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# ANALYSIS OF A DEGENERATED DIFFUSION SVEQIRV EPIDEMIC MODEL WITH GENERAL INCIDENCE IN A SPACE HETEROGENEOUS ENVIRONMENT

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**Abstract** Considering the comprehensive impact of vaccination, quarantine and spatial heterogeneity on diseases dynamics, we formulates an SVEQIRV model with degenerate diffusion. Firstly, we discuss the well-posedness of the model solution. Then, we analyze the dynamic properties of model by using the semigroup theory and the global exponential attractor theory. We use the threshold feature $\lambda^*$ which is the principal eigenvalue of the eigenvalue problem associated with the linearized system at the disease free equilibrium, to describe the transmission dynamics of epidemics. The results show that the disease-free equilibrium is globally asymptotically stable when $\lambda^* < 0$ and the system is uniformly persistent when $\lambda^* > 0$. Finally, some numerical simulations and the sensitivity analysis are conducted to visualize the theoretical results and the effect of vaccination rate on disease dynamics.

**Keywords** Dynamical model, spatial heterogeneity, global exponential attractor, numerical simulations.

**MSC(2010)** 92D30, 35B35, 35K57, 37N25.

## 1. Introduction

At present, the impact of diseases on the human world is becoming more and more serious. WHO has identified infectious diseases as the second most common cause of death worldwide [1]. Therefore, understanding the progression of disease, and developing appropriate prevention and control interventions for the disease are critical. Mathematical models play an important role in understanding and exploring the dynamics of disease progression [2, 3]. In 1927, Kermack and McKendrick used dynamic methods to establish the famous SIR model, and then they proposed the SIS model [4], thereby obtaining the threshold theory to distinguish whether diseases are prevalent, which is an important progress.

In fact, some diseases have incubation periods. For example, influenza, Ebola, COVID-19, etc [5–7]. Therefore, many individuals infected with the virus do not immediately become infected, but rather become exposed. As a result, many works have extended the SIR model to the SEIR model. For instance, Jiao et al. [8] proposed a SEIR epidemic model for home quarantine of susceptible populations. Annas et al. [9] analyzed the spread of COVID-19 in Indonesia by establishing SEIR model. In addition, some research found that vaccination is one of the most important measures to control the spread of diseases [10, 11]. So researchers have incorporated the factor of vaccines into their mathematical models to explore the impact of vaccines on disease control. For example, El Hajji et al. [12] established SVEIR model to study the role of vaccines in controlling the spread of measles. Huo et al. [13] incorporated vaccination into the model and analyzed the impact of vaccination on seasonal influenza in Gansu, China. Poturi et al. [14] based on realistic data, discussed the impact of preventive vaccination in Sierra Leone and the Democratic Republic of the Congo on the Ebola epidemic.

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However, most of the above models are based on ODE models. While these ODE models can well describe many real-world diseases, there are still some situations that cannot be addressed with autonomous ODE models. For example, the random movement of individuals in space is also an influencing factor [15]. Several studies on modelling of infectious diseases such as influenza [16], cholera [17] and COVID-19 [18] have emphasized the importance of individual movement in epidemic dynamics. Therefore, there are some researchers use the reaction-diffusion systems to model individual random movement, see as [19, 25].

On the other hand, more and more people are beginning to consider the impact of spatial heterogeneity on disease transmission, e.g. [20, 21, 24, 26, 30]. Many infectious diseases occur in heterogeneous environments because of different environmental conditions [15]. Constrained by factors such as altitude, temperature, humidity, latitude, climate, and living factors, the diffusion of epidemics in different environments is vastly different [20]. For example, from January to March 2020, the main reason for the rapid spread of the COVID-19 epidemic in China is the conditions of climate and temperature are suitable for the spread of coronavirus. And the gatherings caused a rapid increase in population density and contact in a small area, and it has also created conditions for the spread of the novel coronavirus [21]. Moreover, for mosquito-borne diseases, such as malaria, dengue fever and yellow fever, rising temperatures may increase mosquito populations and bite rates, resulting in wider disease transmission [22]. Hence, regions with different latitudes and longitudes have different climates, and the prevalence of diseases is also different.

As a result, considering the diffusion and spatial heterogeneity of populations is an important factor in epidemic modeling [23]. For example, Wang et al. [24] proposed a reaction-diffusion model with incubation period and nonlinear morbidity in a spatially heterogeneous environment, and theirs results show that the incubation period can significantly enhance the persistence of the disease if the dispersal rate of susceptible hosts or exposed hosts is small or large. Zhu et al. [20] discussed a reaction-diffusion SVIR model in a spatially heterogeneous environment, and obtained the spatial heterogeneity had a great effect on the spread of the disease. Luo et al. [25] considering the individual differences, spatial environment and the temporary acquired immunity, a general multi-group reaction–diffusion epidemic model with nonlinear incidence was proposed and they concluded that the difference of diffusion rate may bring great difficulties to control the disease. In fact, there is a lot of work in this area that we will not describe in detail here, please see [26–29] and corresponding reference .

In recent years, discussions of partially degenerate reaction–diffusion systems that couple partial differential equations with ordinary differential equations have received continuous attentions [24]. For example, Wang et al. [30] established a degenerated reaction–diffusion cholera model with spatial heterogeneity and discussed the impact of the transmission rate of infected individuals on the spread of cholera. Shan et al. [31] discussed a degenerate reaction-diffusion host-pathogen model with general incidence rate, the analysis shows that its theoretical results can be applied to the dynamics of infectious diseases, such as Zika virus, avian influenza, H1N1, and seasonal influenza, and can also be used to assess the risk of disease transmission. For related studies on degenerated reaction–diffusion, see [32, 33].

In addition, different types of incidence rate have been used in the study of infectious diseases [34]. The incidence rate of a disease is a measure of how many people are infected with the disease per unit of time and plays an important role in the dynamics of epidemic models [15]. Traditionally, the incidence rate has been assumed to be bilinear, but disease dynamics do not always follow standard incidence rates, and many epidemiological mechanisms are more suitable for non-linear incidence rates, especially general incidence rates [35]. At present, more and more researchers use the general incidence rate to study the mechanism of disease transmission, can see [15, 35–37].

Based on the discussion above, a more accurate description of the disease transmission model require the consideration vaccination, quarantine and general incidence in a space heterogeneous environment. Furthermore, considering that the effect of many vaccines is not permanent and some diseases relapse, i.e., the recover individuals may changed the sensitive vaccinated individuals such as COVID-19, Ebola, etc. Hence, we propose a partially degenerate reaction–diffusion SVEQIRV systems to model the factors mentioned above. These factors increase the number of equations and the coupling between equations in the system, which poses greater challenges to discuss global dynamics apply the Lyapunov function method.

The article is organized as follows: In Section 2, we give the model. In Section 3, the existence, positive and boundedness of the global solutions are discussed by using comparison principle and Gronwall's inequality. In Section 4, we prove the existence of the global exponential attraction set by using the existence theorem of exponential attractor, and then discuss the global asymptotic stability of the equilibrium and uniform persistence of system. In Section 5, some numerical simulations and the sensitivity analysis are conducted to visualize the theoretical results and the effect of vaccination rate on disease dynamics. Finally, in section 6, Give some conclusions.

## 2. Model formulation

In this section, we assume the total population $N(x,t)$ include six parts, $S(x,t)$ is the density of susceptible individual at position $x$ and time $t$, $V(x,t)$ is the density of vaccinated individual at position $x$ and time $t$, $E(x,t)$ is the density of exposed individual at position $x$ and time $t$, $Q(x,t)$ is the density of quarantined individual at position $x$ and time $t$, $I(x,t)$ is the density of infected individual at position $x$ and time $t$, $R(x,t)$ is the density of recovered individual at position $x$ and time $t$, that is

$$N(x,t) = S(x,t) + V(x,t) + E(x,t) + Q(x,t) + I(x,t) + R(x,t).$$

The transmission dynamics of the disease in the population is shown in Figure 1. We have noticed some phenomenons, for example, most people have been vaccinated with COVID-19 vaccine, but they will also be infected with the virus and become recovered individuals and they are vaccinated individuals. That is to say, the recovered individuals $R$ can return to vaccinated individuals $V$. Similarly, influenza also has this situation. So we assume that some recovered individuals who have been infected with the virus will return to $V$.

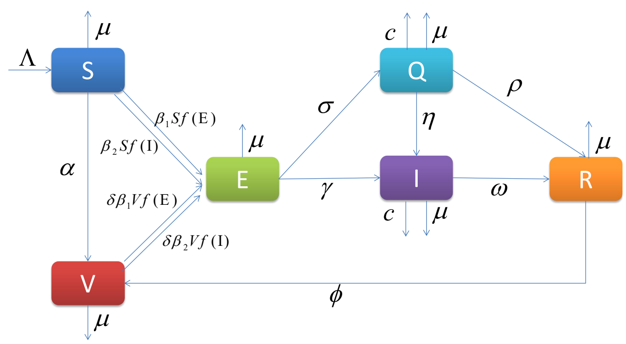


Figure 1. Schematic diagram for the disease transmission

Since the compartment $Q$ is quarantined, so we do not consider the diffusion of the quarantined individuals in this paper. Thus, We can formulate the system:

$$\begin{cases} \dfrac{\partial S}{\partial t} = \nabla \cdot (d_S(x)\nabla S) + \Lambda(x) - \beta_1(x)Sf(E) - \beta_2(x)Sf(I) - [\alpha(x)+\mu(x)]S, \\ \dfrac{\partial V}{\partial t} = \nabla \cdot (d_V(x)\nabla V) + \alpha(x)S - \delta(x)\beta_1(x)Vf(E) - \delta(x)\beta_2(x)Vf(I) - \mu(x)V + \phi(x)R, \\ \dfrac{\partial E}{\partial t} = \nabla \cdot (d_E(x)\nabla E) + \beta_1(x)Sf(E) + \beta_2(x)Sf(I) + \delta(x)\beta_1(x)Vf(E) + \delta(x)\beta_2(x)Vf(I) \\ \qquad - [\sigma(x)+\gamma(x)+\mu(x)]E, \\ \dfrac{\partial Q}{\partial t} = \sigma(x)E - [\rho(x)+\eta(x)+c(x)+\mu(x)]Q, \\ \dfrac{\partial I}{\partial t} = \nabla \cdot (d_I(x)\nabla I) + \gamma(x)E + \eta(x)Q - [\omega(x)+c(x)+\mu(x)]I, \\ \dfrac{\partial R}{\partial t} = \nabla \cdot (d_R(x)\nabla R) + \rho(x)Q + \omega(x)I - [\phi(x)+\mu(x)]R, \\ \dfrac{\partial S}{\partial \mathbf{n}} = \dfrac{\partial V}{\partial \mathbf{n}} = \dfrac{\partial E}{\partial \mathbf{n}} = \dfrac{\partial I}{\partial \mathbf{n}} = \dfrac{\partial R}{\partial \mathbf{n}} = 0, \ x \in \partial\Omega, t > 0, \\ S(x,0) = S_0(x,0) \geq 0, \ V(x,0) = V_0(x,0) \geq 0, \ E(x,0) = E_0(x,0) \geq 0, \\ Q(x,0) = Q_0(x,0) \geq 0, \ I(x,0) = I_0(x,0) \geq 0, \ R(x,0) = R_0(x,0) \geq 0, \ x \in \Omega, \end{cases} \tag{2.1}$$

where $\Omega$ is a bounded domain in $R^m (m \geq 1)$, with smooth boundary $\partial\Omega$ (when $m > 1$), diffusion coefficients

$d_S(x), d_V(x), d_E(x), d_I(x), d_R(x) \in \mathbf{C}^1(\Omega)$ are positive, continuous and uniformly bounded, $\Lambda(x), \alpha(x), \beta_1(x), \beta_2(x)$, $\delta(x), \sigma(x), \eta(x), \rho(x), \omega(x), \mu(x), c(x)$ are positive Hölder continuous functions. And we assume that the initial value $S_0, V_0, E_0, Q_0, I_0, R_0$ are nonnegative continuous functions on $\overline{\Omega}$ and the number of initially infected individuals is positive, i.e. $\int_\Omega I_0(x,0)dx > 0$. The parameters in system (2.1) are given in Table 1.

