# Extended Center Manifold, Global Bifurcation and Approximate Solutions of Chen chaotic dynamical system

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**Abstract**

The study of the chaotic Chen dynamic System (CDS) has been a recent focus in the literature, with numerous works exploring its various chaotic features. However, the majority of these studies have relied primarily on numerical techniques to investigate nonlinear dynamic systems (NLDSs). In this context, our aim is to derive approximate analytical solutions for the CDS by developing an iterative scheme. We have proven the convergence theorem for this scheme, which ensures that our iterative process will converge to the exact solution. Additionally, we introduce a new method for constructing the extended center manifold, a critical component in the analysis of dynamical systems. The characteristics of the global bifurcation of the system components within the parameter space are explored. The error analysis of the iterated solutions demonstrates the efficiency of the present technique. We present both three-dimensional (3D) and two-dimensional (2D) phase portraits of the system. The 3D portrait reveals a feedback loop pattern, while the 2D portrait, which represents the interaction of the system components, exhibits multiple pools and cross pools. Furthermore, we illustrate the global bifurcation by visualizing the components of the CDS against the space-parameters. The sensitivity of CDS to infinitesimal variations in the initial conditions (ICs) is tested. It is found that even minor changes can lead to significant alterations in the system.

Keywords: Chen dynamical system; Global bifurcation; Extended center manifold; Approximate analytical solutions.

## 1 Introduction

The study of the non-linear behavior of systems is essential in almost every field of engineering and science. Chaotic behavior is eminent for elucidating the characteristic properties of systems.

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Chaos assumes a variety of types; Lorenzian chaos, " sandwich", "horseshoe", attractor, strange attractor and wings [1-10]. A sequence of the chaotic chameleon system was studied and it was found that the chaotic system can produce self and hidden attractors [11]. A numerical solutions of low-dimensional chaotic systems, whose statistics exhibit smooth and rough regions via chaotic parameter, was obtained [12]. 3D dynamical system that is derived from the Nosé-Hoover oscillator and fold-Hopf bifurcation, saddle-node bifurcation, transient chaos, and conservative correspond to four cases of equilibrium were studied [13]. The chaotic solutions for the perturbed Fokas system were obtained and displayed with graphs [14].

a Two types of horseshoe chaos spiral and screw chaos are possible. While, sandwich chaos supports a genuine strange attractor. Chaos undergoes different trends, sensitivity to initial conditions, period doubling, degenerate period doubling and indeterminacy. Chaos are experimentally verified in science, engineering and in industry [15,16,17].

The complex dynamical behaviors of the chaotic trajectories of Chen's system were analyzed in detail, with its precise bound [18]. The Lorenz and the Chen systems as two dual systems at the two extremes of their parameter spectrum were introduced. A unified chaotic system bridges the gap between the Lorenz and the Chen system and may also represent the entire family of chaotic systems between them [19]. The effect of state feedback on Wang–Chen system was studied by introducing a further state variable wherein hidden chaotic attractors were revealed [20]. Analytic solution of Chen system was given by differential transformation method (DTM) and the accuracy of the DTM method is then tested against classical numerical methods, such as the Runge–Kutta method were obtained in [21].

Bifurcation occupies a considerable area of research in dynamical systems. In [24], The dynamics of a diffusive Lotka–Volterra type model for two species with nonlocal delay effect and Dirichlet boundary conditions was investigated. The study 3-dimensional Lotka–Volterra systems which can have at least four periodic orbits bifurcating from one of their equilibrium points waqs carried out [25]. Galerkin–Koornwinder (GK) approximations, was applied to the Stuart–Landau (SL) normal form and center manifold which was presented for a broad class of nonlinear systems [26]. An explicit recursive formula was presented for computing the normal form and center manifold of general n-dimensional systems [27]. A partial unfolding for an analog to the fold-Hopf bifurcation in three-dimensional symmetric piecewise linear differential systems was obtained [28]. Time-delayed sampled-data feedback control technique was used to stabilize a class of unstable delayed differential systems asymptotically [29].
The real world applications of chaotic systems are highly diverse, including the study of turbulent flow of fluids, irregularities in heart beat, population dynamics, chemical reactions, and financial mathematics. In particular, the Chen can serve for secure chaotic communications via extended Kalman filter and multi-shift cipher algorithm [30]. On the other side, A criterion for controlling the Chen system was proposed in [31]. The problem of the optimal control for the equilibrium point of chaotic Chen in a simple three-dimensional autonomous system was studied [32].

Indeed, in the literature some methods for approximate analytic solutions were employed. Among them,the the variational iteration method [33], the decomposition method [34], and the series expansion method. The first is based on iteration near a particular solution, the second stands for dealing with linear and nonlinear parts, while the last one's is based on the initial condition [35]. Some relevant works were carried in [36,37,38].

In the present work, approximate analytical solutions of the chaotic Chen systems are found via Picard iteration method, where iteration is carried about the solution of the linearlyzed system

[39,40]. The convergence theorem is proved and a new technique for the extended center manifold is proposed.

The outlines of the paper are as it follows. Mathematical formulation is presented in Sec. 2, while Sec. 3 is concerned with analytical solutions. In Sec.4 global bifurcation is investigated and extended center manifold is constructed in Sec.5. In Sec. 6 Lyapunov exponent values, and solutions-sensitivity against varying ICs are shoe graphically. Secs. 7 and 8 are devoted to discussions and conclusions.

