Journal of Applied Analysis and Computation
Volume *, Number ***, , 1–17

Website:http://www.jaac-online.com
DOI:10.11948/2156-907X

BÄCKLUND TRANSFORMATIONS AND INFINITE NEW EXPLICIT EXACT SOLUTIONS OF A VARIANT BOUSSINESQ EQUATIONS*

Yadong Shang[1,2], Huafei Di[2]

Abstract This paper deals with a variant Boussinesq equations which describes the propagation of shallow water waves in a lake or near an ocean beach. We derive out two hetero-Bäcklund transformations between the variant Boussinesq equations and two linear parabolic equations by using the extended homogeneous balance method. We also obtain two hetero-Bäcklund transformations between the variant Boussinesq equations and Burgers equations. Furthermore, we obtain two hetero-Bäcklund transformation between the variant Boussinesq equations and heat equations. By using these Bäcklund transformations and so-called "seed solution", we obtain a large number of explicit exact solutions of the variant Boussinesq equations. Especially, The infinite explicit exact singular wave solutions of variant Boussinesq equations are obtained for the first time. It is worth noting that these singular wave solutions of variant Boussinesq equations will blow up on some lines or curves in the (x, t) plane. These facts reflect the complexity of the structure of the solution of variant Boussinesq equations. It also reflects the complexity of shallow water wave propagation from one aspect.

Keywords a variant Boussinesq equations, nonlinear transformation,exact linearization, explicit exact solution, singular wave solution.

MSC(2010) 35C07, 37G10.

1. Introduction

The following variant Boussinesq equations

$$\begin{cases} u_t + (uv)_x + v_{xxx} = 0, \\ v_t + (\frac{1}{2}v^2)_x + u_x = 0, \end{cases} \tag{1.1}$$

was derived by Broer [1] to describe the propagation of shallow water waves in a lake or near an ocean beach. It can be converted from the completely integrable

†The corresponding author. Email address: gzydshang@126.com.(Y. Shang)
[1]School of Data Science, Guangzhou Huashang College, Guangzhou, Guangdong, 511300,P R China
[2]School of Mathematics and Information Science, Guangzhou Uinversity, Guangzhou,Guangdong,510006,P R China
*This work was completed with the support of the Natural Science Foundation of China (11801108), Guangdong Basic and Applied Basic Research Foundation (Nos.2021A1515010314,2023A1515030107), the Science and Technology Planning Project Guangzhou City(No.202201010111).

variant of the classical Boussinesq equations

$$\begin{cases} \rho_t + v_x + (\rho v)_x + v_{xxx} = 0, \\ v_t + (\frac{1}{2}v^2)_x + \rho_x = 0, \end{cases} \tag{1.2}$$

by the transformation $u(x,t) = 1 + \rho(x,t)$. As a water wave model, u is the height of free wave surface for fluid in the trough and v is the wave velocity. The same system (1.1) in terms of the potential $\varphi : u = \varphi_x$ was derived by Kaup [2]. He also demonstrated that (1.1) is the compatibility condition for a pair of linear equations. Kupershmidt [3] discussed the symmetry and conservation law of the system (1.1) and the corresponding hierarchy by making use of the theory of non-standard integrable systems developed in Section 2 of his paper. Sachs [4] studied the Painlevé property, rational solutions and its equivalent relation with AKNS system. Ablowitz [5] studied inverse scattering transformation solutions for the system. Wang [6] obtained the solitary wave solutions of variant Boussinesq equations (1.1) by using a homogeneous balance method. Fan and Zhang [7] constructed the Bäcklund transformations of Modified Boussinesq equation (1.1) by an improved homogeneous balance method. Furthermore, Fan [8] obtained the Bäcklund transformation, linearization transformation and three types of similarity reductions by using the generalized homogeneous balance technique. Some new symmetry reductions and similarity solutions of (1.1) were given by using direct reduction method developed by Clarkson and Kruskal in [9]. They obtained two sets explicit exact solutions of rational solutions of (1.1). Yan and Zhang [10] obtained several types of explicit and exact travelling wave solutions by using an improved sine-cosine method. In another paper, Yan and Zhang [11] presented a new generalized transformation based upon the well-known Riccati equation and obtained some new travelling wave solutions, which contain new solitary wave solutions, periodic wave solutions, and the combined formal solitary wave solutions and periodic wave solutions. In [12], Fan and Hon devised a new algebraic method to uniformly construct a series of travelling wave solutions for two variant Boussinesq equations. These solutions include soliton solutions, rational solutions, triangular periodic solutions, Jacobi and Weierstrass doubly periodic wave solutions. In [13], the authors derived the conservation laws of the variant Boussinesq system (1.1) by an interesting method of increasing the order of partial differential equations. For more researches on traveling wave solutions and solitary wave solutions of variant Boussinesq equations, refer to literatures [14–22] and references therein. For more recent researches on the variant Boussinesq equation are recommended for readers to refer to references [23–27]. The study of nonlinear phenomena plays an important role in mathematical physics,chemistry, biology, astrophysics and geophysics. The exact solutions of the nonlinear partial differential equation can not only provide some inspiration for their qualitative research, but also be used to judge the superiority or inferiority of the numerical methods. Some well established methods have been proposed for obtaining exact solutions. Here are some recent typical literature that needs to be mentioned [28–30].

In the literature mentioned above, we find that most of the obtained solutions are traveling wave solutions, which are mainly global smooth solutions. Only a few literatures have obtained some simple singular solutions. As we all know, the structure and dynamical behavior of the solution of the nonlinear evolution equation is very complex. In many cases, the solutions of nonlinear evolution equations have some singularity. Many of the solution of the initial value or/and initial boundary value problem of nonlinear evolution equations will blow up in finite time, even if the initial value is smooth. In order to reveal the structural complexity of the solution of the nonlinear evolution equation, it is necessary to study the singular solution of the equation.

In this paper, we analytically investigate the variant Boussinesq equations. Firstly, we will transform the variant Boussinesq equations (1.1) into a linear equation by introducing a nonlinear function transformation. Secondly, we obtain two hetero-Bäcklund transformations between the variant Boussinesq equations and two linear parabolic equations. We then obtain two hetro-Bäcklund transformations between the variant Boussinesq equations and Burgers equations. Furthermore, we obtain two hetero-Bäcklund transformations between the variant Boussinesq equations and heat equations. Make use of these hetero-Bäcklund transformations and so-called "seed solutions", we construct infinite explicit exact solutions of the variant Boussinesq equations (1.1) by solving the reduced equations. The infinite solutions include single soliton solutions, multi-soliton solutions, rational solutions, and various forms of singular solutions. These singular solutions will blow up on some lines or curves of the (x, t) plane. The results reveal the complexity of the structure of solutions of nonlinear equations.