Table 1. The meanings of the parameters in system (2.1)

| Symbol | Meanings |
|---|---|
| $\Lambda(x)$ | the total recruitment scale at position $x$ |
| $\alpha(x)$ | the vaccination coverage at position $x$ |
| $\beta_1(x)$ | the contact rate of exposed at position $x$ |
| $\beta_2(x)$ | the contact rate of infected at position $x$ |
| $\delta(x)$ | the intensity of contact between the virus carrier and $V$ at position $x$ |
| $\sigma(x)$ | the quarantine rate of exposed at position $x$ |
| $\gamma(x)$ | the incidence rate of exposed at position $x$ |
| $\eta(x)$ | the incidence rate of quarantined at position $x$ |
| $\rho(x)$ | the recovery rate of quarantined at position $x$ |
| $\omega(x)$ | the recovery rate of infected at position $x$ |
| $\mu(x)$ | the natural mortality at position $x$ |
| $c(x)$ | the mortality due to disease at position $x$ |

Next, we give a few definitions. Firstly, $f(E)$ and $f(I)$ is non-negative continuous differentiable on $[0, \infty)$ and it meet the following conditions:

$$(1)\ f(0) = 0,\ 0 \leq \lim_{E\to 0^+} \frac{f(E)}{E} = c_1 < \infty,\ 0 \leq \lim_{I\to 0^+} \frac{f(I)}{I} = c_1 < \infty.$$
$$(2)\ \text{For any } E \geq 0,\ I \geq 0,\ \ f(E) \leq c_1 E,\ f(I) \leq c_1 I.$$

Let $L_1, L_2, L_3, L_5, L_6$ be a linear operator defined by

$$L_1 S(x) := \nabla \cdot (d_S(x)\nabla S), L_2 V(x) := \nabla \cdot (d_V(x)\nabla V), L_3 E(x) := \nabla \cdot (d_E(x)\nabla E),$$
$$L_5 I(x) := \nabla \cdot (d_I(x)\nabla I), L_6 R(x) := \nabla \cdot (d_R(x)\nabla R)$$

on $D(L) = (D(L_1), D(L_2), D(L_3), D(L_5), D(L_6)) \subset \mathbf{X}$ and the domain indicates that the directional derivative of SVEQIR is equal to zero on the boundary. $\mathbf{X} := \mathbf{L}^2(\Omega)$ is Banach space and $\mathbf{L}^2$ space mean the set of all square integrable functions,

$$D(L_1) := \left\{ S \in \mathbf{H}^2(\Omega); \frac{\partial S}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \right\}, D(L_2) := \left\{ V \in \mathbf{H}^2(\Omega); \frac{\partial V}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \right\},$$
$$D(L_3) := \left\{ E \in \mathbf{H}^2(\Omega); \frac{\partial E}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \right\}, D(L_5) := \left\{ I \in \mathbf{H}^2(\Omega); \frac{\partial I}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \right\},$$
$$D(L_6) := \left\{ R \in \mathbf{H}^2(\Omega); \frac{\partial R}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \right\},$$

where $\mathbf{H}^2(\Omega) = \mathbf{H}(\Omega) \times \mathbf{H}(\Omega)$ and $\mathbf{H}$ refers to Hilbert space. So that $L_1, L_2, L_3, L_5, L_6$ is the infinitesimal generator of strongly continuous semigroup $\{e^{tL_1}\}_{t\geq 0}, \{e^{tL_2}\}_{t\geq 0}, \{e^{tL_3}\}_{t\geq 0}, \{e^{tL_5}\}_{t\geq 0}, \{e^{tL_6}\}_{t\geq 0}$ on $\mathbf{H}^2(\Omega)$.

Set space $(\mathbf{H}^2(\Omega))^6 = \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega)$. Define operator $L : (\mathbf{H}^2(\Omega))^6 \to (\mathbf{H}^2(\Omega))^6$, for any $u := (S, V, E, Q, I, R) \in (\mathbf{H}^2(\Omega))^6$, we have

$$L(S,V,E,Q,I,R) := \begin{pmatrix} L_1 S(x,t) \\ L_2 V(x,t) \\ L_3 E(x,t) \\ 0 \\ L_5 I(x,t) \\ L_6 R(x,t) \end{pmatrix}.$$

That is, $L$ is the infinitesimal generator of strongly continuous semigroup $\{e^{tL}\}_{t\geq 0}$.

Let $\mathbf{Y} := (\mathbf{H}^2(\Omega))^6$ be a normed Banach space defined by

$$\|(S,V,E,Q,I,R)^T\|_{\mathbf{Y}} := \max\{\|S\|_{\mathbf{H}^2(\Omega)}, \|V\|_{\mathbf{H}^2(\Omega)}, \|E\|_{\mathbf{H}^2(\Omega)}, \|Q\|_{\mathbf{H}^2(\Omega)}, \|I\|_{\mathbf{H}^2(\Omega)}, \|R\|_{\mathbf{H}^2(\Omega)}\}.$$

We assume $G$ be a nonlinear operator on $\mathbf{Y}$,

$$\begin{aligned} G(S,V,E,Q,I,R) := \big(& g_1(S,V,E,Q,I,R), g_2(S,V,E,Q,I,R), g_3(S,V,E,Q,I,R), \\ & g_4(S,V,E,Q,I,R), g_5(S,V,E,Q,I,R), g_6(S,V,E,Q,I,R)\big)^T, \end{aligned}$$

where

$$\begin{aligned} g_1 &= \Lambda(x) - \beta_1(x)Sf(E) - \beta_2(x)Sf(I) - [\alpha(x)+\mu(x)]S, \\ g_2 &= \alpha(x)S + \phi(x)R - \delta(x)\beta_1(x)Vf(E) - \delta(x)\beta_2(x)Vf(I) - \mu(x)V, \\ g_3 &= \beta_1(x)Sf(E) + \beta_2(x)Sf(I) + \delta(x)\beta_1(x)Vf(E) + \delta(x)\beta_2(x)Vf(I) - [\sigma(x)+\gamma(x)+\mu(x)]E, \\ g_4 &= \sigma(x)E - [\rho(x)+\eta(x)+c(x)+\mu(x)]Q, \\ g_5 &= \gamma(x)E + \eta(x)Q - [\omega(x)+c(x)+\mu(x)]I, \\ g_6 &= \rho(x)Q + \omega(x)I - [\phi(x)+\mu(x)]R. \end{aligned}$$

Let function $F = (g_1, g_2, g_3, g_4, g_5, g_6)$. From operator $L$ and function $F$, we rewrite system (2.1) in the following form

$$\frac{d}{dt}u(t) = Lu(t) + F(u(t)), \tag{2.2}$$

with

$$u(t) = \begin{pmatrix} S(\cdot,t) \\ V(\cdot,t) \\ E(\cdot,t) \\ Q(\cdot,t) \\ I(\cdot,t) \\ R(\cdot,t) \end{pmatrix}, \quad u(0) = \begin{pmatrix} S_0(\cdot) \\ V_0(\cdot) \\ E_0(\cdot) \\ Q_0(\cdot) \\ I_0(\cdot) \\ R_0(\cdot) \end{pmatrix},$$

where $S_0(\cdot) = S(\cdot,0), V_0(\cdot) = V(\cdot,0), E_0(\cdot) = E(\cdot,0), Q_0(\cdot) = Q(\cdot,0), I_0(\cdot) = I(\cdot,0), R_0(\cdot) = R(\cdot,0)$. Because $L$ is the generator of a $C_0$-contractive semigroup on $\mathbf{Y}$, and $F$ is locally Lipschitz continuous, from [38, Corollary 11.3.1], system (2.2) has at least one classical solution, can be represented by

$$u(t) = e^{tL}u_0 + \int_0^t e^{(t-s)L}F(u(s))ds. \tag{2.3}$$

In addition, if $S(x,0), V(x,0), E(x,0), Q(x,0), I(x,0), R(x,0) \in \mathbf{C}^2(\Omega)$, then $u(x,t) \in \mathbf{C}^{2,1}(\Omega \times (0,T))$.

From now on, for any given continuous function $A$ on $\overline{\Omega}$, we define

$$A^* = \max_{x\in\overline{\Omega}} A(x) \text{ and } A_* = \min_{x\in\overline{\Omega}} A(x).$$

Obviously, the system (2.1) has a disease-free equilibrium $P^0 = (S^0(x), V^0(x), 0, 0, 0, 0)$. Next, we illustrate the existence of principal eigenvalues of the eigenvalue problem associated with the linearization of system (2.1) at the disease free equilibrium $P^0$. Linearize the system (2.1) at disease-free equilibrium $P^0$:

$$\begin{cases} \dfrac{\partial E}{\partial t} = \nabla \cdot (d_E(x)\nabla E) + c_1\beta_1(x)E + c_1\beta_2(x)I + c_1\delta(x)\beta_1(x)E + c_1\delta(x)\beta_2(x)I \\ \quad - [\sigma(x) + \gamma(x) + \mu(x)]E, \\ \dfrac{\partial Q}{\partial t} = \sigma(x)E - [\rho(x) + \eta(x) + c(x) + \mu(x)]Q, \\ \dfrac{\partial I}{\partial t} = \nabla \cdot (d_I(x)\nabla I) + \gamma(x)E + \eta(x)Q - [\omega(x) + c(x) + \mu(x)]I, \\ \dfrac{\partial R}{\partial t} = \nabla \cdot (d_R(x)\nabla R) + \rho(x)Q + \omega(x)I - [\phi(x) + \mu(x)]R. \end{cases} \tag{2.4}$$

Let $E = e^{\lambda t}\chi(x), Q = e^{\lambda t}\varphi(x), I = e^{\lambda t}\psi(x), R = e^{\lambda t}\xi(x)$, then system (2.4) can be rewrite as

$$\begin{cases} \lambda\chi(x) = \nabla \cdot (d_E(x)\nabla\chi) + \{c_1\beta_1(x) + c_1\delta(x)\beta_1(x) - [\sigma(x) + \gamma(x) + \mu(x)]\}\chi(x) \\ \qquad + [c_1\beta_2(x) + c_1\delta(x)\beta_2(x)]\psi(x), \\ \lambda\varphi(x) = \sigma(x)\chi(x) - [\rho(x) + \eta(x) + c(x) + \mu(x)]\varphi(x), \\ \lambda\psi(x) = \nabla \cdot (d_I(x)\nabla\psi) + \gamma(x)\chi(x) + \eta(x)\varphi(x) - [\omega(x) + c(x) + \mu(x)]\psi(x), \\ \lambda\xi(x) = \nabla \cdot (d_R(x)\nabla\xi) + \rho(x)\varphi(x) + \omega(x)\psi(x) - [\phi(x) + \mu(x)]\xi(x). \end{cases} \tag{2.5}$$

Define $\Phi(x) = (\chi(x), \varphi(x), \psi(x), \xi(x))^T, m_{11}(x) = c_1\beta_1(x) + c_1\delta(x)\beta_1(x) - (\sigma(x) + \gamma(x) + \mu(x)), m_{22}(x) = \rho(x) + \eta(x) + c(x) + \mu(x), m_{33}(x) = \omega(x) + c(x) + \mu(x), m_{44}(x) = \phi(x) + \mu(x)$, where $m_{ij}(x) \geq 0, i \neq j, x \in \Omega$,

$$D(x) = \begin{pmatrix} d_E(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_I(x) & 0 \\ 0 & 0 & 0 & d_R(x) \end{pmatrix},$$

$$M(x,t) = (m_{ij}(x,t)) = \begin{pmatrix} m_{11}(x) & 0 & c_1\beta_2(x) + c_1\delta(x)\beta_2(x) & 0 \\ \sigma(x) & m_{22} & 0 & 0 \\ \gamma(x) & \eta(x) & m_{33}(x) & 0 \\ 0 & \rho(x) & \omega(x) & m_{44}(x) \end{pmatrix}.$$

Thus system (2.5) can be written

$$\begin{cases} \lambda\Phi(x) = \nabla \cdot (D(x)\nabla\Phi(x)) + M(x,t)\Phi(x), x \in \Omega, \\ \dfrac{\partial \Phi}{\partial \mathbf{n}} = 0. \end{cases} \tag{2.6}$$

According to the Krein-Rutman theorem [39] and [40, Theorem 7.6.1], system (2.6) has a real eigenvalue $\lambda^*$ and positive eigenvector $\Phi^*(x) = (\chi^*(x), \varphi^*(x), \psi^*(x), \xi^*(x))$.

# 3. Positivity and boundedness

In this section, we prove the existence, positivity and boundedness of the global solution.

**Theorem 3.1.** *If $\big(S(x,0),V(x,0),E(x,0),Q(x,0),I(x,0),R(x,0)\big)>0$, for any $x\in\Omega$, then the system(2.1) admits the globally solution $u(x,t)=\big(S(x,t),V(x,t),E(x,t),Q(x,t),I(x,t),R(x,t)\big)$. It is positivity and uniformly bounded in $L^1$ norm. In particular, $E(x,t),I(x,t)$ is ultimately bounded.*

**Proof.** Define

$$\tilde g_1=-\beta_1(x)Sf(E)-\beta_2(x)Sf(I)-[\alpha(x)+\mu(x)]S.$$

Then $\frac{\partial\tilde S}{\partial t}-\nabla\cdot(d_S(x)\nabla\tilde S)=\tilde g_1$, according to the strong extremum principle, $\tilde S>0$. And because $g_1\geq\tilde g_1$, that is

$$\frac{\partial S}{\partial t}-\nabla\cdot(d_S(x)\nabla S)=g_1\geq\tilde g_1=\frac{\partial\tilde S}{\partial t}-\nabla\cdot(d_S(x)\nabla\tilde S).$$

It satisfies the conditions in the theorem [41, Theorem 2.2.1], so we have $S>0$.
Define

$$\tilde g_2=-\delta(x)\beta_1(x)Vf(E)-\delta(x)\beta_2(x)Vf(I)-\mu(x)V.$$

Then $\frac{\partial\tilde V}{\partial t}-\nabla\cdot(d_V(x)\nabla\tilde V)=\tilde g_2$, similarly, $\tilde V>0$. And because $g_2\geq\tilde g_2$, that is

$$\frac{\partial V}{\partial t}-\nabla\cdot(d_V(x)\nabla V)=g_2\geq\tilde g_2=\frac{\partial\tilde V}{\partial t}-\nabla\cdot(d_V(x)\nabla\tilde V).$$

It satisfies the conditions in the theorem [41, Theorem 2.2.1], so we have $V>0$. In the same way, $E>0,Q>0,I>0,R>0$.

