Journal of Applied Analysis and Computation
Volume *, Number *, * *, 1–16
Website:http://www.jaac-online.com
DOI:10.11948/***

DOUBLE PHASE PROBLEM WITH SINGULARITY AND HOMOGENOUS CHOQUARD TYPE TERM

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Abstract In this study, we prove in the context of Musielak Sobolev space that, under various assumptions on the data, two positive non-trivial solutions exist to the double phase problem with a singularity and a homogeneous Choquard type on the right-hand side. Our method relies on the Nehari manifold, the Hardy Littlewood - Sobolev inequality, and some variational approaches. The findings presented here generalize some known results.

Keywords Double phase operator, Singular problem, Choquard term, Existence of solutions, Variational method, Musielak-Orlicz spaces

MSC(2010) 35A15, 35J20, 35J60, 35B38

1. Introduction

The present paper aims to prove the existence of two positive non-trivial solutions to the following problem

$$\begin{cases} \mathcal{L}_{p,q}^{a(x)}\left(u\right)+V\left(x\right)\left(|u|^{p-2}u+a\left(x\right)|u|^{q-2}u\right) \\ =g\left(x\right)|u|^{-\beta-1}u+\eta\left(\displaystyle\int_{\Omega}\frac{H(y,u(y))}{|x-y|^{\mu}}dy\right)h(x,u) \text{ in } \quad x\in\Omega, \\ u=0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \quad x\in\partial\Omega. \end{cases} \tag{1.1}$$

Where $\Omega\subseteq\mathbb{R}^N$ is a smooth bounded domain, $\mathcal{L}_{p,q}^{a(x)}\left(u\right)=-\text{div}\left(|\nabla u|^{p-2}\nabla u+a\left(x\right)|\nabla u|^{q-2}\nabla u\right)$, $0<\mu<N$, $1<p<q<2r<p^*=\frac{Np}{N-p}$, $0<\beta<1, a(.)\in L^\infty\left(\Omega\right)$ with $\min\limits_{x\in\Omega}a\left(x\right)=a_0>0$ and we consider $h:\Omega\times\mathbb{R}\to\mathbb{R}^+\cup\{0\}$ a continuous function with an odd behavior concerning the second variable. This function fulfills the subsequent set of assumptions:

(h_1) h is a positive homogeneous of degree $r-1$, that is,

$$h\left(x,tu\right)=t^{r-1}h\left(x,u\right) \text{ for all } (x,u)\in\Omega\times\mathbb{R}.$$

(h_2) There exists a positive constant C such that

$$|h\left(x,t\right)|\leq C|t|^{r} \quad \text{for any} \quad t\in\mathbb{R}.$$

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(H_1) H is homogeneous of degree r, that is, $H(x, tu) = t^r H(x, u)$ $(t > 0)$ for all $(x, u) \in \overline{\Omega} \times \mathbb{R}$, and h leads to the so-called Euler identity $u\, h(x, u) = r\, H(x, u)$.

The positive continuous function $V : \Omega \to \mathbb{R}$ satisfies

(V_1)
$$0 < \inf_{x\in\mathbb{R}^N} V(x) = V_0 < V_\infty = \liminf_{|x|\to\infty} V(x) < \infty. \tag{1.2}$$

The Choquard equation has been included in a variety of physical models. For example, the polaron model of Fröhlich and Pekar postulates that free electrons in an ionic lattice interact either with phonons associated with lattice deformations or with the polarization it creates on the medium (interaction of an electron with its hole) [10, 25]. Also, to simulate a plasma consisting of a single component, Ph. Choquard developed the Choquard equation in 1976 [16]. The following equation,

$$-\Delta u + V(x)\, u = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu}\, dy \right) |u|^{p-2} u \quad \text{in } \mathbb{R}^N,$$

known in this context as the Schrödinger-Newton equation, is used as a model for self-gravitating matter in the work of Moroz-Penrose-Tod [22]. An appropriate guide to the Choquard-type equation was published in 2017 by Moroz-Van Schaftingen [23], which the reader can consult for more information on applications of the Choquard term.

This wide range of practical applications has prompted the publication of several existing results for various equations incorporating choquard terms over the past few decades. We mention the relevant works of Su-Liu [26], Zhang-Meng-He [29], Gao-Moroz-Yang-Zhao [11], Anthal-Giacomoni- Sreenadh [3], Yao-Sun-fang Wu [28], Maia-Pellacci-Schiera [21], Liu-Liao-Pan-Tang [17], Cingolani-Gallo-Tanaka [6], and Zuo-Choudhuri-Repovš [31].
Following that, we turn our attention to recent research focusing on Choquard double-phase problems within the realm of mathematics. Arora, Fiscella, Mukherjee, and Winkert [4] delved into the examination of ground state solutions' existence for quasilinear elliptic equations governed by the double-phase operator, which incorporates a Choquard term. Exploring the singularity-perturbed double-phase problem with a nonlocal Choquard reaction, Zhang, Zhang, and Rădulescu [30] investigated the multiplicity and concentration phenomena of positive solutions using variational and topological methods. Additional insights can be found in the work of Xie, Wang, and Zhang [27] and the associated references. For double-phase problems devoid of Choquard terms, we highlight the contributions of Aberqi, Benslimane, Elmassoudi, and Ragusa [1], Aberqi, Benslimane, and Knifda [2], Benslimane, Aberqi, and Bennouna [5], Ge and Yuan [13], Liu and Winkert [18], Ge and Pucci [12], Liu and Dai [20], Crespo, Blanco, Papageorgiou, and Winkert [8], as well as Liu, Dai, Papageorgiou, and Winkert [19], along with the relevant references therein.

In this research, motivated by the above works, we will use the variational approach to study the existence of at least two positive non-trivial solutions to the problem (1.1) under conditions $(h_1)-(h_2)$, (H_1) and (V_1). The following are some of the key aspects of this paper:

i) The problem involves the interaction of a double phase operator with a Choquard reaction.

ii) The presence of a non-linear singularity.