2. Exact linearlization of Eqs.(1.1)

Over one hundred years agao, Bäcklund investigated transformation properties of pseudospherical surfaces. As a consequence of this study, Bäcklund derived a transformation which generates a new solution of the Sine-Gordon equation from a given solution. Afterwards, people collectively referred to the relationship between one solution of a partial differential equation and another as the Bäcklund transformation. These two solutions can be solutions of different partial differential equations or systems, or they can be two solutions of the same equation. A hetero-Bäcklund transformation, also called a non-auto-Bäcklund transformation, has been accounted as the relation between the solutions of different partial differential equations or systems, while an auto-Bäcklund transformation, of the same partial differential equation or system itself. The Bäcklund transformation is not only a useful method to find exact solutions to some soliton equation from a trivial 'seed solution' but also related to infinite conservation laws and inverse scattering method.

In order to transform the variant Boussinesq equation (1.1) into linear equation, we assume that Eqs.(1.1) possesses the solution in the form

$$u = \alpha(\ln\varphi(x,t))_{xx} + u_0(x,t), v(x,t) = \beta(\ln\varphi(x,t))_x + v_0(x,t), \tag{2.1}$$

where $\varphi(x, t), u_0(x, t), v_0(x, t)$ are functions of indicated arguments to be determined later and α, β are two constants to be determined.

From (2.1) , we have

$$\begin{aligned}
u_t &= \alpha[\frac{\varphi_{xxt}}{\varphi} - \frac{\varphi_{xx}\varphi_t}{\varphi^2} - 2\frac{\varphi_x\varphi_{xt}}{\varphi^2} + 2\frac{\varphi_x^2\varphi_t}{\varphi^3}] + u_{0t}, \\
v_t &= \beta[\frac{\varphi_{xt}}{\varphi} - \frac{\varphi_x\varphi_t}{\varphi^2}] + v_{0t}, \\
u_x &= \alpha[\frac{\varphi_{xxx}}{\varphi} - 3\frac{\varphi_{xx}\varphi_x}{\varphi^2} + 2\frac{\varphi_x^3}{\varphi^3}] + u_{0x},
\end{aligned} \tag{2.2}$$

$$v_x = \beta[\frac{\varphi_{xx}}{\varphi} - \frac{\varphi_x^2}{\varphi^2}] + v_{0x},$$

$$v_{xxx} = \beta[\frac{\varphi_{xxxx}}{\varphi} - 4\frac{\varphi_x\varphi_{xxx}}{\varphi^2} - 3\frac{\varphi_{xx}^2}{\varphi^2} + 12\frac{\varphi_x^2\varphi_{xx}}{\varphi^3} - 6\frac{\varphi_x^4}{\varphi^4}] + v_{0xxx},$$

Substituting the expressions above (2.2) into the left-hand side of Eqs. (1.1), yields

$$\begin{aligned}
&u_t + (uv)_x + v_{xxx}\\
&= \alpha[\frac{\varphi_{xxt}}{\varphi} - \frac{\varphi_{xx}\varphi_t}{\varphi^2} - 2\frac{\varphi_x\varphi_{xt}}{\varphi^2} + 2\frac{\varphi_x^2\varphi_t}{\varphi^3}] + u_{0t}\\
&+ \alpha\beta[\frac{2\varphi_x^4}{\varphi^4} - \frac{3\varphi_x^2\varphi_{xx}}{\varphi^3} + \frac{\varphi_x\varphi_{xxx}}{\varphi^2}] + \beta u_{0x}\frac{\varphi_x}{\varphi} + \alpha v_0[\frac{\varphi_{xxx}}{\varphi} - 3\frac{\varphi_{xx}\varphi_x}{\varphi^2} + 2\frac{\varphi_x^3}{\varphi^3}]\\
&+ \alpha\beta[\frac{\varphi_{xx}^2}{\varphi^2} - 2\frac{\varphi_x^2\varphi_{xx}}{\varphi^3} + \frac{\varphi_x^4}{\varphi^4}] + (\beta u_0 + \alpha v_{0x})[\frac{\varphi_{xx}}{\varphi} - \frac{\varphi_x^2}{\varphi^2}] + (u_0v_0)_x\\
&+ \beta[\frac{\varphi_{xxxx}}{\varphi} - 4\frac{\varphi_x\varphi_{xxx}}{\varphi^2} - 3\frac{\varphi_{xx}^2}{\varphi^2} + 12\frac{\varphi_x^2\varphi_{xx}}{\varphi^3} - 6\frac{\varphi_x^4}{\varphi^4}] + v_{0xxx},\\
&= (3\alpha\beta - 6\beta)\frac{\varphi_x^4}{\varphi^4} + 2\alpha\frac{\varphi_x^2\varphi_t}{\varphi^3} + (12\beta - 5\alpha\beta)\frac{\varphi_x^2\varphi_{xx}}{\varphi^3}\\
&+ (\alpha\beta - 4\beta)\frac{\varphi_x\varphi_{xxx}}{\varphi^2} + (\alpha\beta - 3\beta)\frac{\varphi_{xx}^2}{\varphi^2} - \alpha[\frac{\varphi_{xx}\varphi_t}{\varphi^2} + 2\frac{\varphi_x\varphi_{xt}}{\varphi^2}]\\
&+ \alpha\frac{\varphi_{xxt}}{\varphi} + \beta\frac{\varphi_{xxxx}}{\varphi} + \left\{\beta u_0\frac{\varphi_x}{\varphi} + \alpha v_0[\frac{\varphi_{xx}}{\varphi} - \frac{\varphi_x^2}{\varphi^2}]\right\}_x + u_{0t} + (u_0v_0)_x + v_{0xxx}.
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&v_t + vv_x + u_x\\
&= \beta[\frac{\varphi_{xt}}{\varphi} - \frac{\varphi_x\varphi_t}{\varphi^2}] + \beta^2[\frac{\varphi_x\varphi_{xx}}{\varphi^2} - \frac{\varphi_x^3}{\varphi^3}] + \alpha[\frac{\varphi_{xxx}}{\varphi} - 3\frac{\varphi_{xx}\varphi_x}{\varphi^2} + 2\frac{\varphi_x^3}{\varphi^3}]\\
&+ (\beta v_0\frac{\varphi_x}{\varphi})_x + v_{0t} + v_0v_{0x} + u_{0x},\\
&= (2\alpha - \beta^2)\frac{\varphi_x^3}{\varphi^3} + [\beta^2 - 3\alpha]\frac{\varphi_{xx}\varphi_x}{\varphi^2} - \beta\frac{\varphi_x\varphi_t}{\varphi^2} + \beta\frac{\varphi_{xt}}{\varphi} + \alpha\frac{\varphi_{xxx}}{\varphi}\\
&+ (\beta v_0\frac{\varphi_x}{\varphi})_x + v_{0t} + v_0v_{0x} + u_{0x}.
\end{aligned} \tag{2.4}$$

