Global smooth solution for the modified anisotropic 3D Boussinesq equations with damping *

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Abstract This paper is mainly concerned with the modified anisotropic three-dimensional Boussinesq equations with damping. We first prove the existence and uniqueness of global solution of velocity anisotropic equations. Then we establish the well-posedness of global solution of temperature anisotropic equations.

Key words Boussinesq equations; anisotropic viscosity; damping

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1 Introduction

In this paper, we investigate the following modified velocity anisotropic three-dimensional Boussinesq equations with damping:

$$\begin{cases} \partial_t u - \Delta_h u + (u\cdot\nabla)u + |u|^{\beta-1}u + \nabla p = \theta e_3, \\ \partial_t \theta - \Delta\theta + (u\cdot\nabla)\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

and the temperature anisotropic three-dimensional Boussinesq equations with damping:

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + |u|^{\beta-1}u + \nabla p = \theta e_3, \\ \partial_t \theta - \Delta_h\theta + (u\cdot\nabla)\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.2}$$

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where $e_3 = (0,0,1)^T$, $t \geq 0$, $x \in \mathbb{R}^3$, u is the velocity fluid, θ is the temperature, p is the pressure, $\beta \geq 1$ is real parameter, $\Delta_h := \partial_1^2 + \partial_2^2$ and ∂_i is the partial derivative in the direction x_i.

Recently, the anisotropic Navier-Stokes equations were investigated in [1, 7, 19, 20, 21, 22, 23, 27]. In [7], Chemin and Zhang proved the local-in-time well-posedness in the anisotropic Sobolev space $H^{0,\frac{1}{2}+\varepsilon}$ for some $\varepsilon > 0$. Meanwhile, if the initial data was sufficiently small, global well-posedness was obtained. In [23], Paicu and Zhang proved the well-posedness for the three dimensional anisotropic Navier-Stokes equations in an appropriate anisotropic Sobolev space.

By using the Friedrichs method, the existence and uniqueness of global-in-time weak and strong solutions of the two-dimensional Boussinesq equations with horizontal viscosity only appearing in one equation were studied in [8]. In [6], Cao and Wu established the global-in-time existence of classical solutions to the 2D anisotropic Boussinesq equations with only vertical dissipation. They proved that the pressure was obtained by separating it into high frequency and low frequency modes via Littlewood-Paley decomposition. The global well-posedness and regularity of solutions of the two dimensional Boussinesq system with anisotropic viscosity and without heat diffusion were established in [11]. Stability and exponential decay for the two-dimensional Boussinesq equations with only horizontal dissipation and horizontal thermal diffusion in the spatial domain $\mathbb{T} \times \mathbb{R}$ were investigated in [10]. Stability and optimal decay for a system of three dimensional Boussinesq modeling anisotropic buoyancy-driven fluids were proved in [25]. By the virtue of damping term, we will prove the well-posedness of system (1.1) and (1.2).

In [5], Cao and Wu proved the global regularity for two-dimensional incompressible magnetohydrodynamic equations without dissipation and magnetic diffusion. In [4], global regularity of classical solutions to the two dimensional incompressible magnetohydrodynamic equations with horizontal dissipation and horizontal magnetic diffusion were studied. By means of anisotropic Littlewood-Paley analysis, Yue and Zhong proved the global well-posedness of the three dimensional incompressible anisotropic magnetohydrodynamics equations in the anisotropic Sobolev spaces of type $H^{0,s_0}(\mathbb{R}^3)$ with $s_0 \geq \frac{1}{2}$ in [26]. The global existence and regularity for a system of the two-dimensional magnetohydrodynamic equations with only directional hyper-resistivity were established in [9].

The Navier-Stokes equations and related models with damping were investigated in [3, 12, 14, 15, 16, 17, 24]. In [2], Bessaih, Trabelsi and Zorgati first introduced the anisotropic Navier-Stokes equations with damping term and proved the existence and uniqueness of global solutions for the modified anisotropic three-dimensional Navier-Stokes equations. In [24], Titi and Trabelsi proved the global well-posedness of solutions to a three-dimensional magnetohydrodynamical model in porous media for $\beta \geq 4$. In [18], the global well-posedness of the three dimensional micropolar equations with partial viscosity and damping was proved for $\beta \geq 4$.

In this paper, our main purpose is to establish the well-posedness for the modified anisotropic $3D$ Boussinesq equations with damping. The main difficulty lies in dealing with the anisotropy estimation. We first prove the existence and uniqueness of global solution of system (1.1) for $\beta \geq 4$ with $u_0 \in H^{0,1}(\mathbb{R}^3)$ and $u_0 \in H^1(\mathbb{R}^3)$, respectively.

Then we get the existence and uniqueness of global solution of system (1.2) for $\beta > 3$. Finally, we prove the unique global smooth solution of system (1.2) for $s \geq 3$.

The outline of the paper is as follows. In section 2, we give some necessary notions and main results. We will prove Theorem 2.1 and Theorem 2.2 in section 3. Then, based on the results in previous sections, the proofs of Theorem 2.3 and Theorem 2.4 are given in section 4.

2 Preliminaries

In this section, we introduce some useful notations and definitions. Denote $x = (x_1, x_2, x_3)$, where $x_h := (x_1, x_2)$ is the horizontal variable and $x_v := x_3$ is the vertical variable. Referring to the Chapter VI in [1, 13], we define the anisotropic Sobolev spaces as follows. For any $s, s' \in \mathbb{R}$, assume that $H^{s,s'}$ is the set of tempered distributions $\psi \in \mathcal{S}'(\mathbb{R}^3)$ such that

$$\|\psi\|_{s,s'}^2 := \int_{\mathbb{R}^3} (1 + |\xi_h|^2)^s (1 + |\xi_3|)^{s'} |\hat{\psi}(\xi)|^2 d\xi < \infty.$$

The space $H^{s,s'}$ endowed with the norm $\|\cdot\|_{s,s'}$ is a Hilbert space. For exponents $p, q \in [1, \infty)$, $L_h^p(L_v^q)$ denotes the space $L^p(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}, L^q(\mathbb{R}_{x_3}))$ which is endowed with the norm

$$\|u\|_{L_h^p(L_v^q)} := \{\int_{\mathbb{R}^2} (\int_{\mathbb{R}} |u(x_h, x_3)|^q dx_3)^{\frac{p}{q}} dx_h\}^{\frac{1}{p}}.$$

The space $L_v^q(L_h^p)$ can be defined similarly. Let $\|\cdot\|_{L^p}$ be the $L^p(\mathbb{R}^3)$ norm for $p \geq 1$. For $s \in \mathbb{R}$, let $H^s := W^{s,2}$ be the usual Sobolev space endowed with the norm

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi.$$

Now we present the main results of this paper.

Theorem 2.1. *Let $\beta \geq 4$, $u_0 \in H^{0,1}(\mathbb{R}^3)$ and $\theta_0 \in H^1(\mathbb{R}^3)$ such that* $\mathrm{div} u_0 = 0$. *The system (1.1) has a unique global solution $(u(t), \theta(t))$ satisfying*

$$u(t) \in L^\infty_{loc}(\mathbb{R}^+; H^{0,1}(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}^+; H^{1,1}(\mathbb{R}^3)) \cap L^{\beta+1}_{loc}(\mathbb{R}^+; L^{\beta+1}(\mathbb{R}^3)),$$
$$\theta(t) \in L^\infty_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}^+; H^2(\mathbb{R}^3)).$$

Theorem 2.2. *Let $\beta \geq 4$, $u_0 \in H^1(\mathbb{R}^3)$ and $\theta_0 \in H^1(\mathbb{R}^3)$ such that* $\mathrm{div} u_0 = 0$. *The system (1.1) has a unique global solution $(u(t), \theta(t))$ satisfying*

$$u(t) \in L^\infty_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^3)) \cap L^{\beta+1}_{loc}(\mathbb{R}^+; L^{\beta+1}(\mathbb{R}^3)), \quad \nabla_h u \in L^2_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^3)),$$
$$\theta(t) \in L^\infty_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}^+; H^2(\mathbb{R}^3)),$$
$$\partial_t u(t) \in L^2_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^3)), \quad \partial_t \theta(t) \in L^2_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^3)).$$

Theorem 2.3. *Let $\beta > 3$, $u_0 \in H^1(\mathbb{R}^3)$ and $\theta_0 \in H^{0,1}(\mathbb{R}^3)$ such that* $\mathrm{div} u_0 = 0$. *The system (1.2) has a unique global solution $(u(t), \theta(t))$ satisfying*

$$u(t) \in L^\infty_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}^+; H^2(\mathbb{R}^3)) \cap L^{\beta+1}_{loc}(\mathbb{R}^+; L^{\beta+1}(\mathbb{R}^3)),$$
$$\theta(t) \in L^\infty_{loc}(\mathbb{R}^+; H^{0,1}(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}^+; H^{1,1}(\mathbb{R}^3)).$$

Theorem 2.4. *Let $\beta > 3$, $s \geq 3$, $u_0 \in H^s(\mathbb{R}^3)$ and $\theta_0 \in H^s(\mathbb{R}^3)$ such that* $\mathrm{div} u_0 = 0$. *The system (1.2) has a unique global smooth solution $(u(t), \theta(t))$ satisfying*

$$u(t) \in L^\infty_{loc}(\mathbb{R}^+; H^s(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}^+; H^{s+1}(\mathbb{R}^3)),$$
$$\theta(t) \in L^\infty_{loc}(\mathbb{R}^+; H^s(\mathbb{R}^3)), \quad \nabla_h \theta(t) \in L^2_{loc}(\mathbb{R}^+; H^s(\mathbb{R}^3)).$$

3 Existence and uniqueness of global solution for the velocity anisotropic system

This section concerns the existence and uniqueness of global solution of system (1.1) for $\beta \geq 4$. We will prove Theorem 2.1 and Theorem 2.2 with different smooth conditions of initial values.