## 2 Mathematical formulation

Chen dynamical system reads [10].

$$\begin{gathered} \dot{x} = a(y-x), \\ \dot{y} = (a+c)x + cy - xz, \\ \dot{z} = xy - bz, \\ , x(0) = x_0,\ y(0) = y_0, z(0) = z_0, \end{gathered} \tag{1}$$

where $x, y$ and $z$ are the state variables, and $a$,$b$ and $c$ are real parameters .

Eq. (1) is rewritten in the matrix form by,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = M \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ -xz \\ xy \end{pmatrix}, M = \begin{pmatrix} -a & a & 0 \\ a+c & c & 0 \\ 0 & 0 & -b \end{pmatrix}, \tag{2}$$

In Eq. (2), we use the transformation,

$$\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{Mt}\mathbf{U},\ \mathbf{U} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \mathbf{U}(0) = \mathbf{u}(0), \tag{3}$$

From Eq. (2) into Eq. (3) , we reduce

$$e^{Mt} \begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} 0 \\ -xz \\ xy \end{pmatrix}, \tag{4}$$

Eq. (4) integrates to

$$\mathbf{U}(t) = \mathbf{u}(0) + \int_0^t e^{-Mt_1} \begin{pmatrix} 0 \\ -x(t)z(t) \\ x(t)y(t) \end{pmatrix} dt_1. \tag{5}$$

In Eq. (5) is criticized and we write ,

$$\mathbf{U}^{(\mathrm{n})}(t) = U^{(0)}(t) + \int_0^t e^{-Mt_1} \begin{pmatrix} 0 \\ -x^{(n-1)}(t_1)z^{(n-1)}(t_1) \\ x^{(n-1)}(t_1)y^{(n-1)}(t_1) \end{pmatrix} dt_1, n \geq 1,$$
$$U^{(0)}(t) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \tag{6}$$

The first approximation is,

$$\mathbf{U}^{(1)}(t) = U^{(0)}(t) + \int_0^t e^{-Mt_1} \begin{pmatrix} 0 \\ -x^{(0)}(t_1)z^{(0)}(t_1) \\ x^{(0)}(t_1)y^{(0)}(t_1) \end{pmatrix} dt_1. \tag{7}$$

Now, we prove the convergence theorem of the iteration scheme Eq. (6). To this issue, we present the following.

We write $U^{(n)} = \{U_i^{(n)}, i = 1,2,3\}$,where $\parallel U^{(n)} \parallel = Max_{i=1,2,3} \parallel U_i^{(n)} \parallel$,$\parallel U_i^{(n)} \parallel = Sup_{t\epsilon\mathbb{R}^+} \mid U^{(n)}(t) \mid$. We assume that the space of solutions

$S = \{U_i^{(n)} : U_i^{(n)} \epsilon C^1(\mathbb{R}^+), \ , i = 1,2,3, \ n\epsilon\mathbb{N} \ \}$ is endowed by the norm $\parallel S \parallel = Max_i \parallel U_i^{(n)} \parallel, \parallel U_i^{(n)} \parallel = Sup_{t\epsilon\mathbb{R}^+} \mid U_i^{(n)}(t) \mid$ ) (cf. (6)).

Define the mapping $\tilde{M} : S \to S, \ M(u_i^{(n-1)}(t)) = u_i^{(n)}(t)$. We proceed the proof of the convergence theorem by the following.

The logarithmic norm of a matrix $M$ is defined by,

$$\mu(M) = Limit_{\delta\to 0} \frac{\parallel I + \delta M \parallel -1}{\delta}, \tag{8}$$

where $\parallel M \parallel$is the matrix norm. Here, we consider $\parallel M \parallel_\infty = Max_{1\leq i\leq n}(\sum_{j=1}^{j=n} \mid m_{ij} \mid)$

**Lemma**. The norm of exponential matrix $M_{m\times m} = (m_{ij})$, $i, j = 1, ..., m$ satisfies [22,23].

$$\parallel exp(-tM) \parallel_\infty < exp(-t\mu_\infty(M)), \tag{9}$$

where $\mu_\infty(M) = Max_i(\mid m_{ii} \mid + \sum_{j=1, j\neq i}^{j=m} \mid m_{ij} \mid)$.

**Theorem** The sequence of solutions $\mathbf{U}^{(\mathbf{n})}$ converges absolutely to the exact solution $\mathbf{U}$as $n \to \infty$.

*Proof*. By using Eqs. (6) , (2) and the lemma, we have,

$$\| \mathbf{U^{(1)}} - \mathbf{U^{(0)}} \| < \| \int_0^t e^{-Mt_1} \begin{pmatrix} 0 \\ -x^{(0)}(t_1)z^{(0)}(t_1) \\ x^{(0)}(t_1)y^{(0)}(t_1) \end{pmatrix} dt_1 \| < \int_0^t \| e^{-Mt_1} \| \| \begin{pmatrix} 0 \\ -x^{(0)}(t_1)z^{(0)}(t_1) \\ x^{(0)}(t_1)y^{(0)}(t_1) \end{pmatrix} \| dt_1$$

$$< \int_0^t \| e^{-\mu t_1 (M)} \| \| \begin{pmatrix} 0 \\ -x^{(0)}(t_1)z^{(0)}(t_1) \\ x^{(0)}(t_1)y^{(0)}(t_1) \end{pmatrix} \| dt_1 = \int_0^t e^{-(|a|+|c|+|b|)t_1} \| \begin{pmatrix} 0 \\ -x^{(0)}(t_1)z^{(0)}(t_1) \\ x^{(0)}(t_1)y^{(0)}(t_1) \end{pmatrix} \| dt_1. \tag{10}$$

