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EXISTENCE RESULTS FOR A NONLINEAR GENERALIZED CAPUTO FRACTIONAL BOUNDARY VALUE PROBLEM*

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Abstract This study gives some new existence results for a three point boundary value problem involving a nonlinear fractional differential equation that incorporates a broad form of the Caputo fractional derivative concerning a new function. Our approach rests upon the fixed point theorems established by Banach, Schafer, and Schauder. Additionally, we substantiate the robustness of our findings by providing an apt illustrative example.

Keywords Fractional differential equation, Caputo fractional derivative, Generalized Caputo fractional derivative and fractional integral, Existence result, Fixed point theorem, Upper and lower solution.

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1. Introduction

Fractional calculus has emerged as a powerful mathematical tool with diverse applications in modeling complex phenomena exhibiting memory effects, anomalous diffusion, and non-local behavior. Fractional differential equations have gained substantial attention in recent decades due to their ability to model complex phenomena in various scientific and engineering disciplines. These equations involve derivatives of non-integer order, providing a more accurate representation of processes that exhibit memory effects and anomalous diffusion. The Caputo fractional derivative, in particular, has proven to be a versatile tool for describing such behaviors. Thus, in recent years, fractional analysis and fractional differential equations have become very popular and gained great importance by the agency of studies and proven applications in many scientific fields such as physics, mathematics, statistics, biology and engineering. When we examine the literature, besides the Caputo fractional derivative, there are so many studies on different operators such as Riemann, Hilfiger, Erdelyi-Kober, Hadamard and generalized fractional derivatives and integrals. We would like to mention some recent monographs devoted to the investigation of boundary value problems for fractional differential equations with many examples and applications, namely [2, 3, 5, 9, 11, 13–17]. In particular we also mention some studies on the existence of Caputo fractional boundary value problems [1, 4, 6, 7, 18, 19]. However, many scholars discovered that some existing fractional operators may not well to describe many phenomena in the real world.

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Hence, a new general definition is proposed recently, so-called Θ-Caputo fractional operator, which could combine the maximum number of definitions of fractional derivatives to a single one by depending upon a nonsingular kernel. The kernel function can provide free arguments to better calibrate a system.

In this paper, we delve into a specific class of fractional boundary value problems associated with this new Θ-Caputo fractional derivative. This derivative encompasses a general form of the traditional Caputo fractional derivative and is defined in terms of a new function Θ. The introduction of Θ allows us to capture a broader range of behaviors and instances, making our study particularly relevant in exploring intricate dynamics. This provides flexibility in finding better solutions to various problems in different application domains.

In [1] Abdo, Panchal and Saeed investigated the existence of positive solutions for the following nonlinear fractional differential equation,

$$\begin{aligned} {}^{c}D_{a^+}^{\beta,\Theta} z(t) &= f(t, z(t)), \qquad t \in [a,b], \\ z_\Theta^{[j]}(a) = z_a^j, \quad j = 0,1,...,m-2; &\quad z_\Theta^{[m-1]}(b) = z_b, \end{aligned}$$

containing a general form of the Caputo fractional derivative according to a new function Θ, where ${}^{c}D_{a^+}^{\beta,\Theta}$ is the Θ-Caputo fractional derivative of order β such that $m-1 < \beta \leq m$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is the continuous function and $z_a^j \quad (j = 0,1,...,m-2)$, z_b are the real constants. By using Banach and Schafer fixed point theorems, the existence of a positive solution is achieved.

Let's take another study in [14] researched the results of existence and uniqueness for the nonlinear Caputo fractional boundary value problem

$$\begin{aligned} {}^{c}D_a^{\beta} z(t) &= f(t, z(t)), \qquad t \in [a,b], \\ z^{(j)}(a) = z_j, \quad j = 0,1,...,m-2; &\quad z^{(m-1)}(b) = z_b, \end{aligned}$$

where $z_0, z_1, ..., z_{m-2}, z_b$ are the real constants, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is the continuous function and ${}^{c}D_a^{\beta}$ is the Caputo fractional derivative of order $m-1 < \beta \leq m$.

Motivated by the research going on in this direction, in this paper, we study the existence of a solution for the following nonlinear fractional differential equation

$$ {}^{c}D_{a^+}^{\beta,\Theta} z(t) = f(t, z(t)), \quad t \in [a,b], \tag{1.1}$$

$$ z_\Theta^{[j]}(a) = z_a^j, \quad j = 0,1,...,m-2; \quad z_\Theta^{[m-2]}(b) = \delta z_\Theta^{[m-2]}(\eta), \tag{1.2}$$

containing a general form of the $\beta-$th order Caputo fractional derivative according to a new function Θ such that Θ is an increasing differentiable function with $\Theta'(t) \neq 0$ for all $t \in [a,b]$. In this problem ${}^{c}D_{a^+}^{\beta,\Theta}$ is the Θ-Caputo fractional derivative of order β with $m-1 < \beta \leq m \quad (m = [\beta]+1)$, $z_a^j \in \mathbb{R}$ for $(j = 0,1,...,m-2)$, $\eta \in (a,b)$, $\delta \in (0,1)$, $z \in C^{m-1}[a,b]$, $z_\Theta^{[m-2]}(t) = \dfrac{(z_\Theta^{[m-3]})'(t)}{\Theta'(t)}$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Our main objective is to establish the existence and uniqueness results for the considered Θ-Caputo fractional boundary value problem (1.1)-(1.2). The search for solutions to fractional differential equations is often complicated by the non-local and non-linear nature of the equations, demanding novel analytical techniques. To this end, we draw upon the foundational fixed point theorems provided by Banach, Schafer, and Schauder. These theorems furnish a solid theoretical framework for proving the existence of solutions.

The remainder of the paper is arranged as follows. In Section 2, we provide a concise overview of essential foundational lemmas and definitions pertinent to the theory of Θ-Caputo fractional calculus. In Section 3, we write the problem (1.1)-(1.2) as an equivalent integral equation, and then, under some assumptions on the nonlinear term f, we establish existence results for (1.1)-(1.2) by means of the fixed point theorems. Also, in this section, to illustrate the validity and applicability of our theoretical findings, we provide a carefully chosen illustrative example that highlights the significance of our approach. This example not only demonstrates the feasibility of our results but also serves as a guide for researchers and practitioners seeking to comprehend the practical implications of our work. Finally, Section 4 concludes the paper with a summary of the achieved results.

In essence, our study aims to contribute to the advancement of the theory and applications of fractional differential equations by tackling the nonlinear generalized Caputo fractional boundary value problem according to a new function. The exploration of this problem not only enhances our understanding of complex dynamics but also holds promise for addressing real-world challenges across diverse domains.

