Asymptotic feature of discrete second-order fuzzy difference equation with quadratic term *

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Abstract

This article explores the asymptotic feature of a discrete second-order fractal FDE (fuzzy difference equation) with quadratic term. Specifically, applying a generalization of division (g-division) of two fuzzy numbers, we obtain dynamical features including the boundedness, persistence, and global behavior of a positive fuzzy solution of the following model

$$x_{n+1} = A + \frac{Bx_n^2}{x_{n-1}^2}, \quad n \in N^+,$$

where the parameters $A, B \in \Re_f^+$(positive fuzzy number) and initial values $x_0, x_{-1} \in \Re_f^+$. Finally, two numerical examples are provided to reveal the validity of our findings.

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1 Introduction

It is well known that difference equations display essentially as discrete analogous of differential equations or delay differential equations, which have a great deal of applications in biology, population dynamics, chemistry, computer science, economics, et al.(see [1-6]). In the past decades, many scholars are especially interested in the study of dynamical features of fractal difference equations (see [7-17]).

For example, Beso, Kalabusic et al. [16] studied a second-order fractal difference equation with quadratic term

$$x_{n+1} = \gamma + \delta\frac{x_n}{x_{n-1}^2}, \quad n \in N^+, \tag{1.1}$$

where the initial conditions $x_0, x_{-1} \in R^+$ and the parameters $\gamma, \delta \in R^+$.

In 2017, Khyat et al.[17] investigated a fractal difference equation with quadratic term in both the denominator and numerator.

$$x_{n+1} = a + \frac{x_n^2}{x_{n-1}^2}, \quad n \in N^+. \tag{1.2}$$

Furthermore, they obtain that the system has a unique positive equilibrium $\overline{x} = a + 1$ which is GAS (globally asymptotically stable) if $a > \sqrt{2}$ and the direction of the Neimark-Sacker bifurcation.

Although the forms of the system of difference equations or difference equations are not complicated, these models are especially hard to fully catch on the dynamical feature of their solutions. In fact, they often include inherent indeterminacy or uncertainty which impacts the dynamical feature of the solution

*Corresponding author. Email: zqianhong68@163.com(Q.Zhang) to these system. To consider these uncertainties comprehensively, scientists utilize fuzzy set founded by Zadeh in 1965, to handle subjective uncertainty or vagueness in mathematical model. Up to now, FDE (fuzzy difference equation) is an efficient model to study the dynamical feature of systems with uncertainty inherently.

FDE is a special kind of difference equation where the initial values of model, the parameters, or system variables are fuzzy numbers, and its' solutions are a FNs sequence (fuzzy numbers sequence). In the past decades, the study on dynamical feature of FDE including stability, boundedness, persistence and oscillation et al. has been a research hot topic in pure mathematics and applied mathematics. More and more researchers are interested in studying the dynamical features of FDE. To the best of our knowledge, due to the lack of proper theoretical tools, there is few literature dealing with the oscillation and periodicity of fuzzy solutions. Many literature focus on the stability, boundedness and persistence of fuzzy solutions for FDE (see [18-37]).

For example, in 2002, using a analogous method of ordinary difference equations and constructing Lyapunov function, Lakshmikantham and Vatsala [18] acquired a comparison theorem of FDE. In 2002, Papaschinopoulos and Papadopoulos [22,23], using Zadeh extension principle, studied global dynamical features of first-order and high-order rational FDE respectively. In 2008, utilizing same approach, Chrysafis, et al. [24] explored a FDE model in finance. In 2016, Mondal et al. [25], applying Lagranges multiplier method, investigated a linear second-order FDE. In 2018, Khastan [26], by virtue of H-difference (Hukuhara difference) of fuzzy numbers, discussed a Logistic FDE and obtained global dynamical features of two corresponding equations.

In addition to above methods, g-division of fuzzy numbers put forward by Stefanini [48] is another effective way to study qualitative features of fractal FDE. Since it can cut down the impression of the fuzzy solutions to FDE model owing to the reduction of the length of support sets. In 2015, using g-division, Zhang et al. [32] discussed a third-order rational FDE. Recently, many authors investigate the dynamical features of FDE model by virtue of g-division (see [36,37,38]). Studies have shown that FDE has a potential to be applied in discrete population dynamics, economics, epidemic, and chemistry etc. Readers can refer to [38-47].

Driven by the above discussion, with the help of g-division, we explore the qualitative features of fuzzy solutions for a second-order fractal FDE analogous to Eq.(1.2).

$$x_{n+1} = A + \frac{Bx_n^2}{x_{n-1}^2},\ n \in N^+, \tag{1.3}$$

where the parameters $A, B \in \Re_f^+$ and initial values $x_i \in \Re_f^+, i = 0, -1$.

The structure of this article is arranged as follows. Section 2 introduces preliminary, some definitions used in the sequel. Section 3 presents the dynamical features of fuzzy solutions of FDE (1.3) by virtue of g-division. Section 4 gives two examples to reveal the validity of the theoretic results. Section 5 makes a general conclusion and future works.

2 Preliminary and definitions

In this section, we give some definitions on fuzzy sets and fuzzy calculus, which can be addressed in [19,20,22].

Definition 2.1. *A mapping $\nu : R \to [0,1]$ is called a FN (fuzzy number) provided that the conditions written below hold true.*
(i) ν is normal, i. e., there is $t \in R$ satisfying $\nu(t) = 1$;
(ii) ν is convex, i. e., $\forall p \in [0,1]$ and $t_1, t_2 \in R$ such that

$$\nu(pt_1 + (1-p)t_2) \geq \min\{\nu(t_1), \nu(t_2)\};$$

(iii) ν is upper semicontinuous;
(iv) The support of ν, $supp\nu = \overline{\bigcup_{\alpha\in(0,1]}[\nu]_\alpha} = \overline{\{t : \nu(t) > 0\}}$ is compact.