Because $F=G$ when $S,V,E,Q,I,R\in R^+$, so $u(t)$ is the solution of system (2.1). This indicates the existence of a solution for system (2.1).

Next, we prove the uniform boundedness. Define

$$U(t)=\int_\Omega[S(x,t)+V(x,t)+E(x,t)+Q(x,t)+I(x,t)+R(x,t)]dx.$$

From system (2.1),

$$\begin{aligned}\frac{dU(t)}{dt}&=\int_\Omega\left[\frac{\partial}{\partial t}S(x,t)+\frac{\partial}{\partial t}V(x,t)+\frac{\partial}{\partial t}E(x,t)+\frac{\partial}{\partial t}Q(x,t)+\frac{\partial}{\partial t}I(x,t)+\frac{\partial}{\partial t}R(x,t)\right]dx\\&=\int_\Omega[\nabla\cdot(d_S(x)\nabla S)+\nabla\cdot(d_V(x)\nabla V)+\nabla\cdot(d_E(x)\nabla E)+\nabla\cdot(d_I(x)\nabla I)+\nabla\cdot(d_R(x)\nabla R)\\&+\Lambda(x)-\mu(x)S-\mu(x)E-(\mu(x)+c(x))Q-(\mu(x)+c(x))I-\mu(x)R]dx\\&\leq\int_\Omega\nabla\cdot(d_S(x)\nabla S)dx+\int_\Omega\nabla\cdot(d_V(x)\nabla V)dx+\int_\Omega\nabla\cdot(d_E(x)\nabla E)dx+\int_\Omega\nabla\cdot(d_I(x)\nabla I)dx\\&+\int_\Omega\nabla\cdot(d_R(x)\nabla R)dx+\int_\Omega[\Lambda^*-\mu_*(S+V+E+Q+I+R)]dx\\&\leq\Lambda^*\mid\Omega\mid-\mu_*U(t).\end{aligned}$$

According to the Gronwall's inequality [20, Lemma 2.3],

$$U(t)\leq U_0e^{-\mu_*t}+\frac{\Lambda^*\mid\Omega\mid}{\mu_*}(1-e^{-\mu_*t}),$$

that is $U(t)\leq\max\{U_0,\frac{\Lambda^*|\Omega|}{\mu_*}\}$, where

$$U_0=\int_\Omega[S_0(x,0)+V_0(x,0)+E_0(x,0)+Q_0(x,0)+I_0(x,0)+R_0(x,0)]dx$$

$$\leq \int_{\Omega} \| S_0(\cdot,0)+V_0(x,0)+E_0(\cdot,0)+Q_0(\cdot,0)+I_0(\cdot,0)+R_0(\cdot,0) \|_{L^\infty(\Omega)} dx$$
$$=\| S_0(\cdot,0)+V_0(\cdot,0)+E_0(\cdot,0)+Q_0(\cdot,0)+I_0(\cdot,0)+R_0(\cdot,0) \|_{L^\infty(\Omega)} | \Omega | .$$

Hence, $U(t)=\int_{\Omega}(S+V+E+Q+I+R)dx$ is bounded. From the positive of the solution for system (2.1)

$$\| S+V+E+Q+I+R \|_{L^1(\Omega)} = \int_{\Omega} | S+V+E+Q+I+R | (x,t)dx$$
$$=\int_{\Omega}(S+V+E+Q+I+R)(x,t)dx = U(t)$$
$$\leq \max\left\{ \| S_0(x,0)+V_0(x,0)+E_0(x,0)+Q_0(x,0)+I_0(x,0)+R_0(x,0) \|_{L^\infty(\Omega)} | \Omega |, \frac{\Lambda^* | \Omega |}{\mu_*}\right\} := K,$$

so $\int_{\Omega}(S+V+E+Q+I+R)dx \leq K$. According [42, Lemma 2.1], there is a positive constant $K^*$ independent of $K$ such that $\| S+V+E+Q+I+R \|_{L^\infty(\Omega)} \leq K^*$. Therefore $S(x,t), V(x,t), E(x,t), I(x,t), Q(x,t), R(x,t)$ is uniformly bounded on $\bar{\Omega}$. From [38, Corollary 11.3.2] , $u(t)$ is global solution of system (2.1).

Finally, we prove $E(x,t), I(x,t)$ is ultimately bounded. According to the method in [25], let $T_i(t) : \mathbf{C}(\overline{\Omega}, R) \to \mathbf{C}(\overline{\Omega}, R)$ for $i=1,2,3,4,5$ are $C_0$-semigroup associated with the operators $\nabla\cdot(d_i\nabla)-\varsigma_i(x)$ subject to the Neumann boundary condition, where $\varsigma_1(x)=\alpha(x)+\mu(x), \varsigma_2(x)=\mu(x), \varsigma_3(x)=\sigma(x)+\gamma(x)+\mu(x), \varsigma_4(x)=\rho(x)+\eta(x)+c(x)+\mu(x), \varsigma_5(x)=\omega(x)+c(x)+\mu(x), \varsigma_6(x)=\phi(x)+\mu(x)$ , $d_1(x)=d_S(x), d_2(x)=d_V(x), d_3(x)=d_E(x), d_4(x)=d_I(x), d_5(x)=d_R(x)$. Then

$$(T_i\varsigma)(x)=\int_{\Omega}\Gamma_i(t,x,y)\varsigma(y)dy,\ i=1,2,3,4,5,$$

for any $t \geq 0$ and $\varsigma \in \mathbf{C}(\overline{\Omega}, R)$, $\Gamma_i(t,x,y)$ are the Green function associated with $\nabla\cdot(d_i\nabla)-\mu(x)$ subject to the Neumann boundary condition. Then there is

$$E(x,t)=T_3(x)E(0,x)+\int_0^t T_3(t)[\beta_1(x)Sf(E)+\beta_2(x)Sf(I)+\delta(x)\beta_1(x)Vf(E)+\delta(x)\beta_2(x)Vf(I)$$
$$-\sigma(x)E-\gamma(x)+\mu(x)E]ds$$
$$\leq K^{'}e^{\alpha_3 t} \| E(0,\cdot) \|_{\mathbf{C}(\overline{\Omega},R)} + \int_0^t\int_{\Omega}\Gamma_3(t,x,y)[\beta_1(x)Sf(E)+\beta_2(x)Sf(I)+\delta(x)\beta_1(x)Vf(E)+$$
$$\delta(x)\beta_2(x)Vf(I)-\sigma(x)E-\gamma(x)+\mu(x)E]dyds$$
$$\leq K^{'}e^{\alpha_3 t} \| E(0,\cdot) \|_{\mathbf{C}(\overline{\Omega},R)} + \int_0^t \sigma_3 e^{-\mu_*(t)}\int_{\Omega}[\beta_1(x)Sf(E)+\beta_2(x)Sf(I)+\delta(x)\beta_1(x)Vf(E)+$$
$$\delta(x)\beta_2(x)Vf(I)-\sigma(x)E-\gamma(x)+\mu(x)E]dyds$$
$$\leq K^{'}e^{\alpha_3 t} \| E(0,\cdot) \|_{\mathbf{C}(\overline{\Omega},R)} + \sigma_3 K^*\int_0^t e^{-\mu_*(t)}\int_{\Omega}[\beta_1(x)Sf(E)+\beta_2(x)Sf(I)+\delta(x)\beta_1(x)Vf(E)$$
$$+\delta(x)\beta_2(x)Vf(I)-\sigma(x)E-\gamma(x)+\mu(x)E]dyds$$
$$\leq K^{'}e^{\alpha_3 t} \| E(0,\cdot) \|_{\mathbf{C}(\overline{\Omega},R)} + \frac{\sigma_3 K^*}{\mu_*}(\beta_1^* c_1 K^* + \beta_2^* c_2 K^* + \delta^*\beta_1^* c_1 K^* + \delta^*\beta_2^* c_2 K^*).$$

Then

$$\|E(\cdot,t)\|_{\mathbf{C}(\overline{\Omega},R)} \leq K^{'}e^{\alpha_3 t} \| E(0,\cdot) \|_{\mathbf{C}(\overline{\Omega},R)} + \frac{\sigma_3 K^*}{\mu_*}(\beta_1^* c_1 K^* + \beta_2^* c_2 K^* + \delta^*\beta_1^* c_1 K^* + \delta^*\beta_2^* c_2 K^*) := M_1.$$

That is

$$\limsup_{t\to\infty} E(x,t) \leq M_1.$$

Similarly, we have

$$\limsup_{t\to\infty} I(x,t) \leq M_2.$$

Let $M = \max\{M_1, M_2\}$ and $M_1, M_2$ dependent on initial conditions, so $E(x,t), I(x,t)$ is ultimately bounded, it satisfy

$$\limsup_{t\to\infty} E(x,t) \le M, \ \limsup_{t\to\infty} I(x,t) \le M.$$

□

# 4. Global exponential attraction set and threshold dynamics

In this section, first of all we obtain the existence of the global exponential attraction set by means of semi-group theory and the global exponential attractor theory in infinite dynamical systems. Then we prove the global asymptotic stability of the disease-free equilibrium and uniform persistence of the system (2.1). Here, we give some definitions:

Let $\mathbf{X}$ be a Banach space with the decomposition

$$\mathbf{X} = \mathbf{X_1} \oplus \mathbf{X_2}; \dim \mathbf{X_1} < \infty,$$

and denote orthogonal projector by $P : \mathbf{X} \to \mathbf{X_1}$ and $(I - P) : \mathbf{X} \to \mathbf{X_2}$. In addition, let $Q(t)_{t\ge 0}$ be a continuous semigroup on $\mathbf{X}$. The following Condition $\mathbf{C}^*$ is an important condition for verifying the global exponential attraction [43]:

**Condition**$(\mathbf{C}^*)$ : For any bounded set $\mathbf{B} \subset \mathbf{X}$, there exist positive constants $t_{\mathbf{B}}$, $C$ and $\alpha$ such that for any $\varepsilon > 0$, there exists a finite dimensional subspace $\mathbf{X_1} \subset \mathbf{X}$ satisfies

$$\| PQ(t)\mathbf{B} \|_{t\ge t_{\mathbf{B}}} \text{ is bounded,}$$

$$\| (I - P)Q(t)\mathbf{B} \| < Ce^{-\alpha t} + \varepsilon \text{ for } t \ge t_{\mathbf{B}},$$

where $P : \mathbf{X} \to \mathbf{X_1}$ is a orthogonal projector.

In the following, define $\mathbf{H} = \mathbf{L}^2(\Omega) \cap \mathbf{C}^{2,1}(\Omega)$, $\mathbf{H}_1 = \mathbf{H}_0^1(\Omega), \mathbf{H}^6 = \mathbf{H} \times \mathbf{H} \times \mathbf{H} \times \mathbf{H} \times \mathbf{H} \times \mathbf{H}$ and $\mathbf{H}_1^6 = \mathbf{H}_1 \times \mathbf{H}_1 \times \mathbf{H}_1 \times \mathbf{H}_1 \times \mathbf{H}_1 \times \mathbf{H}_1$. Note that $\mathbf{H}^6$ and $\mathbf{H}_1^6$ are Banach spaces equipped with norm

$$\| (S, V, E, Q, I, R)^T \|_{\mathbf{H}^6} := \max\{\| S \|_{\mathbf{H}}, \| V \|_{\mathbf{H}}, \| E \|_{\mathbf{H}}, \| Q \|_{\mathbf{H}}, \| I \|_{\mathbf{H}}, \| R \|_{\mathbf{H}}\},$$

and

$$\| (S, V, E, Q, I, R)^T \|_{\mathbf{H}_1^6} := \max\{\| S \|_{\mathbf{H}_1}, \| V \|_{\mathbf{H}_1}, \| E \|_{\mathbf{H}_1}, \| Q \|_{\mathbf{H}_1}, \| I \|_{\mathbf{H}_1}, \| R \|_{\mathbf{H}_1}\}.$$

## 4.1. The existence of global exponential attraction set

In this subsection, we prove the existence of a global exponential attraction set.

**Theorem 4.1.** *System* (2.1) *has a global exponential attraction set* $\mathcal{A}^*$*, which exponentially attracts any bounded set in* $\mathbf{H^6}$.

**Proof.** From the existence of the solution, for any $\varphi = u(0) = (S_0(\cdot), V_0(\cdot), E_0(\cdot), Q_0(\cdot), I_0(\cdot), R_0(\cdot))^T \in \mathbf{H^6}$, then system (2.1) has a global solution $u = (S, V, E, Q, I, R)^T \in C^0([0, \infty), \mathbf{H^6})$. This means that system (2.1) generates an operator semigroup $\overline{Q}(t) = (\overline{Q_1}(t), \overline{Q_2}(t), \overline{Q_3}(t), \overline{Q_4}(t), \overline{Q_5}(t), \overline{Q_6}(t))^T$, and $\overline{Q}(t)\varphi = u(t, \varphi)$.