2. Preliminaries

In this section, we would like to provide a brief overview of the relevant literature in the field of fractional differential equations. Therefore, we give some definitions, notations, lemmas and results for Θ-Caputo fractional derivative [7] which are used throughout this paper.

Definition 2.1. [7] Let $\beta > 0$ such that $m-1 < \beta < m$, $z : [a,b] \to \mathbb{R}$ be an integrable function defined on $[a,b]$ and $\Theta \in C^m[a,b]$ be an increasing differentiable function with $\Theta'(t) \neq 0$ for all $t \in [a,b]$. The $\beta-$th order left-sided Θ-Riemann-Liouville fractional integral of a function z is given by

$$I_{a^+}^{\beta,\Theta} z(t) = \frac{1}{\Gamma(\beta)} \int_a^t \Theta'(s)(\Theta(t)-\Theta(s))^{\beta-1} z(s)ds,$$

where $\Gamma(.)$ is a gamma function.

Definition 2.2. [7] Let $m-1 < \beta < m$, $z : [a,b] \to \mathbb{R}$ be an integrable function and Θ be as defined in Definition 2.1. The left-sided Θ-Riemann-Liouville fractional derivative of order β of a function z is given by

$$D_{a^+}^{\beta,\Theta} z(t) = \left[\frac{1}{\Theta'(t)}\frac{d}{dt}\right]^m I_{a^+}^{m-\beta,\Theta} z(t)$$

where $m = [\beta]+1$ and $[\beta]$ denotes the integer part of the real number β.

Definition 2.3. [7] Let $m-1 < \beta < m$, $z \in C^{m-1}[a,b]$ and Θ be as defined in Definition 2.1. The left-sided Θ-Caputo fractional derivative of function z of order

β is identified as

$$^{c}D_{a^+}^{\beta,\Theta}z(t) = D_{a^+}^{\beta,\Theta}\left[z(t) - \sum_{j=0}^{m-1}\frac{z_{\Theta}^{[j]}(a)}{j!}(\Theta(t)-\Theta(a))^j\right],$$

where $z_{\Theta}^{[j]}(t) = \left[\frac{1}{\Theta'(t)}\frac{d}{dt}\right]^j z(t)$ and $m = [\beta]+1$ for $\beta \notin \mathbb{N}$, $m = \beta$ for $\beta \in \mathbb{N}$. Furthermore if $z \in C^m[a,b]$ and $\beta \notin \mathbb{N}$, then

$$\begin{aligned} ^{c}D_{a^+}^{\beta,\Theta}z(t) &= I_{a^+}^{m-\beta,\Theta}\left[\frac{1}{\Theta'(t)}\frac{d}{dt}\right]^m z(t) \\ &= \frac{1}{\Gamma(m-\beta)}\int_a^t \Theta'(s)(\Theta(t)-\Theta(s))^{m-\beta-1}z_{\Theta}^{[m]}(s)ds. \end{aligned}$$

Thus, if $\beta = m \in \mathbb{N}$, one has $^{c}D_{a^+}^{\beta,\Theta}z(t) = z_{\Theta}^{[m]}(t)$.

Lemma 2.1. *[7] Let $\beta > 0$, then the followings hold for the function $z : [a,b] \to \mathbb{R}$:*

1. *If $z \in C[a,b]$ then $^{c}D_{a^+}^{\beta,\Theta}I_{a^+}^{\beta,\Theta}z(t) = z(t)$.*
2. *If $z \in C^{m-1}[a,b]$, then $I_{a^+}^{\beta,\Theta}\ {}^{c}D_{a^+}^{\beta,\Theta}z(t) = z(t) - \sum_{j=0}^{m-1}c_j[\Theta(t)-\Theta(a)]^j$, where $c_j = \frac{z^{[j]}(a)}{j!}$.*

Lemma 2.2. *[1] Let $\beta > 0$ and $z, \Theta \in C[a,b]$. Then*

1. *$I_{a+}^{\beta,\Theta}(.)$ is bounded from $C[a,b]$ to $C[a,b]$ and linear,*
2. *$I_{a+}^{\beta,\Theta}z(a) = \lim_{t\to a^+} I_{a+}^{\beta,\Theta}z(t) = 0$.*

Lemma 2.3. *[1] Let $\gamma, \beta > 0$ and $z : [a,b] \to \mathbb{R}$. Then*

1. *$I_{a^+}^{\gamma,\Theta}[\Theta(t)-\Theta(a)]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\gamma+\beta)}[\Theta(t)-\Theta(a)]^{\gamma+\beta-1}$,*
2. *$^{c}D_{a^+}^{\gamma,\Theta}[\Theta(t)-\Theta(a)]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\gamma-\beta)}[\Theta(t)-\Theta(a)]^{\beta-\gamma-1}$,*
3. *$^{c}D_{a^+}^{\gamma,\Theta}[\Theta(t)-\Theta(a)]^j = 0, \quad \forall j \in \{0,1,...,m-1\},\ m \in \mathbb{N}$,*
4. *$I_{a^+}^{\gamma,\Theta}I_{a^+}^{\beta,\Theta}z(t) = I_{a^+}^{\gamma+\beta,\Theta}z(t)$.*

Our main results heavily rely on the fundamental and crucial fixed point theorems presented below.

Theorem 2.1. *(Schaefer's Fixed Point Theorem) [10] Let U be a Banach space and $A : U \to U$ be continuous and compact operator. Assume further that the set $\{u \in U : u = \lambda Au,\ for\ some\ \lambda \in (0,1)\}$ is bounded. Then the operator A has a fixed point in U.*

Theorem 2.2. *(Banach Fixed Point Theorem) [8] Let U be a Banach space and K be a closed subset of U. If A is a contraction mapping from K into K, then there exists a unique fixed point u in K such that $A(u) = u$.*

Theorem 2.3. *(Schauder-Tychonov Fixed Point Theorem) [12] Let U be a Banach space and K be a closed, bounded, convex subset of U. If $A : K \to K$ is compact, then A has a fixed point in K.*

3. Main Results

In this section, we establish some sufficient conditions for the existence and uniqueness of the solutions to the problem (1.1)-(1.2). To that end, we first give the following useful result which gives the solution of the linear form of the problem.