The homogeneous Choquard term presented in this work is advantageous in double-phase problems with a singularity term as it provides a mathematical framework to model long-range interactions in systems exhibiting nonlocal behaviour. This term is instrumental in capturing the influence of distinct points, contributing to a more accurate representation of physical phenomena and addressing singularities that may arise in the mathematical formulation of the problem. Consequently, our problem stands out as distinct and more intricate compared to prior research. To the best of our knowledge, the outcomes presented in this work are novel.

The subsequent sections of this paper are structured as follows: The features of the Musielak-Orlicz Sobolev space $W^{1,\mathcal{M}}(\Omega)$ are presented in section 2. The Hardy Littlewood-Sobolev inequality, which played a significant role in our investigation, is also reviewed. Section 3 is then devoted to the demonstration of our principal findings.

2. Preliminaries

We shall review various Musielak-Orlicz space features that can be found in [7, 9, 14, 15, 24] and references to them.
Set the functions $\mathcal{M}(x,t) = t^p + a(x)t^q$ and $\rho(t) = \int_\Omega \mathcal{M}(x,|t|)\,dx$ for $1 < p < q$ and $a(.) \in L^\infty(\Omega)$ and $\min_{x\in\Omega} a(x) = a_0 > 0$.
The Musielak-Orlicz space $L^{\mathcal{M}}(\Omega)$ is defined by

$$L^{\mathcal{M}}(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R}, \text{ measurable and } \int_\Omega \mathcal{M}(x,|u|)\,dx < \infty \right\}$$

endowed with the Luxemburg norm

$$|u|_{\mathcal{M}} = \inf\left\{ \lambda > 0 : \int_\Omega \mathcal{M}\left(x, \frac{|u|}{\lambda}\right) dx \leq 1 \right\}.$$

We use the notation $L^q_a(\Omega)$ to represent the space comprising all measurable functions $u : \Omega \longrightarrow \mathbb{R}$, characterized by the following semi-norm

$$\|u\|_{q,a} = \left(\int_\Omega a(x)\,|u|^q\ dx \right)^{\frac{1}{q}} < \infty.$$

Since $\rho\left(\frac{|u|}{|u|_{\mathcal{M}}}\right) = 1$ whenever $u \neq 0$, we have

$$\min\left(|u|^p_{\mathcal{M}}, |u|^q_{\mathcal{M}}\right) \leq \|u\|^p_p + \|u\|^q_{q,a} \leq \max\left(|u|^p_{\mathcal{M}}, |u|^q_{\mathcal{M}}\right),\ u \in L^{\mathcal{M}}(\Omega). \tag{2.1}$$

While the Musielak-Sobolev spaces $W^{1,\mathcal{M}}(\Omega)$ are defined by

$$W^{1,\mathcal{M}}(\Omega) = \left\{ u \in L^{\mathcal{M}}(\Omega) : |\nabla u| \in L^{\mathcal{M}}(\Omega) \right\},$$

and it is equipped with the norm $\|u\| = |u|_{\mathcal{M}} + |\nabla u|_{\mathcal{M}}$.
We denote by $W^{1,\mathcal{M}}_0(\Omega)$ the completion of $C^\infty_0(\Omega)$ in $W^{1,\mathcal{M}}(\Omega)$ and it can be

equivalently equipped by $\|u\| = |\nabla u|_{\mathcal{M}}$.
$L^{\mathcal{M}}(\Omega)$, $W^{1,\mathcal{M}}(\Omega)$ and $W_0^{1,\mathcal{M}}(\Omega)$ are separable reflexive Banach spaces with the above norms. For more details, see [24].

By Proposition 2.15 in [7], $W_0^{1,\mathcal{M}}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous embedding since $r < p^*$, and by (2.1), we have

$$\min\left(|\nabla u|_{\mathcal{M}}^p, |\nabla u|_{\mathcal{H}}^q\right) \leq \|\nabla u\|_p^p + \|\nabla u\|_{q,a}^q \leq \max\left(|\nabla u|_{\mathcal{M}}^p, |\nabla u|_{\mathcal{M}}^q\right), \ u \in W_0^{1,\mathcal{M}}(\Omega) \tag{2.2}$$

and

$$\|u\|_r \leq C_r |\nabla u|_{\mathcal{M}}. \tag{2.3}$$

We denote $E = W_0^{1,\mathcal{M}}(\Omega)$ and $\|u\|_E = |\nabla u|_{\mathcal{M}}$.
And we define ϱ_E as

$$\varrho_E(u) = \int_\Omega |u|^p \, dx + \int_\Omega a(x) |u|^q \, dx.$$

Lemma 2.1. *(see [8]) Let $u \in E = W_0^{1,\mathcal{M}}(\Omega)$ then*

i) $\|u\|_E = a \Leftrightarrow \varrho_E\left(\frac{u}{a}\right) = 1$.

ii) $\|u\|_E < 1$ *(resp.* $> 1, = 1$*)* $\Leftrightarrow \varrho_E(u) < 1$ *(resp.* $> 1, = 1$*).*

iii) $\|u\|_E < 1 \Rightarrow \|u\|_E^q \leq \varrho_E(u) \leq \|u\|_E^p$ *and* $\|u\|_E > 1 \Rightarrow \|u\|_E^p \leq \varrho_E(u) \leq \|u\|_E^q$.

vi) $\|u\|_E \to 0 \Leftrightarrow \varrho_E(u) \to 0$ *and* $\|u\|_E \to \infty \Leftrightarrow \varrho_E(u) \to \infty$.