First, we prove the operator semigroup $\overline{Q}(t)$ has an absorbing set $B_{\widehat{R}} \subset \mathbf{H^6}$. Make the inner product of $S, V, E, Q, I, R$ and the first, second, third, fourth, fifth and sixth equations of system (2.1) respectively

$$\begin{aligned}
&\langle \nabla \cdot (d_S(x)\nabla S) + \Lambda(x) - \beta_1(x)Sf(E) - \beta_2(x)Sf(I) - [\alpha(x) + \mu(x)]S, S\rangle_{\mathbf{H}} \\
&= \int_\Omega \nabla \cdot (d_S(x)\nabla S)Sdx + \int_\Omega \Lambda(x)Sdx - \int_\Omega \beta_1(x)S^2 f(E)dx - \int_\Omega \beta_2(x)S^2 f(I)dx \\
&\quad - \int_\Omega [\alpha(x) + \mu(x)]S^2 dx \\
&= \int_\Omega \sum_{i=1}^n S \cdot \frac{\partial}{\partial x_i}(d_S(x)\frac{\partial S}{\partial x_i})dx + \int_\Omega \Lambda(x)Sdx - \int_\Omega \beta_1(x)S^2 f(E)dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_\Omega \beta_2(x)S^2 f(I)dx - \int_\Omega [\alpha(x)+\mu(x)]S^2 dx \\
&= \sum_{i=1}^n \int_\Omega S\cdot \frac{\partial}{\partial x_i}(d_S(x)\frac{\partial S}{\partial x_i})dx + \int_\Omega \Lambda(x)Sdx - \int_\Omega \beta_1(x)S^2 f(E)dx \\
&-\int_\Omega \beta_2(x)S^2 f(I)dx - \int_\Omega [\alpha(x)+\mu(x)]S^2 dx \\
&= -\sum_{i=1}^n \int_\Omega d_S(x)(\frac{\partial S}{\partial x_i})^2 dx + \sum_{i=1}^n \int_{\partial\Omega} S\cdot(d_S(x)\frac{\partial S}{\partial x_i})\cdot \mathbf{n}_{x_i} d\mathbf{s} + \int_\Omega \Lambda(x)Sdx \\
&-\int_\Omega \beta_1(x)S^2 f(E)dx - \int_\Omega \beta_2(x)S^2 f(I)dx - \int_\Omega [\alpha(x)+\mu(x)]S^2 dx \\
&= -\int_\Omega d_S(x)\sum_{i=1}^n(\frac{\partial S}{\partial x_i})^2 dx + \int_{\partial\Omega} S(d_S(x)\frac{\partial S}{\partial \mathbf{n}})d\mathbf{s} + \int_\Omega \Lambda(x)Sdx - \int_\Omega \beta_1(x)S^2 f(E)dx \\
&-\int_\Omega \beta_2(x)S^2 f(I)dx - \int_\Omega [\alpha(x)+\mu(x)]S^2 dx \\
&= -\int_\Omega d_S(x) \mid \nabla S \mid^2 dx + \int_\Omega \Lambda(x)Sdx - \int_\Omega \beta_1(x)S^2 f(E)dx - \int_\Omega \beta_2(x)S^2 f(I)dx \\
&-\int_\Omega [\alpha(x)+\mu(x)]S^2 dx \\
&\le -(d_S)_* \int_\Omega \mid \nabla S \mid^2 dx + \Lambda^* \int_\Omega Sdx \\
&= -(d_S)_* \parallel S \parallel^2_{\mathbf{H}^{\frac{1}{2}}} + \Lambda^* \int_\Omega Sdx.
\end{aligned}
$$

$$
\begin{aligned}
&\langle \nabla\cdot(d_V(x)\nabla V) + \alpha(x)S + \phi(x)R - \delta(x)\beta_1(x)Vf(E) - \delta(x)\beta_2(x)Vf(I) - \mu(x)V, V\rangle_{\mathbf{H}} \\
&= \int_\Omega \nabla\cdot(d_V(x)\nabla V)Vdx + \int_\Omega [\alpha(x)S + \phi(x)R]Vdx - \int_\Omega \delta(x)\beta_1(x)V^2 f(E) \\
&\quad - \int_\Omega \delta(x)\beta_2(x)V^2 f(I) - \int_\Omega \mu(x)V^2 dx \\
&\le -(d_V)_* \int_\Omega \mid \nabla V \mid^2 dx + \alpha^* \int_\Omega SVdx + \phi^* \int_\Omega VRdx \\
&= -(d_V)_* \parallel V \parallel^2_{\mathbf{H}^{\frac{1}{2}}} + \alpha^* \int_\Omega SVdx + \phi^* \int_\Omega VRdx.
\end{aligned}
$$

$$
\begin{aligned}
&\langle \nabla\cdot(d_E(x)\nabla E) + \beta_1(x)Sf(E) + \beta_2(x)Sf(I) + \delta(x)\beta_1(x)Vf(E) + \delta(x)\beta_2(x)Vf(I) \\
&\quad - [\sigma(x)+\gamma(x)+\mu(x)]E, E\rangle_{\mathbf{H}} \\
&= \int_\Omega \nabla\cdot(d_E(x)\nabla E)Edx + \int_\Omega \beta_1(x)SEf(E)dx + \int_\Omega \beta_2(x)SEf(I)dx \\
&\quad + \int_\Omega \delta(x)\beta_1(x)VEf(E)dx + \int_\Omega \delta(x)\beta_2(x)VEf(I)dx - \int_\Omega [\sigma(x)+\gamma(x)+\mu(x)]E^2 dx \\
&\le -(d_E)_* \int_\Omega \mid \nabla E \mid^2 dx + \beta_1^* \int_\Omega SEf(E)dx + \beta_2^* \int_\Omega SEf(I)dx + \delta^*\beta_1^* \int_\Omega VEf(E)dx \\
&\quad + \delta^*\beta_2^* \int_\Omega VEf(I)dx \\
&= -(d_E)_* \parallel E \parallel^2_{\mathbf{H}^{\frac{1}{2}}} + \beta_1^* \int_\Omega SEf(E)dx + \beta_2^* \int_\Omega SEf(I)dx + \delta^*\beta_1^* \int_\Omega VEf(E)dx
\end{aligned}
$$

$$+\delta^*\beta_2^*\int_\Omega VEf(I)dx.$$

$$\begin{aligned}&\langle\sigma(x)E-[\rho(x)+\eta(x)+c(x)+\mu(x)]Q,Q\rangle_{\mathbf{H}}\\&=\int_\Omega\sigma(x)EQdx-\int_\Omega[\rho(x)+\eta(x)+c(x)+\mu(x)]Q^2dx\\&\leq-\rho_*\parallel Q\parallel_{\mathbf{H}}^2+\sigma^*\int_\Omega EQdx.\end{aligned}$$

$$\begin{aligned}&\langle\nabla\cdot(d_I(x)\nabla I)+\gamma(x)E+\eta(x)Q-[\omega(x)+c(x)+\mu(x)]I,I\rangle_{\mathbf{H}}\\&=\int_\Omega\nabla\cdot(d_I(x)\nabla I)Idx+\int_\Omega\gamma(x)EIdx+\int_\Omega\eta(x)QIdx-\int_\Omega[\omega(x)+c(x)+\mu(x)]I^2\\&\leq-(d_I)_*\int_\Omega\mid\nabla I\mid^2dx+\gamma^*\int_\Omega EIdx+\eta^*\int_\Omega QIdx\\&=-(d_I)_*\parallel I\parallel_{\mathbf{H}^{\frac12}}^2+\gamma^*\int_\Omega EIdx+\eta^*\int_\Omega QIdx.\end{aligned}$$

$$\begin{aligned}&\langle\nabla\cdot(d_R(x)\nabla R)+\rho(x)Q+\omega(x)I-[\phi(x)+\mu(x)]R,R\rangle_{\mathbf{H}}\\&=\int_\Omega\nabla\cdot(d_R(x)\nabla R)Rdx+\int_\Omega\rho(x)QRdx+\int_\Omega\omega(x)IRdx-\int_\Omega[\phi(x)+\mu(x)]R^2dx\\&\leq-(d_R)_*\int_\Omega\mid\nabla R\mid^2dx+\rho^*\int_\Omega QRdx+\omega^*\int_\Omega IRdx\\&=-(d_R)_*\parallel R\parallel_{\mathbf{H}^{\frac12}}^2+\rho^*\int_\Omega QRdx+\omega^*\int_\Omega IRdx.\end{aligned}$$

Where $\mathbf{H}^{\frac12}$ is the fractional power subspace generated by the sectorial operator $L$ [44]. According to Theorem 3.1, $S,V,E,Q,I,R$ is positive and uniformly bounded, so there exist constant $C_1,C_2,C_3,C_4,C_5,C_6>0$, such that

$$\Lambda^*\int_\Omega Sdx\leq C_1,\quad \alpha^*\int_\Omega SVdx+\phi^*\int_\Omega VRdx\leq C_2,$$

$$\beta_1^*\int_\Omega SEf(E)dx+\beta_2^*\int_\Omega SEf(I)dx+\delta^*\beta_1^*\int_\Omega VEf(E)dx+\delta^*\beta_2^*\int_\Omega VEf(I)dx\leq C_3,$$

$$\sigma^*\int_\Omega EQdx\leq C_4,\quad \gamma^*\int_\Omega EIdx+\eta^*\int_\Omega QIdx\leq C_5,\quad \rho^*\int_\Omega QRdx+\omega^*\int_\Omega IRdx\leq C_6.$$

From $\mathbf{H}^{\frac12}\hookrightarrow\mathbf{H}$, then there exist constant $C>0$, such that $\parallel S\parallel_{\mathbf{H}^{\frac12}}\geq C\parallel S\parallel_{\mathbf{H}},\parallel V\parallel_{\mathbf{H}^{\frac12}}\geq C\parallel V\parallel_{\mathbf{H}}$, $\parallel E\parallel_{\mathbf{H}^{\frac12}}\geq C\parallel E\parallel_{\mathbf{H}},\parallel I\parallel_{\mathbf{H}^{\frac12}}\geq C\parallel I\parallel_{\mathbf{H}},\parallel R\parallel_{\mathbf{H}^{\frac12}}\geq C\parallel R\parallel_{\mathbf{H}}$, for any $S,V,E,Q,I,R\in\mathbf{H}^{\frac12}$.

Because $\frac12\frac{d}{dt}\langle S,S\rangle=\langle S_t,S\rangle_{\mathbf{H}}$, $\frac12\frac{d}{dt}\langle V,V\rangle=\langle V_t,V\rangle_{\mathbf{H}}$, $\frac12\frac{d}{dt}\langle E,E\rangle=\langle E_t,E\rangle_{\mathbf{H}}$, $\frac12\frac{d}{dt}\langle Q,Q\rangle=\langle Q_t,Q\rangle_{\mathbf{H}}$, $\frac12\frac{d}{dt}\langle I,I\rangle=\langle I_t,I\rangle_{\mathbf{H}}$, $\frac12\frac{d}{dt}\langle R,R\rangle=\langle R_t,R\rangle_{\mathbf{H}}$, we have

$$\frac12\frac{d}{dt}\parallel S\parallel_{\mathbf{H}}^2\leq-d_SC^2\parallel S\parallel_{\mathbf{H}}^2+C_1,\ \frac12\frac{d}{dt}\parallel V\parallel_{\mathbf{H}}^2\leq-d_VC^2\parallel V\parallel_{\mathbf{H}}^2+C_2,$$

$$\frac12\frac{d}{dt}\parallel E\parallel_{\mathbf{H}}^2\leq-d_EC^2\parallel E\parallel_{\mathbf{H}}^2+C_3,\ \frac12\frac{d}{dt}\parallel Q\parallel_{\mathbf{H}}^2\leq-\rho\parallel Q\parallel_{\mathbf{H}}^2+C_4,$$

$$\frac12\frac{d}{dt}\parallel I\parallel_{\mathbf{H}}^2\leq-d_IC^2\parallel I\parallel_{\mathbf{H}}^2+C_4,\ \frac12\frac{d}{dt}\parallel R\parallel_{\mathbf{H}}^2\leq-d_RC^2\parallel R\parallel_{\mathbf{H}}^2+C_6.$$

From Gronwall's inequality [20, Lemma 2.3], let $\gamma_1=2d_SC^2,\gamma_2=2d_VC^2,\gamma_3=2d_EC^2,\gamma_4=2\rho,\gamma_5=2d_IC^2,\gamma_6=$

$2d_R C^2$, then

$$\| S \|_{\mathbf{H}}^2 \le e^{-\gamma_1 t} \| S_0(x) \|_{\mathbf{H}}^2 + \frac{2C_1}{\gamma_1}(1-e^{-\gamma_1 t}),$$

$$\| V \|_{\mathbf{H}}^2 \le e^{-\gamma_2 t} \| V_0(x) \|_{\mathbf{H}}^2 + \frac{2C_2}{\gamma_2}(1-e^{-\gamma_2 t}),$$

$$\| E \|_{\mathbf{H}}^2 \le e^{-\gamma_3 t} \| V_0(x) \|_{\mathbf{H}}^2 + \frac{2C_3}{\gamma_3}(1-e^{-\gamma_3 t}),$$

$$\| Q \|_{\mathbf{H}}^2 \le e^{-\gamma_4 t} \| Q_0(x) \|_{\mathbf{H}}^2 + \frac{2C_4}{\gamma_4}(1-e^{-\gamma_4 t}),$$

$$\| I \|_{\mathbf{H}}^2 \le e^{-\gamma_5 t} \| I_0(x) \|_{\mathbf{H}}^2 + \frac{2C_5}{\gamma_5}(1-e^{-\gamma_5 t}),$$

$$\| R \|_{\mathbf{H}}^2 \le e^{-\gamma_6 t} \| R_0(x) \|_{\mathbf{H}}^2 + \frac{2C_6}{\gamma_6}(1-e^{-\gamma_6 t}).$$

Therefore, if admits $(\widehat{R})^2 > \max\{\frac{2C_1}{\gamma_1}, \frac{2C_2}{\gamma_2}, \frac{2C_3}{\gamma_3}, \frac{2C_4}{\gamma_4}, \frac{2C_5}{\gamma_5}, \frac{2C_6}{\gamma_6}\}$, then there exist $t_0 > 0$, such that for any $t \ge t_0$, $u(t,\varphi) \subset \mathbf{B}_{\widehat{R}}$ and $\mathbf{B}_{\widehat{R}}$ is a closed unit ball with a radius of $\widehat{R}$. So $\mathbf{B}_{\widehat{R}} \subset \mathbf{H}^6$ is the absorbing set.