In the present case, when $4a + 2b > \sqrt{4a^2 - m^2}$, then from Eq. (8), there exist $\varepsilon_0 < 1$ and $T_0$ such that,

$$\| \mathbf{U^{(1)}} - \mathbf{U^{(0)}} \| < \varepsilon_0 < 1, \quad t > T_0. \tag{11}$$

Define a mapping $\tilde{M}$ with $\tilde{M}(U^{(i)}) = U^{(i+1)}$. By the same way for $\| \mathbf{U^{(2)}} - \mathbf{U^{(1)}} \| = \| \tilde{M}(\mathbf{U}^{(1)}) - \tilde{M}(\mathbf{U}^{(0)}) \|$ it holds that there exists $\varepsilon_1 < 1$ and $T_1$such that,

$$\begin{array}{c} \| \tilde{M}(\mathbf{U}^{(1)}) - \tilde{M}(\mathbf{U}^{(0)}) \| < \varepsilon_1 < 1, \quad t > T_1, \\ \| \tilde{M}(\mathbf{U}^{(2)}) - \tilde{M}(\mathbf{U}^{(1)}) \| < \varepsilon_2 < 1, \quad t > T_2, \\ . \\ . \end{array} \tag{12}$$

and by induction it holds that,

$$\| \tilde{M}(\mathbf{U}^{(n)}) - \tilde{M}(\mathbf{U}^{n-1}) \| < \varepsilon_n < 1, \quad t > T_n. \tag{13}$$

From Eqs. (11)-(13), there exist $\epsilon = Min_j \varepsilon_j$and $T = Max_j T_j$ such that,

$$\| \tilde{M}(U^{(n)}) - \tilde{M}(U^{n-1}) \| < \varepsilon_j < \varepsilon < 1, \quad t > T > T_j,\ j = 1, ..., n. \tag{14}$$

Thus,$\tilde{M}$ is a contraction mapping. This completes the proof □.

Corollary. The sequence of solutions $u^{(n)}$ converges absolutely on $[0, \infty]$ to the exact solution $\mathbf{u}$ of Eq. (2) (or(1)) .

# 3 Approximate analytic solutions

By using Eq. (7), we calculate the approximate solutions $\{x^{(0)} y^{(0)}, \boldsymbol{z}^{(0)}\}$, and we get,

$$x^{(0)} = \frac{1}{2\sqrt{(a+c)(5a+c)}} e^{-\frac{1}{2}t\left(\sqrt{(a+c)(5a+c)}+a-c\right)} \left(\left(c\left(-e^{t\sqrt{(a+c)(5a+c)}}\right) + \sqrt{(a+c)(5a+c)} + c \right.\right.$$
$$\left.\left. e^{t\sqrt{(a+c)(5a+c)}} + 1\right) x_0 - a(c_1 - 2c_2)\left(e^{t\sqrt{(a+c)(5a+c)}} - 1\right)\right), \tag{15}$$

$$y^{(0)} = \frac{1}{2\sqrt{(a+c)(5a+c)}} e^{-\frac{1}{2}t\left(\sqrt{(a+c)(5a+c)}+a-c\right)} \left(y_0\sqrt{(a+c)(5a+c)}e^{t\sqrt{(a+c)(5a+c)}} + y_0\sqrt{(a+c)(5a+c)} \right.$$
$$\left. (2x_0+y_0)(a+c)e^{t\sqrt{(a+c)(5a+c)}} - (2x_0+y_0)(a+c)\right), \tag{16}$$

$$z^{(0)} = z_0 e^{-bt}. \tag{17}$$

For periodic solutions, we set $m^2 = -(a+c)(5a+c)$ or ( $c = -3a \pm \sqrt{4a^2-m^2}$) so that Eqs. (15)-(17) become,

$$x^{(0)} = \frac{e^{\frac{(c-a)}{2}t}\left(x_0 m\cos\left(\frac{mt}{2}\right) - \sin\left(\frac{mt}{2}\right)\left(a(x_0-2y_0)+cx_0\right)\right)}{m} \tag{18}$$

$$y^{(0)} = \frac{e^{\frac{(c-a)}{2}t}\left(x_0 m\cos\left(\frac{mt}{2}\right) - \sin\left(\frac{mt}{2}\right)\left(a(x_0-2y_0)+cx_0\right)\right)}{m} \tag{19}$$

$$z^{(0)} = z_0 e^{-bt}. \tag{20}$$

By sing Eqs. (6) into (18)-(20), the solutions $\{x^{(1)}y^{(1)}, \boldsymbol{z}^{(1)}\}$are