Lemma 2.2. *(see [14]) Let $u \in W_0^{1,\mathcal{M}}$ then*

i) $W_0^{1,\mathcal{M}}(\Omega) \hookrightarrow L^{\mathcal{M}}(\Omega)$.

ii) $L^{\mathcal{M}}(\Omega) \hookrightarrow L_a^q(\Omega)$

iii) $W_0^{1,\mathcal{M}}(\Omega) \hookrightarrow L^r(\Omega)$ *is continuous for all $r \in [1, p^*]$ and compact for $r \in [p, p^*)$.*

Proposition 2.1. *(see [15]) Let $t, r > 1$ and $0 < \mu < N$ with $\frac{1}{t} + \frac{1}{r} + \frac{\mu}{N} = 2$, $h \in L^t\left(\mathbb{R}^N\right)$ and $k \in L^r\left(\mathbb{R}^N\right)$. There exists a constant $C(t, r, \mu, N)$ independent of h and k, such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{h(x)\, k(y)}{|x-y|^\mu} \, dx \, dy \leq C(t, r, \mu, N) \|h\|_{L^t(\mathbb{R}^N)} \|k\|_{L^r(\mathbb{R}^N)}. \tag{2.4}$$

If $t = r = \frac{2N}{2N-\mu}$, then $C(t, r, \mu, N) = \pi^{\frac{\mu}{2}} \frac{\Gamma\left(\frac{N-\mu}{2}\right)}{\Gamma\left(N - \frac{\mu}{2}\right)} \left(\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)}\right)^{-1+\frac{\mu}{N}}$.
In this case there is equality in (2.4) if and only if $h \equiv Ck$ and $k(x) = A\left(\gamma^2 + |x-a|^2\right)^{-N+\frac{\mu}{2}}$ for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}$.

In the following, we denote

$$\|u\|_{s,a} = \int_\Omega a(x) |u|^s dx, \quad \text{and} \quad \|u\|_{s,Va} = \int_\Omega V(x)\, a(x) |u|^s dx, \text{ for every } s > 1.$$

3. Main result

Definition 3.1. We say that a function $u \in E$ is a weak solution to problem (1.1) if

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right)\nabla\varphi dx + \int_\Omega V(x)\left(|u|^{p-2}u + a(x)|u|^{q-2}u\right)\varphi\, dx$$
$$= \int_\Omega g(x)|u|^{-\beta-1}u\varphi dx + \eta\int_\Omega\int_\Omega \frac{H(y,u)\,h(x,u)}{|x-y|^\mu}\varphi(x)\ dx\ dy$$

for all $\varphi \in E$.

Theorem 3.1. *Let $(h_1) - - (V_1)$ hold. Then there exists $\eta_* > 0$ such that for any $\eta \in (0,\eta_*)$, the problem (1.1) admits at least two nonnegative weak solutions.*

Consider the functional $J_\eta : E \to \mathbb{R}$ associated to the problem (1.1) defined by

$$J_\eta(u) = \int_\Omega \left(\frac{1}{p}|\nabla u|^p\ dx + \frac{1}{q}a(x)|\nabla u|^q\right)\ dx + \int_\Omega V(x)\left(\frac{1}{p}|u|^p\ dx + \frac{1}{q}a(x)|u|^q\right)\ dx$$
$$-\frac{1}{1-\beta}\int_\Omega g(x)|u|^{1-\beta}dx - \frac{\eta}{2}\int_{\Omega\times\Omega}\frac{H(y,u)\,H(x,u)}{|x-y|^\mu}\ dx\ dy.$$

We have $J_\eta \in \mathcal{C}^1(E,\mathbb{R})$, and

$$\begin{aligned}\left\langle J_\eta'(u),\varphi\right\rangle &= \int_\Omega \left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right)\nabla\varphi\ dx\\ &+ \int_\Omega V(x)\left(|u|^{p-2}u + a(x)|u|^{q-2}u\right)\varphi\ dx\\ &- \int_\Omega g(x)|\ u|^{-\beta-1}u\varphi dx - \eta\int_\Omega \frac{1}{|x|^\mu} * H(y,u)\ dy\int_\Omega h(x,u)\varphi\ dx\end{aligned}$$

for all $u,\varphi \in E = W_0^{1,\mathcal{M}}(\Omega)$.

Remark 3.1.
- Since f is an odd function, we can easily have $J_\eta(u) = J_\eta(|u|)$, and the minimizer of J_η will be a nonegative function.
- One can easily notice that any critical point of J_η is a weak solution of problem (1.1).

We define the fibering map $\Phi_\eta : \mathbb{R}^+ \to \mathbb{R}$ as $\Phi_\eta(t) = J_\eta(tu)$,

$$\begin{aligned}\Phi_\eta'(t) &= t^{p-1}\|\nabla u\|_p^p + t^{q-1}\|\nabla u\|_{q,a}^q + t^{p-1}\|u\|_{p,V}^p + t^{q-1}\|u\|_{q,Va}^q\\ &- t^{-\beta}\int_\Omega g(x)|u|^{1-\beta}dx - \eta r\int_{\Omega\times\Omega} t^{2r-1}\frac{H(y,u)H(x,u)}{|x-y|^r}dxdy,\end{aligned}$$

$$\begin{aligned}\Phi_\eta''(t) &= \left((p-1)t^{p-2}\|\nabla u\|_p^p + (q-1)t^{q-2}\|\nabla u\|_{q,a}^q\right)\\ &+ \left((p-1)t^{p-2}\|u\|_{p,V}^p + (q-1)t^{q-2}\|\nabla u\|_{q,Va}^q\right)\\ &+ \beta t^{-\beta-1}\int_\Omega g(x)|u|^{1-\beta}dx - \eta r(2r-1)\int_{\Omega\times\Omega} t^{2r-2}\frac{H(y,u)H(x,u)}{|x-y|^\mu}dxdy\end{aligned}$$

and

$$\mathcal{N}_\eta = \{u \in E \setminus \{0\}; < J'_\eta(u), u >= 0\}.$$

It's obvious that $tu \in \mathcal{N}_\eta$ if and only if $\Phi''_\eta(t) = 0$ so, $u \in \mathcal{N}_\eta$ if and only if $\Phi''_\eta(1) = 0$. Let's divide $\mathcal{N}_\eta$ into three subsets as follows:

$$\begin{aligned}
\mathcal{N}_\eta^+ &= \{u \in E \setminus \{0\}; \Phi''_\eta(1) > 0\},\\
\mathcal{N}_\eta^0 &= \{u \in E \setminus \{0\}; \Phi''_\eta(1) = 0\},\\
\mathcal{N}_\eta^- &= \{u \in E \setminus \{0\}; \Phi''_\eta(1) < 0\}.
\end{aligned}$$