Then we verify **Condition (C*)**. Because $L_1 = \nabla \cdot (d_S(x)\nabla) : \mathbf{H}_1 \to \mathbf{H}$ is a symmetric sector operator, the eigenvectors $\{e_j\}_{j\in\mathbb{N}}$ corresponding to the eigenvalues $\{\lambda_{S,j}\}_{j\in\mathbb{N}}$ are the complete orthonormal on $\mathbf{H}$, that is for any $S \in \mathbf{H}$,

$$S = \sum_{k=1}^{\infty} x_k e_k, \| S \|_{\mathbf{H}}^2 = \sum_{k=1}^{\infty} x_k^2.$$

Furthermore, for any $N_S > 0$, there exists a integer $K_S \ge 1$ such that $-N_S \ge \lambda_{S,j}$, for any $j \ge K_S + 1$. Let

$$\mathbf{H}_1^{K_S} = span\{e_1, e_2, \cdots, e_{K_S}\} \text{ and } \mathbf{H}_2^{K_S} = (\mathbf{H}_1^{K_S})^{\perp}.$$

Then, for any $S \in \mathbf{H}$, it can be decomposed into

$$S = PS + (I-P)S := S_1 + S_2, \quad S_1 = \sum_{i=1}^{K_S} x_i e_i \in \mathbf{H}_1^{K_S}, S_2 = \sum_{i=K_S+1}^{\infty} x_j e_j \in \mathbf{H}_2^{K_S},$$

where $P : \mathbf{H} \to \mathbf{H}_1^{K_S}$ is the orthogonal projector. For any $V, E, Q, I, R \in \mathbf{H}$, there's a similar decomposition.

Because $\overline{Q_t}$ has a bounded absorption set $\mathbf{B}_{\widehat{R}} \subset \mathbf{H}^6$, for any bounded set $\mathbf{B} \subset \mathbf{H}^6$, there is $t_{\mathbf{B}} > 0$, without losing generality, assuming $t_{\mathbf{B}} > t_0$, such that

$$(S(t, S_0(x)), V(t, V_0(x)), E(t, E_0(x)), Q(t, Q_0(x)), I(t, I_0(x)), R(t, R_0(x))) \subset \mathbf{B}_{\widehat{R}},$$

where

$$S(t, S_0(x)) = \overline{Q_1}(t)S_0(x), V(t, V_0(x)) = \overline{Q_2}(t)V_0(x), E(t, E_0(x)) = \overline{Q_3}(t)E_0(x),$$
$$Q(t, Q_0(x)) = \overline{Q_4}(t)Q_0(x), I(t, I_0(x)) = \overline{Q_5}(t)I_0(x), R(t, R_0(x)) = \overline{Q_6}(t)R_0(x).$$

For any $(S_0(x), V_0(x), E_0(x), Q_0(x), I_0(x), R_0(x)) \in \mathbf{B}$, $t \ge t_0$, then

$$\| S(t, S_0(x)) \|_{\mathbf{H}}^2 = \| \overline{Q_1}(t)S_0(x) \|_{\mathbf{H}}^2 \le \widehat{R^2}, \text{ for any } t \ge t_{\mathbf{B}},$$
$$\| V(t, V_0(x)) \|_{\mathbf{H}}^2 = \| \overline{Q_2}(t)V_0(x) \|_{\mathbf{H}}^2 \le \widehat{R^2}, \text{ for any } t \ge t_{\mathbf{B}},$$
$$\| E(t, E_0(x)) \|_{\mathbf{H}}^2 = \| \overline{Q_3}(t)E_0(x) \|_{\mathbf{H}}^2 \le \widehat{R^2}, \text{ for any } t \ge t_{\mathbf{B}},$$
$$\| Q(t, Q_0(x)) \|_{\mathbf{H}}^2 = \| \overline{Q_4}(t)Q_0(x) \|_{\mathbf{H}}^2 \le \widehat{R^2}, \text{ for any } t \ge t_{\mathbf{B}},$$
$$\| I(t, I_0(x)) \|_{\mathbf{H}}^2 = \| \overline{Q_5}(t)I_0(x) \|_{\mathbf{H}}^2 \le \widehat{R^2}, \text{ for any } t \ge t_{\mathbf{B}},$$
$$\| R(t, R_0(x)) \|_{\mathbf{H}}^2 = \| \overline{Q_6}(t)R_0(x) \|_{\mathbf{H}}^2 \le \widehat{R^2}, \text{ for any } t \ge t_{\mathbf{B}},$$

so $\| P\overline{Q}(t)\varphi \|_{\mathbf{H}^6} \leq \widehat{R}$, for any $t \geq t_{\mathbf{B}}$, that is $\| P\overline{Q}(t)\mathbf{B} \|_{\mathbf{H}^6 t \geq t_{\mathbf{B}}}$ is bounded.

The first equation from system (2.1), take the inner product with $S_2$, then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\langle S, S_2\rangle_{\mathbf{H}} = \langle S_t, S_2\rangle_{\mathbf{H}}\\
&= \langle \nabla\cdot(d_S(x)\nabla S) + \Lambda(x) - \beta_1(x)Sf(E) - \beta_2(x)Sf(I) - [\alpha(x)+\mu(x)]S, S_2\rangle_{\mathbf{H}}\\
&= \langle \nabla\cdot(d_S(x)\nabla S), S_2\rangle_{\mathbf{H}} + \langle \Lambda(x) - \beta_1(x)Sf(E) - \beta_2(x)Sf(I) - [\alpha(x)+\mu(x)]S, S_2\rangle_{\mathbf{H}}\\
&= \langle \nabla\cdot(d_S(x)\nabla S_1) + \nabla\cdot(d_S(x)\nabla S_2), S_2\rangle_{\mathbf{H}}\\
&\quad + \langle \Lambda(x) - \beta_1(x)Sf(E) - \beta_2(x)Sf(I) - [\alpha(x)+\mu(x)]S, S_2\rangle_{\mathbf{H}}\\
&\leq \langle \nabla\cdot(d_S(x)\nabla S_2), S_2\rangle_{\mathbf{H}} + \langle \Lambda(x) - \beta_1(x)Sf(E) - \beta_2(x)Sf(I) - [\alpha(x)+\mu(x)]S, S_2\rangle_{\mathbf{H}},
\end{aligned}$$

and

$$\begin{aligned}
\langle \nabla\cdot(d_S(x)\nabla S_2), S_2\rangle_{\mathbf{H}} &= -(d_S)_* \| S_2 \|^2_{\mathbf{H}^{\frac{1}{2}}} = (d_S)_* \sum_{j=K_S+1}^{\infty} \lambda_{S,j} x_j^2\\
&\leq -(d_S)_* N_S \sum_{j=K_S+1}^{\infty} x_j^2 = -(d_S)_* N_S \| S_2 \|^2_{\mathbf{H}}.
\end{aligned}$$

Because $\| S \|_{\mathbf{H}} \leq \widehat{R}, \| E \|_{\mathbf{H}} \leq \widehat{R}, \| I \|_{\mathbf{H}} \leq \widehat{R}$ for $t \geq t_{\mathbf{B}}$,

$$\begin{aligned}
&\langle \Lambda(x) - \beta_1(x)Sf(E) - \beta_2(x)Sf(I) - [\alpha(x)+\mu(x)]S, S_2\rangle_{\mathbf{H}}\\
&\leq \| \Lambda(x) - \beta_1(x)Sf(E) - \beta_2(x)Sf(I) - [\alpha(x)+\mu(x)]S \|_{\mathbf{H}} \| S_2 \|_{\mathbf{H}}\\
&\leq (\| \Lambda(x) \|_{\mathbf{H}} + \| \beta_1(x)SC_1E \|_{\mathbf{H}} + \| \beta_2(x)SC_2I \|_{\mathbf{H}} + \| [\alpha(x)+\mu(x)]S \|_{\mathbf{H}})\widehat{R}\\
&\leq \{\Lambda^* + \beta_1^* C_1 \widehat{R}^2 + \beta_2^* C_2 \widehat{R}^2 + [\alpha(x)+\mu(x)]\widehat{R}\}\widehat{R} := C_{S,\widehat{R}}.
\end{aligned}$$

So $\frac{d}{dt} \| S_2 \|^2_{\mathbf{H}} \leq -2d_S N_S \| u_2 \|^2_{\mathbf{H}} + 2C_{S,\widehat{R}}$. Thus, for any $N_S > 0,\ t > t_{\mathbf{B}}$,

$$\| S_2 \|^2_{\mathbf{H}} \leq e^{-2d_S N_S (t-t_{\mathbf{B}})} \| S_2(t_{\mathbf{B}}) \|^2_{\mathbf{H}} + \frac{C_{S,\widehat{R}}}{d_S N_S}(1 - e^{-2d_S N_S (t-t_{\mathbf{B}})}).$$

Similarly,

$$\begin{aligned}
\| V_2 \|^2_{\mathbf{H}} &\leq e^{-2d_V N_V (t-t_{\mathbf{B}})} \| V_2(t_{\mathbf{B}}) \|^2_{\mathbf{H}} + \frac{C_{V,\widehat{R}}}{d_V N_V}(1 - e^{-2d_V N_V (t-t_{\mathbf{B}})}),\\
\| E_2 \|^2_{\mathbf{H}} &\leq e^{-2d_E N_E (t-t_{\mathbf{B}})} \| E_2(t_{\mathbf{B}}) \|^2_{\mathbf{H}} + \frac{C_{E,\widehat{R}}}{d_E N_E}(1 - e^{-2d_E N_E (t-t_{\mathbf{B}})}),\\
\| Q_2 \|^2_{\mathbf{H}} &\leq e^{-2\rho N_Q (t-t_{\mathbf{B}})} \| Q_2(t_{\mathbf{B}}) \|^2_{\mathbf{H}} + \frac{C_{Q,\widehat{R}}}{\rho N_Q}(1 - e^{-2d_Q N_E (t-t_{\mathbf{B}})}),\\
\| I_2 \|^2_{\mathbf{H}} &\leq e^{-2d_I N_I (t-t_{\mathbf{B}})} \| I_2(t_{\mathbf{B}}) \|^2_{\mathbf{H}} + \frac{C_{I,\widehat{R}}}{d_I N_I}(1 - e^{-2d_I N_I (t-t_{\mathbf{B}})}),\\
\| R_2 \|^2_{\mathbf{H}} &\leq e^{-2d_R N_R (t-t_{\mathbf{B}})} \| R_2(t_{\mathbf{B}}) \|^2_{\mathbf{H}} + \frac{C_{R,\widehat{R}}}{d_R N_R}(1 - e^{-2d_R N_R (t-t_{\mathbf{B}})}).
\end{aligned}$$

Therefore, **Condition**$(\mathbf{C}^*)$ holds. According to [43, Theorem 4.1], the system (2.1) has a global exponential absorption set $\mathcal{A}^*$.

□

## 4.2. Threshold dynamics

After obtaining the global exponential attraction set, we will analyze the dynamic properties of the system (2.1).