$$x^{(1)} = \frac{1}{m}\left(e^{\frac{1}{2}t\left(\sqrt{4a^2-m^2}-4a\right)}\left(x_0\left(m\cos\left(\frac{mt}{2}\right) - \left(\sqrt{4a^2-m^2}-2a\right)\sin\left(\frac{mt}{2}\right)\right) + 2ay_0\sin\left(\frac{mt}{2}\right)\right)\right.$$
$$+\frac{1}{b(b^2+m^2)} a e^{\frac{1}{2}t\left(\sqrt{4a^2-m^2}-4a-2b\right)}\left(m\left(e^{bt}-1\right)\cos\left(\frac{mt}{2}\right)\left(x_0\left(-\sqrt{4a^2-m^2}+2a+b\right)+2ay_0\right)\right.$$
$$\left.\left.+\sin\left(\frac{mt}{2}\right)\left(x_0\left(b\sqrt{4a^2-m^2}\left(e^{bt}+1\right) - 2ab\left(e^{bt}+1\right) - 2b^2e^{bt} - m^2\left(e^{bt}-1\right)\right) - 2aby_0\left(e^{bt}+1\right)\right)\right) z_0\right), \tag{21}$$

$$y^{(1)} = \frac{1}{2m}\left(2e^{\frac{1}{2}t\left(\sqrt{4a^2-m^2}-4a\right)}\left(\left(\sqrt{4a^2-m^2}-2a\right)(2x_0+y_0)\sin\left(\frac{mt}{2}\right) + my_0\cos\left(\frac{mt}{2}\right)\right)\right.$$
$$+\frac{1}{b(b^2+m^2)} e^{\frac{1}{2}t\left(\sqrt{4a^2-m^2}-4a-2b\right)}\left(\left(2m\left(e^{bt}-1\right)\left(2a\left(\sqrt{4a^2-m^2}-b\right)+b\left(\sqrt{4a^2-m^2}-b\right)-4a^2\right)\cos\left(\frac{mt}{2}\right)\right.\right.$$
$$\left.-2b\left(e^{bt}+1\right)\left(2a\left(\sqrt{4a^2-m^2}-b\right)+b\sqrt{4a^2-m^2}-4a^2+m^2\right)\sin\left(\frac{mt}{2}\right)\right)x_0$$
$$+2a\left(m\left(e^{bt}-1\right)\left(\sqrt{4a^2-m^2}-2a-b\right)\cos\left(\frac{mt}{2}\right) - \left(-2ab\left(e^{bt}+1\right) - 2b^2\right.\right.$$
$$\left.\left.\left.\left. b\sqrt{4a^2-m^2}\left(e^{bt}+1\right) + m^2 + \left(e^{bt}-1\right)\right)\sin\left(\frac{mt}{2}\right)\right)y_0\right)z_0\right), \tag{22}$$

$$
\begin{aligned}
z^{(1)} = -\tfrac{1}{Q}\Big(\Big(e^{t(-(a+b))}\Big(e^{at}m^2\Big(-2x_0^2\left(\sqrt{4a^2-m^2}-2a\right)\left(2\sqrt{4a^2-m^2}-6a+b\right) \\
2ay_0^2(2a-b)-2x_0y_0(2a-b)\left(2\sqrt{4a^2-m^2}-6a+b\right)\Big) \\
+e^{t\left(\sqrt{4a^2-m^2}-3a+b\right)}\Big(m\sqrt{4a^2-m^2}-4a+b\sin(mt)\Big(-2\left(\sqrt{4a^2-m^2}-2a\right) \\
x_0^2\left(2\sqrt{4a^2-m^2}-6a+b\right)+8ax_0y_0\left(\sqrt{4a^2-m^2}-2a\right)+2ay_0^2(2a-b)\Big) \\
+2-\sqrt{4a^2-m^2}+4a-b\sqrt{4a^2-m^2}-2a\cos(mt)\Big(-\Big(m^2-\Big(\sqrt{4a^2-m^2}-2a\Big) \\
\sqrt{4a^2-m^2}-4a+b\Big)x_0^2-4ax_0y_0\left(\sqrt{4a^2-m^2}-4a+b\right)+ay_0^2(6a-b)\Big) \\
+\left(\left(\sqrt{4a^2-m^2}-4a+b\right)^2+m^2\right)\Big(2x_0^2\left(\sqrt{4a^2-m^2}-2a\right)^2 \\
-2x_0y_0\left(4a\sqrt{4a^2-m^2}-8a^2+m^2\right)-2ay_0^2\left(\sqrt{4a^2-m^2}-2a\right)\Big)\Big)\Big)\Big)\Big), \\
Q = 2m^2\left(\left(\sqrt{4a^2-m^2}-4a+b\right)^3+m^2\left(\sqrt{4a^2-m^2}-4a+b\right)\right)+z_0e^{-bt}.
\end{aligned}
\tag{23}
$$