For $\alpha \in (0,1]$, the $\alpha-$cuts of a FN ν is described by $[\nu]_\alpha = \{t \in R : \nu(t) \geq \alpha\}$ and for $\alpha = 0$, the support of ν is written as $\text{supp}\nu = [\nu]_0 = \overline{\{t \in R | \nu(t) > 0\}}$, $[\nu]_\alpha$ is closed interval. A FN is positive provided that $\text{supp}\nu \subset (0,\infty)$.

If ν is a positive real number, then ν is a trivial FN with $[\nu]_\alpha = [\nu,\nu], \alpha \in (0,1]$.

Definition 2.2. *Suppose that u, v are FN, the distance between u and v is written as*

$$D(u,v) = \sup_{\alpha \in [0,1]} \max\{|u_{l,\alpha} - v_{l,\alpha}|, |u_{r,\alpha} - v_{r,\alpha}|\}. \tag{2.1}$$

The collection of all FNs (positive FNs) is denoted by $\Re_f(\Re_f^+)$. Then $(\Re_f, D)$ is a Banach space.

For $u, v \in \Re_f$, $[u]_\alpha = [u_{l,\alpha}, u_{r,\alpha}], [v]_\alpha = [v_{l,\alpha}, v_{r,\alpha}]$, $\lambda \in R$, the sum $u+v$, the scalar product λu, the multiplication uv are defined respectively by

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha, [\lambda u] = \lambda[u]_\alpha, \alpha \in [0,1].$$

$$[uv]_\alpha = [\min\{u_{l,\alpha}v_{l,\alpha}, u_{l,\alpha}v_{r,\alpha}, u_{r,\alpha}v_{l,\alpha}, u_{r,\alpha}v_{r,\alpha}\}, \max\{u_{l,\alpha}v_{l,\alpha}, u_{l,\alpha}v_{r,\alpha}, u_{r,\alpha}v_{l,\alpha}, u_{r,\alpha}v_{r,\alpha}\}].$$

Definition 2.3.[48] *For $u, v \in \Re_f$ with $[u]_\alpha = [u_{l,\alpha}, u_{r,\alpha}], [v]_\alpha = [v_{l,\alpha}, v_{r,\alpha}]$, and $0 \notin [v]_\alpha, \forall \alpha \in [0,1]$. The g-division ($\div_g$) is an operator that attains the FN $w = u \div_g v$ with $[w]_\alpha = [w_{l,\alpha}, w_{r,\alpha}]$ (here $[u]_\alpha^{-1} = [1/u_{r,\alpha}, 1/u_{l,\alpha}]$), that is*

$$[w]_\alpha = [u]_\alpha \div_g [v]_\alpha \Longleftrightarrow \begin{cases} (i) \quad [u]_\alpha = [v]_\alpha [w]_\alpha, \\ or \\ (ii) \quad [v]_\alpha = [u]_\alpha [w]_\alpha^{-1}, \end{cases} \tag{2.2}$$

where w is a proper FN ($w_{l,\alpha}$ is non-decreasing, $w_{r,\alpha}$ is non-increasing and $w_{l,1} \leq w_{r,1}$).

Remark 2.1. *In this paper, we assume that $u \div_g v = w \in \Re_f^+$ exists. The following two cases happen.*
Case (i). if $u_{l,\alpha}v_{r,\alpha} \leq u_{r,\alpha}v_{l,\alpha}, \forall \alpha \in [0,1]$, then $w_{l,\alpha} = \frac{u_{l,\alpha}}{v_{l,\alpha}}, w_{r,\alpha} = \frac{u_{r,\alpha}}{vr,\alpha}$,
Case (ii). if $u_{l,\alpha}v_{r,\alpha} \geq u_{r,\alpha}v_{l,\alpha}, \forall \alpha \in [0,1]$, then $w_{l,\alpha} = \frac{u_{r,\alpha}}{v_{r,\alpha}}, w_{r,\alpha} = \frac{u_{l,\alpha}}{vl,\alpha}$.

Definition 2.4. *A positive FNs sequence (x_n) is bounded (resp., persists) provided that there is a positive constant S (resp., R) satisfying*

$$supp\ x_n \subset [0,S) (resp.\ supp\ x_n \subset (R,\infty]), n \in N.$$

A positive FNs sequence (x_n) is bounded and persists provided that there are positive constants $R, S > 0$ satisfying

$$supp\ x_n \subset [R,S], n \in N.$$

A positive FNs sequence (x_n) is unbounded provided that the norm $\|x_n\|$ is an unbounded sequence.

Definition 2.5. *We call x_n a positive solution of (1.3) if (x_n) is a positive FNs sequence satisfying (1.3). A positive FN x is a positive fixed point of (1.3) if*

$$x = A + \frac{Bx^2}{x^2}.$$

Let (x_n) be a positive FNs sequence, $x \in \Re_f^+$, we say x_n converges x in term of metric D provided that $\lim_{n\to\infty} D(x_n, x) = 0$.

3 Main results

3.1 Existence of solution

To explore the qualitative features of the positive solution to (1.3), firstly, we will discuss the existence of positive fuzzy solution of (1.3). The following lemma is used in the sequel.