Lemma 3.1. *Let $m-1 < \beta \leq m$ and the function $g : [a,b] \to \mathbb{R}$ be continuous function. A function $z \in C^{m-1}[a,b]$ is a solution of the following three point fractional boundary value problem*

$$ {}^cD^{\beta,\Theta}_{a^+} z(t) = g(t) \tag{3.1}$$

$$ z^{[j]}_{\Theta}(a) = z^j_a, \quad j = 0,1,...,m-2, \quad z^{[m-2]}_{\Theta}(b) = \delta z^{[m-2]}_{\Theta}(\eta), \tag{3.2}$$

if and only if $z(t)$ satisfies the following fractional integral equation;

$$\begin{aligned} z(t) = &\sum_{j=0}^{m-3} \frac{z^j_a}{j!}(\Theta(t)-\Theta(a))^j + \left[\frac{1}{(m-2)!} + \frac{(\delta-1)(\Theta(t)-\Theta(a))}{(m-1)!\Delta}\right] z^{m-2}_a(\Theta(t)-\Theta(a))^{m-2} \\ &+ \int_a^b \Theta'(s)G(t,s)g(s)ds, \end{aligned}$$

where

$$\Delta := \Theta(b) - \Theta(a) - \delta(\Theta(\eta) - \Theta(a))$$

and

$$G(t,s) = \begin{cases} G_1(t,s), & s \leq \eta, \\ G_2(t,s), & s \geq \eta, \end{cases}$$

such that

$$G_1(t,s) = \begin{cases} \frac{(\Theta(t)-\Theta(a))^{m-1}}{(m-1)!\Delta\Gamma(\beta-m+2)}[(\Theta(\eta)-\Theta(s))^{\beta-m+1} - (\Theta(b)-\Theta(s))^{\beta-m+1}] & \\ +\frac{1}{\Gamma(\beta)}(\Theta(t)-\Theta(s))^{\beta-1}, & s \leq t, \\ \frac{(\Theta(t)-\Theta(a))^{m-1}}{(m-1)!\Delta\Gamma(\beta-m+2)}[(\Theta(\eta)-\Theta(s))^{\beta-m+1} - (\Theta(b)-\Theta(s))^{\beta-m+1}], & s \geq t, \end{cases}$$

and

$$G_2(t,s) = -\begin{cases} \frac{(\Theta(t)-\Theta(a))^{m-1}}{(m-1)!\Delta\Gamma(\beta-m+2)}(\Theta(b)-\Theta(s))^{\beta-m+1} - \frac{1}{\Gamma(\beta)}(\Theta(t)-\Theta(s))^{\beta-1}, & s \leq t, \\ \frac{(\Theta(t)-\Theta(a))^{m-1}}{(m-1)!\Delta\Gamma(\beta-m+2)}(\Theta(b)-\Theta(s))^{\beta-m+1}, & s \geq t. \end{cases}$$

Proof. First, assume that $z \in C^{m-1}[a,b]$ is a solution of the problem (3.1)-(3.2). Then Lemma 2.1 implies that

$$\begin{aligned} z(t) &= c_0 + c_1(\Theta(t)-\Theta(a)) + c_2(\Theta(t)-\Theta(a))^2 + ... + c_{m-1}(\Theta(t)-\Theta(a))^{m-1} \\ &\quad + \frac{1}{\Gamma(\beta)}\int_a^t \Theta'(s)(\Theta(t)-\Theta(s))^{\beta-1} g(s)ds \\ &= \sum_{j=0}^{m-1} c_j(\Theta(t)-\Theta(a))^j + \frac{1}{\Gamma(\beta)}\int_a^t \Theta'(s)(\Theta(t)-\Theta(s))^{\beta-1} g(s)ds. \end{aligned}$$

We need to show the correctness of the conditions in (3.2). It is clear that $z(a) = z_a$. Also with the direct computations, we get

$$\begin{aligned} z_\Theta^{[1]}(t) &= \frac{z'(t)}{\Theta'(t)} \\ &= \sum_{j=1}^{m-1} j c_j(\Theta(t)-\Theta(a))^{j-1} + \frac{1}{\Gamma(\beta-1)}\int_a^t \Theta'(s)(\Theta(t)-\Theta(s))^{\beta-2} g(s)ds \end{aligned}$$

and so $z_\Theta^{[1]}(a) = z_a^1$. By iterating this procedure, we reach the conclusion that

$$\begin{aligned} z_\Theta^{[m-2]}(t) = \frac{\left(z_\Theta^{[m-3]}(t)\right)'}{\Theta'(t)} &= \sum_{j=m-2}^{m-1} j(j-1)...(j-(m-2))c_j(\Theta(t)-\Theta(a))^{j-m+2} \\ &\quad + \frac{1}{\Gamma(\beta-m+2)}\int_a^t \Theta'(s)(\Theta(t)-\Theta(s))^{\beta-m+1} g(s)ds \\ &= (m-2)!\, c_{m-2} + (m-1)!\, c_{m-1}(\Theta(t)-\Theta(a)) \\ &\quad + \frac{1}{\Gamma(\beta-m+2)}\int_a^t \Theta'(s)(\Theta(t)-\Theta(s))^{\beta-m+1} g(s)ds. \end{aligned}$$

From the boundary condition $z_\Theta^{[j]}(a) = z_a^j$, we have

$$c_j = \frac{{z_a}^j}{j!}, \quad j = 0,1,...,m-2,$$

and using the second boundary condition

$$z_\Theta^{[m-2]}(b) = \delta z_\Theta^{[m-2]}(\eta), \qquad 0<\delta<1,$$

we have

$$\begin{aligned} c_{m-1} =& \frac{1}{\Delta(m-1)!}\left\{(\delta-1){z_a}^{m-2} - \frac{1}{\Gamma(\beta-m+2)}\left[\int_a^b \Theta'(s)(\Theta(b)-\Theta(s))^{\beta-m+1} g(s)ds \right.\right. \\ & \left.\left. - \int_a^\eta \Theta'(s)(\Theta(\eta)-\Theta(s))^{\beta-m+1} g(s)ds\right]\right\}. \end{aligned}$$