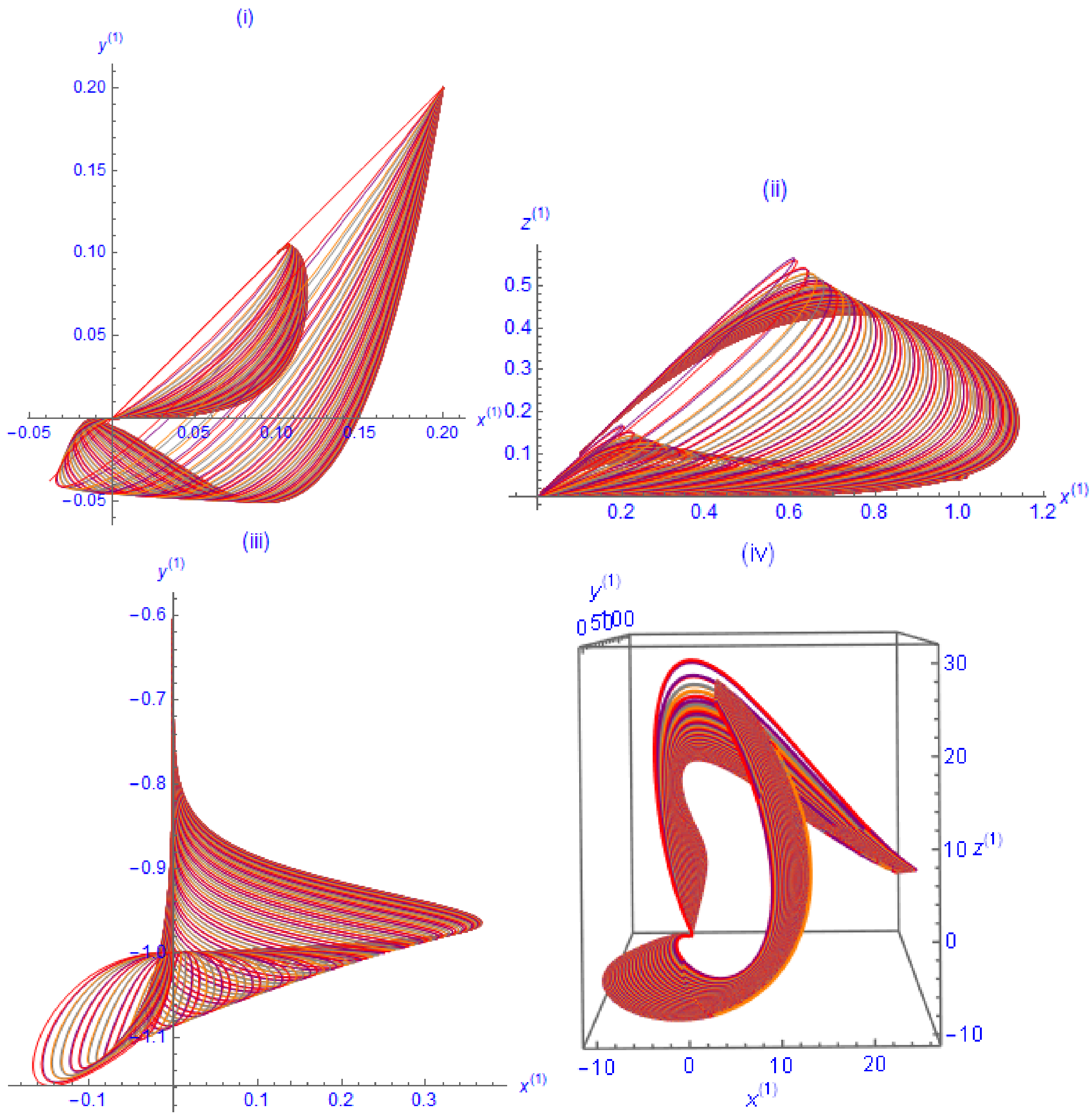
Now, we make a comparison between the first approximation Eqs. (21)-(23) and the zero approximations Eqs. (15)-(17), by evaluating the absolute errors.

| Time | Zero approxmation | First approxmation | Absolute error |
|---|---|---|---|
| $t=1$ | $x^{(0)}=0.00978,\ y^{(0)}=0.00569,\ z^{(0)}=0.0409$ | $x^{(1)}=0.0097608,\ y^{(1)}=0.00531,\ z^{(1)}=0.04101$ | $2\times10^{-4}, 3\times10^{-3}, 6\times10^{-4}$ |
| $t=2$ | $x^{(0)}=0.00923,\ y^{(0)}=0.00257442,\ z^{(0)}=0.0335$ | $x^{(1)}=0.00917,\ y^{(1)}=0.00111,\ z^{(1)}=0.03361$ | $6\times10^{-4}, 5\times10^{-3},\ 9\times10^{-4}$ |
| $t=3$ | $x^{(0)}=0.0085,\ y^{(0)}=0.00038,\ z^{(0)}=0.0274$ | $x^{(1)}=0.0084,\ y^{(1)}=-0.00027,\ z^{(1)}=0.028$ | $6\times10^{-3}, 5\times10^{-3},\ 8\times10^{-4}$ |
| $t=4$ | $x^{(0)}=0.00764,\ y^{(0)}=-0.0011,\ z^{(0)}=0.0225$ | $x^{(1)}=0.0075,\ y^{(1)}=-0.00174,\ z^{(1)}=0.023$ | $1\times10^{-3}, 6\times10^{-3},\ 6\times10^{-4}$ |
| $t=5$ | $x^{(0)}=0.00675,\ y^{(0)}=-0.002063,\ z^{(0)}=0.0184$ | $x^{(1)}=0.0065,\ y^{(1)}=-0.00262,\ z^{(1)}=0.0184$ | $2\times10^{-3}, 5\times10^{-3},\ 4\times10^{-4}$ |

Table 1. When $a=0.1, b=0.2, \gamma=0.1, \mu=0.1, x_0=0.01, y_0=0.01, z_0=0.05$ and $m=0.2$.

After this table, we find that the errors are satisfaction-ally and so, the method used here is efficient.

We return to the results in Eq. (21)-(23), where they are displayed in Figs. 1-(i)-(iv) for phase portraits, in 2D and 3D.