Lemma 3.1. [19] Let $g : R^+ \times R^+ \times R^+ \times R^+ \to R^+$ be continuous mapping, $A_i \in \Re_f^+, i = 1, 2, 3, 4,$ then

$$[g(A_1, A_2, A_3, A_4)]_\alpha = g([A_1]_\alpha, [A_2]_\alpha, [A_3]_\alpha, [A_4]_\alpha), \quad \alpha \in (0, 1]. \tag{3.1}$$

Theorem 3.1. *Consider Eq. (1.3), in which $A, B \in \Re_f^+$. Then, for any $x_{-1}, x_0 \in \Re_f^+$, there is a unique positive fuzzy solution x_n of Eq. (1.3) with initial values x_{-1}, x_0.*

Proof. Assume that there is a positive FNs sequence (x_n) meeting with (1.3) for any initial values $x_i \in \Re_f^+, i = -1, 0$. Consider $\alpha-$cuts, $\alpha \in (0, 1]$,

$$[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}], \quad [B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}], \quad [x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], n \in N. \tag{3.2}$$

Applying Lemma 3.1, Eq. (1.3) transforms to the following form

$$\begin{aligned} [x_{n+1}]_\alpha &= [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[A + \frac{Bx_n^2}{x_{n-1}^2}\right]_\alpha = [A]_\alpha + \frac{[B]_\alpha \times [x_n^2]_\alpha}{[x_{n-1}^2]_\alpha} \\ &= [A_{l,\alpha}, A_{r\alpha}] + \frac{[B_{l,\alpha}L_{n,\alpha}^2, B_{r,\alpha}R_{n,\alpha}^2]}{[L_{n-1,\alpha}^2, R_{n-1,\alpha}^2]} \end{aligned}$$

By virtue of g-division, there are two cases.

Case (i)

$$[x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[A_{l,\alpha} + \frac{B_{l,\alpha}L_{n,\alpha}^2}{L_{n-1,\alpha}^2}, A_{r,\alpha} + \frac{B_{r,\alpha}R_{n,\alpha}^2}{R_{n-1,\alpha}^2}\right] \tag{3.3}$$

Case (ii)

$$[x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[A_{l,\alpha} + \frac{B_{r,\alpha}R_{n,\alpha}^2}{R_{n-1,\alpha}^2}, A_{r,\alpha} + \frac{B_{l,\alpha}L_{n,\alpha}^2}{L_{n-1,\alpha}^2}\right] \tag{3.4}$$

If Case (i) holds true, it follows that, for $n \in N, \alpha \in (0, 1]$,

$$\begin{cases} L_{n+1,\alpha} = A_{l,\alpha} + \frac{B_{l,\alpha}L_{n,\alpha}^2}{L_{n-1,\alpha}^2}, \\ R_{n+1,\alpha} = A_{r,\alpha} + \frac{B_{r,\alpha}R_{n,\alpha}^2}{R_{n-1,\alpha}^2}. \end{cases} \tag{3.5}$$

In fact, for any initial values $(L_{j,\alpha}, R_{j,\alpha}), j = -1, 0, \alpha \in (0, 1]$, there is a unique solution $(L_{n,\alpha}, R_{n,\alpha})$. Now it need to prove that $[L_{n,\alpha}, R_{n,\alpha}], \alpha \in (0, 1]$, here $(L_{n,\alpha}, R_{n,\alpha})$ is the solution of system (3.5), for initial conditions $(L_{j,\alpha}, R_{j,\alpha}), j = -1, 0$, ascertains the solution x_n of (1.3) with initial conditions $x_j, j = -1, 0$, satisfying

$$[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad n \in N, \ \alpha \in (0, 1]. \tag{3.6}$$

Since $x_j \in \Re_f^+, j = -1, 0$, for any $\alpha_1, \alpha_2 \in (0, 1], \alpha_1 \leq \alpha_2$, one has

$$0 < L_{j,\alpha_1} \leq L_{j,\alpha_2} \leq R_{j,\alpha_2} \leq R_{j,\alpha_1}, j = -1, 0. \tag{3.7}$$

One affirms that

$$L_{n,\alpha_1} \leq L_{n,\alpha_2} \leq R_{n,\alpha_2} \leq R_{n,\alpha_1}, n \in N. \tag{3.8}$$

Applying mathematical induction. For $n = 0, 1, 2$, it is clear that assertion (3.8) is true. Assume that, for $n \leq k, k \in N^+$, the assertion (3.8) is true. Then, from (3.5) and (3.7), it has

$$\begin{aligned} L_{k+1,\alpha_1} &= A_{l,\alpha_1} + \frac{B_{l,\alpha_1}L_{k,\alpha_1}^2}{L_{k-1,\alpha_1}^2} \leq A_{l,\alpha_2} + \frac{B_{l,\alpha_2}L_{k,\alpha_2}^2}{L_{k-1,\alpha_2}^2} = L_{k+1,\alpha_2} \\ &= A_{l,\alpha_2} + \frac{B_{l,\alpha_2}L_{k,\alpha_2}^2}{L_{k-1,\alpha_2}^2} \leq A_{r,\alpha_2} + \frac{B_{r,\alpha_2}R_{k,\alpha_2}^2}{R_{k-1,\alpha_2}^2} = R_{k+1,\alpha_2} \\ &= A_{r,\alpha_2} + \frac{B_{r,\alpha_2}R_{k,\alpha_2}^2}{R_{k-1,\alpha_2}^2} \leq A_{r,\alpha_1} + \frac{B_{r,\alpha_1}R_{k,\alpha_1}^2}{R_{k-1,\alpha_1}^2} = R_{k+1,\alpha_1} \end{aligned}$$

Hence assertion (3.8) holds true.