**Theorem 4.2.** *For any* $\big(S(x,0),V(x,0),E(x,0),Q(x,0),I(x,0),R(x,0)\big)>0$, $x\in\Omega$, *the following statements are true:*
(1) *If* $\lambda^*<0$, *then*

$$\lim_{t\to\infty}S(x,t)=S^0(x),\ \lim_{t\to\infty}V(x,t)=V^0(x),\ \lim_{t\to\infty}E(x,t)=0,$$
$$\lim_{t\to\infty}Q(x,t)=0,\ \lim_{t\to\infty}I(x,t)=0,\ \lim_{t\to\infty}R(x,t)=0$$

*on* **H**. *The disease-free equilibrium of the system* (2.1) *is globally asymptotically stable.*
(2) *If* $\lambda^*>0$, *then there exist a function* $h(x)>0$ *that does not depend on the initial value, such that for any* $x\in\overline{\Omega}$ *and solution* $(S,V,E,Q,I,R)$ *of the system* (2.1) *satisfied*

$$\liminf_{t\to\infty}S(x,t)\geq h(x),\liminf_{t\to\infty}V(x,t)\geq h(x),\liminf_{t\to\infty}E(x,t)\geq h(x),$$
$$\liminf_{t\to\infty}Q(x,t)\geq h(x),\liminf_{t\to\infty}I(x,t)\geq h(x),\liminf_{t\to\infty}R(x,t)\geq h(x).$$

*The system* (2.1) *is uniformly persistent.*

**Proof.** (1) if $\lambda^*<0$, from the first equation of system (2.1),

$$\begin{cases}\dfrac{\partial S}{\partial t}\leq\nabla\cdot(d_S(x)\nabla S)+\Lambda(x)-[\alpha(x)+\mu(x)]S,\ x\in\Omega,t>0.\\ \dfrac{\partial S}{\partial\mathbf{n}}=0,\ x\in\Omega.\end{cases}$$

By the comparison principle [45] and [25, Lemma 2], we have

$$\limsup_{t\to\infty}S(x,t)\leq S^0(x)\text{ uniformly for }x\in\overline{\Omega}.$$

Without loss of the generality, we assume that $S(x,t)\leq S^0(x)+\varrho$ and $\varrho$ is a sufficiently small constant, for any $t\geq0,\ x\in\Omega$. First of all, from system (2.1),

$$\begin{cases}\dfrac{\partial E}{\partial t}\leq\nabla\cdot(d_E(x)\nabla E)+[c_1\beta_1(x)+c_1\delta(x)\beta_1(x)](S^0+\varrho)E-[\sigma(x)+\gamma(x)+\mu(x)]E\\ \qquad+[c_1\beta_2(x)+c_1\delta\beta_2(x)](S^0+\varrho)I,\\ \dfrac{\partial Q}{\partial t}\leq\sigma(x)E-[\rho(x)+\eta(x)+c(x)+\mu(x)]Q,\\ \dfrac{\partial I}{\partial t}\leq\nabla\cdot(d_I(x)\nabla I)+\gamma(x)E+\eta(x)Q-[\omega(x)+c(x)+\mu(x)]I,\\ \dfrac{\partial R}{\partial t}\leq\nabla\cdot(d_R(x)\nabla R)+\rho(x)Q+\omega(x)I-[\phi(x)+\mu(x)]R.\end{cases}$$

Then we can write the corresponding comparison equation

$$\begin{cases}\dfrac{\partial\widetilde{E}}{\partial t}=\nabla\cdot(d_E(x)\nabla\widetilde{E})+[c_1\beta_1(x)+c_1\delta(x)\beta_1(x)](S^0+\varrho)\widetilde{E}-[\sigma(x)+\gamma(x)+\mu(x)]\widetilde{E}\\ \qquad+[c_1\beta_2(x)+c_1\delta(x)\beta_2(x)](S^0+\varrho)\widetilde{I},\\ \dfrac{\partial\widetilde{Q}}{\partial t}=\sigma(x)\widetilde{E}-[\rho(x)+\eta(x)+c(x)+\mu(x)]\widetilde{Q},\\ \dfrac{\partial\widetilde{I}}{\partial t}=\nabla\cdot(d_I(x)\nabla\widetilde{I})+\gamma(x)\widetilde{E}+\eta(x)\widetilde{Q}-[\omega(x)+c(x)+\mu(x)]\widetilde{I},\\ \dfrac{\partial\widetilde{R}}{\partial t}=\nabla\cdot(d_R(x)\nabla\widetilde{R})+\rho(x)\widetilde{Q}+\omega(x)\widetilde{I}-[\phi(x)+\mu(x)]\widetilde{R}.\\ \dfrac{\partial\widetilde{E}}{\partial\mathbf{n}}=\dfrac{\partial\widetilde{Q}}{\partial\mathbf{n}}=\dfrac{\partial\widetilde{I}}{\partial\mathbf{n}}=\dfrac{\partial\widetilde{R}}{\partial\mathbf{n}}=0,\ x\in\Omega.\end{cases}\tag{4.1}$$

So we define $(\widetilde{E}(x,t), \widetilde{Q}(x,t), \widetilde{I}(x,t), \widetilde{R}(x,t)) = (Me^{\lambda^* t}\chi^*(x), Me^{\lambda^* t}\varphi^*(x), Me^{\lambda^* t}\psi^*(x), Me^{\lambda^* t}\xi^*(x))$ is solutions of system (4.1), $\lambda^*$ is the eigenvalue of system (2.5) with corresponding positive eigenvector $(\chi^*, \varphi^*, \psi^*, \xi^*)$, $M$ is large enough, such that $E(x,0) \le \widetilde{E}(x,0), Q(x,0) \le \widetilde{Q}(x,0), I(x,0) \le \widetilde{I}(x,0), R(x,0) \le \widetilde{R}(x,0)$.

According to the comparison principle [45], we have

$$E(x,t) \le \widetilde{E}(x,t), Q(x,t) \le \widetilde{Q}(x,t), I(x,t) \le \widetilde{I}(x,t), R(x,t) \le \widetilde{R}(x,t), \text{ for any } x \in \Omega, t \ge 0.$$

Because when $\lambda^* < 0, t \to \infty$, $\widetilde{E}(x,t) \to 0, \widetilde{Q}(x,t) \to 0, \widetilde{I}(x,t) \to 0, \widetilde{R}(x,t) \to 0$, for any $x \in \Omega$. So $E(x,t) \to 0, Q(x,t) \to 0, I(x,t) \to 0, R(x,t) \to 0$, for any $x \in \Omega$ with $t \to \infty$. Next we claim $S(\cdot,t) \to S^0(x)$ uniformly on as $t \to \infty$, $V(\cdot,t) \to V^0(x)$ uniformly on as $t \to \infty$. Furthermore, we get the limit equations as follows:

$$\begin{cases} \dfrac{\partial \hat{S}}{\partial t} - \nabla \cdot (d_S(x)\nabla \hat{S}) = \Lambda(x) - [\alpha(x) + \mu(x)]\hat{S}, \ x \in \Omega, t \ge T, \\ \dfrac{\partial \hat{S}}{\partial \mathbf{n}} = 0, \ x \in \partial\Omega, \\ \hat{S}(x,T) = S(x,T), \ x \in \Omega. \end{cases} \tag{4.2}$$

According to [25, Lemma 2] and the theory of asymptotically autonomous semiflows [46],

$$\lim_{t\to\infty} S(x,t) = S^0(x) \text{ uniformly on } x \in \overline{\Omega}.$$

Similarly,

$$\lim_{t\to\infty} V(x,t) = V^0(x) \text{ uniformly on } x \in \overline{\Omega}.$$

So the disease-free equilibrium of the system (2.1) is globally asymptotically stable.

(2) If $\lambda^* > 0$, according to Theorem 3.1, there exists constant $M > 0$, for any solution $(S(\cdot,t), V(\cdot,t), E(\cdot,t), Q(\cdot,t), I(\cdot,t), R(\cdot,t))$, and time $t_0 > 0$, we have $E(\cdot,t) \le M, I(\cdot,t) \le M$. Thus, from the first equation of system (2.1), we have

$$\frac{\partial S(x,t)}{\partial t} \ge \nabla \cdot (d_S(x)\nabla S(x,t)) + \Lambda(x) - [\beta_1^*(x)c_1M + \beta_2^*(x)c_1M + \alpha^*(x) + \mu^*(x)]S(x,t), \text{ for any } t \ge t_0.$$

By the comparison principle [45] and [25, Lemma 2], $S(x,t)$ has a positive lower bound. Hence, $S(x,t)$ of system (2.1) has a positive lower bound. Similarly, $V(x,t)$ is uniformly persistent. Here we assume

$$\lim_{t\to\infty} S(x,t) = S_-^*(x) > 0, \quad \lim_{t\to\infty} V(x,t) = V_-^*(x) > 0.$$

Define $(E_-(x,t), Q_-(x,t), I_-(x,t), R_-(x,t)) = (\varepsilon\chi^*(x), \varepsilon\varphi^*(x), \varepsilon\psi^*(x), \varepsilon\xi^*(x))$, and $\chi^* \gg 0, \varphi^* \gg 0, \psi^* \gg 0, \xi^* \gg 0$, $\varepsilon > 0$ is a sufficiently small constant. Next, substitute $\varepsilon\chi^*(x), \varepsilon\varphi^*(x), \varepsilon\psi^*(x), \varepsilon\xi^*(x)$ into the third, fourth, fifth and sixth equations of system (2.1), we have

$$\begin{aligned} &\varepsilon\nabla \cdot (d_E(x)\nabla\chi^*) + \beta_1(x)Sf(\varepsilon\chi^*) + \beta_2(x)Sf(\varepsilon\psi^*) + \delta(x)\beta_1(x)Vf(\varepsilon\chi^*) + \delta(x)\beta_2(x)Vf(\varepsilon\psi^*) \\ &\quad - \varepsilon[\sigma(x) + \gamma(x) + \mu(x)]\chi^* - \frac{\partial(\varepsilon\chi^*)}{\partial t} \\ &= \varepsilon\nabla \cdot (d_E(x)\nabla\chi^*) + c_1\varepsilon\beta_1(x)\chi^* + c_1\varepsilon\beta_2(x)\psi^* + c_1\varepsilon\beta_1(x)\delta(x)\chi^* + c_1\varepsilon\beta_2(x)\delta(x)\psi^* \\ &\quad - \varepsilon[\sigma(x) + \gamma(x) + \mu(x)]\chi^* + \beta_1(x)Sf(\varepsilon\chi^*) + \beta_2(x)Sf(\varepsilon\psi^*) + \delta(x)\beta_1(x)Vf(\varepsilon\chi^*) \\ &\quad + \delta(x)\beta_2(x)Vf(\varepsilon\psi^*) - c_1\varepsilon\beta_1(x)\chi^* - c_1\varepsilon\beta_2(x)\psi^* - c_1\varepsilon\beta_1(x)\delta(x)\chi^* - c_1\varepsilon\beta_2(x)\delta(x)\psi^* \\ &= \varepsilon\lambda^*\chi^* + \beta_1(x)[Sf(\varepsilon\chi^*) - c_1\varepsilon\chi^*] + \beta_2(x)[Sf(\varepsilon\psi^*) - c_1\varepsilon\psi^*] \\ &\quad + \beta_1(x)\delta(x)[Vf(\varepsilon\chi^*) - c_1\varepsilon\chi^*] + \beta_2(x)\delta(x)[Vf(\varepsilon\psi^*) - c_1\varepsilon\psi^*] \\ &= \varepsilon\lambda^*\chi^* + \beta_1(x)c_1\chi^*\left[\frac{Sf(\varepsilon\chi^*)}{c_1\chi^*} - \varepsilon\right] + \beta_2(x)c_1\psi^*\left[\frac{Sf(\varepsilon\psi^*)}{c_1\psi^*} - \varepsilon\right] \\ &\quad + \beta_1(x)\delta(x)c_1\chi^*\left[\frac{Vf(\varepsilon\chi^*)}{c_1\chi^*} - \varepsilon\right] + \beta_2(x)\delta(x)c_1\psi^*\left[\frac{Vf(\varepsilon\psi^*)}{c_1\psi^*} - \varepsilon\right] > 0. \ (\varepsilon > 0 \textit{ is sufficiently small constant}) \end{aligned}$$

$$\begin{aligned}&\varepsilon\sigma(x)\chi^* - \varepsilon[\rho(x)+\eta(x)+c(x)+\mu(x)]\varphi^* - \frac{\partial(\varepsilon\varphi^*)}{\partial t}\\&= \varepsilon\{\sigma(x)\chi^* - [\rho(x)+\eta(x)+c(x)+\mu(x)]\varphi^*\}\\&= \varepsilon\lambda^*\varphi^* > 0. \ (\varepsilon > 0 \textit{ is sufficiently small constant})\end{aligned}$$

$$\begin{aligned}&\varepsilon\nabla\cdot(d_I(x)\nabla\psi^*) + \varepsilon\gamma(x)\chi^* + \varepsilon\eta(x)\varphi^* - \varepsilon[\omega(x)+c(x)+\mu(x)]\psi^* - \frac{\partial(\varepsilon\psi^*)}{\partial t}\\&= \varepsilon\{\nabla\cdot(d_I(x)\nabla\psi^*) + \gamma(x)\chi^* + \eta(x)\varphi^* - [\omega(x)+c(x)+\mu(x)]\psi^*\}\\&= \varepsilon\lambda^*\psi^* > 0. \ (\varepsilon > 0 \textit{ is sufficiently small constant})\end{aligned}$$

$$\begin{aligned}&\varepsilon\nabla\cdot(d_R(x)\nabla\xi^*) + \varepsilon\rho(x)\varphi^* + \varepsilon\omega(x)\psi^* - \varepsilon[\phi(x)+\mu(x)]\xi^* - \frac{\partial(\varepsilon\xi^*)}{\partial t}\\&= \varepsilon\{\nabla\cdot(d_R(x)\nabla\xi^*) + \rho(x)\varphi^* + \omega(x)\psi^* - [\phi(x)+\mu(x)]\xi^*\}\\&= \varepsilon\lambda^*\xi^* > 0. \ (\varepsilon > 0 \textit{ is sufficiently small constant})\end{aligned}$$

Therefore, $(\varepsilon\chi^*(x), \varepsilon\varphi^*(x), \varepsilon\psi^*(x), \varepsilon\xi^*(x))$ is the lower solution of the third, fourth, fifth and sixth equations of system (2.1).

Let $0 < h(x) < \min\{S_-^*(x), V_-^*(x), E_-^*(x), Q_-^*(x), I_-^*(x), R_-^*(x)\}$, then

$$\liminf_{t\to\infty} S(x,t) \geq h(x),\ \liminf_{t\to\infty} V(x,t) \geq h(x),\ \liminf_{t\to\infty} E(x,t) \geq h(x),$$

$$\liminf_{t\to\infty} Q(x,t) \geq h(x),\ \liminf_{t\to\infty} I(x,t) \geq h(x),\ \liminf_{t\to\infty} R(x,t) \geq h(x).$$

Thus, the system (2.1) is uniformly persistent.