3.1 Proof of Theorem 2.1

We first consider the case that the initial value $u_0 \in H^{0,1}(\mathbb{R}^3)$ and $\theta_0 \in H^1(\mathbb{R}^3)$. To prove Theorem 2.1, we firstly need to give some priori estimates in the following.
Taking the L^2 inner product of the second equation of (1.1) with θ, we get

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2} + \|\nabla\theta\|^2_{L^2} = 0.$$

Integrating over $[0, t]$, it yields that

$$\|\theta(t)\|^2_{L^2} + 2\int_0^t \|\nabla\theta\|^2_{L^2} ds = \|\theta_0\|^2_{L^2}. \tag{3.1}$$

Taking the L^2 inner product of the first equation of (1.1) with u, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \|\nabla_h u\|^2_{L^2} + \|u\|^{\beta+1}_{L^{\beta+1}} &= \int_{\mathbb{R}^3} \theta e_3 u dx \leq \|u\|_{L^2}\|\theta\|_{L^2} \\ &\leq \|u\|^2_{L^2} + \|\theta\|^2_{L^2} \leq \|u\|^2_{L^2} + \|\theta_0\|^2_{L^2}.\end{aligned}$$

Applying Gronwall inequality, we get

$$\|u(t)\|^2_{L^2} + \int_0^t (\|\nabla_h u\|^2_{L^2} + \|u\|^{\beta+1}_{L^{\beta+1}}) ds \leq C(t, u_0, \theta_0). \tag{3.2}$$

Taking the L^2 inner product of the second equation of (1.1) with $-\Delta\theta$, it yields that, for $\beta \geq 4$,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2_{L^2} + \|\Delta\theta\|^2_{L^2} = \int_{\mathbb{R}^3} (u \cdot \nabla\theta)\Delta\theta dx$$

$$
\begin{aligned}
&\le \|u\|_{L^{\beta+1}}\|\nabla\theta\|_{\frac{2(\beta+1)}{\beta-1}}\|\Delta\theta\|_{L^2}\\
&\le C\|u\|_{L^{\beta+1}}\|\nabla\theta\|_{L^2}^{\frac{\beta-2}{\beta+1}}\|\Delta\theta\|_{L^2}^{\frac{\beta+4}{\beta+1}}\\
&\le \frac{1}{2}\|\Delta\theta\|_{L^2}^2 + C\|u\|_{L^{\beta+1}}^{\frac{2(\beta+1)}{\beta-2}}\|\nabla\theta\|_{L^2}^2\\
&\le \frac{1}{2}\|\Delta\theta\|_{L^2}^2 + C(1+\|u\|_{L^{\beta+1}}^{\beta+1})\|\nabla\theta\|_{L^2}^2. \qquad (3.3)
\end{aligned}
$$

Consequently

$$
\frac{d}{dt}\|\nabla\theta\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 \le C(1+\|u\|_{L^{\beta+1}}^{\beta+1})\|\nabla\theta\|_{L^2}^2. \qquad (3.4)
$$

By Gronwall inequality again, it is easy to get that

$$
\|\nabla\theta(t)\|_{L^2}^2 + \int_0^t \|\Delta\theta\|_{L^2}^2 ds \le C(t, u_0, \theta_0). \qquad (3.5)
$$

Taking the L^2 inner product of the first equation of (1.1) with $-\partial_3^2 u$ and the integration by parts, it yields that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_3 u\|_{L^2}^2 + \|\nabla_h\partial_3 u\|_{L^2}^2 + \||u|^{\frac{\beta-1}{2}}\partial_3 u\|_{L^2}^2 + \frac{4(\beta-1)}{(\beta+1)^2}\|\partial_3|u|^{\frac{\beta-1}{2}}\|_{L^2}^2&\\
= \int_{\mathbb{R}^3}(u\cdot\nabla)u\partial_3^2 u dx - \int_{\mathbb{R}^3}\theta e_3\partial_3^2 u dx&\\
:= I_1(t) + I_2(t).& \qquad (3.6)
\end{aligned}
$$

For $I_1(t)$, by integration by parts, we have for $\beta > 3$

$$
\begin{aligned}
I_1(t) &= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\partial_3 u_i\partial_i u_j\partial_3 u_j dx - \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i\partial_i\partial_3 u_j\partial_3 u_j dx\\
&= -\sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_3 u_i\partial_i u_j\partial_3 u_j dx - \sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_3 u_3\partial_3 u_j\partial_3 u_j dx\\
&= -\sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_3 u_i\partial_i u_j\partial_3 u_j dx + \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_i u_i\partial_3 u_j\partial_3 u_j dx\\
&= \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} u_j\partial_3 u_j\partial_i\partial_3 u_i dx + \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} u_j\partial_3 u_i\partial_i\partial_3 u_j dx\\
&\quad - 2\sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} u_i\partial_3 u_j\partial_i\partial_3 u_j dx\\
&\le \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}|u_j||\partial_3 u_j||\partial_i\partial_3 u_i| dx + \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}|u_j||\partial_3 u_i||\partial_i\partial_3 u_j| dx
\end{aligned}
$$

$$
\begin{aligned}
&+2\sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}|u_i||\partial_3 u_j||\partial_i\partial_3 u_j|dx\\
&\le 4\||u||\partial_3 u|^{\frac{2}{\beta-1}}\|_{L^{\beta-1}}\||\partial_3 u|^{\frac{\beta-3}{\beta-1}}\|_{L^{\frac{2(\beta-1)}{\beta-3}}}\|\nabla_h\partial_3 u\|_{L^2}\\
&\le C\||u|^{\frac{\beta-1}{2}}\partial_3 u\|_{L^2}^{\frac{2}{\beta-1}}\|\partial_3 u\|_{L^2}^{\frac{\beta-3}{\beta-1}}\|\nabla_h\partial_3 u\|_{L^2}\\
&\le \frac{1}{2}\|\nabla_h\partial_3 u\|_{L^2}^2+\frac{1}{2}\||u|^{\frac{2}{\beta-1}}\partial_3 u\|_{L^2}^2+C\|\partial_3 u\|_{L^2}^2. \qquad (3.7)
\end{aligned}
$$

For $I_2(t)$, applying the Hölder inequality and Young inequality, we have

$$
I_2(t)\le\|\partial_3\theta\|_{L^2}\|\partial_3 u\|_{L^2}\le\|\partial_3\theta\|_{L^2}^2+\|\partial_3 u\|_{L^2}^2\le\|\nabla\theta\|_{L^2}^2+\|\partial_3 u\|_{L^2}^2. \qquad (3.8)
$$

Inserting the estimates of (3.7) and (3.8) into (3.6), we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_3 u\|_{L^2}^2+\frac{1}{2}\|\nabla_h\partial_3 u\|_{L^2}^2+\frac{1}{2}\||u|^{\frac{\beta-1}{2}}\partial_3 u\|_{L^2}^2+\frac{4(\beta-1)}{(\beta+1)^2}\|\partial_3|u|^{\frac{\beta-1}{2}}\|_{L^2}^2&\\
\le (C+1)\|\partial_3 u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2.& \qquad (3.9)
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
\frac{d}{dt}\|\partial_3 u\|_{L^2}^2+\|\nabla_h\partial_3 u\|_{L^2}^2+\||u|^{\frac{\beta-1}{2}}\partial_3 u\|_{L^2}^2+\|\partial_3|u|^{\frac{\beta-1}{2}}\|_{L^2}^2&\\
\le C(\|\partial_3 u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2).& \qquad (3.10)
\end{aligned}
$$

By Gronwall inequality, we have

$$
\|\partial_3 u\|_{L^2}^2+\int_0^t(\|\nabla_h\partial_3 u\|_{L^2}^2+\||u|^{\frac{\beta-1}{2}}\partial_3 u\|_{L^2}^2+\|\partial_3|u|^{\frac{\beta-1}{2}}\|_{L^2}^2)ds\le C(t,u_0,\theta_0). \qquad (3.11)
$$

Next, we will prove the uniqueness of strong solutions of (1.1). Let $(v(t),\theta_1(t))$ and $(w(t),\theta_2(t))$ be two solutions of system (1.1) with the same initial data. Setting $(u,p,\theta)=(v-w,p_1-p_2,\theta_1-\theta_2)$, we get the following form:

$$
\begin{cases}
\partial_t u+(v\cdot\nabla)u+(u\cdot\nabla)w-\Delta_h u+|v|^{\beta-1}v-|w|^{\beta-1}w+\nabla p=\theta e_3,\\
\partial_t\theta-\Delta\theta+(v\cdot\nabla)\theta+(u\cdot\nabla)\theta_2=0,\\
\nabla\cdot u=0.
\end{cases} \qquad (3.12)
$$

Multiplying both sides of the first equation of (3.12) by u and the second equation of (3.12) by θ, respectively, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2+\|\theta\|_{L^2}^2)+\|\nabla_h u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\int_{\mathbb{R}^3}(|v|^{\beta-1}v-|w|^{\beta-1}w)udx&\\
=-\int_{\mathbb{R}^3}(u\cdot\nabla)wudx-\int_{\mathbb{R}^3}(u\cdot\nabla)\theta_2\theta dx+\int_{\mathbb{R}^3}\theta e_3 udx&\\
:=J_1(t)+J_2(t)+J_3(t).& \qquad (3.13)
\end{aligned}
$$

Inspired by [3, 14, 15, 16], it yields that

$$\int_{\mathbb{R}^3} (|v|^{\beta-1}v - |w|^{\beta-1}w)udx \geq 0. \tag{3.14}$$

By Sobolev embedding $\dot{H}_h^{\frac{1}{2}} \hookrightarrow L_h^4$, we get for all $u \in L_v^2 \cap \dot{H}_h^1$

$$\begin{aligned} \|u\|_{L_v^2(L_h^4)}^2 &\leq C\int_{\mathbb{R}} \|u\|_{L_h^2}\|\nabla_h u\|_{L_h^2}dx_3 \\ &\leq C\|u\|_{L^2}\|\nabla_h u\|_{L^2}. \end{aligned}$$

Bearing in mind that $\partial_3 u_3 = -\mathrm{div}_h u_h$, we get

$$\|\partial_3 u_3\|_{L^2} \leq C\|\nabla_h u\|_{L^2}.$$

For $J_1(t)$, by Hölder inequality and Gagliardo-Nirenberg inequality, $J_1(t)$ can be estimated as follows

$$\begin{aligned} J_1(t) &= -\sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} u_i\partial_i w_j u_j dx - \sum_{j=1}^{3}\int_{\mathbb{R}^3} u_3\partial_3 w_j u_j dx \\ &\leq \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}} \|u_i\|_{L_h^4}\|\partial_i w_j\|_{L_h^2}\|u_j\|_{L_h^4}dx_3 + \sum_{j=1}^{3}\int_{\mathbb{R}} \|u_3\|_{L_h^2}\|\partial_3 w_j\|_{L_h^4}\|u_j\|_{L_h^4}dx_3 \\ &\leq C\|\nabla_h w\|_{L_v^\infty(L_h^2)}\|u\|_{L_v^2(L_h^4)}^2 + C\|u_3\|_{L_v^\infty(L_h^2)}\|\partial_3 w\|_{L_v^2(L_h^4)}\|u\|_{L_v^2(L_h^4)} \\ &\leq C\|\nabla_h w\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3 w\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}\|\nabla_h u\|_{L^2} \\ &+ C\|u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3 w\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3 w\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{1}{2}} \\ &\leq C\|\nabla_h w\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3 w\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}\|\nabla_h u\|_{L^2} \\ &+ C\|u\|_{L^2}\|\nabla_h u\|_{L^2}\|\partial_3 w\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3 w\|_{L^2}^{\frac{1}{2}} \\ &\leq \frac{1}{2}\|\nabla_h u\|_{L^2}^2 + C(\|\nabla_h w\|_{L^2}^2 + \|\partial_3 w\|_{L^2}^2 + \|\nabla_h\partial_3 w\|_{L^2}^2)\|u\|_{L^2}^2. \end{aligned} \tag{3.15}$$