Furthermore, from (3.5), it has

$$L_{1,\alpha} = A_{l,\alpha} + \frac{B_{l,\alpha} L_{0,\alpha}^2}{L_{-1,\alpha}^2}, \quad R_{1,\alpha} = A_{r,\alpha} + \frac{B_{r,\alpha} R_{0,\alpha}^2}{R_{-1,\alpha}^2}, \quad \alpha \in (0,1]. \tag{3.9}$$

Since the parameters $A, B \in \Re_f^+$ and the initial values $x_j \in \Re_f^+, j = -1, 0$, and then we have that $L_{0,\alpha}, R_{0,\alpha}, L_{-1,\alpha}, R_{-1,\alpha}$ are left continuous. Thus from (3.9), it has that $L_{1,\alpha}, R_{1,\alpha}$ are also left continuous. By mathematical induction method, it gets that $L_{n,\alpha}, R_{n,\alpha}, n \in N$, are left continuous.

Next it need to show that the support, supp$x_n = \overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]}$ is compact. In fact, we will deduce that $\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]$ is bounded.

Because of $x_j \in \Re_f^+, j = -1, 0$, and the parameters $A, B \in \Re_f^+$, there are constants $M_A > 0, N_A > 0, N_B > 0, M_B > 0, M_j > 0, N_j > 0, j = -1, 0$, satisfying, for each $\alpha \in (0, 1]$,

$$\begin{cases} [A_{l,\alpha}, A_{r,\alpha}] \subset [M_A, N_A], \\ [B_{l,\alpha}, B_{r,\alpha}] \subset [M_B, N_B], \\ [L_{j,\alpha}, R_{j,\alpha}] \subset [M_j, N_j]. \end{cases} \tag{3.10}$$

From (3.9) and (3.10), one has

$$[L_{1,\alpha}, R_{1,\alpha}] \subset \left[M_A + \frac{M_B M_0^2}{N_{-1}^2}, N_A + \frac{N_B N_0^2}{M_{-1}^2}\right], \alpha \in (0,1]. \tag{3.11}$$

From which it is obvious that

$$\bigcup_{\alpha\in(0,1]} [L_{1,\alpha}, R_{1,\alpha}] \subset \left[M_A + \frac{M_B M_0^2}{N_{-1}^2}, N_A + \frac{N_B M_0^2}{M_{-1}^2}\right], \quad \alpha \in (0,1]. \tag{3.12}$$

Then, it has that $\overline{\bigcup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]}$ is compact and $\overline{\bigcup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]} \subset (0, \infty)$. By mathematical induction, one has $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]}$ is compact, and

$$\overline{\bigcup_{\alpha\in(0,1]} [L_{n,\alpha}, R_{n,\alpha}]} \subset (0, \infty) \ n = 1, 2, \cdots. \tag{3.13}$$

Therefore, (3.8), (3.13), and $L_{n,\alpha}, R_{n,\alpha}$ are left continuous, so $[L_{n,\alpha}, R_{n,\alpha}]$ determines a positive FNs sequence x_n satisfying (3.6).

Since for all $\alpha \in (0, 1]$,

$$[x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[A_{l,\alpha} + \frac{B_{l,\alpha} L_{n,\alpha}^2}{L_{n-1,\alpha}^2}, A_{r,\alpha} + \frac{B_{r,\alpha} R_{n,\alpha}^2}{R_{n-1,\alpha}^2}\right] = \left[A + \frac{B x_n^2}{x_{n-1}^2}\right]_\alpha,$$

Therefore x_n is the solution of (1.3) with initial values $x_i, i = -1, 0$.

Let $\overline{x}_n$ be another positive fuzzy solution of (1.3) with the initial vales $x_i, i = -1, 0$. Arguing as above, it can easily get that

$$[\overline{x}_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad n \in N, \quad \alpha \in (0,1]. \tag{3.14}$$

From (3.6) and (3.14), so $[x_n]_\alpha = [\overline{x}_n]_\alpha, \alpha \in (0, 1], n \in N$, and $x_n = \overline{x}_n, n \in N$.

It can be proved similarly if Case (ii) holds true. This completes the proof of Theorem 3.1.

3.2 Dynamical features of solution

In this subsection, to obtain the qualitative features of the positive fuzzy solutions to (1.3), based on g-division operation, let us to discuss two cases (3.3) and (3.4), respectively.

Provided that Case (i) happens, we give the following lemma.

Lemma 3.2 *Consider the second-order difference equation*

$$y_{n+1} = a + \frac{cy_n^2}{y_{n-1}^2}, \quad n \in N, \tag{3.15}$$

here $a, c \in (0, +\infty), y_i \in (0, +\infty), i = -1, 0$. *Then there are following assertions.*

(i) The positive solution $y_n, n \in N$, *of Eq. (3.15) is bounded.*

(ii) The positive equilibrium point $\overline{y} = a + c$ *of Eq. (3.15) is GAS if* $a > 3c$.

Proof. (i) From (3.15), it is easy to see $y_n > a$ for $n \geq 1$. And also, we have

$$\begin{aligned} y_n &= a + \frac{cy_{n-1}^2}{y_{n-2}^2} = a + \frac{c}{y_{n-2}^2}\left(a + \frac{cy_{n-2}^2}{y_{n-3}^2}\right)^2 = a + c\left(\frac{a}{y_{n-2}} + \frac{cy_{n-2}}{y_{n-3}^2}\right)^2 \\ &= a + c\left(\frac{a}{y_{n-2}} + \frac{c}{y_{n-3}^2}\left(a + \frac{cy_{n-3}^2}{y_{n-4}^2}\right)\right)^2 \\ &= a + c\left(\frac{a}{y_{n-2}} + \frac{ac}{y_{n-3}^2} + \frac{c^2}{y_{n-4}^2}\right)^2 \\ &\leq a + c\left(1 + \frac{c}{a} + \frac{c^2}{a^2}\right)^2 \end{aligned} \tag{3.16}$$

Therefore the assertion (i) holds true.