Lemma 3.1. *J_η is coercive and bounded below on $\mathcal{N}_\eta$.*

Proof. Let $u \in \mathcal{N}_\eta$, using (h_2) and iii) of Lemma 2.2, there exists a positive constant $c > 0$ such that

$$\int_{\Omega\times\Omega} \frac{H(y,u)H(x,u)}{|x-y|^\mu} dxdy \leq C\|H(.,u(.))\|^2_{L^s(\Omega)} \leq c\Big(\int_\Omega |u|^{rs} dx\Big)^{\frac{2}{s}} = C\|u\|^{2r}_{L^{sr}}, \tag{3.1}$$

where $\frac{1}{s} = 1 - \frac{\mu}{2N}$.
Since $p \leq sr \leq p^*$, we get

$$\int_{\Omega\times\Omega} \frac{H(y,u)H(x,u)}{|x-y|^\mu} dxdy \leq C\|u\|^{2r}.$$

On the other hand, using the Hölder inequality and the embedding iii) of Lemma 2.2, we obtain

$$\int_\Omega g(x)|u|^{1-\beta} dx \leq \|g\|_{L^\infty(\Omega)} |\Omega|^{\frac{p-1+\beta}{p}} \|u\|^{1-\beta}_{L^p(\Omega)} \leq C\|g\|_{L^\infty(\Omega)} |\Omega|^{\frac{p-1+\beta}{p}} \|u\|^{1-\beta}.$$

Return now to $J_\eta(u)$, since $u \in \mathcal{N}_\eta$, then

$$\|\nabla u\|^p + \|\nabla u\|^q_{q,a} + \|u\|^p_{p,V} + \|u\|^q_{q,Va} - \int_\Omega g(x)|u|^{1-\beta} dx = \eta r \int_{\Omega\times\Omega} \frac{H(y,u)H(x,u)}{|x-y|^r} dxdy, \tag{3.2}$$

thus

$$\begin{aligned}
J_\eta(u) &= \frac{1}{p}\left(\|\nabla u\|^p_p + \|u\|^p_{p,V}\right) + \frac{1}{q}\left(\|\nabla u\|^q_{q,a} + \|u\|^q_{q,Va}\right) - \frac{1}{1-\beta}\int_\Omega g(x)|u|^{1-\beta} dx\\
&\quad - \frac{1}{2r}\left(\|\nabla u\|^p_p + \|u\|^p_{p,V} + \|\nabla u\|^q_{q,a} + \|u\|^q_{q,Va} - \int_\Omega g(x)|u|^{1-\beta} dx\right)\\
&= \frac{2r-p}{2rp}\left(\|\nabla u\|^p_p + \|u\|^p_{p,V}\right) + \frac{2r-q}{2rq}\left(\|\nabla u\|^q_{q,a} + \|u\|^q_{q,Va}\right)\\
&\quad + \frac{1-\beta-2r}{2r(1-\beta)}\int_\Omega g(x)|u|^{1-\beta} dx\\
&\geq \frac{2r-q}{2rq}\min(1,V_0)\varrho(u) - \frac{(2r+\beta-1)}{2r(1-\beta)} C\|g\|_{L^\infty(\Omega)} |\Omega|^{\frac{p-1+\beta}{p}} \|u\|^{1-\beta}.
\end{aligned}$$

Taking $\|u\| > 1$, then

$$J_\eta(u) \geq \frac{2r-q}{2rq}\min(1,V_0)\|u\|^p - \frac{(2r+\beta-1)}{2r(1-\beta)} C\|g\|_{L^\infty(\Omega)} |\Omega|^{\frac{p-1+\beta}{p}} \|u\|^{1-\beta}.$$

The fact that $0 < \beta < 1$ implies that J_η is coercive. $\square$

Lemma 3.2. *Under the assumptions* $(h_1) - -(V_1)$*, there exists* $\eta_0 > 0$ *such that* $\mathcal{N}_\eta^0 = \emptyset$*, for any* $\eta \in (0, \eta_0)$*.*

Proof. Through a method of contradiction, we assume that for every value of η, there exists $u \in E \setminus \{0\}$ such that $< J'_\eta(u), u >= 0$ and $\Phi''_\eta(1) = 0$. Then

$$\|\nabla u\|^p + \|\nabla u\|_{q,a}^q + \|u\|_{p,V}^p + \|u\|_{q,Va}^q = \int_\Omega g(x)|u|^{1-\beta}dx + \eta r \int_{\Omega\times\Omega} \frac{H(y,u)H(x,u)}{|x-y|^r}dxdy, \tag{3.3}$$

and

$$\begin{aligned}&(p-1)\|\nabla u\|^p + (q-1)\|\nabla u\|_{q,a}^q + (p-1)\|u\|_{p,V}^p + (q-1)\|u\|_{q,Va}^q \\ &= -\beta \int_\Omega g(x)|u|^{1-\beta}dx + \eta r(2r-1)\int_{\Omega\times\Omega} \frac{H(y,u)H(x,u)}{|x-y|^\mu}dxdy.\end{aligned} \tag{3.4}$$

Multiplying (3.3) with β and adding it to (3.4) yields

$$\begin{aligned}&(p+\beta-1)\|\nabla u\|^p + (q+\beta-1)\|\nabla u\|_{q,a}^q + (p+\beta-1)\|u\|_{p,V}^p + (q+\beta-1)\|u\|_{q,Va}^q \\ &= \eta r(2r+\beta-1)\int_{\Omega\times\Omega} \frac{H(y,u)H(x,u)}{|x-y|^\mu}dxdy.\end{aligned} \tag{3.5}$$

Subtracting (3.4) from (3.3) multiplied by $(2r-1)$, we obtain

$$\begin{aligned}&(2r-p)\|\nabla u\|_p^p + (2r-q)\|\nabla u\|_{q,q}^q + (2r-p)\|u\|_{p,V}^p + (2r-q)\|u\|_{q,Va}^q \\ &= (2r+\beta-1)\int_\Omega g(x)|u|^{1-\beta}dx.\end{aligned} \tag{3.6}$$