To simplify expression (2.3), (2.4) and determine α, β of (2.1), setting the coefficients of $\frac{\varphi_x^4}{\varphi^4}$ in (2.3) and $\frac{\varphi_x^3}{\varphi^3}$ in (2.4) to zero yields a system of algebraic equations for α and β

$$3\alpha\beta - 6\beta = 0, \tag{2.5}$$

$$2\alpha - \beta^2 = 0. \tag{2.6}$$

Considering α, β are non-zero, solving (2.5) and (2.6), we get

$$\alpha = 2, \beta = \pm 2. \tag{2.7}$$

Making use of (2.7), expressions (2.3) and (2.4) can be simplified as

$$\begin{aligned}
&u_t + (uv)_x + v_{xxx}\\
&= \frac{2\varphi_x^2}{\varphi^3}(\alpha\varphi_t + \beta\varphi_{xx} + \alpha v_0\varphi_x) - \frac{2\varphi_x}{\varphi^2}(\alpha\varphi_{xt} + \beta\varphi_{xxx} + \alpha v_0\varphi_{xx} + \beta u_0\varphi_x)
\end{aligned} \tag{2.8}$$

$$-\frac{\varphi_{xx}}{\varphi^2}(\alpha\varphi_t+\beta\varphi_{xx}+\alpha v_0\varphi_x)+\partial_x(\alpha\varphi_{xt}+\beta\varphi_{xxx}+\alpha v_0\varphi_{xx}+\beta u_0\varphi_x)\frac{1}{\varphi}$$
$$+(\beta u_0-\alpha v_{0x})\frac{\varphi_x^2}{\varphi^2}+u_{0t}+(u_0v_0)_x+v_{0xxx}.$$

and

$$\begin{aligned} &v_t+vv_x+u_x \\ &=[-\alpha\varphi_{xx}-\beta\varphi_t-\beta v_0\varphi_x]\frac{\varphi_x}{\varphi^2}+(\beta\varphi_t+\alpha\varphi_{xx}+\beta v_0\varphi_x)_x\frac{1}{\varphi}+v_{0t}+v_0v_{0x}+u_{0x}. \end{aligned} \tag{2.9}$$

It is easy to find that $u(x,t), v(x,t)$ are the solution of (1.1), as long as $\varphi(x,t)$ satisfy the following compatibility conditions

$$\begin{aligned} &\alpha\varphi_t+\beta\varphi_{xx}+\alpha v_0\varphi_x=0, \\ &\alpha\varphi_{xt}+\beta\varphi_{xxx}+\alpha v_0\varphi_{xx}+\beta u_0\varphi_x=0, \\ &\beta\varphi_t+\alpha\varphi_{xx}+\beta v_0\varphi_x=0, \end{aligned} \tag{2.10}$$

while $u_0(x,t)$ and $v_0(x,t)$ satisfy the following compatibility conditions

$$\begin{aligned} &u_{0t}+(u_0v_0)_x+v_{0xxx}=0, \\ &v_{0t}+v_0v_{0x}+u_{0x}=0, \\ &\beta u_0-\alpha v_{0x}=0. \end{aligned} \tag{2.11}$$

From (2.10), taking into account (2.5), (2.6), the third equations of (2.11), we know that $\varphi(x,t)$ only needs to satisfy the linear parabolic equation

$$\alpha\varphi_t+\beta\varphi_{xx}+\alpha v_0\varphi_x=0. \tag{2.12}$$

In summary, we have the following conclusion:

Theorem 1. *If the functions $v_0(x,t)$ is the solutions of Burgers equation*

$$v_{0t}+v_0v_{0x}\pm v_{0xx}=0 \tag{2.13}$$

then there exist two hetero-Bäcklund transformations

$$u(x,t)=2(\ln\varphi(x,t))_{xx}\pm v_{0x}(x,t), v(x,t)=\pm 2(\ln\varphi(x,t))_x+v_0(x,t), \tag{2.14}$$

between the variant Boussinesq equations (1.1) and the linear parabolic equations

$$\varphi_t\pm\varphi_{xx}+v_0\varphi_x=0. \tag{2.15}$$

Remark 1. Taking $\alpha=\beta=2$, from Theorem 1, we derive a hetero-Bäcklund transformation

$$u(x,t)=2(\ln\varphi(x,t))_{xx}+u_0(x,t), v(x,t)=2(\ln\varphi(x,t))_x+v_0(x,t), \tag{2.16}$$

where φ satisfy

$$\varphi_t+\varphi_{xx}+v_0\varphi_x=0, \tag{2.17}$$

while u_0, v_0 is solution of the following equations

$$\begin{aligned} &v_{0t}+v_0v_{0x}+u_{0x}=0, \\ &u_0-v_{0x}=0. \end{aligned} \tag{2.18}$$

This result is in agreement with the results obtained by [7-8]. However, the last conditional equation $u_0 = v_{0x}$ was omitted in the compatibility conditions (4.8) of [7], all the compatibility conditions for u_0 and v_0 were even omitted in [8].

Remark 2. Taking $\alpha = 2, \beta = -2$, from Theorem 1, we derive another hetero-Bäcklund transformation

$$u(x,t) = 2(\ln \varphi(x,t))_{xx} + u_0(x,t), v(x,t) = -2(\ln \varphi(x,t))_x + v_0(x,t), \tag{2.19}$$

where φ satisfy

$$\varphi_t + v_0\varphi_x - \varphi_{xx} = 0, \tag{2.20}$$

While u_0, v_0 is solution of the following equations

$$\begin{aligned} &v_{0t} + v_0 v_{0x} + u_{0x} = 0, \\ &u_0 + v_{0x} = 0. \end{aligned} \tag{2.21}$$

The Bäcklund transformation (2.19) with conditions (2.20) is obviously different from the Bäcklund transformation (2.16) with compatibility conditions (2.17). This is a new Bäcklund transformation obtained in this article. This result is a supplement to the results of literature [7-8].

Remark 3. From Theorem 1, setting $u = \pm v_x$, we find that the variant Boussinesq equations (1.1) can be transformed into Burgers equation

$$v_t + vv_x \pm v_{xx} = 0. \tag{2.22}$$

Remark 4. Furthermore, taking $v_0(x,t) = \varphi(x,t)$, we obtain two hetero-Bäcklund transformations

$$u(x,t) = 2(\ln \varphi(x,t))_{xx} \pm \varphi_x(x,t), v(x,t) = \pm 2(\ln \varphi(x,t))_x + \varphi(x,t), \tag{2.23}$$

between the variant Boussinesq equations (1.1) and the Burgers equations

$$\varphi_t \pm \varphi_{xx} + \varphi\varphi_x = 0. \tag{2.24}$$

Remark 5. When taking $v_0 = 0$ in Theorem 1, we obtain two hetero-Bäcklund transformations

$$u(x,t) = 2(\ln \varphi(x,t))_{xx}, v(x,t) = \pm 2(\ln \varphi(x,t))_x \tag{2.25}$$

between the variant Boussinesq equations (1.1) and the heat equations

$$\varphi_t \pm \varphi_{xx} = 0. \tag{2.26}$$

By using the different Bäcklund transformations obtained above and starting from different "seed solutions", we can obtain a large number of explicit exact solutions of the variant Boussinesq equations (1.1).