On the other hand, by Hölder inequality and Gagliardo-Nirenberg inequality again, together with Young inequality, $J_2(t)$ can be estimated by

$$\begin{aligned} J_2(t) &\leq \|u\|_{L^2}\|\theta\|_{L^6}\|\nabla\theta_2\|_{L^3} \\ &\leq C\|u\|_{L^2}\|\nabla\theta\|_{L^2}\|\nabla\theta_2\|_{L^2}^{\frac{1}{2}}\|\Delta\theta_2\|_{L^2}^{\frac{1}{2}} \\ &\leq \frac{1}{2}\|\nabla\theta\|_{L^2}^2 + C\|\nabla\theta_2\|_{L^2}\|\Delta\theta_2\|_{L^2}\|u\|_{L^2}^2 \\ &\leq \frac{1}{2}\|\nabla\theta\|_{L^2}^2 + C(\|\nabla\theta_2\|_{L^2}^2 + \|\Delta\theta_2\|_{L^2}^2)\|u\|_{L^2}^2. \end{aligned} \tag{3.16}$$

For $J_3(t)$, by Hölder inequality and Young inequality, we get

$$J_3(t) \leq \|u\|_{L^2}\|\theta\|_{L^2} \leq \|u\|_{L^2}^2 + \|\theta\|_{L^2}^2. \tag{3.17}$$

Putting all the results (3.14)-(3.17) into (3.13), it yields that

$$\frac{d}{dt}(\|u\|_{L^2}^2 + \|\theta\|_{L^2}^2) + \|\nabla_h u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 \le C(1 + \|\nabla_h w\|_{L^2}^2 + \|\partial_3 w\|_{L^2}^2 + \|\nabla_h \partial_3 w\|_{L^2}^2 + \|\nabla\theta_2\|_{L^2}^2 + \|\Delta\theta_2\|_{L^2}^2)(\|u\|_{L^2}^2 + \|\theta\|_{L^2}^2). \tag{3.18}$$

By Gronwall inequality, it is easy to get

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 \le C(\|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2)e^{\int_0^t(1+\|\nabla_h w\|_{L^2}^2+\|\partial_3 w\|_{L^2}^2+\|\nabla_h\partial_3 w\|_{L^2}^2+\|\nabla\theta_2\|_{L^2}^2+\|\Delta\theta_2\|_{L^2}^2)ds}. \tag{3.19}$$

Then, by (3.2), (3.5) and (3.11), the uniqueness of the solution is proved, and then the proof of Theorem 2.1 is completed.

3.2 Proof of Theorem 2.2

In this subsection, we get a higher regularity about the solution of system (1.1) with a more smooth initial value.

Step 1. Taking the L^2 inner product of the first equation of (1.1) with $-\Delta u$ and integration by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\nabla\nabla_h u\|_{L^2}^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \frac{4(\beta-1)}{(\beta+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2 &= \int_{\mathbb{R}^3}(u\cdot\nabla)u\Delta u dx - \int_{\mathbb{R}^3}\theta e_3\Delta u dx \\ &:= I_3(t) + I_4(t).\end{aligned} \tag{3.20}$$

For $I_3(t)$, integration by parts, we have

$$\begin{aligned}I_3(t) &= -\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u\nabla u\partial_k u dx \\ &= -\sum_{k=1}^{3}\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_i u\partial_k u dx \\ &= \sum_{k=1}^{3}\sum_{i=1}^{3}\int_{\mathbb{R}^3}u\partial_k u_i\partial_k\partial_i u dx \\ &= \sum_{i=1}^{2}\sum_{k=1}^{3}\int_{\mathbb{R}^3}u\partial_k u_i\partial_k\partial_i u dx + \sum_{k=1}^{2}\int_{\mathbb{R}^3}u\partial_k u_3\partial_k\partial_3 u dx + \int_{\mathbb{R}^3}u\partial_3 u_3\partial_3\partial_3 u dx \\ &:= I_{31}(t) + I_{32}(t) + I_{33}(t).\end{aligned} \tag{3.21}$$

For $I_{31}(t)$, by Hölder inequality and Young inequality, for $\beta > 3$, $I_{31}(t)$ can be estimated by

$$I_{31}(t) \le \int_{\mathbb{R}^3}|u||\nabla u||\nabla\nabla_h u|dx$$

$$
\begin{aligned}
&\le \||u||\nabla u|^{\frac{2}{\beta-1}}\|_{L^{\beta-1}}\||\nabla u|^{\frac{\beta-3}{\beta-1}}\|_{L^{\frac{2(\beta-1)}{\beta-3}}}\|\nabla\nabla_h u\|_{L^2}\\
&\le \frac{1}{8}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\frac{1}{8}\|\nabla\nabla_h u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2. \qquad (3.22)
\end{aligned}
$$

For $I_{32}(t)$, by Hölder inequality, Gagliardo-Nirenberg inequality and Young inequality, for $\beta>3$, $I_{32}(t)$ can be estimated by

$$
\begin{aligned}
I_{32}(t) &\le \int_{\mathbb{R}^3}|u||\nabla_h u|\nabla_h\partial_3 u|dx\\
&\le \int_{\mathbb{R}^3}|u||\nabla u||\nabla\nabla_h u|dx\\
&\le \||u||\nabla u|^{\frac{2}{\beta-1}}\|_{L^{\beta-1}}\||\nabla u|^{\frac{\beta-3}{\beta-1}}\|_{L^{\frac{2(\beta-1)}{\beta-3}}}\|\nabla\nabla_h u\|_{L^2}\\
&\le \frac{1}{8}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\frac{1}{8}\|\nabla\nabla_h u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2. \qquad (3.23)
\end{aligned}
$$

For $I_{33}(t)$, since $\partial_3 u_3=-\mathrm{div}_h u_h$, we have for $\beta>3$

$$
\begin{aligned}
J_3(t) &= -\sum_{i=1}^{2}\int_{\mathbb{R}^3}u\partial_i u_i\partial_3\partial_3 u dx\\
&= \sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_3 u\partial_i u_i\partial_3 u dx+\sum_{i=1}^{2}\int_{\mathbb{R}^3}u\partial_3 u\partial_i\partial_3 u_i dx\\
&\le \|\partial_3 u\|_{L^2_v(L^4_h)}\|\nabla_h u\|_{L^\infty_v(L^2_h)}\|\partial_3 u\|_{L^2_v(L^4_h)}+\||u||\nabla u|^{\frac{2}{\beta-1}}\|_{L^{\beta-1}}\||\nabla u|^{\frac{\beta-3}{\beta-1}}\|_{L^{\frac{2(\beta-1)}{\beta-3}}}\|\nabla\nabla_h u\|_{L^2}\\
&\le C\|\partial_3 u\|_{L^2}\|\nabla_h\partial_3 u\|_{L^2}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla_h u\|_{L^2}^{\frac{1}{2}}+C\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^{\frac{2}{\beta-1}}\|\nabla u\|_{L^2}^{\frac{\beta-3}{\beta-1}}\|\nabla\nabla_h u\|_{L^2}\\
&\le C\|\partial_3 u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h u\|_{L^2}^{\frac{3}{2}}+\frac{1}{8}\|\nabla\nabla_h u\|_{L^2}^2+\frac{1}{4}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2\\
&\le \frac{1}{4}\|\nabla\nabla_h u\|_{L^2}^2+\frac{1}{4}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C\|\partial_3 u\|_{L^2}^4\|\nabla u\|_{L^2}^2. \qquad (3.24)
\end{aligned}
$$

Putting (3.22)-(3.24) into (3.21), we get

$$
I_3(t)\le \frac{1}{2}\|\nabla\nabla_h u\|_{L^2}^2+\frac{1}{2}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C\|\partial_3 u\|_{L^2}^4\|\nabla u\|_{L^2}^2.
$$

By Hölder inequality and Young inequality, $I_4(t)$ can be estimated by

$$
I_4(t)\le \|\nabla\theta\|_{L^2}\|\nabla u\|_{L^2}\le \|\nabla\theta\|_{L^2}^2+\|\nabla u\|_{L^2}^2.
$$

Adding the estimates of $I_3(t)$ and $I_4(t)$, we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla u\|_{L^2}^2+\|\nabla\nabla_h u\|_{L^2}^2+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\|\nabla|u|^{\frac{\beta-1}{2}}\|_{L^2}^2\\
&\qquad \le C\|\nabla\theta\|_{L^2}^2+C(1+\|\partial_3 u\|_{L^2}^4)\|\nabla u\|_{L^2}^2.
\end{aligned}
$$

By Gronwall inequality, noticing (3.5) and (3.11), we have

$$\|\nabla u\|_{L^2}^2+\int_0^t(\|\nabla\nabla_h u\|_{L^2}^2+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\|\nabla|u|^{\frac{\beta-1}{2}}\|_{L^2}^2)ds\le C(t,u_0,\theta_0). \tag{3.25}$$

Step 2. Taking the L^2 inner product of the first equation of (1.1) with $\partial_t u$ and the integration by parts, we obtain

$$\begin{aligned}\|\partial_t u\|_{L^2}^2+\frac{1}{2}\frac{d}{dt}\|\nabla_h u\|_{L^2}^2+\frac{1}{\beta+1}\frac{d}{dt}\|u\|_{L^{\beta+1}}^{\beta+1}&=(\theta e_3,\partial_t u)-((u\cdot\nabla)u,\partial_t u)\\
&\le\|\theta\|_{L^2}\|\partial_t u\|_{L^2}+\||u||\nabla u|^{\frac{2}{\beta-1}}\|_{L^{\beta-1}}\||\nabla u|^{\frac{\beta-3}{\beta-1}}\|_{L^{\frac{2(\beta-1)}{\beta-3}}}\|\partial_t u\|_{L^2}\\
&\le\frac{1}{2}\|\partial_t u\|_{L^2}^2+C(\|\theta\|_{L^2}^2+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}^2).\end{aligned}$$

Then we have

$$\|\partial_t u\|_{L^2}^2+\frac{d}{dt}\|\nabla_h u\|_{L^2}^2+\frac{d}{dt}\|u\|_{L^{\beta+1}}^{\beta+1}\le C(\|\theta\|_{L^2}^2+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}^2).$$

Integrating on $[0,t]$, we get

$$\begin{aligned}&\|\nabla_h u(t)\|_{L^2}^2+\|u(t)\|_{L^{\beta+1}}^{\beta+1}+\int_0^t\|\partial_t u\|_{L^2}^2ds\\
&\le C\int_0^t(\|\theta\|_{L^2}^2+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}^2)ds+\|\nabla_h u_0\|_{L^2}^2+\|u_0\|_{L^{\beta+1}}^{\beta+1}\\
&\le C(t,u_0,\theta_0).\end{aligned} \tag{3.26}$$