Defining the functional $\mathcal{T}_\eta : \mathcal{N} \to \mathbb{R}$ as

$$\begin{aligned}\mathcal{T}_\eta(u) =& \frac{1}{r(2r+\beta-1)}\Big[(p+\beta-1)\|\nabla u\|_p^p + (q+\beta-1)\|\nabla u\|_{q,a}^q \\ &+ (p+\beta-1)\|u\|_{p,V}^p + (q+\beta-1)\|u\|_{q,Va}^q\Big] - \eta\int_{\Omega\times\Omega} \frac{H(y,u)H(x,u)}{|x-y|^\mu}dxdy.\end{aligned}$$

From (3.3) and (3.4), we see that $\mathcal{T}_\eta(u) = 0$ for all $u \in \mathcal{N}^0$.
we get

$$\begin{aligned}\mathcal{T}_\eta(u) &\geq \frac{(p+\beta-1)}{r(2r+\beta-1)}\|\nabla u\|_p^p - \eta\int_{\Omega\times\Omega} \frac{H(y,u)H(x,u)}{|x-y|^\mu}dxdy \\ &\geq \frac{(p+\beta-1)}{r(2r+\beta-1)}\|\nabla u\|_p^p - \eta c\|u\|_{L^{sr}}^{2r}.\end{aligned}$$

Since $p < 2r$ and using Poincarée inequality, we obtain that

$$\mathcal{T}_\eta(u) \geq \frac{(p+\beta-1)}{r(2r+\beta-1)}\|\nabla u\|_p^p - \eta C\|\nabla u\|_p^{2r}$$

then

$$\mathcal{T}_\eta(u) \geq \|\nabla u\|_p^{2r}\left[\frac{(p+\beta-1)}{r(2r+\beta-1)}\|\nabla u\|_p^{p-2r} - 2\eta C\right]. \tag{3.7}$$

Using (3.6), we get

$$\begin{aligned}(2r-p)\|\nabla u\|_p^p &\leq (2r+\beta-1)\int_\Omega g(x)|u|^{1-\beta}dx\\ &\leq (2r+\beta-1)\|g\|_\infty\|u\|_p^{1-\beta},\end{aligned}$$

and by the Poincarée inequality, we have

$$(2r-p)\|\nabla u\|_p^p \leq C(2r+\beta-1)\|g\|_\infty\|\nabla u\|_p^{1-\beta},$$

then

$$\|\nabla u\|_p^p \leq \frac{C(2r+\beta-1)}{(2r-p)}\|g\|_\infty\|\nabla u\|_p^{1-\beta}.$$

Thus

$$\|\nabla u\|_p \leq \left(\frac{C(2r+\beta-1)}{(2r-p)}\|g\|_\infty\right)^{\frac{1}{p-\beta+1}}. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\mathcal{T}_\eta(u) \geq \|\nabla u\|_p^{2r}\left[\frac{(p+\beta-1)}{r(2r+\beta-1)}\left(\frac{C(2r+\beta-1)}{(2r-p)}\|g\|_\infty\right)^{\frac{p-2r}{p-\beta+1}} - 2\eta C\right].$$

Taking

$$\eta_0 = \frac{(p+\beta-1)}{2r(2r+\beta-1)C}\left[\frac{C(2r+\beta-1)}{(2r-p)}\|g\|_\infty\right]^{\frac{p-2r}{p-\beta+1}},$$

then for any $\eta \in (0, \eta_0)$, we have $\mathcal{T}_\eta(u) > 0$ which contradicts that $\mathcal{T}_\eta(u) = 0$ for all $u \in \mathcal{N}^0$. $\square$

Lemma 3.3. *For $u \in E$, there exists $\eta_1 > 0$ such that for any $\eta \in (0, \eta_1)$. Then, there exists $t_{max} = t_{max}(u) > 0$, $t_2 = t_2(u), t_1 = t_1(u) > 0$, with $t_1 < t_{max} < t_2$ such that $t_1 u \in \mathcal{N}_\eta^+$, $t_2 u \in \mathcal{N}_\eta^-$, and $J_\eta(t_1 u) = \min_{0\leq t\leq t_{max}} J_\eta(tu)$, $J_\eta(t_2 u) = \min_{t>t_{max}} J_\eta(tu)$.*

Proof. We can write $\Phi_\eta'(t)$ as

$$\Phi_\eta'(t) = t^{2r-1}\left(\Psi(t) - \eta r\int_{\Omega\times\Omega}\frac{H(y,u)H(x,u)}{|x-y|^r}dxdy\right), \tag{3.9}$$

where

$$\Psi(t) = t^{p-2r}\left(\|\nabla u\|_p^p + \|u\|_{p,V}^p\right) + t^{q-2r}\left(\|\nabla u\|_{q,a}^q + \|u\|_{q,Va}^q\right) - t^{-\beta-2r+1}\int_\Omega g(x)|u|^{1-\beta}dx.$$

Evidently, the condition for $tu \in \mathcal{N}$ is equivalent to the statement that $\Psi(t) = \eta r\int_{\Omega\times\Omega}\frac{H(y,u)H(x,u)}{|x-y|^r}dxdy.$

$$\lim_{t\to 0^+}\Psi(t) = -\infty, \quad \lim_{t\to+\infty}\Psi(t) = 0 \text{ and } \Psi(t) > 0, \text{ for } t \text{ large enough.}$$

Let $tu \in E \setminus \{0\}$, then

$$0 = \Psi'(t) = (p-2r)\, t^{p-2r-1} \left(\|\nabla u\|_p^p + \|u\|_{p,V}^p \right) + (q-2r)\, t^{q-2r-1} \left(\|\nabla u\|_{q,a}^q + \|u\|_{q,Va}^q \right)$$
$$-(-\beta - 2r + 1) t^{-\beta-2r} \int_\Omega g(x)|u|^{1-\beta} dx,$$

equivalent to

$$\begin{aligned} &(p-2r)t^{p+\beta-1} \left(\|\nabla u\|_p^p + \|u\|_{p,V}^p \right) + (q-2r)t^{q+\beta-1} \left(\|\nabla u\|_{q,a}^q + \|u\|_{q,Va}^q \right) \\ &= (-\beta - 2r + 1) \int_\Omega g(x)|u|^{1-\beta} dx. \end{aligned} \tag{3.10}$$