Then, the solution of the problem (3.1)-(3.2) is given

$$z(t) = \sum_{j=0}^{m-3} \frac{{z_a}^j}{j!}(\Theta(t)-\Theta(a))^j + \left[\frac{1}{(m-2)!} + \frac{(\Theta(t)-\Theta(a))(\delta-1)}{\Delta(m-1)!}\right]{z_a}^{m-2}(\Theta(t)-\Theta(a))^{m-2}$$

$$- \frac{(\Theta(t)-\Theta(a))^{m-1}}{\Gamma(\beta-m+2)\Delta(m-1)!}\left\{\int_a^b \Theta'(s)((\Theta(b)-\Theta(s))^{\beta-m+1}g(s)ds\right.$$
$$\left.-\int_a^\eta \Theta'(s)(\Theta(\eta)-\Theta(s))^{\beta-m+1}g(s)ds\right\}+\frac{1}{\Gamma(\beta)}\int_a^t \Theta'(s)(\Theta(t)-\Theta(s))^{\beta-1}g(s)ds$$
$$=\sum_{j=0}^{m-3}\frac{{z_a}^j}{j!}(\Theta(t)-\Theta(a))^j+\left[\frac{1}{(m-2)!}+\frac{(\Theta(t)-\Theta(a))(\delta-1)}{\Delta(m-1)!}\right]{z_a}^{m-2}(\Theta(t)-\Theta(a))^{m-2}$$
$$+\int_a^b \Theta'(s)G(t,s)g(s)ds,$$

where $G(t,s)$ is defined in Lemma 3.1. □

Lemma 3.2. *The Green's function $G(t,s)$ is continuous on $t,s\in[a,b]$. Also the functions $G_i(t,s)$ $(i=1,2)$ given by Lemma 3.1 satisfy the followings*

(i) $|G_1(t,s)|\le \frac{(\Theta(b)-\Theta(a))^{m-1}}{\Gamma(\beta-m+1)\Delta(m-1)!}(\Theta(b)-\Theta(\eta))^{\beta-m+1}+\frac{1}{\Gamma(\beta)}(\Theta(b)-\Theta(a))^{\beta-1}$ *for any* $t,s\in[a,b]$,

(ii) $|G_2(t,s)|\le \frac{(\Theta(b)-\Theta(a))^{\beta}}{\Gamma(\beta-m+1)\Delta(m-1)!}+\frac{1}{\Gamma(\beta)}(\Theta(b)-\Theta(a))^{\beta-1}$ *for any* $t,s\in[a,b]$.

Proof. We can easily see that the function $G(t,s)$ is continuous on $t,s\in[a,b]$, so we can omit this part.

Now we show that (i) holds. If $s\le\eta\le t,\ 0<\beta-m+1\le 1,\ 0<\delta<1$ and $\Gamma(.)$ is positive function then

$$|G_1(t,s)|\le\frac{(\Theta(b)-\Theta(a))^{m-1}}{\Gamma(\beta-m+1)\Delta(m-1)!}[(\Theta(b)-\Theta(s))^{\beta-m+1}-(\Theta(\eta)-\Theta(s))^{\beta-m+1}]+\frac{1}{\Gamma(\beta)}(\Theta(b)-\Theta(a))^{\beta-1}.$$

Since the function $f(t)=(\Theta(b)-t)^{\beta-m+1}-(\Theta(\eta)-t)^{\beta-m+1}$ is increasing, for $s\le\eta$ we have $\Theta(s)\le\Theta(\eta)$ and so $f(\Theta(s))\le f(\Theta(\eta))$. Thus we get desired result. When $s\ge t$ we can easily see that the inequality is satisfied.

Next, we show that (ii) holds. If $s\le t$, since $\beta-1\le m-1$ and $m\ge 2$ then

$$|G_2(t,s)|\le\frac{(\Theta(b)-\Theta(a))^{\beta}}{\Gamma(\beta-m+1)\Delta(m-1)!}+\frac{1}{\Gamma(\beta)}(\Theta(b)-\Theta(a))^{\beta-1}.$$

So we have desired inequality. Similarly, for $s\ge t$ we easily obtain. □

Corollary 3.1. *The Green's function $G(t,s)$ for the problem (1.1)-(1.2) satisfies the following inequality*

$$|G(t,s)|\le(\Theta(b)-\Theta(a))^{\beta-1}\left[\frac{\Theta(b)-\Theta(a)}{\Gamma(\beta-m+1)\Delta(m-1)!}+\frac{1}{\Gamma(\beta)}\right]:=M.$$

Proof. It is easy from Lemma 3.2. □

Theorem 3.1. *Assume that*

(U1) $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ *is continuous function and there exists* $0<k_0<1$ *such that*

$$|f(t,z_1)-f(t,z_2)|\le k_0|z_1-z_2|,\ t\in[a,b],\ z_1,z_2\in\mathbb{R}.$$

If

$$Mk_0(\Theta(b)-\Theta(a))<1,$$

then the boundary value problem (1.1)-(1.2) has a unique solution on $[a,b]$*, where* M *is the upper bound of* $G(t,s)$ *which is given in Corollary 3.1.*

Proof. In view of Lemma 3.1, the function $z\in C^{m-1}[a,b]$ is a solution to (1.1)-(1.2) if z satisfies

$$\begin{aligned} z(t) = &\sum_{j=0}^{m-3}\frac{{z_a}^j}{j!}(\Theta(t)-\Theta(a))^j+\left[\frac{1}{(m-2)!}+\frac{(\Theta(t)-\Theta(a))(\delta-1)}{\Delta(m-1)!}\right]{z_a}^{m-2}(\Theta(t)-\Theta(a))^{m-2}\\ &+\int_a^b\Theta'(s)G(t,s)f(s,z(s))ds. \end{aligned}$$

Set

$$P=\{\ z\in C^{m-1}[a,b]:\ {}^cD_{a^+}^{\beta,\Theta}z\in C[a,b]\ \} \tag{3.3}$$

and define the operator $T:P\to P$ by

$$\begin{aligned} (Tz)(t) = &\sum_{j=0}^{m-3}\frac{{z_a}^j}{j!}(\Theta(t)-\Theta(a))^j+\left[\frac{1}{(m-2)!}+\frac{(\Theta(t)-\Theta(a))(\delta-1)}{\Delta(m-1)!}\right]{z_a}^{m-2}(\Theta(t)-\Theta(a))^{m-2}\\ &+\int_a^b\Theta'(s)G(t,s)f(s,z(s))ds. \end{aligned}\tag{3.4}$$