3. Explicit exact solutions of Eqs.(1.1)

In this section, we focus on finding more new explicit exact solutions of the variant Boussinesq equations.

Taking the solution $v_0(x,t)$ of Burgers equation (1.1) as constant $-c_1$, it is easy to know that equation (2.15) has a solution $\varphi(x,t) = x+c_1t+c_2$. Thus the Bäcklund transformation (2.14) gives the following explicit exact traveling wave solutions of variant Boussinesq equations (1.1)

$$\begin{cases} u(x,t) = -\frac{2}{(x+c_1t+c_2)^2}, \\ v(x,t) = -c_1 \pm \frac{2}{x+c_1t+c_2}. \end{cases} \tag{3.1}$$

where c_1, c_2 are two arbitrary constants. The rational solutions (3.1) here contain the solutions obtained by the direct reduction method in [8,9] as a special case (the case of the sign of (3.1) is positive). It should be pointed out that a minus sign is omitted in front of u in the solution expression in [8].

In the case of $v_0 = Constant$, we make a variable transformation $w(x,t) = e^{\pm(\frac{v_0}{2}x - \frac{v_0^2}{4}t)}\varphi(x,t)$, then the Bäcklund transformation (2.14) becomes Bäcklund transformation (2.25), and $w(x,t)$ is the solution of equation (2.26). It is easy to verify that $\varphi(x,t) = (x + c_1t + c_2)\exp\mp(\frac{c_1}{2}x + \frac{c_1^2}{4}t)$ is a solution of (2.15), so the rational solutions (27) can also be obtained from the Bäcklund transformation (2.23).

In [9], Fan and Zhang obtained two set of explicit exact solutions of variant Boussinesq equations (1.1)

$$u = 0, \quad v = \frac{x - C_1}{t + C_2}.$$

and

$$u = \frac{(x + C_1)^2}{9(t + C_2)^2}, \quad v = \frac{2(x + C_1)}{3(t + C_2)}.$$

It is easy to verify that $\varphi(x,t) = \frac{x+C_1}{t+C_2}$ is a solution of Burgers equation (2.24)(one can also see [31, 32]). By using $u = v_x, v = \varphi(x,t)$, we know that variant Boussinesq equations (1.1) have two sets of new explicit exact solutions

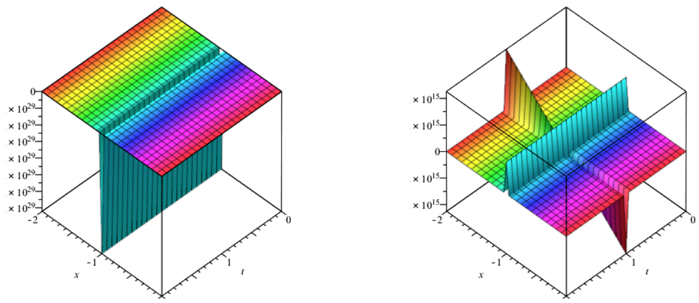
$$\begin{cases} u(x,t) = \pm\frac{1}{t+C_2}, \\ v(x,t) = \frac{x+C_1}{t+C_2}. \end{cases} \tag{3.2}$$

From the hetero-Bäcklund transformations (2.23) we can also obtain two sets of new explicit exact solutions of variant Boussinesq equations (1.1)

$$\begin{cases} u(x,t) = -\frac{2}{(x+C_1)^2} \pm \frac{1}{t+C_2}, \\ v(x,t) = \pm\frac{2}{(x+C_1)} + \frac{x+C_1}{t+C_2}. \end{cases} \tag{3.3}$$

In [33], S. Hood obtained the exact solution of Burgers equation $\varphi(x,t) = \frac{x}{t} + \frac{\lambda_4}{x+\lambda_3 t}, \lambda_4 = 0, 2.$ by using direct method of Clarkson and Kruskal ((1.23) and (3.41) of [33]). More general, we can find the solution of Burgers equation

$$\varphi(x,t) = \frac{\lambda_4}{x + \lambda_3 t + c_1 + c_2c_3} + \frac{x + c_1}{t + c_2}, \lambda_4 = 0, 2. \tag{3.4}$$



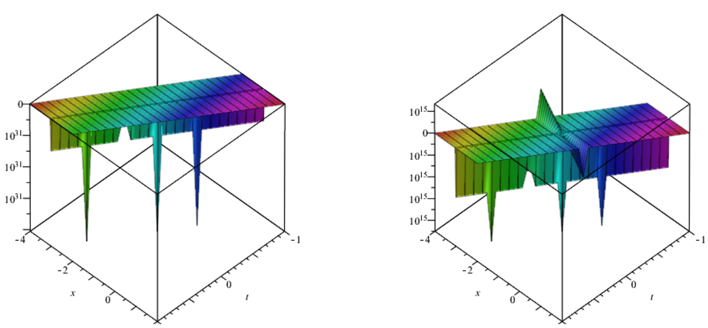
(a) $u(x,t)$ (b) $v(x,t)$

Figure 1. The surface plots of (3.3) when $C_1 = 1$, $C_2 = -1$

It should be noted the solution (3.4) here contains the solution (3.39) of [33] as a special case. By $u = v_x, v = \varphi(x,t)$ we obtain another set of solutions of variant Boussinesq equations

$$\begin{cases} u(x,t) = -\frac{2}{(x+\lambda_3 t+c_1+c_2c_3)^2} + \frac{1}{t+c_2}, \\ v(x,t) = \frac{2}{x+\lambda_3 t+c_1+c_2c_3} + \frac{x+c_1}{t+c_2}. \end{cases} \tag{3.5}$$

From the hetero-Bäcklund transformations (2.23) and (3.4) we can another new explicit exact solutions of variant Boussinesq equations (1.1).



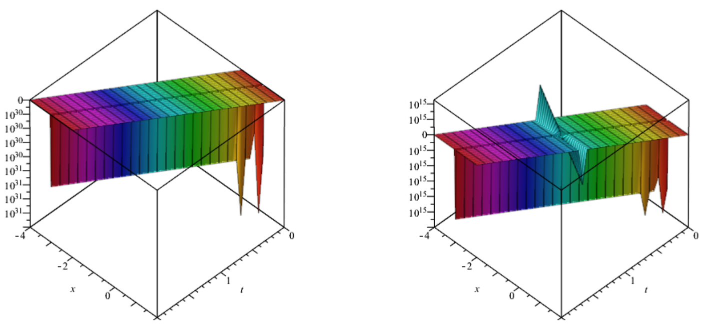
(a) $u(x,t)$ (b) $v(x,t)$

Figure 2. The surface plots of (3.5) when $c_1 = 1$, $c_2 = 0$, $\lambda_3 = 2$

The explicit exact solutions (3.1),(3.2),(3.3), and (3.5) are extensions and supplements to the corresponding results in References [8.9].