Taking the L^2 inner product of the second equation of (1.1) with $\partial_t\theta$ and the integration by parts, we have for $\beta\ge4$

$$\begin{aligned}&\|\partial_t\theta\|_{L^2}^2+\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_{L^2}^2=-((u\cdot\nabla)\theta,\partial_t\theta)\\
&\le\|\partial_t\theta\|_{L^2}\|u\|_{L^{\beta+1}}\|\nabla\theta\|_{L^{\frac{2(\beta+1)}{\beta-1}}}\\
&\le C\|\partial_t\theta\|_{L^2}\|u\|_{L^{\beta+1}}\|\nabla\theta\|_{L^2}^{\frac{\beta-2}{\beta+1}}\|\Delta\theta\|_{L^2}^{\frac{3}{\beta+1}}\\
&\le\frac{1}{2}\|\partial_t\theta\|_{L^2}^2+C\|u\|_{L^{\beta+1}}^2\|\nabla\theta\|_{L^2}^{\frac{2(\beta-2)}{\beta+1}}\|\Delta\theta\|_{L^2}^{\frac{6}{\beta+1}}\\
&\le\frac{1}{2}\|\partial_t\theta\|_{L^2}^2+\frac{1}{2}\|\Delta\theta\|_{L^2}^2+C\|u\|_{L^{\beta+1}}^{\frac{2(\beta+1)}{\beta-2}}\|\nabla\theta\|_{L^2}^2\\
&\le\frac{1}{2}\|\partial_t\theta\|_{L^2}^2+\frac{1}{2}\|\Delta\theta\|_{L^2}^2+C(1+\|u\|_{L^{\beta+1}}^{\beta+1})\|\nabla\theta\|_{L^2}^2.\end{aligned} \tag{3.27}$$

Consequently

$$\frac{d}{dt}\|\nabla\theta\|_{L^2}^2+\|\partial_t\theta\|_{L^2}^2\le\|\Delta\theta\|_{L^2}^2+C(1+\|u\|_{L^{\beta+1}}^{\beta+1})\|\nabla\theta\|_{L^2}^2. \tag{3.28}$$

Integrating on $[0,t]$, by (3.2) and (3.5), we obtain

$$\|\nabla\theta\|_{L^2}^2+\int_0^t\|\partial_t\theta\|_{L^2}^2ds\le C(t,u_0,\theta_0), \tag{3.29}$$

which completes the proof of Theorem 2.2.

4 Well-posedness of global solution for the temperature anisotropic system

In this section, we will prove that system (1.2) has a unique global solution with different smooth conditions of initial values.

4.1 Proof of Theorem 2.3

We first consider the case that the initial value $u_0 \in H^1(\mathbb{R}^3)$ and $\theta_0 \in H^{0,1}(\mathbb{R}^3)$. We will get the existence and uniqueness of global solution of system (1.2) for $\beta > 3$.
Taking the L^2 inner product of the second equation of (1.2) with θ and the integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \|\nabla_h \theta\|_{L^2}^2 = 0.$$

Integrating on $[0, t]$, we obtain

$$\|\theta(t)\|_{L^2}^2 + 2\int_0^t \|\nabla_h \theta\|_{L^2}^2 ds = \|\theta_0\|_{L^2}^2. \tag{4.1}$$

Similarly, taking the L^2 inner product of the first equation of (1.2) with u, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|u\|_{L^{\beta+1}}^{\beta+1} &= (\theta e_3, u)\\ &\le \|\theta\|_{L^2}\|u\|_{L^2} \le \|\theta_0\|_{L^2}^2 + \|u\|_{L^2}^2.\end{aligned}$$

By Gronwall inequality, we obtain

$$\|u(t)\|_{L^2}^2 + \int_0^t (\|\nabla u\|_{L^2}^2 + \|u\|_{L^{\beta+1}}^{\beta+1}) ds \le C(t, u_0, \theta_0). \tag{4.2}$$

Taking the L^2 inner product of the first equation of (1.2) with $-\Delta u$ and the integration by parts, for $\beta > 3$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \frac{4(\beta-1)}{(\beta+1)^2}\|\nabla |u|^{\frac{\beta+1}{2}}\|_{L^2}^2\\ = \int_{\mathbb{R}^3}(u\cdot\nabla)u\Delta u dx - \int_{\mathbb{R}^3}\theta e_3 \Delta u dx\\ := K_1(t) + K_2(t).\end{aligned} \tag{4.3}$$

For $K_1(t)$, by Hölder inequality and Young inequality, for $\beta > 3$, we get

$$\begin{aligned}K_1(t) &\le \frac{1}{4}\|\Delta u\|_{L^2}^2 + C\int_{\mathbb{R}^3}|u|^2|\nabla u|^{\frac{4}{\beta-1}}|\nabla u|^{2-\frac{4}{\beta-1}} dx\\ &\le \frac{1}{4}\|\Delta u\|_{L^2}^2 + \frac{1}{2}\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2.\end{aligned} \tag{4.4}$$

For $K_2(t)$, by Hölder and Young's inequalities, we have

$$K_2(t) \le \|\Delta u\|_{L^2}\|\theta\|_{L^2} \le \frac{1}{4}\|\Delta u\|_{L^2}^2 + \|\theta\|_{L^2}^2. \tag{4.5}$$

Putting (4.4) and (4.5) into (4.3), we have

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \|\nabla |u|^{\frac{\beta+1}{2}}\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^2 + C\|\theta\|_{L^2}^2. \tag{4.6}$$

By Gronwall inequality, we obtain

$$\|\nabla u\|_{L^2}^2 + \int_0^t (\|\Delta u\|_{L^2}^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \|\nabla |u|^{\frac{\beta+1}{2}}\|_{L^2}^2) ds \le C(t, u_0, \theta_0). \tag{4.7}$$

Taking the L^2 inner product of the second equation of (1.2) with $-\partial_3^2\theta$ and the integration by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_3\theta\|_{L^2}^2 + \|\nabla_h\partial_3\theta\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3} \partial_3 u \cdot \nabla\theta\partial_3\theta dx \\
&= -\sum_{i=1}^{2}\int_{\mathbb{R}^3} \partial_3 u_i\partial_i\theta\partial_3\theta dx - \int_{\mathbb{R}^3} \partial_3 u_3\partial_3\theta\partial_3\theta dx \\
&\le \|\partial_3 u\|_{L_v^2(L_h^4)}\|\nabla_h\theta\|_{L_v^\infty(L_h^2)}\|\partial_3\theta\|_{L_v^2(L_h^4)} + \|\partial_3 u\|_{L_v^\infty(L_h^2)}\|\partial_3\theta\|_{L_v^2(L_h^4)}^2 \\
&\le C\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3\theta\|_{L^2}^{\frac{1}{2}}\|\partial_3\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3\theta\|_{L^2}^{\frac{1}{2}} \\
&+ C\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_3\theta\|_{L^2}\|\nabla_h\partial_3\theta\|_{L^2} \\
&\le \frac{1}{2}\|\nabla_h\partial_3\theta\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\|\nabla_h\theta\|_{L^2}\|\partial_3\theta\|_{L^2} + C(\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)\|\partial_3\theta\|_{L^2}^2 \\
&\le \frac{1}{2}\|\nabla_h\partial_3\theta\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla_h\theta\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)\|\partial_3\theta\|_{L^2}^2.
\end{aligned}$$

Consequently

$$\frac{d}{dt}\|\partial_3\theta\|_{L^2}^2 + \|\nabla_h\partial_3\theta\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^2\|\nabla_h\theta\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)\|\partial_3\theta\|_{L^2}^2.$$

By Gronwall inequality, we obtain

$$\|\partial_3\theta\|_{L^2}^2 + \int_0^t \|\nabla_h\partial_3\theta\|_{L^2}^2 ds \le C(t, u_0, \theta_0). \tag{4.8}$$

Now, we will prove the uniqueness of solutions of system (1.2). Let $(\bar{u}, \bar{p}, \bar{\theta})$ and $(\tilde{u}, \tilde{p}, \tilde{\theta})$ be two solution of (1.2) with the same initial data. Assume that $u = \bar{u} - \tilde{u}$, $p = \bar{p} - \tilde{p}$, $\theta = \bar{\theta} - \tilde{\theta}$, it is easy to get the following form:

$$\begin{cases} \partial_t u - \Delta u + (\bar{u}\cdot\nabla)u + (u\cdot\nabla)\tilde{u} + |\bar{u}|^{\beta-1}\bar{u} - |\tilde{u}|^{\beta-1}\tilde{u} + \nabla p = \theta e_3, \\ \partial_t\theta - \Delta_h\theta + (\bar{u}\cdot\nabla)\theta + (u\cdot\nabla)\tilde{\theta} = 0, \\ \nabla\cdot u = 0. \end{cases} \tag{4.9}$$

Taking first the L^2 inner product of the first equation of (4.9) with u and the second equation with θ, respectively, the integration by parts, and then taking all results into account, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2+\|\theta\|_{L^2}^2)+&\|\nabla u\|_{L^2}^2+\|\nabla_h\theta\|_{L^2}^2+\int_{\mathbb{R}^3}(|\bar{u}|^{\beta-1}\bar{u}-|\tilde{u}|^{\beta-1}\tilde{u})udx\\ &=-\int_{\mathbb{R}^3}(u\cdot\nabla)\tilde{u}udx+\int_{\mathbb{R}^3}\theta e_3udx-\int_{\mathbb{R}^3}(u\cdot\nabla)\tilde{\theta}\theta dx\\ &:=\sum_{i=3}^{5}K_i(t).\end{aligned}\tag{4.10}$$

Inspired by [3, 14, 15, 16], it is easy to prove that

$$\int_{\mathbb{R}^3}(|\bar{u}|^{\beta-1}\bar{u}-|\tilde{u}|^{\beta-1}\tilde{u})udx\geq 0.\tag{4.11}$$

For $K_3(t)$, by Hölder inequality, Gagliardo-Nirenberg inequality and Young inequality, we have

$$\begin{aligned}K_3(t)&\leq\|u\|_{L^4}^2\|\nabla\tilde{u}\|_{L^2}\leq C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla\tilde{u}\|_{L^2}\\ &\leq\frac{1}{4}\|\nabla u\|_{L^2}^2+C\|\nabla\tilde{u}\|_{L^2}^4\|u\|_{L^2}^2.\end{aligned}\tag{4.12}$$

For $K_4(t)$, by Hölder inequality and Young inequality, we have

$$K_4(t)\leq\|\theta\|_{L^2}\|u\|_{L^2}\leq\|\theta\|_{L^2}^2+\|u\|_{L^2}^2.\tag{4.13}$$

For $K_5(t)$, we get

$$\begin{aligned}K_5(t)&=-\sum_{i=1}^{2}\int_{\mathbb{R}^3}u_i\partial_i\tilde{\theta}\theta dx-\int_{\mathbb{R}^3}u_3\partial_3\tilde{\theta}\theta dx\\ &\leq\|u\|_{L_v^2(L_h^4)}\|\nabla_h\tilde{\theta}\|_{L_v^\infty(L_h^2)}\|\theta\|_{L_v^2(L_h^4)}+\|u\|_{L_v^\infty(L_h^2)}\|\partial_3\tilde{\theta}\|_{L_v^2(L_h^4)}\|\theta\|_{L_v^2(L_h^4)}\\ &\leq C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\tilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3\tilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\theta\|_{L^2}^{\frac{1}{2}}\\ &+C\|u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_3\tilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3\tilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\theta\|_{L^2}^{\frac{1}{2}}\\ &\leq\frac{1}{4}\|\nabla u\|_{L^2}^2+\frac{1}{2}\|\nabla_h\theta\|_{L^2}^2+C\|u\|_{L^2}\|\nabla_h\tilde{\theta}\|_{L^2}\|\nabla_h\partial_3\tilde{\theta}\|_{L^2}\|\theta\|_{L^2}\\ &+C\|u\|_{L^2}\|\partial_3\tilde{\theta}\|_{L^2}\|\nabla_h\partial_3\tilde{\theta}\|_{L^2}\|\theta\|_{L^2}\\ &\leq\frac{1}{4}\|\nabla u\|_{L^2}^2+\frac{1}{2}\|\nabla_h\theta\|_{L^2}^2\\ &+C(\|\nabla_h\tilde{\theta}\|_{L^2}^2+\|\partial_3\tilde{\theta}\|_{L^2}^2+\|\nabla_h\partial_3\tilde{\theta}\|_{L^2}^2)(\|u\|_{L^2}^2+\|\theta\|_{L^2}^2).\end{aligned}\tag{4.14}$$