Figs. 1 (i)-(v). The phase portraits (2D-parametricplot) for each pair of $x^{(1)}, y^{(1)}$, and $z^{(1)}$ the. Fig. (iv) show 3D parametric plot in 3D-dimensions. When. $a{=}2, b{=}0.1, x_0{=}0.1, y_0{=}0.2, z_0{=}0.8$. Fig. 1-(i) shows two-layer shell, Fig. 1 (ii) shows shell, Figs. -(iii) show scrolling loop chaos.

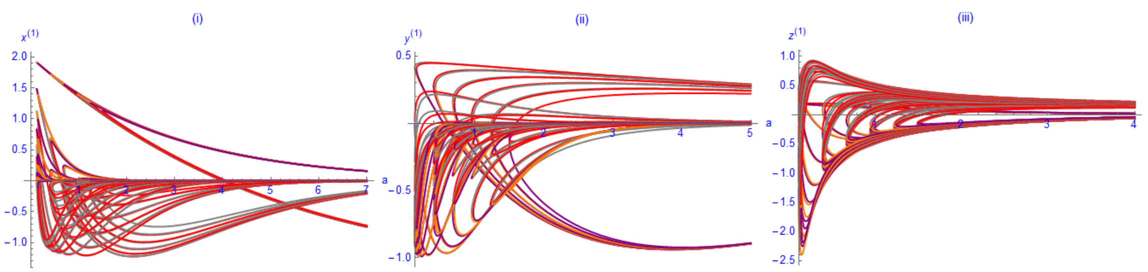
Fig. 1 (iv) shows hook chaos.

# 4 Global bifurcation

We distinguish between local and global bifurcation (GB). Local bifurcation is concerned with the study of different kinds of bifurcations near the equilibrium points. While the Global type stands for investigations the behavior of the system components in the parameter- space.

Now, we investigate the global bifurcation by displaying the solutions $u^{(1)}$,$v^{(1)}$, and $w^{(1)}$ against parameters $\alpha, \gamma$ and $\nu$.

**Case (i).** By using Eqs. (21) and (22), $x^{(1)}$and $y^{(1)}$ are displayed in Figs. 2 (i)-(iii) against $\gamma$ and $\nu$ .




Figs. 2 (i)-(iii). Show the bifurcations of $x^{(1)}y^{(1)}, \boldsymbol{z}^{(1)}$ Eq. (21)-(23) against $a$, when $b = 0.1, x_0 = 2, y_0 = -5, z_0 = 0.1$.

Figs. 2 (i)-(iii) show mixing of saddle node bifurcation, super-crtical bifurcation, pitch fork bifurcation, and sub-critical pitch fork bifurcation, and sub-critical pitch fork bifurcation respectively.

# 5 Extended Center manifold

In the literature's the study of extended center manifold (ECM) is not complete in the sense that there was not explicit determination for the ECM. Here, we present a simple technique to construct the extended center manifold. It is based on:

(i) Transforming the Chen system, in the neighborhood of an equilibrium point, to another system.

(ii) In the new system, some assumptions are adapted to derive the equation governing the ECM,

(iii) It is solved and yield to the explicit solution.

Consider the NLDS.

$$\dot{x} = f_1(x, y, z), \quad \dot{y} = f_2(x, y, z), \quad , \dot{z} = f_3(x, y, z), \tag{24}$$

and we assume that $P = (0, 0, 0)$ is the equilibrium point of Eq. (1), so we write,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + M \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \; M = \begin{pmatrix} -a & a & 0 \\ a+c & c & 0 \\ 0 & 0 & -b \end{pmatrix}, \tag{25}$$

For extended center manifold (ECM) , $det\, M \neq 0$,when $a \neq -2c$.

To construct ECM $\Gamma$, we proceed by setting $a=\alpha-2c$,and $\Gamma \epsilon H^{(4)}=\{(x,y,z,\alpha),\ x,y,z\,\epsilon\mathbb{R},\alpha\neq 0\}$, and $\dot{\alpha}=0$. From (25), we have,

$$x=x_1+x_2,\ y=x_3,\ z=x_1\left(\frac{\alpha}{c}-2\right)+x_2. \tag{26}$$

Eq. (25) can be rewritten,

$$\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}=M^{-1}\begin{pmatrix} x\\ y\\ z\end{pmatrix},\ M^{-1}=\begin{pmatrix} \frac{c}{3c-\alpha} & 0 & \frac{c}{\alpha-3c}\\ \frac{c}{\alpha-3c}+1 & 0 & \frac{c}{3c-\alpha}\\ 0 & 1 & 0\end{pmatrix}. \tag{27}$$

Or,

$$x_1=\frac{c(x-z)}{3c-\alpha},\ x_2=\frac{c(x-z)}{\alpha-3c}+x,\ x_3=y.$$

From Eq. (27) , we have,

$$\begin{pmatrix} \dot{x}_1\\ \dot{y}_1\\ \dot{z}_1\end{pmatrix}=M^{-1}\begin{pmatrix} \dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}=M^{-1}\begin{pmatrix} f_1(x,y,z)\\ f_2(x,y,z)\\ f_3(x,y,z)\end{pmatrix}. \tag{28}$$