We first show that the operator T which is given in (3.4) is well defined operator, that is, $T(P)\subseteq P$. For this reason, we consider a function $z\in C^{m-1}[a,b]$. It is evident that $Tz\in C^{m-1}[a,b]$. Also, we have

$$\begin{aligned} {}^cD_{a^+}^{\beta,\Theta}(Tz(t)) ={}& {}^cD_{a^+}^{\beta,\Theta}\left[\sum_{j=0}^{m-3}\frac{{z_a}^j}{j!}(\Theta(t)-\Theta(a))^j\right.\\ &+\left[\frac{1}{(m-2)!}+\frac{(\Theta(t)-\Theta(a))(\delta-1)}{\Delta(m-1)!}\right]{z_a}^{m-2}(\Theta(t)-\Theta(a))^{m-2}\\ &\left.+\int_a^b\Theta'(s)G(t,s)f(s,z(s))ds\right]\\ ={}& {}^cD_{a^+}^{\beta,\Theta}\sum_{j=0}^{m-3}\frac{{z_a}^j}{j!}(\Theta(t)-\Theta(a))^j\ +\ {}^cD_{a^+}^{\beta,\Theta}I_a^{\beta,\Theta}f(t,z(t)) \end{aligned}$$

$$\begin{aligned} &+{}^cD_{a^+}^{\beta,\Theta}\left[\frac{1}{(m-2)!}{z_a}^{m-2}(\Theta(t)-\Theta(a))^{m-2}\right]+{}^cD_{a^+}^{\beta,\Theta}\left[\frac{\delta-1}{\Delta(m-1)!}{z_a}^{m-2}(\Theta(t)-\Theta(a))^{m-1}\right]\\ &+\left\{\int_a^\eta\Theta'(s)\left[(\Theta(\eta)-\Theta(s))^{\beta-m}-(\Theta(b)-\Theta(s))^{\beta-m}\right]f(s,z(s))ds\right.\\ &\left.-\int_\eta^b\Theta'(s)\left(\Theta(b)-\Theta(s)\right)^{\beta-m}f(s,z(s))ds\right\}\frac{{}^cD_{a^+}^{\beta,\Theta}(\Theta(t)-\Theta(a))^{m-1}}{\Gamma(\beta-m-1)\Delta(m-1)!} \end{aligned}$$

$=f(t,z(t))$.

Since $f(t,z(t))$ is continuous on $[a,b]$, then we have ${}^{c}D_{a^+}^{\beta;\Theta}(Tz(t)) \in C[a,b]$.

Let $z_1, z_2 \in P$, since

$$\begin{aligned}
|Tz_1(t) - Tz_2(t)| &= \left| \int_a^b \Theta'(s)G(t,s)f(s,z_1(s))ds - \int_a^b \Theta'(s)G(t,s)f(s,z_2(s))ds \right| \\
&= \left| \int_a^b \Theta'(s)G(t,s)(f(s,z_1(s)) - f(s,z_2(s))ds \right| \\
&\leq \int_a^b |\Theta'(s)|\,|G(t,s)|\,|(f(s,z_1(s)) - f(s,z_2(s))|\,ds \\
&\leq M \int_a^b \Theta'(s)ds\, k_0||z_1 - z_2||
\end{aligned}$$

for all $t \in [a,b]$, we get

$$||Tz_1 - Tz_2|| \leq Mk_0(\Theta(b) - \Theta(a))||z_1 - z_2||.$$

From the condition $Mk_0(\Theta(b)-\Theta(a)) < 1$, the operator T is a contraction mapping from P to P. By virtue of the Banach fixed point theorem, there exists a unique fixed point $z \in P$ such that $Tz(t) = z(t)$.
Therefore, z is the unique solution for the problem (1.1)-(1.2) on $[a,b]$. □

The next result relies on the Schaefer's fixed point theorem.

Theorem 3.2. *Assume that*

(U2) $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a positive constant k_1 such that

$$|f(t,z(t))| \leq k_1|z| \text{ for all } (t,z) \in [a,b] \times \mathbb{R}.$$

If

$$Mk_1(\Theta(b) - \Theta(a)) < 1 \tag{3.5}$$

then the boundary value problem (1.1)-(1.2) has at least one solution on $[a,b]$, where M is given in Corollary 3.1.

Proof. We consider the set P and the operator $T : P \to P$ defined by (3.3) and (3.4), respectively. The demonstration will unfold through a series of sequential steps.
Step 1. We show that T is a continuous operator. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in P such that $z_n \to z$ in P, as $n \to \infty$. Then by the equation (3.4) and for every $t \in [a,b]$, we obtain

$$\begin{aligned}
|Tz_n(t) - Tz(t)| &= \left| \int_a^b \Theta'(s)G(t,s)\left(f(s,z_n(s)) - f(s,z(s))\right)ds \right| \\
&\leq \int_a^b \Theta'(s)|G(t,s)||f(s,z_n(s)) - f(s,z(s))|ds \\
&\leq M\|f(s,z_n(s)) - f(s,z(s))\|(\Theta(b) - \Theta(a)).
\end{aligned}$$

Thus we get

$$||Tz_n - Tz|| \leq M(\Theta(b) - \Theta(a))||f(s, z_n(s)) - f(s, z(s))||.$$

Using the continuity of the function f, it follows that $||Tz_n - Tz|| \to 0$ as $n \to \infty$, which implies that T is a continuous operator.
Step 2. We see that T maps bounded sets into uniformly bounded sets in P. For this reason, we show that for all $r_1 > 0$ there exists some $r_2 > 0$ such that for all $z \in B_{r_1} := \{z \in P : ||z|| < r_1\}$, $||Tz|| \leq r_2$ is satisfied.
Indeed, let $z \in B_{r_1}$, for all $t \in [a, b]$, we have

$$\begin{aligned}
|Tz(t)| = &\Bigg| \sum_{j=0}^{m-3} \frac{{z_a}^j}{j!} (\Theta(t) - \Theta(a))^j \\
&+ \left[\frac{1}{(m-2)!} + \frac{(\Theta(t) - \Theta(a))(\delta - 1)}{\Delta(m-1)!} \right] {z_a}^{m-2} (\Theta(t) - \Theta(a))^{m-2} + \int_a^b \Theta'(s) G(t, s) f(s, z(s)) ds \Bigg| \\
\leq &\sum_{j=0}^{m-3} \frac{|{z_a}^j|}{j!} (\Theta(t) - \Theta(a))^j \\
&+ \left[\frac{1}{(m-2)!} + \frac{(\Theta(t) - \Theta(a))(1 - \delta)}{\Delta(m-1)!} \right] |{z_a}^{m-2}| (\Theta(t) - \Theta(a))^{m-2} + \int_a^b \Theta'(s) |G(t, s)||f(s, z(s))| ds \\
\leq &\sum_{j=0}^{m-3} \frac{|{z_a}^j|}{j!} (\Theta(b) - \Theta(a))^j \\
&+ \left[\frac{1}{(m-2)!} + \frac{(\Theta(b) - \Theta(a))(1 - \delta)}{\Delta(m-1)!} \right] |{z_a}^{m-2}| (\Theta(b) - \Theta(a))^{m-2} + k_1 r_1 M \int_a^b \Theta'(s) ds \\
= &\sum_{j=0}^{m-3} \frac{|{z_a}^j|}{j!} (\Theta(b) - \Theta(a))^j \\
&+ \left[\frac{1}{(m-2)!} + \frac{(\Theta(b) - \Theta(a))(1 - \delta)}{\Delta(m-1)!} \right] |{z_a}^{m-2}| (\Theta(b) - \Theta(a))^{m-2} + k_1 r_1 M (\Theta(b) - \Theta(a)).
\end{aligned}$$