Putting (4.11)-(4.14) into (4.10), we have

$$\frac{d}{dt}(\|u\|_{L^2}^2+\|\theta\|_{L^2}^2)+\|\nabla u\|_{L^2}^2+\|\nabla_h\theta\|_{L^2}^2$$

$$\leq C(1+\|\nabla\tilde{u}\|_{L^2}^4+\|\nabla_h\tilde{\theta}\|_{L^2}^2+\|\partial_3\tilde{\theta}\|_{L^2}^2+\|\nabla_h\partial_3\tilde{\theta}\|_{L^2}^2)(\|u\|_{L^2}^2+\|\theta\|_{L^2}^2). \quad (4.15)$$

By virtue of Gronwall inequality, we obtain

$$\begin{aligned}
&\|u(t)\|_{L^2}^2+\|\theta(t)\|_{L^2}^2+\int_0^t(\|\nabla u\|_{L^2}^2+\|\nabla_h\theta\|_{L^2}^2)ds \\
&\quad\leq(\|u_0\|_{L^2}^2+\|\theta_0\|_{L^2}^2)e^{\int_0^t(1+\|\nabla\tilde{u}\|_{L^2}^4+\|\nabla_h\tilde{\theta}\|_{L^2}^2+\|\partial_3\tilde{\theta}\|_{L^2}^2+\|\nabla_h\partial_3\tilde{\theta}\|_{L^2}^2)ds}. \qquad (4.16)
\end{aligned}$$

The uniqueness of the solution of system (1.2) is proved. This completes the proof of Theorem 2.3.

4.2 Proof of Theorem 2.4

In this subsection we suppose higher regularities on the initial values, i.e. $u_0\in H^s(\mathbb{R}^3)$ and $\theta_0\in H^s(\mathbb{R}^3)$. In this case the existence and uniqueness of global smooth solution of system (1.2) can be obtained. To begin with, we deduce some priori estimates.

Step 1. Taking the L^2 inner product of the second equation of (1.2) with $-\Delta\theta$ and the integration by parts, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_{L^2}^2+\|\nabla\nabla_h\theta\|_{L^2}^2&=\int_{\mathbb{R}^3}(u\cdot\nabla)\theta\Delta\theta dx\\
&\leq\|\nabla u\|_{L_v^\infty(L_h^2)}\|\nabla\theta\|_{L_v^2(L_h^4)}^2\\
&\leq C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}\|\nabla\nabla_h\theta\|_{L^2}\\
&\leq C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}\|\nabla\nabla_h\theta\|_{L^2}\\
&\leq\frac{1}{2}\|\nabla\nabla_h\theta\|_{L^2}^2+C(\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2)\|\nabla\theta\|_{L^2}^2.
\end{aligned}$$

Consequently

$$\frac{d}{dt}\|\nabla\theta\|_{L^2}^2+\|\nabla\nabla_h\theta\|_{L^2}^2\leq C(\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2)\|\nabla\theta\|_{L^2}^2.$$

By Gronwall inequality and (4.7), we have

$$\|\nabla\theta(t)\|_{L^2}^2+\int_0^t\|\nabla\nabla_h\theta\|_{L^2}^2ds\leq C(t,u_0,\theta_0). \qquad (4.17)$$

Step 2. Taking the L^2 inner product of the first equation of (1.2) with $\partial_t u$ and the integration by parts, for $\beta>3$, we have

$$\begin{aligned}
&\|\partial_t u\|_{L^2}^2+\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\frac{1}{\beta+1}\frac{d}{dt}\|u\|_{L^{\beta+1}}^{\beta+1}\\
&=-\int_{\mathbb{R}^3}(u\cdot\nabla)u\partial_t u dx+\int_{\mathbb{R}^3}\theta e_3\partial_t u dx\\
&\leq\||u||\nabla u|^{\frac{2}{\beta-1}}\|_{L^{\beta-1}}\||\nabla u|^{\frac{\beta-3}{\beta-1}}\|_{L^{\frac{2(\beta-1)}{\beta-3}}}\|\partial_t u\|_{L^2}+\|\theta\|_{L^2}\|\partial_t u\|_{L^2}
\end{aligned}$$

$$\leq \frac{1}{2}\|\partial_t u\|_{L^2}^2 + C(\|\theta\|_{L^2}^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2).$$

Moreover,

$$\|\partial_t u\|_{L^2}^2 + \frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|u\|_{L^{\beta+1}}^{\beta+1}) \leq C(\|\theta\|_{L^2}^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2).$$

Integrating on $[0,t]$, we get

$$\|\nabla u(t)\|_{L^2}^2 + \|u(t)\|_{L^{\beta+1}}^{\beta+1} + \int_0^t \|\partial_t u\|_{L^2}^2 ds \leq C(t, u_0, \theta_0). \tag{4.18}$$

Taking the L^2 inner product of the second equation of (1.2) with $\partial_t\theta$ and the integration by parts, we have

$$\begin{aligned}
&\|\partial_t\theta\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla_h\theta\|_{L^2}^2 = -\int_{\mathbb{R}^3}(u\cdot\nabla)\theta\partial_t\theta dx\\
&\leq \|u\|_{L^3}\|\nabla_h\theta\|_{L^6}\|\partial_t\theta\|_{L^2} + \|u\|_{L_v^\infty(L_h^4)}\|\partial_3\theta\|_{L_v^2(L_h^4)}\|\partial_t\theta\|_{L^2}\\
&\leq C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h\theta\|_{L^2}\|\partial_t\theta\|_{L^2} + C\|u\|_{L_h^4(L_v^2)}^{\frac{1}{2}}\|\partial_3 u\|_{L_h^4(L_v^2)}^{\frac{1}{2}}\|\partial_3\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3\theta\|_{L^2}^{\frac{1}{2}}\|\partial_t\theta\|_{L^2}\\
&\leq \frac{1}{4}\|\partial_t\theta\|_{L^2}^2 + C(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)\|\nabla\nabla_h\theta\|_{L^2}^2\\
&+ C\|u\|_{L_h^4(L_v^2)}^{\frac{1}{2}}\|\partial_3 u\|_{L_h^4(L_v^2)}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h\theta\|_{L^2}^{\frac{1}{2}}\|\partial_t\theta\|_{L^2}\\
&\leq \frac{1}{4}\|\partial_t\theta\|_{L^2}^2 + C(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)\|\nabla\nabla_h\theta\|_{L^2}^2\\
&+ C\|u\|_{L^2}^{\frac{1}{4}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{4}}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h\theta\|_{L^2}^{\frac{1}{2}}\|\partial_t\theta\|_{L^2}\\
&\leq \frac{1}{2}\|\partial_t\theta\|_{L^2}^2 + C(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + 1)\|\nabla\nabla_h\theta\|_{L^2}^2\\
&+ C\|u\|_{L^2}\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}\|\nabla\theta\|_{L^2}^2\\
&\leq \frac{1}{2}\|\partial_t\theta\|_{L^2}^2 + C(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + 1)\|\nabla\nabla_h\theta\|_{L^2}^2\\
&+ C\|\Delta u\|_{L^2}^2 + C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4\|\nabla\theta\|_{L^2}^4.
\end{aligned}$$

Consequently

$$\begin{aligned}
\|\partial_t\theta\|_{L^2}^2 + \frac{d}{dt}\|\nabla_h\theta\|_{L^2}^2 \leq C(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + 1)\|\nabla\nabla_h\theta\|_{L^2}^2 + C\|\Delta u\|_{L^2}^2\\
+ C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4\|\nabla\theta\|_{L^2}^4.
\end{aligned}$$

Integrating on $[0,t]$, we get

$$\|\nabla_h\theta(t)\|_{L^2}^2 + \int_0^t \|\partial_t\theta\|_{L^2}^2 ds \leq C(t, u_0, \theta_0). \tag{4.19}$$

Step 3. Applying the operator ∂_t to the first equation of (1.2), and then taking the L^2 inner product to the result with $\partial_t u$ and the integration by parts, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_t u\|_{L^2}^2 + \|\nabla\partial_t u\|_{L^2}^2 + \int_{\mathbb{R}^3} \partial_t(|u|^{\beta-1}u)\partial_t u dx
&= -\int_{\mathbb{R}^3}(\partial_t u\cdot\nabla)u\partial_t u dx + \int_{\mathbb{R}^3}\partial_t(\theta e_3)\partial_t u dx\\
&\le \|\nabla u\|_{L^2}\|\partial_t u\|_{L^4}^2 + \|\partial_t\theta\|_{L^2}\|\partial_t u\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}\|\partial_t u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_t u\|_{L^2}^{\frac{3}{2}} + \|\partial_t\theta\|_{L^2}\|\partial_t u\|_{L^2}\\
&\le \frac{1}{2}\|\nabla\partial_t u\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^4)\|\partial_t u\|_{L^2}^2 + C\|\partial_t\theta\|_{L^2}^2.
\end{aligned}
$$

Moveover,

$$
\frac{d}{dt}\|\partial_t u\|_{L^2}^2 + \|\nabla\partial_t u\|_{L^2}^2 \le C(1+\|\nabla u\|_{L^2}^4)\|\partial_t u\|_{L^2}^2 + C\|\partial_t\theta\|_{L^2}^2.
$$

Noticing (4.7) and (4.19), by Gronwall inequality we have

$$
\|\partial_t u(t)\|_{L^2}^2 + \int_0^t \|\nabla\partial_t u\|_{L^2}^2 ds \le C(t, u_0, \theta_0). \tag{4.20}
$$

