{(n)}(x), b^{(n)}(x))$ strongly converges to $(V_0(x), I_0(x))$ in H. In addition, we have the following estimate

$$\begin{aligned} &\|a^{(n)}(x) - V_0(x)\|^2_{L^2(0,1)} + \|b^{(n)}(x) - I_0(x)\|^2_{L^2(0,1)} \\ \le\ &\delta^{n+1}[\|a^{(0)}(x) - V_0(x)\|^2_{L^2(0,1)} + \|b^{(0)}(x) - I_0(x)\|^2_{L^2(0,1)}]. \end{aligned} \tag{7.11}$$

Theorem 7.1 and Theorem 7.2 are the same with Theorem 3.1 and Theorem 3.2 of [34] respectively, you can find detailed proofs of these results in [34].

7.2. State reconstruction for the discrete system

In light of the discrete system (3.1), we discretize (7.1) as follows:

$$\begin{cases} C_j \frac{d}{dt}\delta_{\frac{1}{2}} V_j(t) + \delta_x I_j(t) = 0,\ j = 0,1,2,\cdots N+1, \\ L_j \frac{d}{dt}\delta_{\frac{1}{2}} I_j(t) + \delta_x V_j(t) = 0, \\ V_0(t) = 0, \\ I_{N+1}(t) = u_h(t), \\ O_h(t) = V_{N+1}(t), \\ V_j(0) = V_{0j},\ I_j(0) = I_{0j}, \end{cases} \tag{7.12}$$

in which $u_h(t)$ and $O_h(t)$ are new input and output, the other notations are the same as those of section 3. This section mainly gives the iterative sequence for the discrete reconstruction problem. To this end, the forward observer

$$\begin{cases} C_j \frac{d}{dt}\delta_{\frac{1}{2}} V^{(0)}_{j+}(t) + \delta_x I^{(0)}_{j+}(t) = 0,\ j = 0,1,\cdots N+1, \\ L_j \frac{d}{dt}\delta_{\frac{1}{2}} I^{(0)}_{j+}(t) + \delta_x V^{(0)}_{j+}(t) = 0, \\ V^{(0)}_{0+}(t) = 0, \\ I^{(0)}_{(N+1)+}(t) = u_h(t) + k(V^{(0)}_{(N+1)+}(t) - O_h(t)),\ t \in [0,T] \\ V^{(0)}_{j+}(0) = V^{(0)}_j,\ I^{(0)}_{j+}(0) = I^{(0)}_j, \end{cases} \tag{7.13}$$

and the backward observer

$$\begin{cases} C_j \frac{d}{dt}\delta_{\frac{1}{2}} V_{j-}^{(0)}(t) + \delta_x I_{j-}^{(0)}(t) = 0, \ j = 0, 1, \cdots N+1, \\ L_j \frac{d}{dt}\delta_{\frac{1}{2}} I_{j-}^{(0)}(t) + \delta_x V_{j-}^{(0)}(t) = 0, \\ V_{0-}^{(0)}(t) = 0, \\ I_{(N+1)-}^{(0)}(t) = u_h(t) - k(V_{(N+1)-}^{(0)}(t) - O_h(t)), \ t \in [0, T] \\ V_{j-}^{(0)}(T) = V_{j+}^{(0)}(T), \ I_{j-}^{(0)}(T) = I_{j+}^{(0)}(T), \end{cases} \tag{7.14}$$

are designed as before. Let$(u_j^+(t), v_j^+(t)) = (V_{j+}^{(0)}(t) - V_j(t), I_{j+}^{(0)}(t) - I_j(t))$ be the forward error, then it can be obtained

$$\begin{cases} C_j \frac{d}{dt}\delta_{\frac{1}{2}} u_j^+(t) + \delta_x v_j^+(t) = 0, \ j = 0, 1, \cdots N+1, \\ L_j \frac{d}{dt}\delta_{\frac{1}{2}} v_j^+(t) + \delta_x u_j^+(t) = 0, \\ u_0^+(t) = 0, \\ v_{N+1}^+(t) = k u_{N+1}^+(t), \ t \in [0, T] \\ u_j^+(0) = V_j^{(0)} - V_{0j}, \ v_j^+(0) = I_j^{(0)} - I_{0j}. \end{cases} \tag{7.15}$$