(ii) Suppose that $\overline{y}$ is an equilibrium point of Eq. (3.15), it gets that $\overline{y} = a + c$. Then we can obtain the linearized equation associated with (3.15) at $\overline{y}$ is

$$y_{n+1} + \frac{2c}{a+c}y_n - \frac{2c}{a+c}y_{n-1} = 0, \quad n \in N. \tag{3.17}$$

Since $a > 3c$, it can get $\frac{4c}{a+c} < 1$. With the help of Theorem 1.3.7 [3], the equilibrium point of (3.15) $\overline{y}$ is locally asymptotically stable.

Furthermore, the proof is similar to Theorem 2 [12], we have that $\lim_{n\to\infty} y_n = a + c$. Therefore, the positive equilibrium point $\overline{y} = a + c$ of (3.15) is GAS.

Theorem 3.2 *Consider FDE (1.3), in which the initial conditions* $x_j \in \Re_f^+, j = -1, 0$, *and the parameters* $A, B \in \Re_f^+$. *Provided that*

$$\frac{B_{l,\alpha}}{B_{r,\alpha}} \leq \frac{L_{n-1,\alpha}^2 R_{n,\alpha}^2}{L_{n,\alpha}^2 R_{n-1,\alpha}^2}, \forall \alpha \in (0, 1], \tag{3.18}$$

then the results below hold true.

(i) Every positive fuzzy solution x_n *of (1.3) is persistent and bounded.*

(ii) If furthermore

$$A_{l,\alpha} > 3B_{l,\alpha}, \quad A_{r,\alpha} > 3B_{r,\alpha}, \quad \forall \alpha \in (0, 1]. \tag{3.19}$$

Then every positive fuzzy solution x_n *of (1.3) is GAS.*

Proof. (i) Since condition (3.18) holds true, then x_n is a positive fuzzy solution of (1.3) satisfying (3.3). It follows from (3.5) that

$$M_A \leq L_{n,\alpha}, \quad M_A \leq R_{n,\alpha}, \quad n \in N^+, \quad \alpha \in (0, 1]. \tag{3.20}$$

Then from $A_{l,\alpha} \geq M_A$, with the help of Lemma 3.2, one has

$$[L_{n,\alpha}, R_{n,\alpha}] \subset [M_A, T], \quad n \geq 3, \tag{3.21}$$

here

$$T = N_A + N_B \left(1 + \frac{N_B}{N_A} + \frac{N_B^2}{N_A^2}\right)^2.$$

Therefore, for $n \geq 3$, it has $\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}] \subset [M_A, T]$, and $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \subseteq [M_A, T]$. Thus the assertion (i) is true.

(ii) From condition (3.18), we have (3.5) holds true. Let $\bar{x}, [\bar{x}]_\alpha = [L_\alpha, R_\alpha]$, be the equilibrium of (1.3), namely

$$L_\alpha = A_{l,\alpha} + \frac{B_{l,\alpha}L_\alpha^2}{L_\alpha^2}, \quad R_\alpha = A_{r,\alpha} + \frac{B_{r,\alpha}R_\alpha^2}{R_\alpha^2}, \quad \alpha \in (0, 1]. \tag{3.22}$$

It is clear that, for $\alpha \in (0, 1]$,

$$L_\alpha = A_{l,\alpha} + B_{l,\alpha}, \quad R_\alpha = A_{r,\alpha} + B_{r,\alpha}.$$

Noting (3.19) and Applying (ii) of Lemma 3.2, we have that the equilibrium $\bar{x}$ is GAS.

Next, suppose that Case (ii) happens, for $n \in N, \alpha \in (0, 1]$, it has

$$L_{n+1,\alpha} = A_{l,\alpha} + \frac{B_{r,\alpha}R_{n,\alpha}^2}{R_{n-1,\alpha}^2}, \quad R_{n+1,\alpha} = A_{r,\alpha} + \frac{B_{l,\alpha}L_{n,\alpha}^2}{L_{n-1,\alpha}^2}. \tag{3.23}$$

To discuss dynamical features of (1.3) under Case (ii), we need the following lemmas.

Lemma 3.3. *Consider the following two-dimensional couple difference systems*

$$y_{n+1} = a + \frac{dz_n^2}{z_{n-1}^2}, \quad z_{n+1} = b + \frac{cy_n^2}{y_{n-1}^2}, \quad n \in N, \tag{3.24}$$

where the initial conditions $y_i, z_i, i = -1, 0$ are positive real numbers. If

$$b + c > 4d, \quad a + d > 4c. \tag{3.25}$$

Then there exists a unique positive equilibrium $(\bar{y}, \bar{z}) = (a + d, b + c)$ that is locally asymptotically stable.