Applying the operator ∂_t to the second equation of (1.2), and then taking the L^2 inner product to the result with $\partial_t\theta$ and the integration by parts, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_t\theta\|_{L^2}^2 + \|\nabla_h\partial_t\theta\|_{L^2}^2 = -\int_{\mathbb{R}^3}\partial_t u\nabla\theta\partial_t\theta dx\\
&\le C\|\partial_t u\|_{L_v^\infty(L_h^2)}\|\nabla_h\theta\|_{L_v^2(L_h^4)}\|\partial_t\theta\|_{L_v^2(L_h^4)} + \|\partial_t u\|_{L_v^\infty(L_h^2)}\|\partial_3\theta\|_{L_v^2(L_h^4)}\|\partial_t\theta\|_{L_v^2(L_h^4)}\\
&\le C\|\partial_t u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_t u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\nabla_h\theta\|_{L^2}^{\frac{1}{2}}\|\partial_t\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_t\theta\|_{L^2}^{\frac{1}{2}}\\
&+ C\|\partial_t u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_t u\|_{L^2}^{\frac{1}{2}}\|\partial_3\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3\theta\|_{L^2}^{\frac{1}{2}}\|\partial_t\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_t\theta\|_{L^2}^{\frac{1}{2}}\\
&\le C\|\partial_t u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_t u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h\theta\|_{L^2}^{\frac{1}{2}}\|\partial_t\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_t\theta\|_{L^2}^{\frac{1}{2}}\\
&+ C\|\partial_t u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_t u\|_{L^2}^{\frac{1}{2}}\|\partial_3\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h\theta\|_{L^2}^{\frac{1}{2}}\|\partial_t\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_t\theta\|_{L^2}^{\frac{1}{2}}\\
&\le \frac{1}{2}\|\nabla_h\partial_t\theta\|_{L^2}^2 + \|\nabla\partial_t u\|_{L^2}^2 + \|\nabla\nabla_h\theta\|_{L^2}^2 + C\|\partial_t u\|_{L^2}^2\|\nabla_h\theta\|_{L^2}^2\|\partial_t\theta\|_{L^2}^2\\
&+ C\|\partial_t u\|_{L^2}^2\|\partial_3\theta\|_{L^2}^2\|\partial_t\theta\|_{L^2}^2\\
&\le \frac{1}{2}\|\nabla_h\partial_t\theta\|_{L^2}^2 + \|\nabla\partial_t u\|_{L^2}^2 + \|\nabla\nabla_h\theta\|_{L^2}^2 + (\|\partial_t u\|_{L^2}^4 + \|\nabla_h\theta\|_{L^2}^4 + \|\partial_3\theta\|_{L^2}^4)\|\partial_t\theta\|_{L^2}^2.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\frac{d}{dt}\|\partial_t\theta\|_{L^2}^2 + \|\nabla_h\partial_t\theta\|_{L^2}^2 &\le C(\|\nabla\partial_t u\|_{L^2}^2 + \|\nabla\nabla_h\theta\|_{L^2}^2)\\
&+ C(\|\partial_t u\|_{L^2}^4 + \|\nabla_h\theta\|_{L^2}^4 + \|\partial_3\theta\|_{L^2}^4)\|\partial_t\theta\|_{L^2}^2.
\end{aligned} \tag{4.21}
$$

By (4.8), (4.17), (4.19), (4.20) and Gronwall inequality, it yields that

$$\|\partial_t\theta\|_{L^2}^2+\int_0^t\|\nabla_h\partial_t\theta\|_{L^2}^2ds\le C(t,u_0,\theta_0). \tag{4.22}$$

Step 4. Taking the L^2 inner product of the first equation of (1.2) with $-\Delta u$ and the integration by parts, we have

$$\begin{aligned}\|\Delta u\|_{L^2}^2+&\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\frac{4(\beta-1)}{(\beta+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2\\&=\int_{\mathbb{R}^3}\partial_t u\Delta udx+\int_{\mathbb{R}^3}(u\cdot\nabla)u\Delta udx-\int_{\mathbb{R}^3}\theta e_3\Delta udx\\&\le C(\|\partial_t u\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\theta\|_{L^2}^2)+\frac12\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\frac12\|\Delta u\|_{L^2}^2.\end{aligned}$$

Moreover,

$$\|\Delta u\|_{L^2}^2+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2\le C(\|\partial_t u\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\theta\|_{L^2}^2).$$

By (4.1), (4.7), (4.20), we have

$$\|\Delta u\|_{L^2}^2+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2\le C(t,u_0,\theta_0). \tag{4.23}$$

Taking the L^2 inner product of the second equation of (1.2) with $-\Delta_h\theta$ and the integration by parts, we have

$$\begin{aligned}\|\Delta_h\theta\|_{L^2}^2&=\int_{\mathbb{R}^3}\partial_t\theta\Delta_h\theta dx+\int_{\mathbb{R}^3}(u\cdot\nabla)\theta\Delta_h\theta dx\\&\le\|\partial_t\theta\|_{L^2}\|\Delta_h\theta\|_{L^2}+\|u\|_{L^\infty}\|\nabla\theta\|_{L^2}\|\Delta_h\theta\|_{L^2}\\&\le\frac12\|\Delta_h\theta\|_{L^2}^2+C\|\partial_t\theta\|_{L^2}^2+C\|u\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2\\&\le\frac12\|\Delta_h\theta\|_{L^2}^2+C(\|\partial_t\theta\|_{L^2}^2+\|u\|_{L^\infty}^4+\|\nabla\theta\|_{L^2}^4).\end{aligned}$$

Consequently

$$\|\Delta_h\theta\|_{L^2}^2\le C(\|\partial_t\theta\|_{L^2}^2+\|u\|_{L^\infty}^4+\|\nabla\theta\|_{L^2}^4). \tag{4.24}$$

By (4.2), (4.7), (4.23), we have

$$\|u\|_{L^\infty}\le\|u\|_{H^1}^{\frac12}\|u\|_{H^2}^{\frac12}\le C(t,u_0,\theta_0). \tag{4.25}$$

Also, by (4.17), (4.22), (4.25), we have

$$\|\Delta_h\theta\|_{L^2}^2\le C(t,u_0,\theta_0). \tag{4.26}$$

Step 5. Applying the operator ∂_t to the first equation of (1.2)， we can get

$$\partial_t\partial_t u+\partial_t(u\cdot\nabla u)-\Delta\partial_t u+\partial_t(|u|^{\beta-1}u)+\nabla\partial_t p=\partial_t(\theta e_3). \tag{4.27}$$

Taking the L^2 inner product of the above equation with $-\Delta\partial_t u$ and the integration by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\partial_t u\|_{L^2}^2+\|\Delta\partial_t u\|_{L^2}^2 &\le |\int_{\mathbb{R}^3}\partial_t(u\cdot\nabla u)\cdot\Delta\partial_t u dx|+|\int_{\mathbb{R}^3}\partial_t(|u|^{\beta-1}u)\Delta\partial_t u dx|\\ &\quad+|\int_{\mathbb{R}^3}\partial_t(\theta e_3)\Delta\partial_t u dx|\\ &:=K_6(t)+K_7(t)+K_8(t).\end{aligned}\tag{4.28}$$

For $K_6(t)$, by Hölder inequality, we have

$$\begin{aligned}K_6(t)&\le|\int_{\mathbb{R}^3}\partial_t u\nabla u\Delta\partial_t u dx|+|\int_{\mathbb{R}^3}u\nabla\partial_t u\Delta\partial_t u dx|\\ &\le\|\Delta\partial_t u\|_{L^2}\|\partial_t u\|_{L^6}\|\nabla u\|_{L^3}+\|u\|_{L^\infty}\|\nabla\partial_t u\|_{L^2}\|\Delta\partial_t u\|_{L^2}\\ &\le\frac{1}{4}\|\Delta\partial_t u\|_{L^2}^2+C\|\nabla u\|_{L^3}^2\|\nabla\partial_t u\|_{L^2}^2+C\|u\|_{L^\infty}^2\|\nabla\partial_t u\|_{L^2}^2\\ &\le\frac{1}{4}\|\Delta\partial_t u\|_{L^2}^2+C(\|u\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\|u\|_{L^\infty}^2)\|\nabla\partial_t u\|_{L^2}^2.\end{aligned}\tag{4.29}$$

For $K_7(t)$, it is obvious that

$$K_7(t)\le C\|\Delta\partial_t u\|_{L^2}\|u\|_{L^\infty}^{\beta-1}\|\partial_t\theta\|_{L^2}\le\frac{1}{8}\|\Delta\partial_t u\|_{L^2}^2+C\|u\|_{L^\infty}^{2(\beta-1)}\|\partial_t u\|_{L^2}^2.\tag{4.30}$$

For $K_8(t)$, by Hölder inequality and Young inequality, we have

$$K_8(t)\le\|\partial_t\theta\|_{L^2}\|\Delta\partial_t u\|_{L^2}\le\frac{1}{8}\|\Delta\partial_t u\|_{L^2}^2+C\|\partial_t\theta\|_{L^2}^2.\tag{4.31}$$

Putting (4.29)-(4.31) into (4.28), we obtain

$$\begin{aligned}\frac{d}{dt}\|\nabla\partial_t u\|_{L^2}^2+\|\Delta\partial_t u\|_{L^2}^2&\le C\|\partial_t\theta\|_{L^2}^2+C\|u\|_{L^\infty}^{2(\beta-1)}\|\partial_t u\|_{L^2}^2\\ &\quad+C(\|u\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\|u\|_{L^\infty}^2)\|\nabla\partial_t u\|_{L^2}^2.\end{aligned}\tag{4.32}$$

By virtue of Gronwall inequality, we can get from (4.2), (4.20), (4.22), (4.23) and (4.25) that

$$\|\nabla\partial_t u\|_{L^2}^2+\int_0^t\|\Delta\partial_t u\|_{L^2}^2 ds\le C(t,u_0,\theta_0).\tag{4.33}$$

Applying the operator $\nabla\partial_3$ to the second equation of (1.2), and then taking the L^2 inner product to the result with $\nabla\partial_3\theta$ and the integration by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\partial_3\theta\|_{L^2}^2+\|\nabla\nabla_h\partial_3\theta\|_{L^2}^2&=-\int_{\mathbb{R}^3}\nabla\partial_3(u\cdot\nabla\theta)\nabla\partial_3\theta dx\\ &=-\int_{\mathbb{R}^3}\nabla(\partial_3 u\nabla\theta+u\nabla\partial_3\theta)\nabla\partial_3\theta dx\end{aligned}$$

$$= -\int_{\mathbb{R}^3} \nabla\partial_3 u \nabla\theta \nabla\partial_3\theta dx - \int_{\mathbb{R}^3} \partial_3 u \nabla\nabla\theta \nabla\partial_3\theta dx - \int_{\mathbb{R}^3} \nabla u \nabla\partial_3\theta \nabla\partial_3\theta dx$$
$$:= K_9(t) + K_{10}(t) + K_{11}(t). \tag{4.34}$$

For $K_9(t)$, by Sobolev inequality and Young inequality, it yields that

$$\begin{aligned}
K_9(t) &\le \|\nabla\partial_3 u\|_{L^2} \|\nabla\theta\|_{L^\infty_v(L^4_h)} \|\nabla\partial_3\theta\|_{L^2_v(L^4_h)} \\
&\le C\|\nabla\partial_3 u\|_{L^2} \|\nabla\theta\|^{\frac{1}{2}}_{L^4_h(L^2_v)} \|\nabla\partial_3\theta\|^{\frac{1}{2}}_{L^4_h(L^2_v)} \|\nabla\partial_3\theta\|_{L^2_v(L^4_h)} \\
&\le C\|\nabla\partial_3 u\|_{L^2} \|\nabla\theta\|^{\frac{1}{2}}_{L^2_v(L^4_h)} \|\nabla\partial_3\theta\|^{\frac{1}{2}}_{L^2_v(L^4_h)} \|\nabla\partial_3\theta\|_{L^2_v(L^4_h)} \\
&\le C\|\Delta u\|_{L^2} \|\nabla\theta\|^{\frac{1}{4}}_{L^2} \|\nabla\nabla_h\theta\|^{\frac{1}{4}}_{L^2} \|\nabla\partial_3\theta\|^{\frac{3}{4}}_{L^2} \|\nabla\nabla_h\partial_3\theta\|^{\frac{3}{4}}_{L^2} \\
&\le \frac{1}{4}\|\nabla\nabla_h\partial_3\theta\|^2_{L^2} + C\|\nabla\nabla_h\theta\|^2_{L^2} + C\|\Delta u\|^8_{L^2} + \|\nabla\theta\|^{\frac{2}{3}}_{L^2} \|\nabla\partial_3\theta\|^2_{L^2}.
\end{aligned} \tag{4.35}$$