Hence there exists an

$$\begin{aligned}
r_2 := &\sum_{j=0}^{m-3} \frac{|{z_a}^j|}{j!} (\Theta(b) - \Theta(a))^j + \left[\frac{\Delta(m-1) + (\Theta(b) - \Theta(a))(1 - \delta)}{\Delta(m-1)!} \right] (\Theta(b) - \Theta(a))^{m-2} |{z_a}^{m-2}| \\
&+ k_1 r_1 M (\Theta(b) - \Theta(a))
\end{aligned}$$

such that $\|Tz\| \leq r_2$. Thus $\{Tz\}$ is uniformly bounded set.

Step 3. We show that T maps bounded sets into equi-continuous sets of P. Let B_{r_1} be a bounded set of P as in Step 2 and $z \in B_{r_1}$.
Consequently for $t_1, t_2 \in [a, b]$ with $t_1 < t_2$, we have

$$\begin{aligned}
|Tz(t_2) - Tz(t_1)| = &\Bigg| \sum_{j=0}^{m-3} \frac{{z_a}^j}{j!} (\Theta(t_2) - \Theta(a))^j \\
&+ \left[\frac{1}{(m-2)!} + \frac{(\Theta(t_2) - \Theta(a))(\delta - 1)}{\Delta(m-1)!} \right] {z_a}^{m-2} (\Theta(t_2) - \Theta(a))^{m-2}
\end{aligned}$$

$$\begin{aligned}
&+\int_a^b \Theta'(s)G(t_2,s)f(s,z(s))ds-\sum_{j=0}^{m-3}\frac{{z_a}^j}{j!}(\Theta(t_1)-\Theta(a))^j\\
&-\left[\frac{1}{(m-2)!}+\frac{(\Theta(t_1)-\Theta(a))(\delta-1)}{\Delta(m-1)!}\right]{z_a}^{m-2}(\Theta(t_1)-\Theta(a))^{m-2}\\
&-\int_a^b \Theta'(s)G(t_1,s)f(s,z(s))ds\Big|\\
\le&\sum_{j=0}^{m-3}\frac{|{z_a}^j|}{j!}\left|(\Theta(t_2)-\Theta(a))^j-(\Theta(t_1)-\Theta(a))^j\right|\\
&+\frac{|{z_a}^{m-2}|}{(m-2)!}\left|(\Theta(t_2)-\Theta(a))^{m-2}-(\Theta(t_1)-\Theta(a))^{m-2}\right|\\
&+\frac{(1-\delta)|{z_a}^{m-2}|}{\Delta(m-1)!}\left|(\Theta(t_2)-\Theta(a))^{m-1}-(\Theta(t_1)-\Theta(a))^{m-1}\right|\\
&+\int_a^b \Theta'(s)|G(t_2,s)-G(t_1,s)||f(s,z(s)|ds\\
\le&\sum_{j=0}^{m-3}\frac{|{z_a}^j|}{j!}\left|(\Theta(t_2)-\Theta(a))^j-(\Theta(t_1)-\Theta(a))^j\right|\\
&+\frac{|{z_a}^{m-2}|}{(m-2)!}\left|(\Theta(t_2)-\Theta(a))^{m-2}-(\Theta(t_1)-\Theta(a))^{m-2}\right|\\
&+\frac{(1-\delta)|{z_a}^{m-2}|}{\Delta(m-1)!}\left|(\Theta(t_2)-\Theta(a))^{m-1}-(\Theta(t_1)-\Theta(a))^{m-1}\right|\\
&+k_1r_1\int_a^b \Theta'(s)|G(t_2,s)-G(t_1,s)|ds.
\end{aligned}$$

Using the continuity of the function $G(t,s)$, as $t_1\to t_2$, the right side of the above inequality tends to zero. Therefore, we can conclude that $T:P\to P$ is a completely continuous operator with the Arzela-Ascoli theorem.
Step 4. We see that the set

$$S=\{z\in P: z=\lambda Tz,\ \text{for some } \lambda\in(0,1)\}$$

is bounded.
Let $z\in S$ and $\lambda\in(0,1)$ be such that $z=\lambda Tz$. By Step 2, for all $t\in[a,b]$, we have

$$\begin{aligned}
|Tz(t)|\le&\sum_{j=0}^{m-3}\frac{|{z_a}^j|}{j!}(\Theta(b)-\Theta(a))^j+\left[\frac{\Delta(m-1)+(\Theta(b)-\Theta(a))(1-\delta)}{\Delta(m-1)!(\Theta(b)-\Theta(a))^{2-m}}\right]|{z_a}^{m-2}|\\
&+k_1\|z\|M(\Theta(b)-\Theta(a)),
\end{aligned}$$

by the inequality (3.5). Since $\lambda\in(0,1),\ z\le Tz$ and hence we get

$$\begin{aligned}
\|z\|\le\|Tz\|\le&\sum_{j=0}^{m-3}\frac{|{z_a}^j|}{j!}(\Theta(b)-\Theta(a))^j+\left[\frac{\Delta(m-1)+(\Theta(b)-\Theta(a))(1-\delta)}{\Delta(m-1)!(\Theta(b)-\Theta(a))^{2-m}}\right]|{z_a}^{m-2}|\\
&+k_1M\|z\|(\Theta(b)-\Theta(a))
\end{aligned}$$

and so

$$\|z\| \leq \frac{\displaystyle\sum_{j=0}^{m-3} \frac{|{z_a}^j|}{j!}(\Theta(b)-\Theta(a))^j + \left[\frac{\Delta(m-1)+(\Theta(b)-\Theta(a))(1-\delta)}{\Delta(m-1)!(\Theta(b)-\Theta(a))^{2-m}}\right]|{z_a}^{m-2}|}{1-k_1M(\Theta(b)-\Theta(a))}.$$

Thus, confirming the boundedness of S, we then employ Schaefer's fixed point theorem to establish the existence of at least one fixed point z of the operator T within the set P. This particular fixed point z serves as the sought-after solution for (1.1)-(1.2) over the interval $[a,b]$, thus culminating in the fulfillment of our proof. □

Theorem 3.3. *Assume that there exist nonnegative functions $e, g \in C[a,b]$ such that*

$$|f(t,z(t))| \leq e(t) + g(t)z(t), \ \forall t \in [a,b], \ z \in \mathbb{R}.$$

Then the problem (1.1)-(1.2) has at least one solution, provided that

$$l \geq \frac{\displaystyle\sum_{j=0}^{m-3} \frac{|{z_a}^j|}{j!}(\Theta(b)-\Theta(a))^j + \frac{\Delta(m-1)+(\Theta(b)-\Theta(a))(1-\delta)}{\Delta(m-1)!(\Theta(b)-\Theta(a))^{2-m}}|{z_a}^{m-2}| + M\|e\|(\Theta(b)-\Theta(a))}{1-M\|g\|(\Theta(b)-\Theta(a))}$$

for a constant $l > 0$.