Next, we will obtain more explicit exact solutions of variant Boussinesq equations (1.1) by using the solution of the heat conduction equation (2.26) and the Bäcklund transform (2.25). We only discuss the case of positive sign in (2.26).

Case 1. Taking the static solution of Eq.(2.26) as $\varphi(x,t) = ax + b, a, b$ are two arbitrary constants that are not all zero, from the Bäcklund transformation (2.25),



(a) $u(x,t)$ (b) $v(x,t)$

Figure 3. The surface plots of (3.5) when $c_1 = 1$, $c_2 = -1$, $\lambda_3 = c_3 = 2$

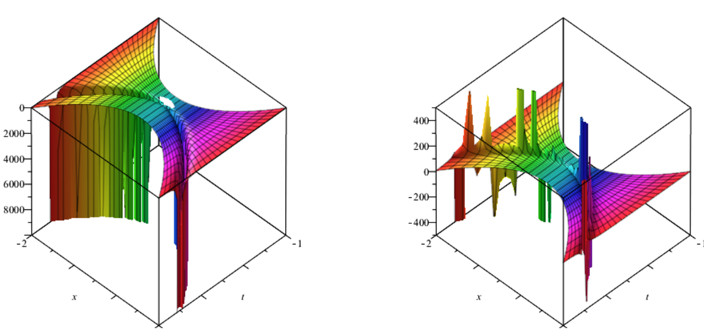
we obtain a set of explicit exact solutions of Eqs. (1.1) as follows

$$\begin{cases} u(x,t) = \frac{-2a^2}{(ax+b)^2}, \\ v(x,t) = \frac{2a}{ax+b}. \end{cases} \tag{3.6}$$

the solutions (3.6) are two sets of static solutions variant Boussinesq equations (1.1).

Case 2. Taking the solution of Eq. (2.26) as $\varphi(x,t) = a(x^2-2t)+bx+c, a, b, c$ are three arbitrary constants that are not all zero, from the Bäcklund transformation (2.25), we obtain a set of explicit exact solutions of Eqs.(1.1) as follows

$$\begin{cases} u(x,t) = 2\frac{2ac-b^2-2abx-2a^2(x^2+2t)}{[a(x^2-2t)+bx+c]^2}, \\ v(x,t) = 2\frac{2ax+b}{a(x^2-2t)+bx+c}. \end{cases} \tag{3.7}$$



(a) $u(x,t)$ (b) $v(x,t)$

Figure 4. The surface plots of (3.7) when $a = b = c = 1$

Case 3. Taking the solution of Eq. (2.26) as $\varphi(x,t) = ax^3+bx^2+cx+d-(6ax+2b)t, a, b, c, d$ are four arbitrary constants that are not all zero, from the Bäcklund transformation (2.25), we obtain a set of explicit exact solutions of Eqs.(1.1) as follows

$$\begin{cases} u(x,t) = -2\frac{3a^2x^4+4abx^3+2b^2x^2+(2bc-6ad)x+c^2-2bd+36a^2t^2+(4b^2-12ac)t}{[ax^3+bx^2+cx+d-(6ax+2b)t]^2}, \\ v(x,t) = 2\frac{3ax^2+2bx+c-6at}{ax^3+bx^2+cx+d-(6ax+2b)t}. \end{cases} \tag{3.8}$$

Case 4. Taking the solution of Eq.(2.26) as $\varphi(x,t) = ax^4 + bx^3 + cx^2 + dx + e - (12ax^2 + 6bx + 2c)t + 12at^2, a, b, c, d, e$ are five arbitrary constants that are not all zero, from the Bäcklund transformation (2.25), we obtain a set of explicit exact solutions of Eqs.(1.1) as follows

$$\begin{cases} u(x,t) = 2\frac{Q(x,t)}{[ax^4+bx^3+cx^2+dx+e-(12ax^2+6bx+2c)t+12at^2]^2}, \\ v(x,t) = 2\frac{4ax^3+3bx^2+2cx+d-(24ax+6b)t}{ax^4+bx^3+cx^2+dx+e-(12ax^2+6bx+2c)t+12at^2}. \end{cases} \tag{3.9}$$

where $Q(x,t) = -4a^2x^6 - 6abx^5 + 24a^2x^4t - (3b^2 + 2ac)x^4 + 24abx^3t - 144a^2x^2t^2 + (4ad - 4bc)x^3 + 24acx^2t - 72abxt^2 - 288a^2t^3 + 2(6ae - c^2)x^2 + 36(2ac - b^2)t^2 + 24adxt + (6be - 2cd)x + (12bd - 24ae - 4c^2)t + 2ce - d^2$.

Remark 6. In general, we can suppose that Eq. (2.26) has a solution in the following form

$$\varphi(x,t) = \Sigma_{i=0}^{k} P_i(x)t^i, \quad k \geq 1. \tag{3.10}$$

where $P_i(x)$ are the polynomial of its variable x. $P_i(x)$ can be determined by solving the following ordinary differential equations

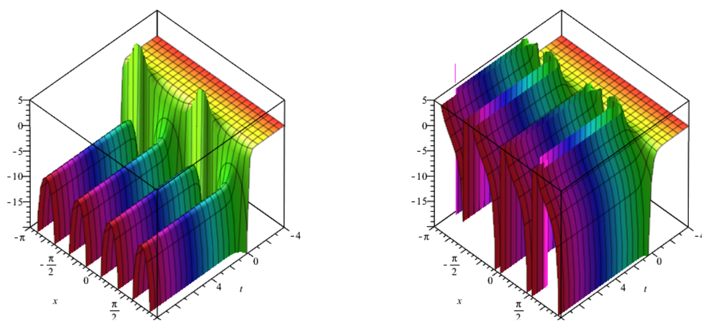
$$\begin{cases} P_0''(x) + P_1(x) = 0, \\ P_1''(x) + 2P_2(x) = 0, \\ . \\ . \\ . \\ P_{k-1}''(x) + kP_k(x) = 0, \\ P_k''(x) = 0, \end{cases} \tag{3.11}$$

By combining $P_i(x)$, the expression of $\varphi(x,t)$ (3.10), and Bäcklund transformation (2.25), we can construct infinite explicit exact solutions of variant Boussinesq equations (1.1).

Remark 6. It should be pointed out that all these singular wave solutions of variant Boussinesq equations will blow up on some lines or curves in the (x,t) plane.