□

# 5. Numerical simulation

In this section, we discuss the extinction and uniform persistence of the system (2.1) through numerical simulations. Then, we perform sensitivity analysis on the parameter vaccination rate.

## 5.1. The extinction and uniform persistence of system

For numerical simulation, we select the following incidence functions:

$$f(E) = \frac{0.03\cdot E}{1+0.15\cdot E}; f(I) = \frac{0.09\cdot I}{1+0.15\cdot I}.$$

Firstly, we simulate the situation with the constant coefficient. Figures 2 and 3 simulate the extinction and uniform persistence of system (2.1). The parameter selection in Figures 2 is $\Lambda = 10, \beta_1 = 0.04, \beta_2 = 0.06, \alpha = 0.8, \delta = 0.7, \phi = 0.7, \gamma = 0.46, \sigma = 0.32, \eta = 0.48, \rho = 0.25, \omega = 0.7, \mu = 0.1595, c = 0.1$, and the initial condition is $S(x,0) = 200\cdot\exp(-10(x-5)^2), V(x,0) = 10\cdot\exp(-10(x-5)^2), E(x,0) = 20\cdot\exp(-10(x-5)^2), Q(x,0) = 10\cdot\exp(-10(x-5)^2), I(x,0) = 5\cdot\exp(-10(x-5)^2), R(x,0) = 0$. The parameter selection in Figures 3 is $\Lambda = 10, \beta_1 = 0.3, \beta_2 = 0.7, \alpha = 0.5, \delta = 0.8, \phi = 0.7, \gamma = 0.46, \sigma = 0.32, \eta = 0.35, \rho = 0.25, \omega = 0.7, \mu = 0.15, c = 0.2$, and the initial condition is $S(x,0) = 20000\cdot\exp(-10(x-5)^2), V(x,0) = 100\cdot\exp(-10(x-5)^2), E(x,0) = 100\cdot\exp(-10(x-5)^2), Q(x,0) = 300\cdot\exp(-10(x-5)^2), I(x,0) = 200\cdot\exp(-10(x-5)^2), R(x,0) = 0$.

Next, we simulate the global asymptotic stability of the disease-free equilibrium, the parameters are shown in Table 2. Here we select the initial value $S(x,0) = 20000\cdot 0.92\cdot\exp(-10(x-5)^2), V(x,0) = 100\cdot 0.94\cdot\exp(-10(x-5)^2), E(x,0) = 800\cdot 0.04\cdot\exp(-10(x-5)^2), Q(x,0) = 300\cdot 0.03\cdot\exp(-10(x-5)^2), I(x,0) = 700\cdot 0.02\cdot\exp(-10(x-5)^2), R(x,0) = 0$.

Figures 4 and 5 simulate the extinction of the system (2.1). By comparing with Figures 2, we can see that due to the heterogeneity of the coefficients, the susceptible and vaccinated individual tend to be stable after the

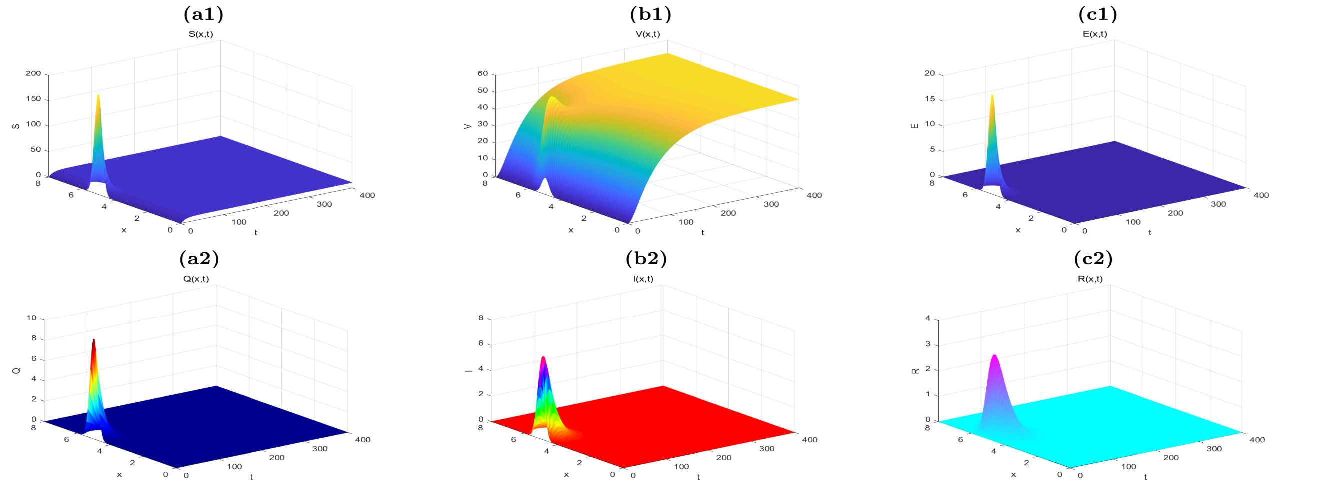


Figure 2. The time evolution of the extinction of the system (2.1) with the constant coefficient.

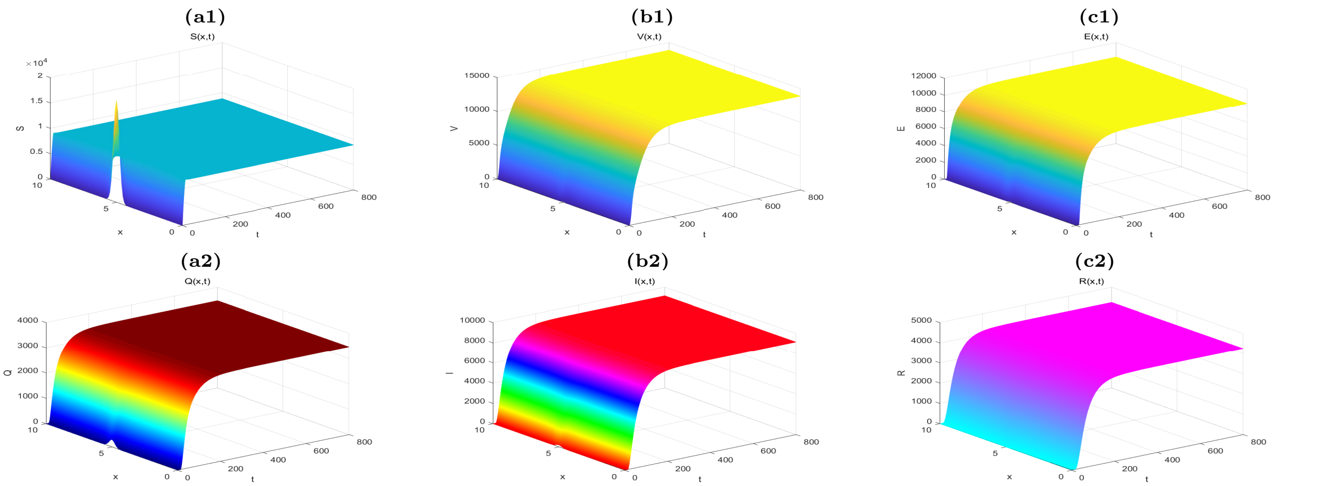


Figure 3. The time evolution of the uniform persistence of the system (2.1) with the constant coefficient.

Table 2. Values of all parameters in model (2.1)

| Parameter | Value | Parameter | Value |
|---|---|---|---|
| $\Lambda(x)$ | 10 | $\alpha(x)$ | $0.8(1+0.5sin(2\pi x))$ |
| $\beta_1(x)$ | $0.04(1+0.5sin(2\pi x))$ | $\beta_2(x)$ | $0.06(1+0.5sin(2\pi x))$ |
| $\delta(x)$ | $0.7(1+0.5sin(2\pi x))$ | $\phi(x)$ | $0.7(1+0.5sin(2\pi x))$ |
| $\gamma(x)$ | $0.46(1+0.5sin(2\pi x))$ | $\sigma(x)$ | $0.32(1+0.5sin(2\pi x))$ |
| $\eta(x)$ | $0.48(1+0.5sin(2\pi x))$ | $\rho(x)$ | $0.25(1+0.5sin(2\pi x))$ |
| $\omega(x)$ | $0.7(1+0.5sin(2\pi x))$ | $\mu(x)$ | $0.1595(1+0.5sin(2\pi x))$ |
| $c(x)$ | $0.1(1+0.5sin(2\pi x))$ | $d_S(x)$ | $0.06+0.005\sin(2\pi x)$ |
| $d_V(x)$ | $0.08+0.005\sin(2\pi x)$ | $d_E(x)$ | $0.05+0.005\sin(2\pi x)$ |
| $d_I(x)$ | $0.02+0.005\sin(2\pi x)$ | $d_R(x)$ | $0.08+0.005\sin(2\pi x)$ |

Table 3. Values of all parameters in model (2.1)

| Parameter | Value | Parameter | Value |
|---|---|---|---|
| $\Lambda(x)$ | $10000$ | $\alpha(x)$ | $0.5(1+0.5sin(2\pi x))$ |
| $\beta_1(x)$ | $0.3(1+0.5sin(2\pi x))$ | $\beta_2(x)$ | $0.7(1+0.5sin(2\pi x))$ |
| $\delta(x)$ | $0.8(1+0.5sin(2\pi x))$ | $\phi(x)$ | $0.7(1+0.5sin(2\pi x))$ |
| $\gamma(x)$ | $0.46(1+0.5sin(2\pi x))$ | $\sigma(x)$ | $0.32(1+0.5sin(2\pi x))$ |
| $\eta(x)$ | $0.35(1+0.5sin(2\pi x))$ | $\rho(x)$ | $0.25(1+0.5sin(2\pi x))$ |
| $\omega(x)$ | $0.3(1+0.5sin(2\pi x))$ | $\mu(x)$ | $0.15(1+0.5sin(2\pi x))$ |
| $c(x)$ | $0.2(1+0.5sin(2\pi x))$ | $d_S(x)$ | $0.06+0.005\sin(2\pi x)$ |
| $d_V(x)$ | $0.08+0.005\sin(2\pi x)$ | $d_E(x)$ | $0.05+0.005\sin(2\pi x)$ |
| $d_I(x)$ | $0.02+0.005\sin(2\pi x)$ | $d_R(x)$ | $0.08+0.005\sin(2\pi x)$ |

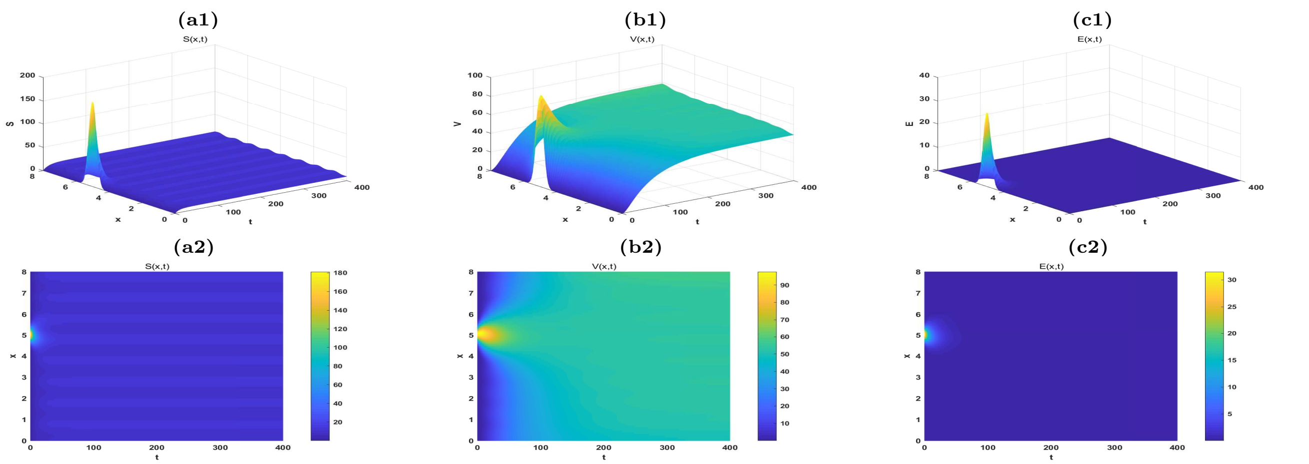


Figure 4. The time evolution of the extinction of the system (2.1).