Proof. It is clear from (3.24) that the unique positive equilibrium is $(\bar{y}, \bar{z}) = (a+d, b+c)$. The linearized equation of (3.24) at $(\bar{y}, \bar{z})$ is

$$\Psi_{n+1} = G\Psi_n \tag{3.26}$$

here $\Psi_n = (y_n, y_{n-1}, z_n, z_{n-1})^T$,

$$G = \begin{pmatrix} 0 & 0 & \frac{2d}{b+c} & -\frac{2d}{b+c} \\ 1 & 0 & 0 & 0 \\ \frac{2c}{a+d} & -\frac{2c}{a+d} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Let $\lambda_i, i = 1, 2, 3, 4$, be the eigenvalues of Jacobian matrix G, $Q = \text{diag}(q_1, q_2, q_3, q_4)$, here $q_1 = q_3 = 1, q_i = q_{2+i} = 1 - i\varepsilon (i = 2)$, and

$$0 < \varepsilon < \min\left\{\frac{b + c - 4d}{2(b + c - 2d)}, \frac{a + d - 4c}{2(a + d - 2c)}\right\}. \tag{3.27}$$

It is clear that Q is invertible. Calculating QGQ^{-1}, one has

$$QGQ^{-1} = \begin{pmatrix} 0 & 0 & \frac{2d}{b+c}q_1q_3^{-1} & -\frac{2d}{b+c}q_1q_4^{-1} \\ q_2q_1^{-1} & 0 & 0 & 0 \\ \frac{2c}{a+d}q_3q_1^{-1} & -\frac{2c}{a+d}q_3q_2^{-1} & 0 & 0 \\ 0 & 0 & q_4q_3^{-1} & 0 \end{pmatrix}$$

Since $q_1 > q_2 > 0, q_3 > q_4 > 0$, one has

$$q_2 q_1^{-1} < 1, q_4 q_3^{-1} < 1.$$

Furthermore, noting (3.27), we have

$$\frac{2d}{b+c} q_1 q_3^{-1} + \frac{2d}{b+c} q_1 q_4^{-1} = \frac{2d}{b+c}\left(1 + \frac{1}{1-2\varepsilon}\right) < 1,$$

$$\frac{2c}{a+d} q_3 q_1^{-1} + \frac{2c}{a+d} q_3 q_2^{-1} = \frac{2c}{a+d}\left(1 + \frac{1}{1-2\varepsilon}\right) < 1.$$

In fact, the eigenvalues of G is same as those of QGQ^{-1}, then

$$\begin{aligned}\max_{1\leq i\leq 4} |\lambda_i| &\leq \|QGQ^{-1}\|_\infty \\ &= \max\left\{q_2 q_1^{-1}, q_4 q_3^{-1}, \frac{2d}{b+c} q_1 q_3^{-1} + \frac{2d}{b+c} q_1 q_4^{-1}, \frac{2c}{a+d} q_3 q_1^{-1} + \frac{2c}{a+d} q_3 q_2^{-1}\right\} < 1.\end{aligned}$$

So the equilibrium $(\bar{y}, \bar{z})$ of (3.23) is locally asymptotically stable.

Theorem 3.3. *Consider FDE (1.3), here both the initial conditions $x_i (i = -1, 0)$ and the parameters $A, B \in \Re_f^+$. If the following condition holds true, for $\forall\alpha \in (0, 1]$,*

$$\begin{cases} \frac{B_{l,\alpha}}{B_{r,\alpha}} \geq \frac{L_{n-1,\alpha}^2 R_{n,\alpha}^2}{L_{n,\alpha}^2 R_{n-1,\alpha}^2} \\ A_{r,\alpha} - A_{l,\alpha} \geq B_{r,\alpha} - B_{l,\alpha} \\ A_{l,\alpha} + B_{r,\alpha} > 4B_{l,\alpha} \\ A_{r,\alpha} + B_{l,\alpha} > 4B_{r,\alpha}. \end{cases} \tag{3.28}$$

Then system (1.3) has a unique positive fuzzy equilibrium $\bar{x}$, where $[\bar{x}]_\alpha = [A_{l,\alpha} + B_{r,\alpha}, A_{r,\alpha} + B_{l,\alpha}]$, and it is locally stable.

Proof. Let $\bar{x}$ be the equilibrium point of (1.3), from the first inequality of (3.28), we can get

$$L_\alpha = A_{l,\alpha} + \frac{B_{r,\alpha} R_\alpha^2}{R_\alpha^2}, \quad R_\alpha = A_{r,\alpha} + \frac{B_{l,\alpha} L_\alpha^2}{L_\alpha^2}. \tag{3.29}$$

Hence it implies that

$$L_\alpha = A_{l,\alpha} + B_{r,\alpha}, \quad R_\alpha = A_{r,\alpha} + B_{l,\alpha}, \quad \alpha \in (0, 1]. \tag{3.30}$$

Suppose that x_n is a positive fuzzy solution of (1.3). Since Eq. (3.2) and the second inequality to the last inequality of (3.28) are satisfied, applying Lemma 3.3 to system (3.23), we can obtain that the positive equilibrium point $\bar{x}$ is locally stable. This completes the proof of Theorem 3.3.

4 Two examples

In this section, we give two numerical examples to verify the validity of theoretical results.