Similarly, let$(u_j^-(t), v_j^-(t)) = (V_{j-}^{(0)}(T-t) - V_j(T-t), I_{j-}^{(0)}(T-t) - I_j(T-t))$ be the backward error then it satisfies

$$\begin{cases} C_j \frac{d}{dt}\delta_{\frac{1}{2}} u_j^-(t) - \delta_x v_j^-(t) = 0, \ j = 0, 1, 2, \cdots N+1, \\ L_j \frac{d}{dt}\delta_{\frac{1}{2}} v_j^-(t) - \delta_x u_j^-(t) = 0, \\ u_0^-(t) = 0, \\ v_{N+1}^-(t) = -k u_{N+1}^-(t), \ t \in [0, T] \\ u_j^-(0) = V_{j-}^{(0)}(T) - V_j(T), \ v_j^-(0) = I_{j-}^{(0)}(T) - I_j(T). \end{cases} \tag{7.16}$$

The state spaces of (7.12)-(7.16) are also be X_h and let

$$u_h^{\pm}(t) = (u_1^{\pm}(t), u_2^{\pm}(t), ..., u_{(N+1)}^{\pm}(t)), \quad v_h^{\pm}(t) = (v_0^{\pm}(t), v_1^{\pm}(t), ..., v_N^{\pm}(t)),$$

definition mapping

$$(u_h^+(t), v_h^+(t)) = F_h(t)(u_h^+(0), v_h^+(0)), \quad (u_h^-(t), v_h^-(t)) = B_h(t)(u_h^-(0), v_h^-(0)). \tag{7.17}$$

Finally, based on the above preparations, we have the following theorem.

Theorem 7.3. *Let $(a_h^{(0)}, b_h^{(0)}) \in X_h$ be any guess value of initial value (V_{0h}, I_{0h}), construct the iterative sequence as follows:*

$$(a_h^{(n)}, b_h^{(n)}) = B_h(T) F_h(T)(a_h^{(n-1)}, b_h^{(n-1)}), \ n \in \mathbb{N}. \tag{7.18}$$

If T is selected so that $\delta_h := \|B_h(T)F_h(T)\|_h < 1$ is true, then as $n \to \infty$, $(a_h^{(n)}, b_h^{(n)})$ converges uniformly strongly to (V_{0h}, I_{0h}) in X_h with respect to the discretization parameter h, and there is an error estimate as follows:

$$\|(a_h^{(n)} - V_{0h}, b_h^{(n)} - I_{0h})\|_h^2 \leq \delta^{n+1} \|(a_h^{(0)} - V_{0h}, b_h^{(0)} - I_{0h})\|_h^2. \tag{7.19}$$

Proof: According to Theorem 3.1, we can know that the system (7.12) is uniformly exponentially stable. It can be described by the space X_h and mapping $F_h(t)$ defined above, there exist normal numbers M_1 and ω_1 independent of h and t that make

$$\|F_h(t)\|_h \le M_1 e^{-\omega_1 t}.$$

Similarly, system (7.16) is uniform exponential stability, there are normal numbers of M_2 and ω_2 independent of h and t that make

$$\|B_h(t)\|_h \le M_2 e^{-\omega_2 t}.$$

On the other hand, by using the proof of Theorem 7.2 and (7.12) - (7.16), we can know that the following relation is valid:

$$(V_{h-}^{(0)} - V_{0h}, I_{h-}^{(0)} - I_{0h}) = B_h(T)F_h(T)(a_h^{(0)} - V_{0h}, b_h^{(0)} - I_{0h}).$$