For $K_{10}(t)$, by Sobolev inequality and Young inequality, we get

$$\begin{aligned}
K_{10}(t) &\le \|\partial_3 u\|_{L^2_v(L^4_h)} \|\nabla\nabla_h\theta\|_{L^\infty_v(L^2_h)} \|\nabla\partial_3\theta\|_{L^2_v(L^4_h)} + \|\nabla\partial_3\theta\|^2_{L^2_v(L^4_h)} \|\partial_3 u\|_{L^\infty_v(L^2_h)} \\
&\le C\|\partial_3 u\|^{\frac{1}{2}}_{L^2} \|\nabla_h\partial_3 u\|^{\frac{1}{2}}_{L^2} \|\nabla\nabla_h\theta\|^{\frac{1}{2}}_{L_2} \|\nabla\nabla_h\partial_3\theta\|^{\frac{1}{2}}_{L_2} \|\nabla\partial_3\theta\|^{\frac{1}{2}}_{L^2} \|\nabla\nabla_h\partial_3\theta\|^{\frac{1}{2}}_{L^2} \\
&+ C\|\nabla\partial_3\theta\|_{L^2} \|\nabla\nabla_h\partial_3\theta\|_{L^2} \|\partial_3 u\|^{\frac{1}{2}}_{L^2} \|\partial_3\partial_3 u\|^{\frac{1}{2}}_{L^2} \\
&\le \frac{1}{8}\|\nabla\nabla_h\partial_3\theta\|^2_{L^2} + C\|\nabla\nabla_h\theta\|^2_{L^2} \\
&+ C(\|\nabla u\|^2_{L^2} + \|\nabla u\|^4_{L^2} + \|\Delta u\|^2_{L^2} + \|\Delta u\|^4_{L^2}) \|\nabla\partial_3\theta\|^2_{L^2}.
\end{aligned} \tag{4.36}$$

Similarly, for $K_{11}(t)$, it turns out that

$$\begin{aligned}
K_{11}(t) &\le \|\nabla\partial_3\theta\|^2_{L^2_v(L^4_h)} \|\nabla u\|_{L^\infty_v(L^2_h)} \\
&\le C\|\nabla\partial_3\theta\|_{L^2} \|\nabla\nabla_h\partial_3\theta\|_{L^2} \|\nabla u\|^{\frac{1}{2}}_{L^2} \|\nabla\partial_3 u\|^{\frac{1}{2}}_{L^2} \\
&\le \frac{1}{8}\|\nabla\nabla_h\partial_3\theta\|^2_{L^2} + C(\|\nabla u\|^2_{L^2} + \|\Delta u\|^2_{L^2}) \|\nabla\partial_3\theta\|^2_{L^2}.
\end{aligned} \tag{4.37}$$

Putting (4.35)-(4.37) into (4.34), we have

$$\begin{aligned}
\frac{d}{dt}\|\nabla\partial_3\theta\|^2_{L^2} + \|\nabla\nabla_h\partial_3\theta\|^2_{L^2} &\le C\|\nabla\nabla_h\theta\|^2_{L^2} + C(\|\nabla u\|^2_{L^2} + \|\nabla\theta\|^2_{L^2} + \|\nabla u\|^4_{L^2} \\
&\quad + \|\Delta u\|^2_{L^2} + \|\Delta u\|^4_{L^2}) \|\nabla\partial_3\theta\|^2_{L^2}.
\end{aligned} \tag{4.38}$$

By virtue of Gronwall inequality, we can get from (4.2), (4.7), (4.17) and (4.23) that

$$\|\nabla\partial_3\theta\|^2_{L^2} + \int_0^t \|\nabla\nabla_h\partial_3\theta\|^2_{L^2} ds \le C(t, u_0, \theta_0). \tag{4.39}$$

Applying the operator ∂_t to the second equation of (1.2), we have

$$\partial_t\partial_t\theta - \Delta_h\partial_t\theta + \partial_t(u \cdot \nabla\theta) = 0. \tag{4.40}$$

Taking the L^2 inner product of (4.40) with $-\partial_3^2\partial_t\theta$ and the integration by parts, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_3\partial_t\theta\|_{L^2}^2+\|\nabla_h\partial_3\partial_t\theta\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}\partial_t(u\cdot\nabla\theta)\partial_3^2\partial_t\theta dx\\
&=\int_{\mathbb{R}^3}(\partial_t u\nabla\theta+u\nabla\partial_t\theta)\partial_3^2\partial_t\theta dx\\
&=-\int_{\mathbb{R}^3}\partial_3\partial_t u\nabla\theta\partial_3\partial_t\theta dx-\int_{\mathbb{R}^3}\partial_t u\nabla\partial_3\theta\partial_3\partial_t\theta dx-\int_{\mathbb{R}^3}\partial_3 u\nabla\partial_t\theta\partial_3\partial_t\theta dx\\
&:=K_{12}(t)+K_{13}(t)+K_{14}(t).
\end{aligned}
\tag{4.41}
$$

For $K_{12}(t)$, by Hölder inequality and Young inequality, it yields that

$$
\begin{aligned}
K_{12}(t)&\le\|\partial_3\partial_t u\|_{L_v^\infty(L_h^2)}\|\partial_3\partial_t\theta\|_{L_v^2(L_h^4)}\|\nabla\theta\|_{L_v^2(L_h^4)}\\
&\le C\|\partial_3\partial_t u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_3\partial_t u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_t\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3\partial_t\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h\theta\|_{L^2}^{\frac{1}{2}}\\
&\le C\|\nabla\partial_t u\|_{L^2}^{\frac{1}{2}}\|\Delta\partial_t u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_t\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3\partial_t\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h\theta\|_{L^2}^{\frac{1}{2}}\\
&\le\frac{1}{8}\|\nabla_h\partial_3\partial_t\theta\|_{L^2}^2+C(\|\Delta\partial_t u\|_{L^2}^2+\|\nabla\nabla_h\theta\|_{L^2}^2)\\
&+C(\|\nabla\partial_t u\|_{L^2}^4+\|\nabla\theta\|_{L^2}^4)\|\partial_3\partial_t\theta\|_{L^2}^2.
\end{aligned}
\tag{4.42}
$$

Similarly, for $K_{13}(t)$, it turns out that

$$
\begin{aligned}
K_{13}(t)&\le\|\partial_t u\|_{L_v^\infty(L_h^2)}\|\nabla\partial_3\theta\|_{L_v^2(L_h^4)}\|\partial_3\partial_t\theta\|_{L_v^2(L_h^4)}\\
&\le C\|\partial_t u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_t u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h\partial_3\theta\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_t\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3\partial_t\theta\|_{L^2}^{\frac{1}{2}}\\
&\le\frac{1}{8}\|\nabla_h\partial_3\partial_t\theta\|_{L^2}^2+C(\|\nabla\partial_3\theta\|_{L^2}^2+\|\nabla\nabla_h\partial_3\theta\|_{L^2}^2)\\
&+C(\|\partial_t u\|_{L^2}^4+\|\nabla\partial_t u\|_{L^2}^4)\|\partial_3\partial_t\theta\|_{L^2}^2.
\end{aligned}
\tag{4.43}
$$

Using a similar method, for $K_{14}(t)$, we get

$$
\begin{aligned}
K_{14}(t)&\le\|\partial_3\partial_t\theta\|_{L_v^2(L_h^4)}^2\|\partial_3 u\|_{L_v^\infty(L_h^2)}+\|\partial_3\partial_t\theta\|_{L_v^2(L_h^4)}\|\nabla_h\partial_t\theta\|_{L_v^\infty(L_h^2)}\|\partial_3 u\|_{L_v^2(L_h^4)}\\
&\le C\|\partial_3\partial_t\theta\|_{L^2}\|\nabla_h\partial_3\partial_t\theta\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\\
&+C\|\partial_3\partial_t\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_t\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3\partial_t\theta\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\\
&\le\frac{1}{4}\|\nabla_h\partial_3\partial_t\theta\|_{L^2}^2+C(\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2)\|\partial_3\partial_t\theta\|_{L^2}^2\\
&+C\|\nabla_h\partial_t\theta\|_{L^2}^2+C(\|\nabla u\|_{L^2}^4+\|\Delta u\|_{L^2}^4)\|\partial_3\partial_t\theta\|_{L^2}^2.
\end{aligned}
\tag{4.44}
$$

Putting (4.42)-(4.44) into (4.41), we obtain

$$
\frac{d}{dt}\|\partial_3\partial_t\theta\|_{L^2}^2+\|\nabla_h\partial_3\partial_t\theta\|_{L^2}^2
$$

$$
\begin{aligned}
&\le C(\|\Delta\partial_t u\|_{L^2}^2+\|\nabla\nabla_h\theta\|_{L^2}^2+\|\nabla\partial_3\theta\|_{L^2}^2+\|\nabla\nabla_h\partial_3\theta\|_{L^2}^2+\|\nabla_h\partial_t\theta\|_{L^2}^2)\\
&+C(\|\nabla\partial_t u\|_{L^2}^4+\|\nabla\theta\|_{L^2}^4+\|\partial_t u\|_{L^2}^4+\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2\\
&+\|\nabla u\|_{L^2}^4+\|\Delta u\|_{L^2}^4)\|\partial_3\partial_t\theta\|_{L^2}^2. \qquad (4.45)
\end{aligned}
$$

By virtue of Gronwall inequality, we can get from (4.17), (4.18), (4.20), (4.22), (4.23), (4.33) and (4.39) that

$$
\|\partial_3\partial_t\theta\|_{L^2}^2+\int_0^t\|\nabla_h\partial_3\partial_t\theta\|_{L^2}^2ds\le C(t,u_0,\theta_0). \qquad (4.46)
$$

Step 6. Applying the operator Δ to the first equation of (1.2), and then taking the L^2 inner product to the result with Δu and the integration by parts, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta u\|_{L^2}^2+\|\nabla\Delta u\|_{L^2}^2\\
&=-\int_{\mathbb{R}^3}\Delta((u\cdot\nabla)u)\Delta udx-\int_{\mathbb{R}^3}\Delta(|u|^{\beta-1}u)\Delta udx+\int_{\mathbb{R}^3}\Delta(\theta e_3)\Delta udx\\
&:=K_{15}(t)+K_{16}(t)+K_{17}(t). \qquad (4.47)
\end{aligned}
$$