Case 5. Taking the solution of Eq. (2.26) as $\varphi(x,t) = A+B\exp(\lambda^2 t)\sin\lambda x, A, B, \lambda$ are non-zero real constants,by using Bäcklund transformation (2.25), we obtain a set of explicit exact solutions of Eqs.(1.1) as follows

$$\begin{cases} u(x,t) = -2\lambda^2 B\frac{[A\sin\lambda x+B\exp(\lambda^2 t)]\exp(\lambda^2 t)}{[A+B\exp(\lambda^2 t)\sin\lambda x]^2}, \\ v(x,t) = 2\lambda B\frac{\exp(\lambda^2 t)\cos\lambda x}{A+B\exp(\lambda^2 t)\sin\lambda x}. \end{cases} \tag{3.12}$$



(a) $u(x,t)$ (b) $v(x,t)$

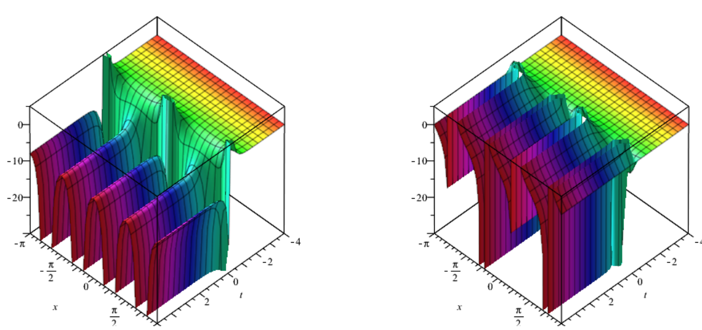
Figure 5. The surface plots of (3.12) when $A=1$, $B=-1$, $\lambda=2$

In specially, when $A=0$, we obtain a set of explicit exact static periodic wave solutions of variant Boussinesq equations (1)

$$\begin{cases} u(x,t) = -2\lambda^2 \csc^2 \lambda x, \\ v(x,t) = 2\lambda \cot \lambda x. \end{cases} \tag{3.13}$$

Case 6. Taking the solution of Eq. (2.26) as $\varphi(x,t) = A + B\exp(\lambda^2 t)\cos \lambda x$, A, B, λ are non-zero real constants, by using Bäcklund transformation (2.25), we obtain the explicit exact solutions of Eqs.(1.1) as follows

$$\begin{cases} u(x,t) = -2\lambda^2 B \frac{[A\cos\lambda x + B\exp(\lambda^2 t)]\exp(\lambda^2 t)}{[A + B\exp(\lambda^2 t)\cos\lambda x]^2}, \\ v(x,t) = -2\lambda B \frac{\exp(\lambda^2 t)\sin\lambda x}{A + B\exp(\lambda^2 t)\cos\lambda x}. \end{cases} \tag{3.14}$$



(a) $u(x,t)$ (b) $v(x,t)$

Figure 6. The surface plots of (3.14) when $A=-1$, $B=2$, $\lambda=-2$

In specially, when $A=0$, we obtain a set of explicit exact static periodic wave solutions of variant Boussinesq equations (1.1)

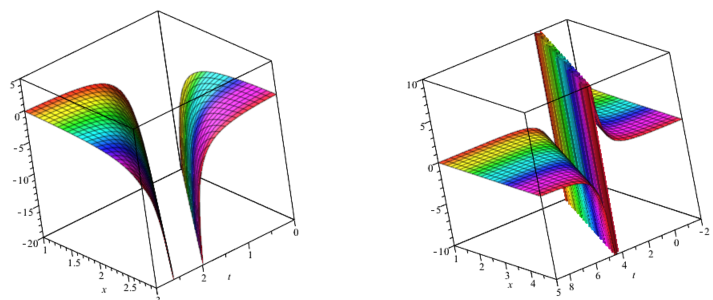
$$\begin{cases} u(x,t) = -2\lambda^2 \sec^2 \lambda x, \\ v(x,t) = -2\lambda \tan \lambda x. \end{cases} \tag{3.15}$$

The solutions (3.12) and (3.14) are all periodic in the spatial direction. The solutions (3.12) and (3.14) can be either globally smooth or singular, depending on the choosing of constants A and B. When $t \to +\infty$, their asymptotic states are static solutions (3.13) and (3.15), respectively.

Case 7. Taking the solution of Eq. (2.26) as $\phi(x,t) = A + B\exp(kx - k^2t), A, B$ are non-zero real constants, we obtain the explicit exact solutions of Eqs.(1.1) as follows

$$\begin{cases} u(x,t) = 2k^2AB\frac{\exp(kx-k^2t)}{[A+B\exp(kx-k^2t)]^2}, \\ v(x,t) = 2Bk\frac{\exp(kx-k^2t)}{A+B\exp(kx-k^2t)}. \end{cases} \tag{3.16}$$

Especially, when $A = B = 1$, the solution (3.16) becomes a set of one-soliton solution, While $A = 1, B = -1$, the solution (3.16) becomes a set of singular traveling wave solution. It is easy to see that the solutions (23) and (24) obtained in Reference [29] are special cases of (3.16) here.



(a) $u(x,t)$ (b) $v(x,t)$

Figure 7. The surface plots of (3.16) when $A = 2$, $B = -1$, $k = 1$

Remark 7. Using the superposition principle, one can obtain more solutions of equation (2.26). Thus we can obtain abundant explicit exact solutions of variant Boussinesq equations (1.1), such as multi-soliton solutions and so on.

Remark 8. If we take the negative sign in equation (2.26), we can also get the corresponding explicit exact solutions,

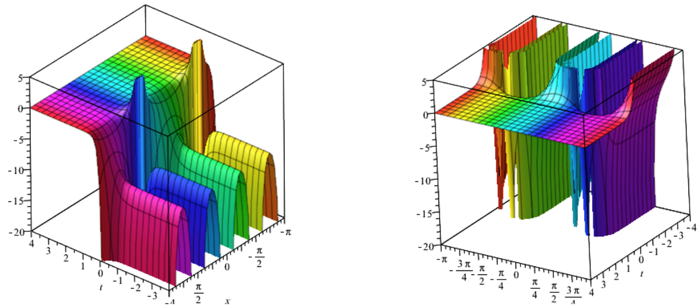
$$\begin{cases} u(x,t) = -2\lambda^2B\frac{[A\sin\lambda x + B\exp(-\lambda^2t)]\exp(-\lambda^2t)}{[A+B\exp(-\lambda^2t)\sin\lambda x]^2}, \\ v(x,t) = -2\lambda B\frac{\exp(-\lambda^2t)\cos\lambda x}{A+B\exp(-\lambda^2t)\sin\lambda x}. \end{cases} \tag{3.17}$$

In specially, when $A = 0$, we obtain a set of explicit exact static periodic wave solutions of variant Boussinesq equations (1.1)

$$\begin{cases} u(x,t) = -2\lambda^2\csc^2\lambda x, \\ v(x,t) = -2\lambda\cot\lambda x. \end{cases} \tag{3.18}$$