Let set

$$\Theta(t) = (p-2r)t^{p+\beta-1} \left(\|\nabla u\|_p^p + \|u\|_{p,V}^p \right) + (q-2r)t^{q+\beta-1} \left(\|\nabla u\|_{q,a}^q + \|u\|_{q,Va}^q \right), \ t > 0.$$

Since $p < q < 2r$, one can has $\lim\limits_{t\to 0^+} \Theta(t) = 0$, $\lim\limits_{t\to+\infty} \Theta(t) = -\infty$, and $\Theta'(t) < 0$. Thus, through the application of the Intermediate Value Theorem, there exists a unique $t_{max} > 0$ such that (3.10) holds.
Moreover, if we consider $\Psi'(t) > 0$, then in place of (3.10) we get, $\Theta(t) > (-\beta - 2r + 1) \int_\Omega g(x)|u|^{1-\beta} dx$, for $t < t_{max}$, since Θ is strictly decreasing. Similarly for $\Psi'(t) < 0$ and $t > t_{max}$. In addition $\Psi(t_{max}) = \max\limits_{t>0} \Psi(t)$. Moreover, we have $\lim\limits_{t\to 0^+} \Psi(t) = 0$, $\lim\limits_{t\to+\infty} \Psi(t) = -\infty$.

Observe that

$$\Psi(t) \geq t^{p-2r} \|\nabla u\|_p^p - t^{-\beta-2r+1} \int_\Omega g(x)|u|^{1-\beta} dx$$
$$\geq t^{p-2r} \|\nabla u\|_p^p - t^{-\beta-2r+1} \|g\|_\infty \|\nabla u\|_p^{1-\beta}.$$

Set

$$\psi(t) = t^{p-2r} \|\nabla u\|_p^p - t^{-\beta-2r+1} \|g\|_\infty \|\nabla u\|_p^{1-\beta},$$

we have

$$\psi'(t) = (p-2r)t^{p-2r-1} \|\nabla u\|_p^p - (-\beta - 2r + 1)t^{-\beta-2r} \|g\|_\infty \|\nabla u\|_p^{1-\beta}.$$

Taking

$$t_0 = \frac{1}{\|\nabla u\|_p} \left(\frac{(-\beta - 2r + 1)\|g\|_\infty}{(p-2r)} \right)^{\frac{1}{p+\beta-1}},$$

then

$$\max_{t>0} \psi(t) = \psi(t_0).$$

It is clear that $\Psi(t) \geq \psi(t)$, and since Ψ is increasing in $(0, t_{max})$, we have

$$\Psi(t_{max}) \geq \Psi(t_0) \geq \psi(t_0),$$

where

$$\psi(t_0) = t_0^{p-2r} \|\nabla u\|_p^p - t_0^{-\beta-2r+1} \|g\|_\infty \|\nabla u\|_p^{1-\beta}$$

$$= \|\nabla u\|_p^{2r}\left(\frac{p+\beta-1}{2r-p}\|g\|_\infty\right)\left[\frac{\beta+2r-1}{2r-p}\|g\|_\infty\right]^{\frac{-\beta-2r+1}{p+\beta-1}} = A\|\nabla u\|_p^{2r},$$

with

$$A = \left(\frac{p+\beta-1}{2r-p}\|g\|_\infty\right)\left[\frac{\beta+2r-1}{2r-p}\|g\|_\infty\right]^{\frac{-\beta-2r+1}{p+\beta-1}}.$$

Return now to

$$\begin{aligned}\Psi(t_{max}) - \eta r\int_{\Omega\times\Omega}\frac{H(y,u)H(x,u)}{|x-y|^r}dxdy &\geq A\|\nabla u\|_p^{2r} - \eta C\|\nabla u\|_p^{2r}\\ &= (A-\eta C)\|\nabla u\|_p^{2r}.\end{aligned}$$

Taking $\eta_1 = \frac{A}{C}$, then for any $\eta \in (0,\eta_1)$, we have

$$\Psi(t_{max}) - \eta r\int_{\Omega\times\Omega}\frac{H(y,u)H(x,u)}{|x-y|^r}dxdy > 0. \tag{3.11}$$

Now, by (3.11) and the variation of Ψ allows us to conclure that there exist $t_1 \in (0, t_{max})$ and $t_2 \in (t_{max}, +\infty)$, with $t_1 < t_{max} < t_2$ such that,
$\Psi(t_1) = \eta r\int_{\Omega\times\Omega}\frac{H(y,u)H(x,u)}{|x-y|^r}dxdy = \Psi(t_2)$ and $\Psi'(t_1) > 0 > \Psi'(t_2)$, that means $t_1 u \in \mathcal{N}_\eta^+$ and $t_2 u \in \mathcal{N}_\eta^-$. □

Let define $\mathcal{J}_\eta^+ = \inf_{u\in\mathcal{N}_\eta^+} J_\eta(u)$, $\mathcal{J}_\eta^- = \inf_{u\in\mathcal{N}_\eta^-} J_\eta(u)$, and taking $\eta_* = \min(\eta_0,\eta_1)$

Proposition 3.1. *For any $\eta \in (0,\eta_*)$, we have*

$$\mathcal{J}_\eta^+ < 0, \tag{3.12}$$

and there exists $u^+ \in \mathcal{N}_\eta^+$ such that

$$\mathcal{J}_\eta^+ = J_\eta(u^+). \tag{3.13}$$

.