In the present case,

$$f_1(x,y,z)=a(y-x),\ f_2(x,y,z)=(a+c)\,x+cy-xz,\ ,f_3(x,y,z)=xy-bz, \tag{29}$$

and by using Eq. (26) into Eq. (29), Eq. (28), becomes,

$$\begin{pmatrix} \dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix}=M^{-1}\begin{pmatrix} \dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}=\begin{pmatrix} F_1(x_1,x_2,x_3)\\ F_2(x_1,x_2,x_3)\\ F_3(x_1,x_2,x_3)\end{pmatrix}, \tag{30}$$

where,

$$\begin{gathered} F_1=(2c-\alpha)(x_1+x_2-x_3),\\ F_2=(x_1+x_2)\left(-\left(x_1\left(\tfrac{\alpha}{c}-2\right)+x_2\right)\right)+(\alpha-c)(x_1+x_2)+cx_3,\\ F_3=x_3(x_1+x_2)-b\left(x_1\left(\tfrac{\alpha}{c}-2\right)+x_2\right). \end{gathered} \tag{31}$$

Or,

$$\begin{gathered} \dot{x}_1=(2c-\alpha)(x_1+x_2-x_3),,\\ \dot{x}_2=(x_1+x_2)\left(-\left(x_1\left(\tfrac{\alpha}{c}-2\right)+x_2\right)\right)+(\alpha-c)(x_1+x_2)+cx_3,\\ \dot{x}_3=x_3(x_1+x_2)-b\left(x_1\left(\tfrac{\alpha}{c}-2\right)+x_2\right). \end{gathered} \tag{32}$$

Now, we write,

$$\begin{gathered} x_2=h_1(x_1)=\alpha^2\beta_3+\alpha\beta_2x_1+\beta_1x_1^2,\\ x_3=h_2(x_1)=\alpha^2\rho_3+\alpha\rho_2x_1+\rho_1x_1^2. \end{gathered} \tag{33}$$

By using Eq. (33) into Eq. (32), lengthy calculations give rise to,

$$\begin{gathered} \beta_1=0,\ \beta_2=\tfrac{\sqrt{17}-5}{4\alpha},\ \beta_3=-\tfrac{\sqrt{17}-17}{16\alpha},\ \rho_1=-\tfrac{1}{\alpha},\ \rho_2=-\tfrac{1}{2\alpha},\ \tfrac{17-2\sqrt{17}}{16\alpha},\\ \rho_3=-\tfrac{1}{2\alpha},\ \tfrac{17-2\sqrt{17}}{16\alpha},b=\tfrac{1}{8}\left(\sqrt{17}-9\right)\alpha,\ c=\tfrac{1}{16}\left(\sqrt{17}+7\right)\alpha,\\ \dot{x}_1(t)=\tfrac{1}{128}\left(\alpha+4x_1(t)\right)\left(\left(\sqrt{17}+17\right)\alpha-4\left(\sqrt{17}+1\right)x_1(t)\right), \end{gathered} \tag{34}$$

By solving Eq.(34),

$$x_1(t) = \frac{\alpha\left(\sqrt{17}e^{\frac{9\alpha t}{16}+\frac{1}{16}\sqrt{17}\alpha t} - e^{8\left(\sqrt{17}+9\right)\alpha\, A}\right)}{4\left(e^{\frac{9\alpha t}{16}+\frac{1}{16}\sqrt{17}\alpha t} + e^{8\left(\sqrt{17}+9\right)\alpha\, A}\right)}. \tag{35}$$

The extended center manifold function $x_1(t) = \tilde{F}(t,\alpha)$ is displayed in figures 3 (i), (ii).

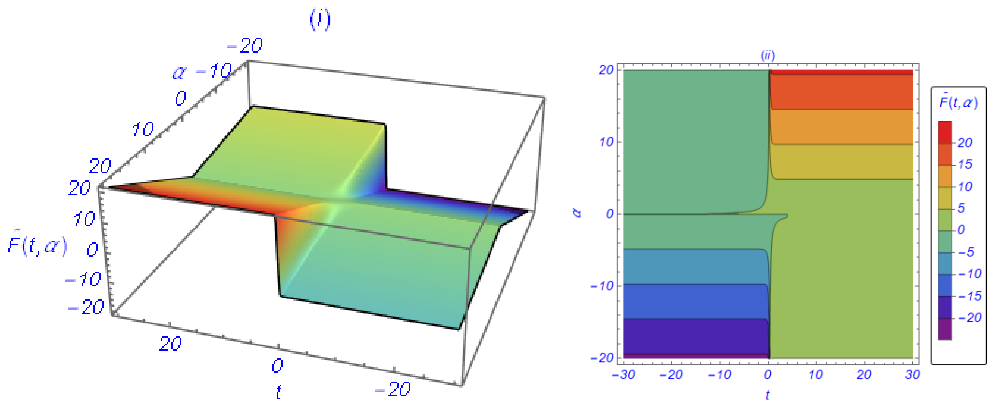



Fig 3 (i), (ii) , show 3D and contour plot of Eq. (35), against $(t,\alpha)$. Fig. 3(i) shows two-fold double kinky solitary waves. Fig. 3 (ii) consolidates the results in Fig. 3 (i).