Proof. Define the set $B_l = \{z \in P : \|z\| < l\}$. Clearly, B_l is a bounded closed convex subset in P. As $B_l \subset P$ and $T : P \to P$ is completely continuous operator, $T : B_l \to B_l$ compact. Next, we show that if $z \in B_l$, we have $Tz \in B_l$.
For any $z \in B_l$, we have $\|z\| < l$.
Hence, we get

$$\begin{aligned}
|Tz(t)| = &\left|\sum_{j=0}^{m-3} \frac{{z_a}^j}{j!}(\Theta(t)-\Theta(a))^j\right. \\
&\left. + \left[\frac{1}{(m-2)!} + \frac{(\Theta(t)-\Theta(a))(\delta-1)}{\Delta(m-1)!}\right]{z_a}^{m-2}(\Theta(t)-\Theta(a))^{m-2} + \int_a^b \Theta'(s)G(t,s)f(s,z(s))ds\right| \\
\leq &\sum_{j=0}^{m-3} \frac{|{z_a}^j|}{j!}(\Theta(t)-\Theta(a))^j + \left[\frac{1}{(m-2)!} + \frac{(\Theta(t)-\Theta(a))(1-\delta)}{\Delta(m-1)!}\right]|{z_a}^{m-2}|(\Theta(t)-\Theta(a))^{m-2} \\
&+ \int_a^b \Theta'(s)|G(t,s)|(e(s)+g(s)z(s))ds
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=0}^{m-3} \frac{|{z_a}^j|}{j!}(\Theta(b)-\Theta(a))^j + \left[\frac{1}{(m-2)!} + \frac{(\Theta(b)-\Theta(a))(1-\delta)}{\Delta(m-1)!}\right]|{z_a}^{m-2}|(\Theta(b)-\Theta(a))^{m-2} \\
&\quad + M(\|e\| + \|g\|\|z\|)(\Theta(b)-\Theta(a)) \\
&\leq \sum_{j=0}^{m-3} \frac{|{z_a}^j|}{j!}(\Theta(b)-\Theta(a))^j + \frac{\Delta(m-1)+(\Theta(b)-\Theta(a))(1-\delta)}{\Delta(m-1)!(\Theta(b)-\Theta(a))^{2-m}}|{z_a}^{m-2}|
\end{aligned}$$

$$+ M(\|e\| + \|g\|l)(\Theta(b) - \Theta(a))$$
$$\leq l.$$

In view of the Schauder's fixed point theorem, there exists a fixed point $z \in B_l$. Therefore the problem (1.1)-(1.2) has at least one solution $z \in P$ such that $\|z\| < l$, for all $t \in [a,b]$. □

In order the present the another existence result for our problem, let's define the lower and upper control functions.

Definition 3.1. Let $p, q \in \mathbb{R}^+$ such that $q > p$. Then for any $z \in [p,q] \subset \mathbb{R}^+$, we define the upper-control function $\bar{f}(t,z) = \sup_{p \leq \zeta \leq z} f(t,\zeta)$ and lower-control function $\underline{f}(t,z) = \inf_{z \leq \zeta \leq q} f(t,\zeta)$. It is clear that functions $\bar{f}(t,z)$ and $\underline{f}(t,z)$ are non-decreasing on $[p,q]$ and satisfies

$$\underline{f}(t,z) \leq f(t,z) \leq \bar{f}(t,z).$$

Now, let we define the lower and upper solutions for the problem (1.1)-(1.2).

Definition 3.2. Let $u, v \in P$ such that $p \leq u \leq v \leq q$ and satisfy

$$\begin{aligned} &{}^cD^{\beta,\Theta}_{a^+} v(t) \geq \bar{f}(t, v(t)),\ t \in [a,b] \\ &v^{[j]}_{\Theta}(a) \geq v^j_a,\ j = 0,1,...,m-2\,;\quad v^{[m-2]}_{\Theta}(b) \geq \delta v^{[m-2]}_{\Theta}(\eta) \end{aligned}$$

or

$$\begin{aligned} v(t) \geq\ & \sum_{j=0}^{m-3} \frac{v^j_a}{j!}(\Theta(t)-\Theta(a))^j + \left[\frac{1}{(m-2)!} + \frac{(\delta-1)(\Theta(t)-\Theta(a))}{(m-1)!\Delta}\right] v^{m-2}_a (\Theta(t)-\Theta(a))^{m-2} \\ &+ \int_a^b \Theta'(s) G(t,s) f(s, v(s)) ds, \end{aligned}$$

and

$$\begin{aligned} &{}^cD^{\beta,\Theta}_{a^+} u(t) \leq \underline{f}(t, u(t)),\ t \in [a,b] \\ &u^{[j]}_{\Theta}(a) \leq u^j_a,\ j = 0,1,...,m-2\,;\quad u^{[m-2]}_{\Theta}(b) \leq \delta u^{[m-2]}_{\Theta}(\eta) \end{aligned}$$

or

$$\begin{aligned} u(t) \leq\ & \sum_{j=0}^{m-3} \frac{u^j_a}{j!}(\Theta(t)-\Theta(a))^j + \left[\frac{1}{(m-2)!} + \frac{(\delta-1)(\Theta(t)-\Theta(a))}{(m-1)!\Delta}\right] u^{m-2}_a (\Theta(t)-\Theta(a))^{m-2} \\ &+ \int_a^b \Theta'(s) G(t,s) f(s, u(s)) ds,\ t \in [a,b]. \end{aligned}$$

Then the functions $u(t)$ and $v(t)$ are called a pair lower and upper solution for the problem (1.1)-(1.2).