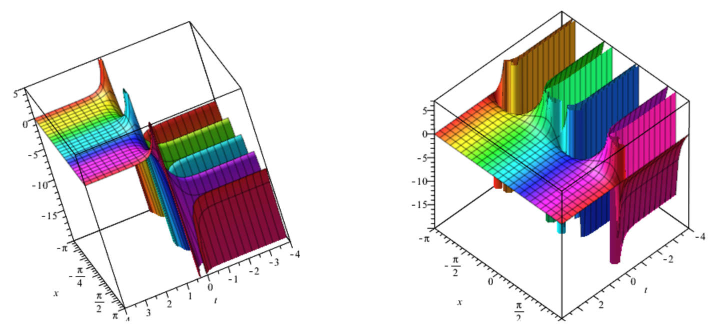
and

$$\begin{cases} u(x,t) = -2\lambda^2B\frac{[A\cos\lambda x + B\exp(-\lambda^2t)]\exp(-\lambda^2t)}{[A+B\exp(-\lambda^2t)\cos\lambda x]^2}, \\ v(x,t) = 2\lambda B\frac{\exp(-\lambda^2t)\sin\lambda x}{A+B\exp(-\lambda^2t)\cos\lambda x}. \end{cases} \tag{3.19}$$



(a) $u(x,t)$ (b) $v(x,t)$

Figure 8. The surface plots of (3.17) when $A = 1$, $B = -1$, $\lambda = 2$



(a) $u(x,t)$ (b) $v(x,t)$

Figure 9. The surface plots of (3.19) when $A = -1$, $B = 2$, $\lambda = -2$

In specially, when $A = 0$, we obtain a set of explicit exact static periodic wave solutions of variant Boussinesq equations (1.1)

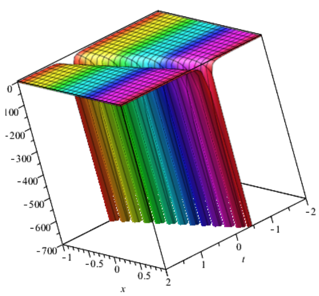
$$\begin{cases} u(x,t) = -2\lambda^2 \sec^2 \lambda x, \\ v(x,t) = 2\lambda \tan \lambda x. \end{cases} \tag{3.20}$$

The solutions (3.17) and (3.19) are all periodic in the spatial direction. The solutions (3.17) and (3.19) can be either globally smooth or singular, depending on the choosing of constants$A$ and B. When $t \to +\infty$,they all asymptotically approach zero.

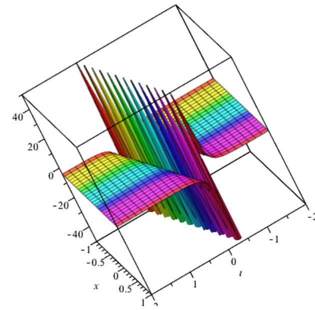
And

$$\begin{cases} u(x,t) = 2k^2 AB \frac{\exp(kx+k^2t)}{[A+B\exp(kx+k^2t)]^2}, \\ v(x,t) = -2Bk \frac{\exp(kx+k^2t)}{A+B\exp(kx+k^2t)}. \end{cases} \tag{3.21}$$

Especially, when $A = B = 1$, the solution (3.21) becomes a set of one-soliton solution, While$A = 1, B = 1$, the solution (3.21) becomes a set of singular traveling wave solution. The solitary wave solution solution (3.21) here is different from the solutions (23) and (24) in Reference [29].



(a) $u(x,t)$



(b) $v(x,t)$

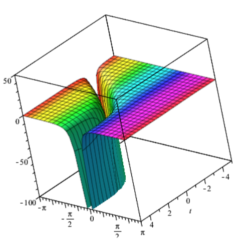
Figure 10. The surface plots of (3.21) when $A=3$, $B=-2$, $k=2$

In addition, equation (1.1) also admits the following explicit exact solutions

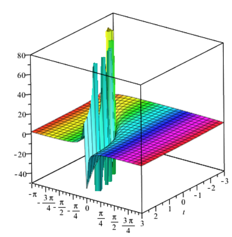
$$\begin{cases} u(x,t) = 2\lambda^2 B \frac{[A\sinh\lambda x - B\exp(\lambda^2 t)]\exp(\lambda^2 t)}{[A+B\exp(\lambda^2 t)\sinh\lambda x]^2}, \\ v(x,t) = -2\lambda B \frac{\exp(\lambda^2 t)\cosh\lambda x}{A+B\exp(\lambda^2 t)\sinh\lambda x}. \end{cases} \tag{3.22}$$

and

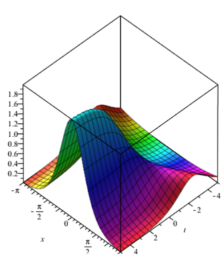
$$\begin{cases} u(x,t) = 2\lambda^2 B \frac{[A\cosh\lambda x + B\exp(\lambda^2 t)]\exp(\lambda^2 t)}{[A+B\exp(\lambda^2 t)\cosh\lambda x]^2}, \\ v(x,t) = -2\lambda B \frac{\exp(\lambda^2 t)\sinh\lambda x}{A+B\exp(\lambda^2 t)\cosh\lambda x}. \end{cases} \tag{3.23}$$



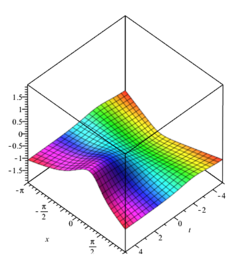
(a) $u(x,t)$ of (3.22)



(b) $v(x,t)$ of (3.22)



(c) $u(x,t)$ of (3.23)



(d) $v(x,t)$ of (3.23)

Figure 11. The surface plots of (3.22) and (3.23) when $A=B=\lambda=1$

In specially, when $A = 0$ in (3.22),((3.23), respectively,) we obtain two set of explicit exact static solutions of variant Boussinesq equations (1.1)

$$\begin{cases} u(x,t) = -2\lambda^2 csch^2 \lambda x, \\ v(x,t) = -2\lambda \coth \lambda x. \end{cases} \tag{3.24}$$

and

$$\begin{cases} u(x,t) = 2\lambda^2 sech^2 \lambda x, \\ v(x,t) = -2\lambda \tanh \lambda x. \end{cases} \tag{3.25}$$

For different values of parameters A and B, solution (3.22)and (3.23) can be either a globally smooth solution or a singular solution. When $t \to +\infty$, their asymptotic states are static solutions (3.24) and (3.25), respectively.

4. Conclusions

In this paper, a variant Boussinesq equation has been investigated. It is shown that the variant Boussinesq equation can be exactly linearized. We derive out two hetero-Bäcklund transformations between the variant Boussinesq equation and two linear parabolic equation by using the simplified homogeneous balance method. We also obtain two hetero-Bäcklund transformation between the variant Boussinesq equation and Burgers equations. Furthermore, we obtain two hetero-Bäcklund transformations between the variant Boussinesq equation and heat equations. By using these Bäcklund transformations and so-called "seed solution", we obtain a large number of explicit exact solutions of the variant Boussinesq equation. These solutions include both globally smooth solutions, such as single soliton solutions, multi soliton solutions, as well as a large number of singular traveling wave solutions and non traveling wave solutions. Especially, The infinite explicit exact singular wave solutions of variant Boussinesq equation are obtained for the first time. It is worth noting that these singular wave solutions of variant Boussinesq equations will blow up on some lines or curves in the (x, t) plane. These facts reflect the complexity of the structure of the solution of variant Boussinesq equations. It also reflects the complexity of shallow water wave propagation from one aspect.