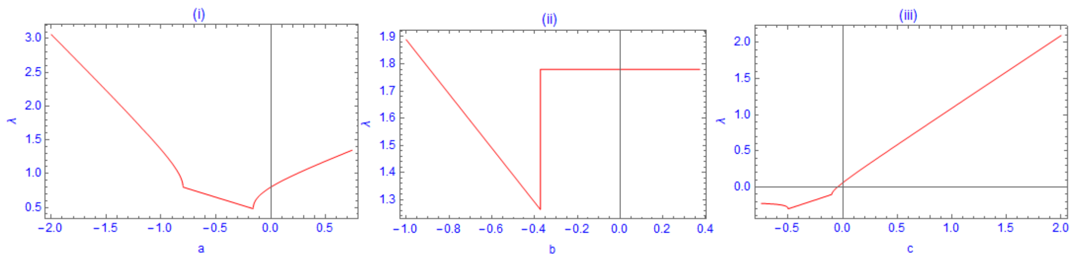
# 6 Lyapunov exponent, and solutions-sensitivity against varying ICs

## 6.1 Lyapunov exponent

The Lyapunov exponent $\lambda$ of the system is defined by,

$$\lambda = \lim_{t\to\infty} \frac{\log\left(\sqrt{\frac{(x^{(1)})^2+(y^{(1)})^2+(z^{(1)})^2}{x_0^2+{y_0}^2+z_0^2}}\right)}{t} \tag{36}$$

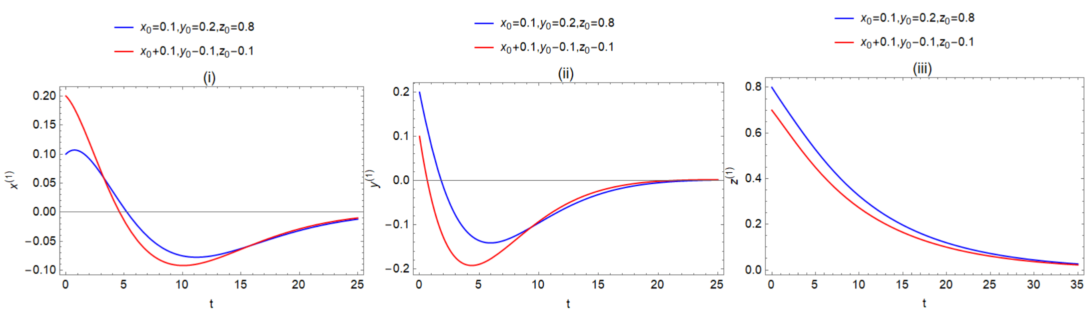
Eq. (36) is used to find $\lambda$ and the results are shown in Figs. 4 (i)-(iii).




Figs. 4 (i) - (ii). When:(i) $b = 1.5, x_0 = 2, y_0 = 1.2, z_0 = 3$, (ii) $a$=0.5, (iii) $\nu{=}-0.2$, $\mu$=0.19, $x_0 = 2.5, y_0 = 1.5, z_0 = 3$,. Fig 3 (i)-(iii) show that the system is unstable by varying $a$,$b$ and it is stable when $c < 0$and unstable when $c > 0$.

## 6.2 Sensitivity against varying ICs

Here, we show the behavior of the system components under small disturbance in the ICs. To this issue, the solutions in Eq. (24) are displayed in Figs. 6 (i)-(iii).




Figs. 5 (i). When $a = 2, b = 0.1, m = 0.2$,

Figs. 5 (i) and (ii) show a remarkable deviation in the two solutions under small disturbance in ICs. So, the system exhibits sensitive change against small variation in the ICs.

# 7 Discussions

In this study, the Chen chaotic system is analyzed using approximate analytical solutions. The behavior of the system is investigated using these solutions, revealing several interesting characteristics. Here,approximate analytical solutions of Chen chaotic system are obtained. The results obtained are used to investigate the system behavior. It is observed that the 2D portrait of the mutual components exhibit multiple-crossing loops, while the 3D portrait shows feedback loops (cf. Figs.1 (i)-(iv)). It is found that the global bifurcation behaves as a mixing of saddle node bifurcation , super-critical pitch fork bifurcation, sub-critical pitch fork bifurcation and imperfect pitch fork bifurcation (cf. Figs. 2 (i)-(iii). It is remarked that the extended center manifold exhibits different patterns formation (cf. Figs 3 (i)-(vi)). It is found that the system is unstable by varying $a$,$b$ while stable when varying $c$ (cf. Figs. 4 (i)-(iii) ). The sensitive depends of the system component, under small disturbance in the ICs, is examined, where the solutions are remarkably higher than before disturbance (cf. Figs. 5 (i)-(iii)).

# 8 Conclusions

In this study, an iteration scheme is constructed to derive approximate analytical solutions for the Chen dynamical system. The efficiency of this technique is confirmed through error analysis, and the convergence theorem is proven. A novel technique is proposed for establishing the extended center manifold. These results are then utilized to examine the behavior of the system components. Both 2D and 3D phase portraits are investigated, revealing that: The mutual components display multiple loops with crossings in the 2D phase portrait. The 3D phase portrait exhibits a double-layer feedback loops pattern. The global bifurcation is explored within the parameters space, demonstrating a mixture of different bifurcation structures.

Finally, the sensitivity of the system to small disturbances in initial conditions is tested using the derived solutions. This comprehensive analysis provides valuable insights into the dynamics and behavior of the Chen dynamical system. As a future work, we intend studying the complex Chen system by following the same insight as in the present work. Also, synchronization will be visualized.

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