For any $n \ge 1$, $V_{h\pm}^{(n)}(t)$ and $I_{h\pm}^{(n)}(t)$ are constructed using forward observer (7.13) and backward observer (7.14):

$$\begin{cases} C_j \frac{d}{dt}\delta_{\frac{1}{2}} V_{j+}^{(n)}(t) + \delta_x I_{j+}^{(n)}(t) = 0, \ j = 0,1,2,\cdots N+1, \\ L_j \frac{d}{dt}\delta_{\frac{1}{2}} I_{j+}^{(n)}(t) + \delta_x V_{j+}^{(n)}(t) = 0, \\ V_{0+}^{(n)}(t) = 0, \\ I_{(N+1)+}^{(n)}(t) = u_h(t) + k(V_{(N+1)+}^{(n)}(t) - V_{N+1}(t)), \\ V_{j+}^{(n)}(0) = V_{j+}^{(n-1)}(0), \ I_{j+}^{(n)}(0) = I_{j+}^{(n-1)}(0). \end{cases} \tag{7.20}$$

$$\begin{cases} C_j \frac{d}{dt}\delta_{\frac{1}{2}} V_{j-}^{(n)}(t) + \delta_x I_{j-}^{(n)}(t) = 0, \ j = 0,1,2,\cdots N+1, \\ L_j \frac{d}{dt}\delta_{\frac{1}{2}} I_{j-}^{(n)}(t) + \delta_x V_{j-}^{(n)}(t) = 0, \\ V_{0-}^{(n)}(t) = 0, \\ I_{(N+1)-}^{(n)}(t) = u_h(t) - k(V_{(N+1)-}^{(n)}(t) - V_{N+1}(t)), \\ V_{j-}^{(n)}(T) = V_{j+}^{(n)}(T), \ I_{j-}^{(n)}(T) = I_{j+}^{(n)}(T). \end{cases} \tag{7.21}$$

Similarly, the following relationship can be obtained:

$$(V_{h-}^{(n)}(0) - V_{0h}, I_{h-}^{(n)}(0) - I_{0h}) = B_h(T)F_h(T)(V_{h-}^{(n-1)}(0) - V_{0h}, I_{h-}^{(n-1)}(0) - I_{0h}).$$

then

$$(a_h^{(n)}, b_h^{(n)}) := (V_{h-}^{(n)}(0), I_{h-}^{(n)}(0))$$

that's the sequence of iterations we're looking for. After a simple calculation

$$(a_h^{(n)} - V_{0h}, b_h^{(n)} - I_{0h}) = [B_h(T)F_h(T)]^{n+1}(a_h^{(0)} - V_{0h}, b_h^{(0)} - I_{0h}),$$

the error estimate (7.19) and the convergence result can be obtained by taking the norm in X_h on both sides

$$\begin{aligned} \|(a_j^{(n)} - V_{0j}, b_j^{(n)} - I_{0j})\|_h^2 &= \|[S_j(T)T_j(T)]^{n+1}(a_j^{(0)} - V_{0j}, b_j^{(0)} - I_{0j})\|_h^2 \\ &\le \|[S_j(T)T_j(T)]^{n+1}\|_h^2 \|(a_j^{(0)} - V_{0j}, b_j^{(0)} - I_{0j})\|_h^2 \\ &\le \delta^{n+1}\|(a_j^{(0)} - V_{0j}, b_j^{(0)} - I_{0j})\|_h^2. \end{aligned} \tag{7.22}$$

Then when $n \to \infty$,

$$\|(a_j^{(n)} - V_{0j}, b_j^{(n)} - I_{0j})\|_h \to 0.$$

So there is

$$(a_j^{(n)}, b_j^{(n)}) \xrightarrow{strong} (V_0 j, I_0 j).$$

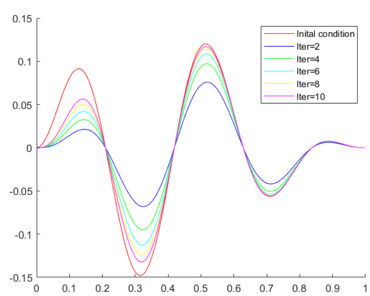
8. Numerical simulations for the state reconstruction