Theorem 3.4. *Assume that $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous function and u, v are a pair of lower and upper solution of (1.1)-(1.2), respectively, then the problem (1.1)-(1.2) has at least one solution such that*

$$u(t) \leq z(t) \leq v(t), \quad t \in [a,b].$$

Proof. Define the set K as follows

$$K = \{z \in P : u(t) \leq z(t) \leq v(t), \quad t \in [a,b]\}.$$

It is clear that K is a convex, bounded and closed subset of the Banach space P endowed with the max norm $\|u\| = \max_{t\in[a,b]} |u(t)|$. As $K \subset P$ and $T : P \to P$ is completely continuous operator, $T : K \to K$ is compact. Next, we show that if $z \in K$, we have $Tz \in K$.
For any $z \in K$, we have $u \leq z \leq v$. Hence we have

$$\begin{aligned}
(Tz)(t) &= \sum_{j=0}^{m-3} \frac{{z_a}^j}{j!}(\Theta(t)-\Theta(a))^j + \left[\frac{1}{(m-2)!} + \frac{(\delta-1)(\Theta(t)-\Theta(a))}{(m-1)!\Delta}\right] z_a^{m-2}(\Theta(t)-\Theta(a))^{m-2} \\
&+ \int_a^b \Theta'(s)G(t,s)f(s,z(s))ds \\
&\leq \sum_{j=0}^{m-3} \frac{{v_a}^j}{j!}(\Theta(t)-\Theta(a))^j + \left[\frac{1}{(m-2)!} + \frac{(\delta-1)(\Theta(t)-\Theta(a))}{(m-1)!\Delta}\right] v_a^{m-2}(\Theta(t)-\Theta(a))^{m-2} \\
&+ \int_a^b \Theta'(s)G(t,s)\bar{f}(s,v(s))ds \\
&\leq v(t)
\end{aligned}$$

and

$$\begin{aligned}
(Tz)(t) &= \sum_{j=0}^{m-3} \frac{{z_a}^j}{j!}(\Theta(t)-\Theta(a))^j + \left[\frac{1}{(m-2)!} + \frac{(\delta-1)(\Theta(t)-\Theta(a))}{(m-1)!\Delta}\right] z_a^{m-2}(\Theta(t)-\Theta(a))^{m-2} \\
&+ \int_a^b \Theta'(s)G(t,s)f(s,z(s))ds \\
&\geq \sum_{j=0}^{m-3} \frac{{u_a}^j}{j!}(\Theta(t)-\Theta(a))^j + \left[\frac{1}{(m-2)!} + \frac{(\delta-1)(\Theta(t)-\Theta(a))}{(m-1)!\Delta}\right] u_a^{m-2}(\Theta(t)-\Theta(a))^{m-2} \\
&+ \int_a^b \Theta'(s)G(t,s)\underline{f}(s,u(s))ds \\
&\geq u(t).
\end{aligned}$$

It follows that,

$$u(t) \leq Tz(t) \leq v(t), \quad t \in [a,b],$$

which implies $Tz \in K$, that proves $T : K \to K$ is compact. By the means of fixed point theorem of Schauder, T has a fixed point in K. Hence the problem (1.1)-(1.2) has at least one solution $z(t)$ in $C[a,b]$.

$\square$

In what follows, we present an example which illustrates one of our results.

Example 3.1. Consider the fractional boundary value problem

$$D_{0^+}^{\frac{7}{2},\Theta} z(t) = \frac{z(t)}{10^2(1+z^2(t))}, \quad t \in [0,2] \tag{3.6}$$

$$z_\Theta^{[0]}(0) = z_0^0,\ z_\Theta^{[1]}(0) = z_0^1,\ z_\Theta^{[2]}(0) = z_0^2,\ z_\Theta^{[2]}(2) = \frac{1}{4} z_\Theta^{[2]}(\frac{1}{2}) \tag{3.7}$$

with $\Theta(t) = \ln(1+t^2)$. Here, $\beta = \frac{7}{2}$, $f(t,z(t)) = \frac{z(t)}{10^2(1+z^2(t))}$ which is continuous on $[0,2]\times\mathbb{R}$, $\delta = \frac{1}{4}$ and $\eta = \frac{1}{2}$. Since $\beta = \frac{7}{2}$, we get $m = 4$. Also it is easy to see that $\Theta'(t) = \frac{2t}{1+t^2} \neq 0$ for all $t \in [0,2]$. We can easily evaluate that

$$\Delta = \ln 5 - \frac{1}{4}\ln\frac{5}{4} = \ln 5 - \frac{1}{4}\ln\frac{5}{4} = \frac{1}{4}\ln(5^3 4)$$

$$\begin{aligned}
|G(t,s)| &\leq (\ln 5)^{\frac{5}{2}}\left[\frac{\ln 5}{3!\frac{1}{4}\ln(5^3 4)\Gamma(\frac{7}{2}-2)} + \frac{1}{\Gamma(\frac{7}{2})}\right] \\
&= (1,6)^{\frac{5}{2}}\left(\frac{4}{3\sqrt{\pi}}\frac{1,6}{6,4} + \frac{8}{15\sqrt{\pi}}\right) \\
&\cong 3,23\,(0,1+0,3) = 1,3 = M
\end{aligned}$$

and

$$|f(t,z)| = \left|\frac{z(t)}{10^2(1+z^2(t))}\right| \leq \frac{1}{10^2}|z(t)|,$$

so $k_1 = \frac{1}{10^2}$. Since $\frac{M\ln 5}{10^2} < 1$, then the problem (3.6)-(3.7) has at least one solution on $[0,2]$ by Theorem 3.2.

4. Conclusion

In this paper, we have delved into the investigation of the existence of solutions for a Θ-Caputo fractional boundary value problem, a class of equations that incorporates the Θ-Caputo fractional derivative. Our exploration into this realm has shed light on the complexities of fractional differential equations and their applications. So we considered a fractional differential equation involving the generalized $\Theta-$Caputo fractional derivative with three-point boundary condition. First of all, we expressed the Green's function of the boundary value problem and summarized the necessary properties of the Green's function for existence theorems. Next, the proofs of the existence theorems are based on applications of the Banach, Schaefer's and Schauder's fixed point theorems. Furthermore, by using lower and upper solution method an existence theorem was proved. At the end of this paper an example is also given. This problem is new and not yet studied. Compared with [1, 14] the boundary conditions of our problem (1.1)-(1.2) includes three points.

Moving forward, avenues for future exploration could involve delving deeper into the application of the $\Theta-$Caputo fractional derivative in various contexts, as well as extending our analysis to more intricate boundary conditions or generalized equations. Additionally, investigating the numerical methods and stability analysis for solving the derived equations could provide valuable insights for both theoretical and practical applications.

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