Proof. Let $u \in \mathcal{N}$, then we have

$$\|\nabla u\|_p^p + \|\nabla u\|_{q,a}^q + \|u\|_{p,V}^p + \|u\|_{q,a}^q - \int_\Omega g(x)|\,u|^{1-\beta}dx - \eta r\int_{\Omega\times\Omega}\frac{H(y,u)H(x,u)}{|x-y|^\mu}dxdy = 0,$$

thus

$$\begin{aligned}J_\eta(u) &= \frac{(1-\beta-p)}{p(1-\beta)}\left(\|\nabla u\|_p^p + \|u\|_{p,V}^p\right) + \frac{(1-\beta-q)}{q(1-\beta)}\left(\|\nabla u\|_{q,a}^q + \|u\|_{q,Va}^q\right)\\ &\quad + \frac{(2r+\beta-1)}{2(1-\beta)}\eta\int_{\Omega\times\Omega}\frac{H(y,u)H(x,u)}{|x-y|^\mu}dxdy\\ &\leq \frac{1}{q(1-\beta)}\left[(1-\beta-p)\left(\|\nabla u\|_p^p + \|u\|_{p,V}^p\right) + (1-\beta-q)\left(\|\nabla u\|_{q,a}^q + \|u\|_{q,Va}^q\right)\right]\\ &\quad + \frac{(2r+\beta-1)}{2(1-\beta)}\eta\int_{\Omega\times\Omega}\frac{H(y,u)H(x,u)}{|x-y|^\mu}dxdy.\end{aligned}$$

Since $u \in \mathcal{N}_\eta^+$, we have

$$(1-\beta-p)\left(\|\nabla u\|_p^p + \|u\|_{p,V}^p\right) + (1-\beta-q)\left(\|\nabla u\|_{q,a}^q + \|u\|_{q,Va}^q\right)$$

$$< -(2r+\beta-1)\, r\eta \int_{\Omega\times\Omega} \frac{H(y,u)H(x,u)}{|x-y|^{\mu}} dx\ dy.$$

Thus

$$J_\eta(u) \le \frac{(2r+\beta-1)(q-2r)}{2q(1-\beta)}\eta \int_{\Omega\times\Omega} \frac{H(y,u)H(x,u)}{|x-y|^{\mu}} dxdy < 0,$$

which implies that

$$\mathcal{J}_\eta^+ = \inf_{u\in\mathcal{N}_\eta^+} J_\eta(u) < 0.$$

Now let $\{u_n\} \subset \mathcal{N}_\eta^+$ such that $\lim_{n\to+\infty} J_\eta(u_n) = \mathcal{J}_\eta^+$. By Lemma 3.1, $\{u_n\}$ is bounded in E, then there exists a subsequence still denoted by $\{u_n\}$ and a function $u^+ \in E$ such that

$$u_n \rightharpoonup u_\eta \ \text{ in } E, \quad u_n \to u^+ \ \text{ in } \ L^{\alpha}(\Omega) \text{ and } \ u_n \to u^+ \ \text{ a.e. in } \ \Omega \quad \text{for any } \alpha \in [1, p^*). \tag{3.14}$$

Let's prove the equi-absolutely continuous of $\int_\Omega g(x)|u_n|^{1-\beta}dx$. Indeed, form (3.14), $\{u_n\}$ is bounded in $L^{\alpha}(\Omega)$, then there exists a positive constant c_1 such that $\|u_n\|_{p^*} \le c_1$ for all $n \in \mathbb{N}$.
Let $F \subset \Omega$ be a measurable set, we have that

$$\int_F g(x)|u_n|^{1-\beta}dx \le \|g\|_{L^\infty} c_1^{1-\beta}|F|^{\frac{p^*-1+\beta}{p^*}},$$

and

$$\int_{F\times F} \frac{H(y,u_n)H(x,u_n)}{|x-y|^{\mu}} dxdy \le c\left(\int_F |u|^{rs}dx\right)^{\frac{2}{s}} \le c c_1^{2r}|F|^{\frac{p^*-rs}{p^*}}.$$

For any $\epsilon > 0$, there exists $\delta > 0$ such that for any measurable set $F \subset \Omega$ with $|F| < \delta$, we have

$$\int_F g(x)|u_n|^{1-\beta}dx \le \epsilon \quad \text{and} \quad \int_{F\times F} \frac{H(y,u_n)H(x,u_n)}{|x-y|^{\mu}} dxdy \le \epsilon, \quad \text{for any} \quad n \in \mathbb{N}.$$

By Vitali's convergence Theorem, we get

$$\lim_{n\to\infty} \int_\Omega g(x)|u_n|^{1-\beta}dx = \int_\Omega g(x)|u^+|^{1-\beta}dx, \tag{3.15}$$

and

$$\lim_{n\to\infty} \int_{\Omega\times\Omega} \frac{H(y,u_n)H(x,u_n)}{|x-y|^{\mu}} dxdy = \int_{\Omega\times\Omega} \frac{H(y,u^+)H(x,u^+)}{|x-y|^{\mu}} dxdy. \tag{3.16}$$

With the same arguments as above, we obtain

$$\lim_{n\to\infty} \int_\Omega V(x)\left(|u_n|^p + a(x)\,|u_n|^q\right)\ dx = \int_\Omega V(x)\left(|u^+|^p + a(x)\,|u^+|^q\right)\ dx, \tag{3.17}$$

the weak lower semicontinuity of the norms leads to

$$J_\eta(u^+) \le \liminf_{n\to+\infty} J_\eta(u_n) < 0 = J_\eta(0),$$

which implies that $u^+ \neq 0$ and by lemma 3.3, there exists t_1 such that $t_1 u^+ \in \mathcal{N}_\eta^+$. Now, suppose that $\liminf_{n\to\infty} \varrho(\nabla u_n) > \varrho(\nabla u^+)$, then using (3.17), (3.15) and (3.16), we have $\liminf_{n\to\infty} \Phi'_{u_n}(t_1) > \Phi'_{u^+}(t_1) = 0$, thus $\Phi'_{u_n}(t_1) > 0$ for n large enough.
On the other hand, $\Phi'_{u_n}(t) < 0$ for $t \in (0,1)$ and $\Phi'_{u_n}(1) = 0$. And as we have $t_1 > 1$ and Φ_{u^+} is decreasing in $(0, t_1]$, we obtain

$$\mathcal{J}_\eta^+ \leq J(t_1 u^+) \leq J(u^+) < \inf_{u\in\mathcal{N}_\eta^+} J_\eta(u) = \mathcal{J}_\eta^+,$$

which is absurd. Hence $u_n \to u^+$ strongly in E and $J_\eta(u^+) = \mathcal{J}_\eta^+$.