In our first example, we take

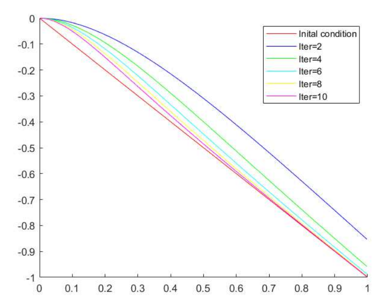
$$V_0(x) = x(1-x)^2 sin(15x), I_0(x) = -x \ (x \in [0,1]).$$

We show in Figures 3-4 several iterations of the iterative method ($n \in 2, 4, 6, 8, 10$).

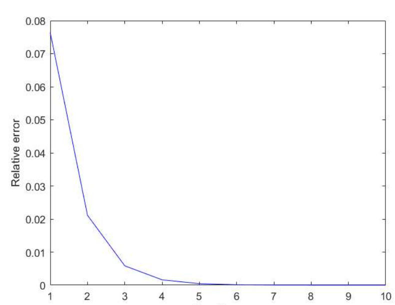
In this test, we take initial guess value$(a_j^{(0)}, b_j^{(0)}) = (0,0)$ of (V_0, I_0) and a mesh size $h = 1/(N+1)$ with $N = 240$. We also discretized the time interval, with a time step $\delta t = \tau/M$, with $M = 1200$, and $\tau = 2.5$. We notice that the reconstructions for V_0 and I_0 become more accurate as the number of iterations increases. We represent in Figure 5 the relative error made after each iteration.



(a) Figure 3. Reconstruction of V_0

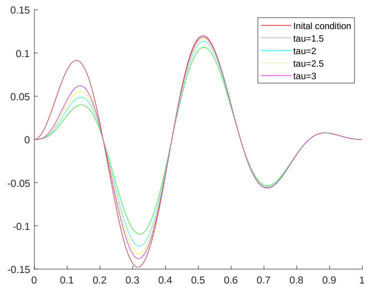


(b) Figure 4. Reconstruction of I_0

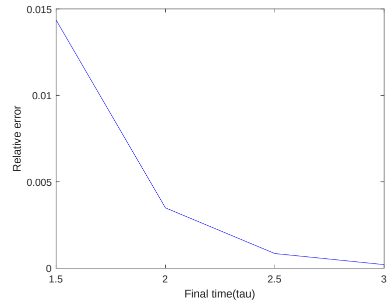


(c) Figure 5. Relative error made after n iterations

We also show how the the time τ plays a role in the method. We take $M = 1200$, $N = 240$, $n = 10$. We represent in Figures 6 - 7 the final reconstruction of the initial condition for V_0 and the relative error for $\tau \in \{1.5, 2, 2.5, 3\}$.



(d) Figure 6. Reconstruction of V_0



(e) Figure 7. Relative error with respect to τ

9. Concluding remarks

Transmission line is a basic structure of circuit and plays an important role in physics and engineering. This paper is devoted to uniformly exponentially stable approximations for the transmission line with varying capacity and inductance. This means that we study it from the viewpoints of the numerical approximating and control theory. It is well-known that there are many discretization methods to discretize the spatial variables. It is nontrivial to pick one that preserves exponential stability among so many semi-discretization methods. On the other hand, if the capacity parameter and the inductance parameter are constant, many existing results can be applied and there is no any challenge. To bypass the troubles brought by variable coefficients, suitable similar transforms are introduced. The uniform exponential stability of the transmission line with varying capacity and inductance is then smoothly obtained based on the method for the wave equations with constant coefficients. We gave an important application of this main result, i.e., the state reconstruction of the transmission line with varying capacity and inductance. Moreover, the uniform exponential stability has potential applications in uniform controllability and other problems. They deserve to be investigated at length in further research.

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