Example 4.1 Consider the following second-order fractal FDE with quadratic terms

$$x_{n+1} = A + \frac{Bx_n^2}{x_{n-1}^2}, \ n \in N, \tag{4.1}$$

where

$$A(t)=\begin{cases} t-6, & 6\le t\le 7\\ -t+8, & 7\le t\le 8\end{cases},\quad B(t)=\begin{cases} 2t-2, & 1\le t\le 1.5\\ -2t+4, & 1.5\le t\le 2\end{cases} \tag{4.2}$$

$$x_{-1}(t)=\begin{cases} 2t-4, & 2\le t\le 2.5\\ -2t+6, & 2.5\le t\le 3\end{cases},\quad x_0(t)=\begin{cases} t-1, & 1\le t\le 2\\ -t+3, & 2\le t\le 3\end{cases} \tag{4.3}$$

From (4.2), we get

$$[A]_\alpha=[6+\alpha,8-\alpha]\,,\ [B]_\alpha=\left[1+\frac{\alpha}{2},2-\frac{\alpha}{2}\right],\ \alpha\in(0,1]. \tag{4.4}$$

From (4.3), we get

$$[x_{-1}]_\alpha=\left[2+\frac{1}{2}\alpha,3-\frac{1}{2}\alpha\right],\ \ [x_0]_\alpha=[1+\alpha,3-\alpha],\ \alpha\in(0,1]. \tag{4.5}$$

Therefore, it follows that

$$\overline{\bigcup_{\alpha\in(0,1]}[A]_\alpha}=[6,8],\ \overline{\bigcup_{\alpha\in(0,1]}[B]_\alpha}=[1,2],\ \overline{\bigcup_{\alpha\in(0,1]}[x_{-1}]_\alpha}=[2,3],\ \overline{\bigcup_{\alpha\in(0,1]}[x_0]_\alpha}=[1,3]. \tag{4.6}$$

From (4.1), considering $\alpha-$cuts, one has

$$L_{n+1,\alpha}=A_{l,\alpha}+\frac{B_{l,\alpha}L^2_{n,\alpha}}{L^2_{n-1,\alpha}},\ \ R_{n+1,\alpha}=A_{r,\alpha}+\frac{B_{r,\alpha}R^2_{n,\alpha}}{R^2_{n-1,\alpha}},\ \alpha\in(0,1]. \tag{4.7}$$

Therefore, $A_{l,\alpha}>3B_{l,\alpha},A_{r,\alpha}>3B_{r,\alpha},\forall\alpha\in(0,1]$, and $x_i\in\Re^+_f(i=0,-1)$, so by virtue of Theorem 3.2, one has that the positive fuzzy solution x_n of Eq.(4.1) is bounded and persistent.

On the other hand, there exists a unique positive equilibrium $\overline{x}=(7,8.5,10)$. Moreover, $\lim_{n\to\infty}x_n=\overline{x}$. (see Fig.1-Fig.3)

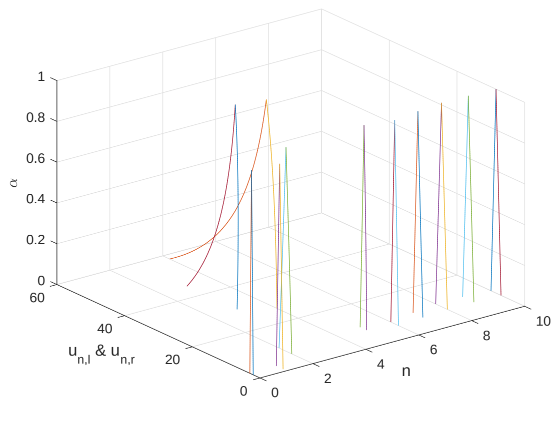

Fig.1. The Dynamics of system (4.7).

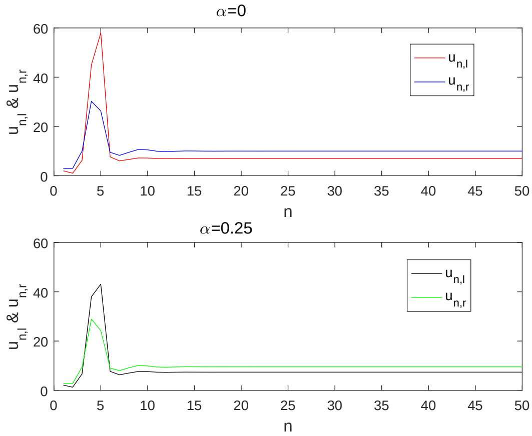

Fig.2. The solution of system (4.7) at $\alpha = 0$ and $\alpha = 0.25$.

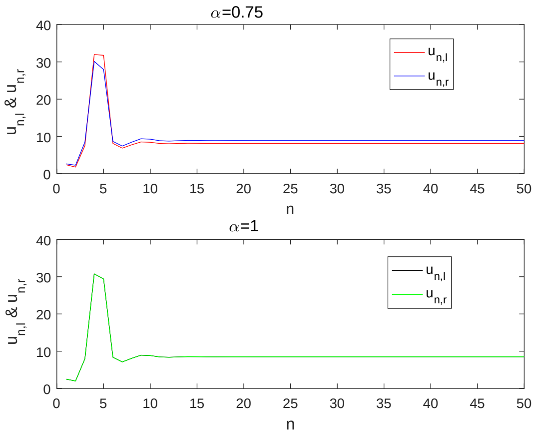

Fig.3. The solution of system (4.7) at $\alpha = 0.75$ and $\alpha = 1$.