Acknowledgments

The authors are very grateful to the referees for useful remarks and suggestions that greatly improved the presentation of this manuscript. We also would like to thank Dr. Quting Chen for plotting the figures of some solutions for this article.

References

[1] L. J. F. Broer, Approximate equations for long water waves, Appl. Sci. Res. 1975,31(5),377-395.

[2] D. J. Kaup, A higher-order water wave equation and the method for solving it, Prog. Theor. Phys,1975, 54(2),396-408.

[3] B. A. Kupershmidt, Mathematics of dispersive water waves, Commun. Math. Phys, 1985,99,51-73.

[4] R. L. Sachs, On the integrable variant of the Boussinesq system: Painlevé property, rational solutions, a related many-body system, and equivalence with the AKNS hierarchy, Phys. D, 1988,30(1-2),1-27.

[5] M. J. Ablowitz, P. A. Clarkson, Solitons, nonlinear evolution equations and inverse scattering, Cambridge University Press, 1991.

[6] M. L. Wang, Solitray wave solutions for variant Boussinesq equations, Phys. Lett, A 1995,199,169-172.

[7] E. G. Fan, H. Q. Zhang, A new approach to Bäcklund transformations of nonlinear evolution equations, Appl. Math. Mech, 1998,19(7),645-650.

[8] E. G. Fan, Two new applications of the homogeneous balance method, Phys. Lett, A 2000,265,353-357.

[9] E. G. Fan, H. Q. Zhang, Symmetry reductions and similarity solutions of Variant Boussinesq equation systems,(Chinese) Acta. Math. Sci, 1999,19(4),373-378.

[10] Z. Y. Yan, H. Q. Zhang, New explicit and exact travelling wave solutions for a system of variant Boussinesq equations in mathematical physics, Phys. Lett, A 1999,252(6),291-296.

[11] Z. Y. Yan, H. Q. Zhang, New explicit solitary wave solutions and periodic wave solutions for Whitham-Broer-Kaup equation in shallow water, Phys. Lett, A 2001,285,355-362.

[12] E. G. Fan, Y. C. Hon, A series of travelling wave solutions for two variant Boussinesq equations in shallow water waves, Chaos, Solitons and Fractals, 2003,15,559-566.

[13] R. Naz, F. Mahomed,T. Hayat, Conservation laws for the third-order variant Boussinesq system, App. Math. Lett, 2010,23(8),8830886.

[14] E. G. Fan, H. Q. Zhang, Bäcklund transformation and exact solution-s for Whitham-Broer-Kaup equations in shallow water, Appl Math Mech, 1998,19(8),713-716.

[15] J. F. Zhang, Multi-solitary wave solutions for variant Boussinesq equations and Kupershmidt equations, Appl Math Mech, 2000,21(2),193-198.

[16] E. Yomba, The extended Fan's sub-equation method and its application to KdV-MKdV, BKK, and variant Boussinesq equations, Phys. Lett, A 2005, 336,463-476.

[17] Y. B. Yuan, D. M. Pu, S. M. Li, Bifurcations of travelling wave solutions in variant Boussinesq equations, Appl Math Mech, 2006,27(7),811-822

[18] P. Zhang, New exact solutions to breaking soliton equations and Whitham-Broer-Kaup equations, Appl. Math Comput, 2010,217,1688-1696.

[19] W. G. Zhang, Q. Liu, X. Li, B. L. Guo, Shape analysis of bounded traveling wave solutions and solution to the generalized Whitham-Broer-Kaup equation with dissipation terms, Chin. Ann. Math, 2012,33B(2),281-308.

[20] W. G. Zhang, Q. Liu, Z. M. Li, X. Li, Bounded traveling wave solutions of variant Boussinesq equation with a dissipation term and dissipation effect, Acta Math Sci, 2014,34B(3),941-959.

[21] H. Guo, T. Z. Xu, S. J. Yang, G. W. Wang, Aanlytical study of solitons for the variant Boussinesq equations, Nonliinear Dyn, 2017,88,1139-1146.

[22] X. F. Yang, Z. C. Deng, Q. J. Li, Y. Wei, Exact combined traveling wave solutions and multi-sympletic structure of the varianr Boussinesq Whitham-Broer-Kaup type equations, Commun Nonlin Sci Simul, 2016,36,1-13.

[23] J. E. Okeke, R. Narain, K. S. Govinder, New exact solutions of a generalised Boussinesq equation with damping term and a system of variant Boussinesq equations via double reduction theory,J. Appl Anal Comp, 2018,8(2),471-485

[24] L. Akinyemia, M. Senolb, O. S. Iyiola, Exact solutions of the generalized multidimensional mathematical physics models via sub-equation method, Math. Comput. Simula, 2021,182,211-233.

[25] B. Bibekananda, M. Hemanta, Z. D. Dia, Exact solution of the time fractional variant Boussinesq-Burgers equations, Appl. Math.2021,66(3),437-449.

[26] D. Evgueni, Travelling waves in the Boussinesq type systems, J. Math. Pures Appl. 2022,163(9),1-10.

[27] W. H. Liu, Z. F. Zhang, Lie symmetry analysis, analytical solutions and conservation laws to the coupled time fractional variant Boussinesq equations, Waves Random Complex Media, 2021,31(1),182-197.

[28] M. Sarit, G. Arindam and R. C. Asesh, Exact solutions and symmetry analysis of a new equation invariant under scaling of dependent variable, Phys.Scr, 2019,94(8),085212(7pp).

[29] G. Arindam, M. Sarit, The first integral method and some nonlinear models, Comput. Appl. Math, 2021,40,79(16 pp).

[30] M. Sarit, G. Arindam and R. C. Asesh, Exact solutions and symmetry snalysis of a Boussinesq type equation for longitudinal waves through a magneto-electro-elastic circular rod, Int. J. Appl.Comput. Math,2021,7,171(14pp).

[31] D. J. Arrigo, P. Broadbridge, and J. M. Hill, Nonclassical symmetry solutions andthe methods of Bluman-Cole and Clarkson-Kruskal,J. Math. Phys. 1993,34,4692-4703.

[32] B. L. Guo, Nonlinear evolution equations, Shanghai Scientific and Technological Education Publishing House, Shanghai,1995.

[33] S. Hood, New exact solutions of Burgers'sequation-an extension to the direct-method of Clarkson and Kruskal, J. Math. Phys. 1995,36,1971-1990.