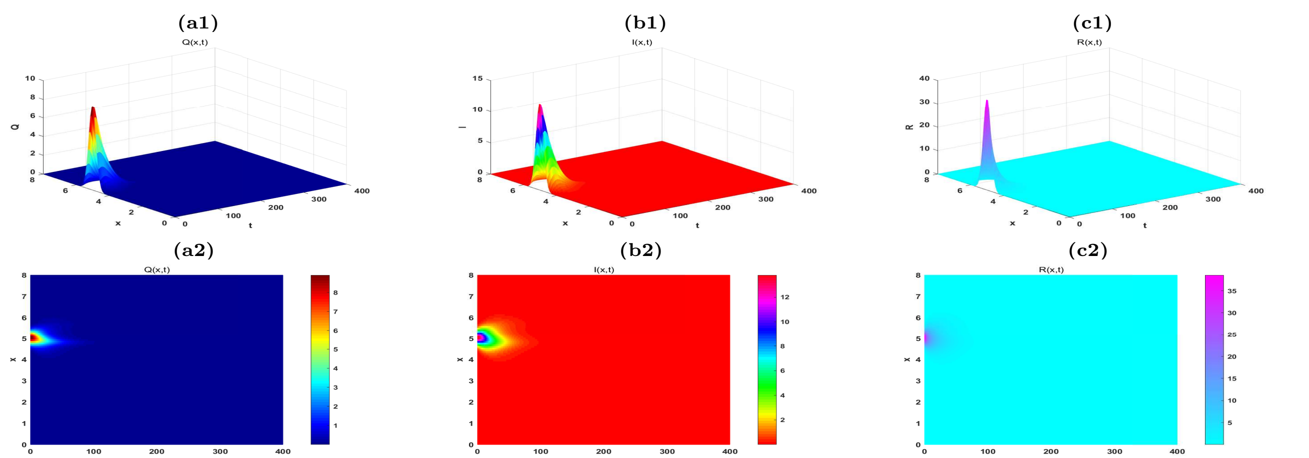


Figure 5. The time evolution of the extinction of the system (2.1).

fluctuation, and show the phenomenon of non-smoothness. Other groups tend to zero after a sharp rise and a sharp decline, which reflects the extinction of the disease.

Then, we discuss the uniform persistence of system (2.1). Parameter's selection is shown in Table 3. Here we select the initial value $S(x,0) = 20000 \cdot 0.92 \cdot \exp(-10(x-5)^2), V(x,0) = 100 \cdot 0.94 \cdot \exp(-10(x-5)^2), E(x,0) = 7000 \cdot 0.04 \cdot \exp(-10(x-5)^2), Q(x,0) = 3000 \cdot 0.03 \cdot \exp(-10(x-5)^2), I(x,0) = 5000 \cdot 0.02 \cdot \exp(-10(x-5)^2), R(x,0) = 0$.

Figures 6 and 7 simulate the system (2.1) is uniformly persistent. By comparing with Figures 3, we can see that due to the heterogeneity of the coefficient, the number of people increases sharply and then becomes stable, presenting irregular and oscillating phenomena. That is, the system (2.1) has attractive set.

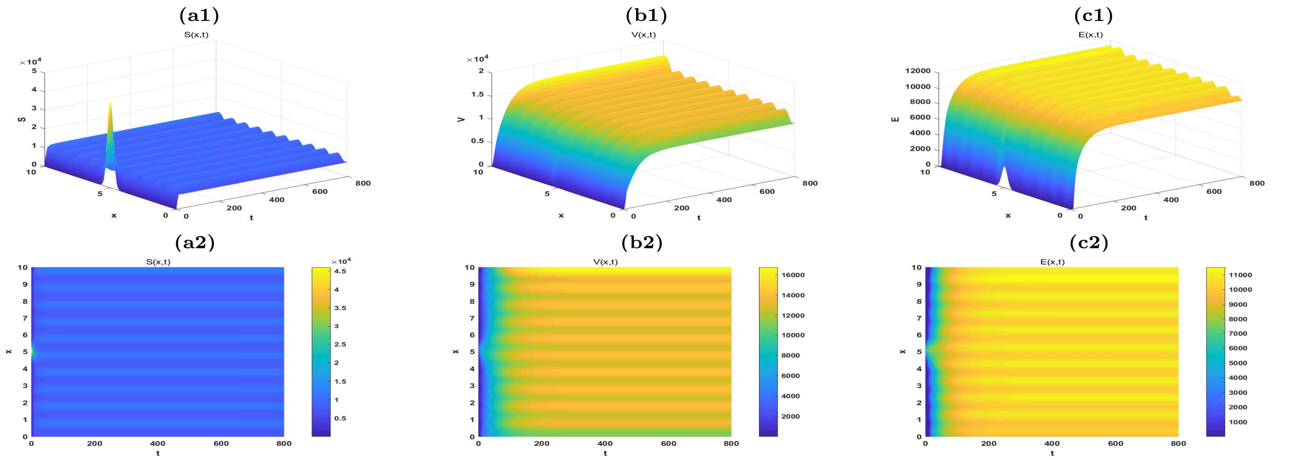


Figure 6. The time evolution of the uniform persistence of the system (2.1).

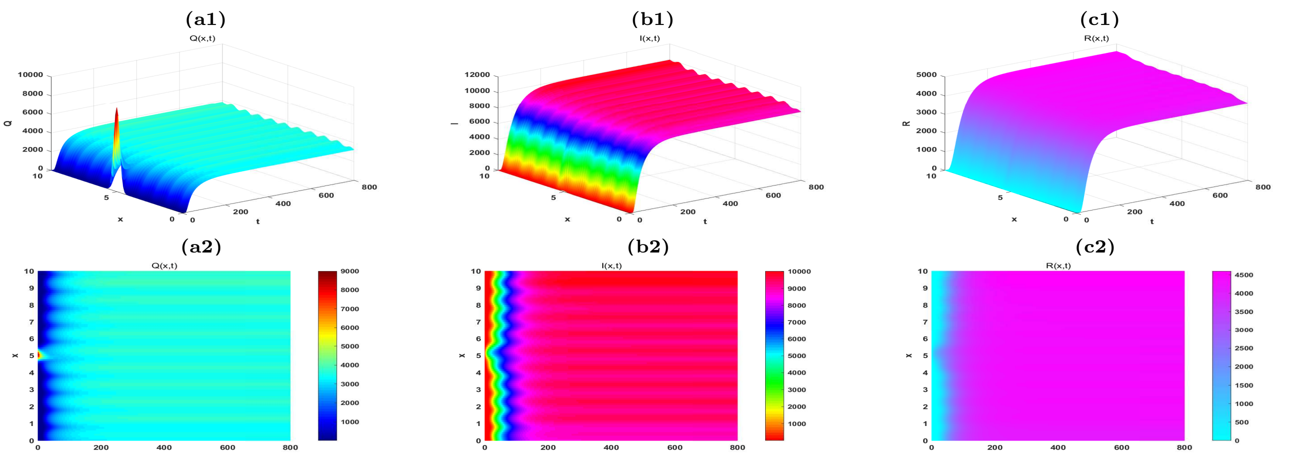


Figure 7. The time evolution of the uniform persistence of the system (2.1).

## 5.2. The sensitivity analysis

To investigate the impact of vaccination rates on disease transmission, we will conduct sensitivity analysis in this subsection.

Choosing the parameter values as in Table 2, we discuss the changes in the density of infected individuals over time under different vaccination rates. During the extinction of the disease, Figures 8 $(b)$ and $(d)$ compare the changes in the peak value of infected individuals with vaccination rates of $\alpha = 0.2 \cdot (1 + 0.5sin(2\pi x))$ and $\alpha = 0.9 \cdot (1 + 0.5sin(2\pi x))$, respectively. It can be seen that at the same time $t$ and location $x$, the higher the vaccination rate, the smaller the number of infected individuals. This means that in the case of spatial heterogeneity, increasing the vaccination rate can reduce the peak value of infected individuals. Under the condition of spatial heterogeneity and other parameters being the same as Table 3, Figures 9 $(b)$ and $(d)$ compare the changes in the peak value of infected individuals with vaccination rates of $\alpha = 0.5 \cdot (1+0.5sin(2\pi x))$ and $\alpha = 0.9 \cdot (1+0.5sin(2\pi x))$, respectively. It can be seen that at the same time $t$ and location $x$, the higher the vaccination rate, the lower the number of infected individuals. Which means that in the case of spatial heterogeneity, increasing the vaccination rate can reduce the peak of infected individuals.

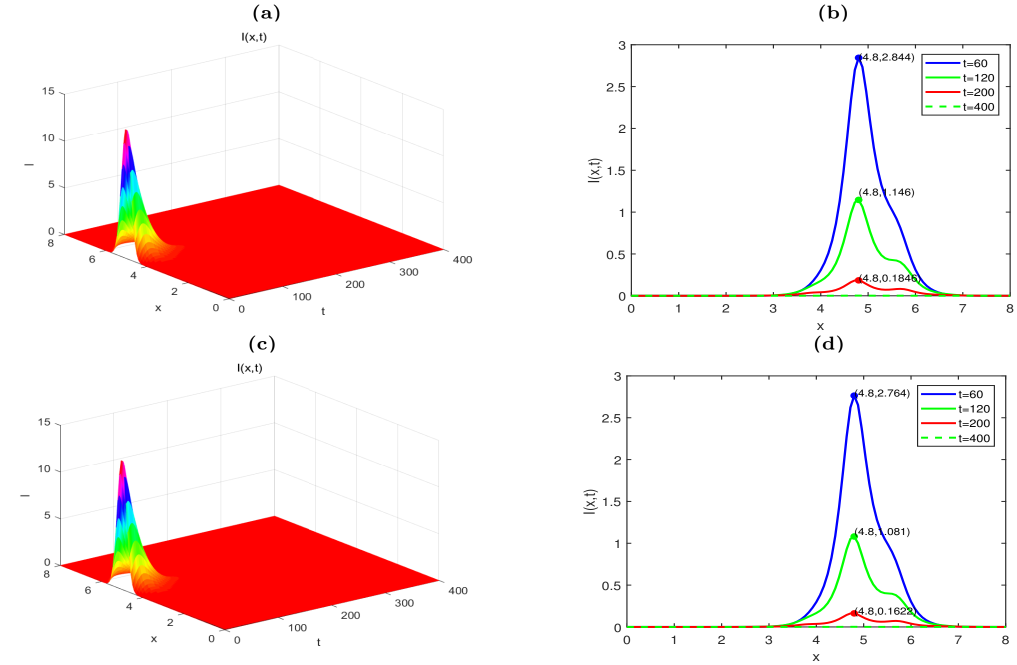


Figure 8. The effect of different levels of vaccination rates on the number of $I(x,t)$ when system is extinct

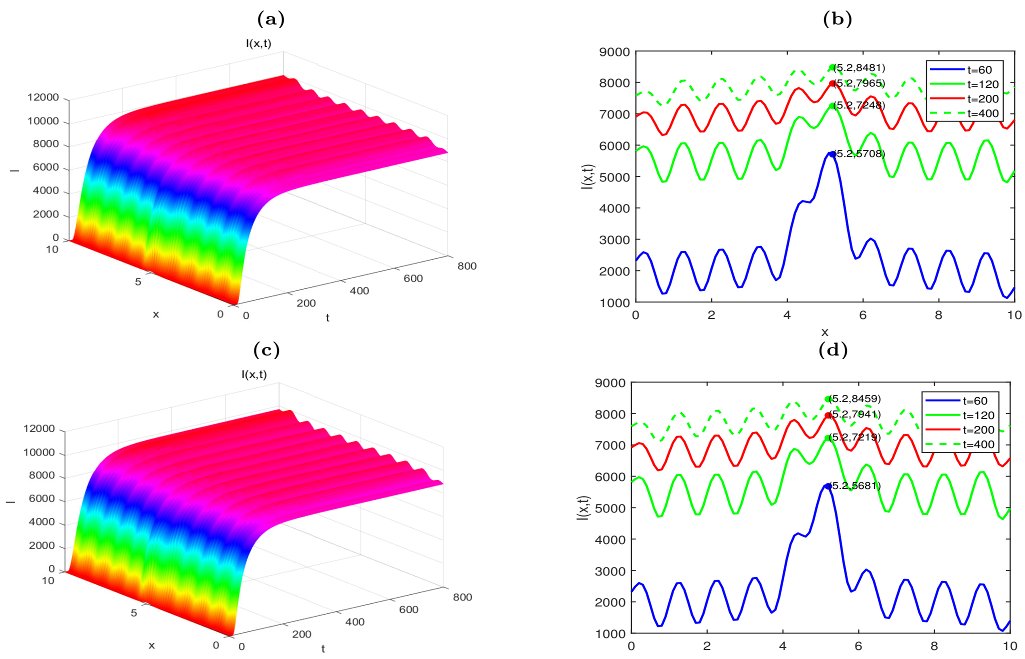


Figure 9. The effect of different levels of vaccination rates on the number of $I(x,t)$ when system is uniformly persistent

# 6. Conclusion

We have established a degenerated diffusion SVEQIRV epidemic model with general incidence in a space heterogeneous environment, analyzed the well-posedness and dynamic properties. Theoretical results show that the disease-free equilibrium is globally asymptotically stable when $\lambda^* < 0$ and the system is uniformly persistent when $\lambda^* > 0$. The numerical simulation also described this result. Firstly, we found that the image of the solution is turbulent and exhibits an unsmooth state, which is caused by the spatial heterogeneity of diffusion coefficients. Secondly, this turbulent phenomenon is controlled within a certain range, which is the range of the global exponential attraction set. Then we can also see that the image rises or falls very quickly in the initial stage, and then quickly stabilizes. It is confirmed that the solution of system (2.1) is globally exponentially attractive. Finally, in the sensitivity analysis subsection, we found that the peak of infected individuals decreases as the vaccination rate increases.

On the whole, for complex reaction-diffusion models, the semigroup theory and infinite dimensional dynamical system method can also be used to analyze the dynamic properties conveniently. However, our model still has some shortcomings, such as not taking into account the time-delay effect, seasonal change of the periodic environment and so on, which are all questions worthy of our investigation in the future.

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**The conflict of interest disclosure**

We all author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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