Example 4.2 Consider Eq.(4.1), where

$$A(t) = \begin{cases} \frac{1}{2}t - 3, & 6 \le t \le 8 \\ -\frac{1}{2}t + 5, & 8 \le t \le 10 \end{cases}, \quad B(t) = \begin{cases} 4t - 8, & 2 \le t \le 2.25 \\ -4t + 10, & 2.25 \le t \le 2.5 \end{cases} \tag{4.8}$$

$$x_{-1}(t) = \begin{cases} t - 2, & 2 \le t \le 3 \\ -t + 4, & 3 \le t \le 4 \end{cases}, \quad x_0(t) = \begin{cases} t - 3, & 3 \le t \le 4 \\ -t + 5, & 4 \le t \le 5 \end{cases} \tag{4.9}$$

From (4.8), we get

$$[A]_\alpha = [6 + 2\alpha, 10 - 2\alpha], \ [B]_\alpha = \left[2 + \frac{1}{4}\alpha, 2.5 - \frac{1}{4}\alpha\right], \ \alpha \in (0, 1]. \tag{4.10}$$

From (4.9), we get

$$[x_{-1}]_\alpha = [2 + \alpha, 4 - \alpha], \quad [x_0]_\alpha = [3 + \alpha, 5 - \alpha], \ \alpha \in (0, 1]. \tag{4.11}$$

Therefore, it follows that

$$\overline{\bigcup_{\alpha \in (0,1]} [A]_\alpha} = [6, 10], \ \overline{\bigcup_{\alpha \in (0,1]} [B]_\alpha} = [2, 2.5], \ \overline{\bigcup_{\alpha \in (0,1]} [x_{-1}]_\alpha} = [2, 4], \ \overline{\bigcup_{\alpha \in (0,1]} [x_0]_\alpha} = [3, 5]. \tag{4.12}$$

From (4.1), considering $\alpha-$cuts, one has

$$L_{n+1,\alpha} = A_{l,\alpha} + \frac{B_{r,\alpha} R_{n,\alpha}^2}{R_{n-1,\alpha}^2}, \quad R_{n+1,\alpha} = A_{r,\alpha} + \frac{B_{l,\alpha} L_{n,\alpha}^2}{L_{n-1,\alpha}^2}, \ \alpha \in (0,1]. \tag{4.13}$$

It is obvious that condition (3.28) holds true, and $x_i \in \Re_f^+ (i = 0, -1)$, so by virtue of Theorem 3.3, there exists a unique positive equilibrium $\overline{x} = (8.5, 10.25, 12)$. Furthermore it is locally stable. (see Fig.4-Fig.6)

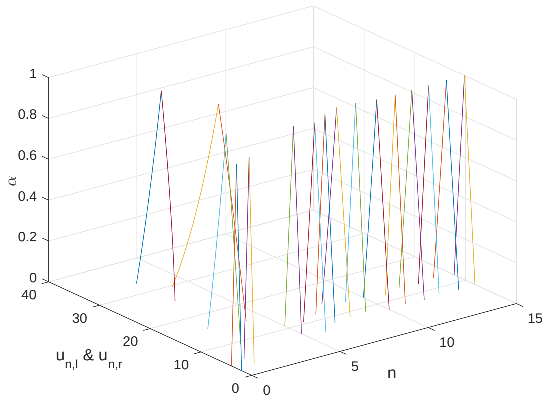

Fig.4. The Dynamics of system (4.13).

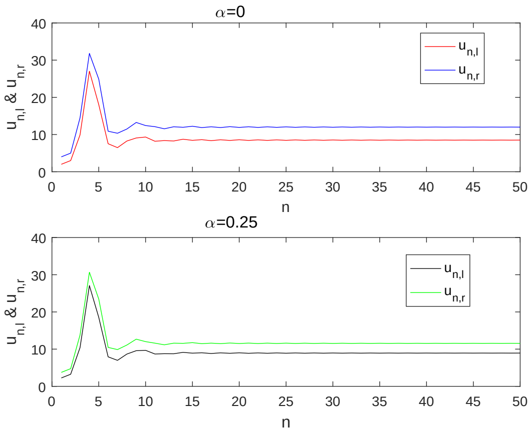

Fig.5. The solution of system (4.13) at $\alpha = 0$ and $\alpha = 0.25$.

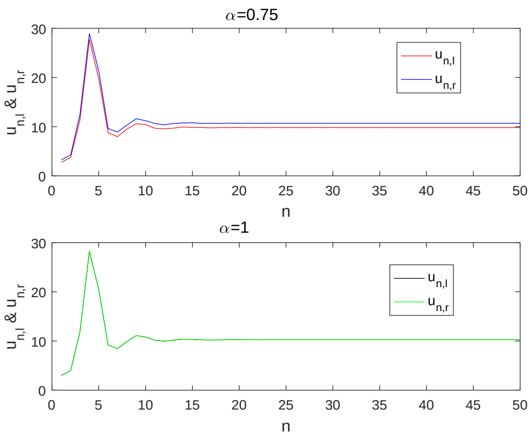

Fig.6. The solution of system (4.13) at $\alpha = 0.75$ and $\alpha = 1$.

5 Conclusion and future work

This work discusses FDE $x_{n+1} = A + \frac{Bx_n^2}{x_{n-1}^2}$ in term with g-division of fuzzy numbers. Firstly, the existence of a positive solution to (1.3) is obtained. Secondly, main results are included in as follows.

(i) The positive fuzzy solution of system (1.3) is persistence and bounded under condition (3.18), every positive fuzzy solution x_n tend to the unique equilibrium x if $A_{l,\alpha} > 3B_{l,\alpha}, A_{r,\alpha} > 3B_{r,\alpha}, \alpha \in (0, 1]$ as $n \to \infty$. (ii) If conditions (3.28) hold true, then the unique positive fuzzy equilibrium x is locally stable. Finally, two illustrative examples are provided to verify the effectiveness of theoretical analysis. In next work, we will further study the following fuzzy difference equation $x_{n+1} = A + \frac{Bx_n^p}{x_{n-1}^p}, p \in (0, +\infty)$. We give an open problem: which condition is the parameter p satisfied with, then the positive fuzzy solution is bounded and persistence.

Data Availability

The data used to support the findings of this study are included within the article.

Competing interests

The authors declare that they have no competing interests.

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