By virtue of Hölder inequality and Young inequality, $K_{15}(t)$, $K_{16}(t)$, $K_{17}(t)$ can be estimated by

$$
\begin{aligned}
K_{15}(t)&\le\|\nabla u\|_{L^4}^2\|\nabla\Delta u\|_{L^2}+\|u\|_{L^\infty}\|\Delta u\|_{L^2}\|\nabla\Delta u\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{3}{2}}\|\nabla\Delta u\|_{L^2}+\|u\|_{L^\infty}\|\Delta u\|_{L^2}\|\nabla\Delta u\|_{L^2}\\
&\le\frac{1}{4}\|\nabla\Delta u\|_{L^2}^2+C(\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\|u\|_{L^\infty}^2)\|\Delta u\|_{L^2}^2, \qquad (4.48)\\
K_{16}(t)&\le C\|\nabla u\|_{L^2}\|u\|_{L^\infty}^{\beta-1}\|\nabla\Delta u\|_{L^2}\\
&\le\frac{1}{8}\|\nabla\Delta u\|_{L^2}^2+C\|u\|_{L^\infty}^{2(\beta-1)}\|\nabla u\|_{L^2}^2, \qquad (4.49)\\
K_{17}(t)&\le\|\nabla\theta\|_{L^2}\|\nabla\Delta u\|_{L^2}\\
&\le\frac{1}{8}\|\nabla\Delta u\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2. \qquad (4.50)
\end{aligned}
$$

Putting (4.48)-(4.50) into (4.47), we have

$$
\begin{aligned}
\frac{d}{dt}\|\Delta u\|_{L^2}^2+\|\nabla\Delta u\|_{L^2}^2&\le C(\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\|u\|_{L^\infty}^2)\|\Delta u\|_{L^2}^2\\
&+C\|u\|_{L^\infty}^{2(\beta-1)}\|\nabla u\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2. \qquad (4.51)
\end{aligned}
$$

By virtue of Gronwall inequality, we can get from (4.7), (4.17), (4.23) and (4.25) that

$$
\|\Delta u(t)\|_{L^2}^2+\int_0^t\|\nabla\Delta u\|_{L^2}^2ds\le C(t,u_0,\theta_0). \qquad (4.52)
$$

Applying the operator Δ to the second equation of (1.2), and then taking the L^2 inner product to the result with $\Delta\theta$ and the integration by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta\theta\|_{L^2}^2+\|\Delta\nabla_h\theta\|_{L^2}^2&=-\int_{\mathbb{R}^3}\Delta((u\cdot\nabla)\theta)\Delta\theta dx\\&\leq C\|\Delta u\|_{L^3}\|\nabla\theta\|_{L^6}\|\Delta\theta\|_{L^2}+C\|\nabla u\|_{L_v^\infty(L_h^2)}\|\Delta\theta\|_{L_v^2(L_h^4)}^2\\&\leq C\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\Delta\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta\theta\|_{L^2}^2+C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\Delta\theta\|_{L^2}\|\Delta\nabla_h\theta\|_{L^2}\\&\leq\frac{1}{2}\|\Delta\nabla_h\theta\|_{L^2}^2+C(\|\Delta u\|_{L^2}^{\frac{2}{3}}+\|\Delta\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}\|\Delta u\|_{L^2})\|\Delta\theta\|_{L^2}^2\\&\leq\frac{1}{2}\|\Delta\nabla_h\theta\|_{L^2}^2+C(1+\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\|\Delta\nabla u\|_{L^2}^2)\|\Delta\theta\|_{L^2}^2.\end{aligned}$$

Then we have

$$\frac{d}{dt}\|\Delta\theta\|_{L^2}^2+\|\Delta\nabla_h\theta\|_{L^2}^2\leq C(1+\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\|\Delta\nabla u\|_{L^2}^2)\|\Delta\theta\|_{L^2}^2.$$

By virtue of Gronwall inequality, we can get from (4.7), (4.23) and (4.52) that

$$\|\Delta\theta(t)\|_{L^2}^2+\int_0^t\|\Delta\nabla_h\theta\|_{L^2}^2ds\leq C(t,u_0,\theta_0).\tag{4.53}$$

Lemma 4.1. *Assume that $s\geq 3$ and $(u_0,\theta_0)\in H^s$. Then we have $u\in L^\infty([0,T],H^s)\cap L^2([0,T],H^{s+1})$, $\theta\in L^\infty([0,T],H^s)$, $\nabla_h\theta\in L^2([0,T],H^s)$, and there exists a positive $C(t,u_0,\theta_0)$ such that*

$$\|u\|_{H^s}^2+\|\theta\|_{H^s}^2+\int_0^T(\|u\|_{H^{s+1}}^2+\|\nabla_h\theta\|_{H^s}^2)ds\leq C(t,u_0,\theta_0).$$

Proof. First from (4.1), (4.2), (4.52) and (4.53), we know

$$\begin{aligned}u\in L^\infty([0,T],H^2)\cap L^2([0,T],H^3),\\\theta\in L^\infty([0,T],H^2),\quad\nabla_h\theta\in L^2([0,T],H^2).\end{aligned}$$

For $s\geq 3$, assume that we have gotten

$$u\in L^\infty([0,T],H^{s-1})\cap L^2([0,T],H^s),\tag{4.54}$$

$$\theta\in L^\infty([0,T],H^{s-1}),\quad\nabla_h\theta\in L^2([0,T],H^{s-1}).\tag{4.55}$$

Based on above assumption, we prove

$$\begin{aligned}u\in L^\infty([0,T],H^s)\cap L^2([0,T],H^{s+1}),\\\theta\in L^\infty([0,T],H^s),\quad\nabla_h\theta\in L^2([0,T],H^s).\end{aligned}$$

Applying the operator Λ^s to the first equation of (1.2), and then taking the L^2 inner product to the result with $\Lambda^s u$ and the integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s\nabla u\|_{L^2}^2=-\int_{\mathbb{R}^3}\Lambda^s((u\cdot\nabla)u)\Lambda^s udx-\int_{\mathbb{R}^3}\Lambda^s(|u|^{\beta-1}u)\Lambda^s udx$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^3}\Lambda^s(\theta e_3)\Lambda^s u dx\\
&:= K_{15}(t)+K_{16}(t)+K_{17}(t).
\end{aligned}\tag{4.56}
$$

For $K_{17}(t)$, by Hölder inequality and Young inequality, we can deduce that

$$
K_{17}(t)\le \|\Lambda^{s-1}\theta\|_{L^2}\|\Lambda^s\nabla u\|_{L^2}\le \frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+C\|\Lambda^{s-1}\theta\|_{L^2}^2. \tag{4.57}
$$

For $K_{15}(t)$ and $K_{16}(t)$, by a similar method, it turns out that

$$
\begin{aligned}
|K_{15}(t)|&=|\int_{\mathbb{R}^3}\Lambda^{s-1}(u\cdot\nabla u)\Lambda^{s+1}udx|\\
&\le \frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+C\|\Lambda^{s-1}(u\cdot\nabla u)\|_{L^2}^2\\
&\le \frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+C(\|u\|_{L^\infty}^2\|\Lambda^s u\|_{L^2}^2+\|\nabla u\|_{L^3}^2\|\Lambda^{s-1}u\|_{L^6}^2)\\
&\le \frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+C(\|u\|_{L^\infty}^2+\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2)\|\Lambda^s u\|_{L^2}^2
\end{aligned}\tag{4.58}
$$

and

$$
\begin{aligned}
K_{16}(t)&\le \|\Lambda^{s-1}(|u|^{\beta-1}u)\|_{L^2}\|\Lambda^s\nabla u\|_{L^2}\\
&\le \frac{1}{4}\|\Lambda^s\nabla u\|_{L^2}^2+C\|u\|_{L^\infty}^{2(\beta-1)}\|\Lambda^{s-1}u\|_{L^2}^2.
\end{aligned}\tag{4.59}
$$

Putting (4.57)-(4.59) into (4.56), we have

$$
\begin{aligned}
\frac{d}{dt}\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s\nabla u\|_{L^2}^2&\le C(\|u\|_{L^\infty}^2+\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2)\|\Lambda^s u\|_{L^2}^2\\
&\quad+C\|\Lambda^{s-1}\theta\|_{L^2}^2+C\|u\|_{L^\infty}^{2(\beta-1)}\|\Lambda^{s-1}u\|_{L^2}^2.
\end{aligned}\tag{4.60}
$$

By virtue of Gronwall inequality, we can get from (4.7), (4.23), (4.25) and (4.54)-(4.55) that

$$
\|\Lambda^s u\|_{L^2}^2+\int_0^t\|\Lambda^s\nabla u\|_{L^2}^2ds\le C(t,u_0,\theta_0). \tag{4.61}
$$

Applying the operator Λ^s to the second equation of (1.2), and then taking the L^2 inner product to the result with $\Lambda^s\theta$ and the integration by parts, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Lambda^s\theta\|_{L^2}^2+\|\Lambda^s\nabla_h\theta\|_{L^2}^2&=-\int_{\mathbb{R}^3}\Lambda^s((u\cdot\nabla)\theta)\Lambda^s\theta dx\\
&=\int_{\mathbb{R}^3}(\Lambda^s((u\cdot\nabla)\theta)-(u\cdot\nabla)\Lambda^s\theta)\Lambda^s\theta dx\\
&\le C\|\Lambda^s u\|_{L^6}\|\nabla\theta\|_{L^3}\|\Lambda^s\theta\|_{L^2}+C\|\nabla u\|_{L^\infty}\|\Lambda^s\theta\|_{L^2}^2\\
&\le C\|\Lambda^s\nabla u\|_{L^2}\|\nabla\theta\|_{L^3}\|\Lambda^s\theta\|_{L^2}+C\|\nabla u\|_{L^\infty}\|\Lambda^s\theta\|_{L^2}^2
\end{aligned}
$$

$$\leq \|\Lambda^s\nabla u\|_{L^2}^2 + C(\|\theta\|_{L^2}^2 + \|\Lambda^{s-1}\theta\|_{L^2}^2 + \|\nabla u\|_{L^\infty})\|\Lambda^s\theta\|_{L^2}^2.$$

Consequently

$$\frac{d}{dt}\|\Lambda^s\theta\|_{L^2}^2 + \|\Lambda^s\nabla_h\theta\|_{L^2}^2 \leq C\|\Lambda^s\nabla u\|_{L^2}^2 + C(\|\theta\|_{L^2}^2 + \|\Lambda^{s-1}\theta\|_{L^2}^2 + \|\nabla u\|_{L^\infty})\|\Lambda^s\theta\|_{L^2}^2.$$

By virtue of Gronwall inequality, we can get from (4.1), (4.25), (4.55) and (4.61) that

$$\|\Lambda^s\theta\|_{L^2}^2 + \int_0^t \|\Lambda^s\nabla_h\theta\|_{L^2}^2 ds \leq C(t, u_0, \theta_0). \tag{4.62}$$

This completes the proof of Lemma 4.1. By standard method, this completes the proof of Theorem 2.4.

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