□

Proposition 3.2. *For any $\eta \in (0, \eta_*)$, we have*

$$\mathcal{J}_\eta^- > 0, \tag{3.18}$$

and there exists

$$u^- \in \mathcal{N}_\eta^- \text{ such that } \mathcal{J}_\eta^- = J_\eta(u^-). \tag{3.19}$$

Proof. Let $u \in \mathcal{N}$, then we have

$$\|\nabla u\|_p^p + \|\nabla u\|_{q,a}^q + \|u\|_{p,V}^p + \|u\|_{q,a}^q - \int_\Omega g(x)|\,u|^{1-\beta}dx - \eta r\int_{\Omega\times\Omega} \frac{H(y,u)H(x,u)}{|x-y|^\mu}dxdy = 0,$$

thus

$$\begin{aligned} J_\eta(u) &= \frac{(1-\beta-p)}{p(1-\beta)}\left(\|\nabla u\|_p^p + \|u\|_{p,V}^p\right) + \frac{(1-\beta-q)}{q(1-\beta)}\left(\|\nabla u\|_{q,a}^q + \|u\|_{q,Va}^q\right) \\ &\quad + \frac{(2r+\beta-1)}{2(1-\beta)}\eta\int_{\Omega\times\Omega}\frac{H(y,u)H(x,u)}{|x-y|^\mu}dxdy \\ &> \frac{1}{2r(1-\beta)}\left[(1-\beta-p)\left(\|\nabla u\|_p^p + \|u\|_{p,V}^p\right) + (1-\beta-q)\left(\|\nabla u\|_{q,a}^q + \|u\|_{q,Va}^q\right)\right] \\ &\quad + \frac{(2r+\beta-1)}{2(1-\beta)}\eta\int_{\Omega\times\Omega}\frac{H(y,u)H(x,u)}{|x-y|^\mu}dxdy. \end{aligned}$$

Since $u \in \mathcal{N}_\eta^-$, we have

$$\begin{aligned} &(1-\beta-p)\left(\|\nabla u\|_p^p + \|u\|_{p,V}^p\right) + (1-\beta-q)\left(\|\nabla u\|_{q,a}^q + \|u\|_{q,Va}^q\right) \\ &> -\left(2r+\beta-1\right) r\eta\int_{\Omega\times\Omega}\frac{H(y,u)H(x,u)}{|x-y|^\mu}dxdy, \end{aligned}$$

thus

$$J_\eta(u) > 0,$$

and

$$\mathcal{J}_\eta^- = \inf_{u\in\mathcal{N}_\eta^-} J_\eta(u) \geq 0.$$

Let now $\{u_n\}_n \subset \mathcal{N}_\eta^-$ as a sequence such that $\lim_{n\to\infty} J_\eta(u_n) = \inf_{u\in\mathcal{N}_\eta^-} J_\eta(u) = \mathcal{J}_\eta^-$. By Lemma 3.1, $\{u_n\}_n$ is bounded in E and then, there exists $u^- \in E$ such that up to a subsequence

$$u_n \rightharpoonup u^- \text{ in } E \ \text{ and } u_n \to u^- \text{ in } L^\alpha(\Omega) \ \text{ for } \alpha \in [1, p^*).$$

And as above, we find that

$$\lim_{n\to\infty} \int_\Omega g(x)|u_n|^{1-\beta} dx = \int_\Omega g(x)|u^-|^{1-\beta} dx, \tag{3.20}$$

$$\lim_{n\to\infty} \int_{\Omega\times\Omega} \frac{H(y,u_n)H(x,u_n)}{|x-y|^\mu} dxdy = \int_{\Omega\times\Omega} \frac{H(y,u^-)H(x,u^-)}{|x-y|^\mu} dxdy, \tag{3.21}$$

and

$$\lim_{n\to\infty} \int_\Omega V(x)\left(|u_n|^p + a(x)|u_n|^q\right) dx = \int_\Omega V(x)\left(|u^-|^p + a(x)|u^-|^q\right) dx. \tag{3.22}$$

Let assume that $u_n \nrightarrow u^-$ strongly in E, then we have

$$\int_\Omega \left(\frac{1}{p}|\nabla u^-|^p + \frac{1}{q}a(x)|\nabla u^-|^q\right) dx < \liminf_{n\to\infty} \int_\Omega \left(\frac{1}{p}|\nabla u_n|^p + \frac{1}{q}a(x)|\nabla u_n|^q\right) dx.$$

On the other hand, there exists $t_2 > 0$ such that $t_2 u^- \in \mathcal{N}_\eta^-$. Moreover, as $u_n \in \mathcal{N}_\eta^-$, the map $t \to J_\eta(tu_n)$ attains its maximum at $t = 1$.

$$J_\eta(t_2 u^-) < \liminf_{n\to\infty} J_\eta(t_2 u_n) \le \liminf_{n\to\infty} J_\eta(u_n) = \mathcal{J}_\eta^-,$$

which contradicts the fact that $t_2 u^- \in \mathcal{N}_\eta^-$. Therefore $u_n \to u^-$ strongly in E and $J_\eta(u^-) = \mathcal{J}_\eta^-$. □

Proposition 3.3. *Let $(h_1) - -(V_1)$ hold. Let $s \in W^{1,\mathcal{M}}(\Omega)$ and $\eta \in \left(0, \overline{\eta}\right]$. Then, there exists $c_2 > 0$ such that for all $t \in [0, c_2]$ we have*

$$J_\eta\left(u^+\right) \le J_\eta\left(u^+ + ts\right).$$

Proof. Let introduce $\zeta_s : [0; +\infty) \to \mathbb{R}$ given by

$$\begin{aligned}\zeta_s(t) = (p-1)\left(\|\nabla u + t\nabla s\|^p + \|u + ts\|_{p,V}^p\right) + (q-1)\left(\|\nabla u + t\nabla s\|_{q,a}^q + \|u + ts\|_{q,Va}^q\right)\\ + \beta \int_\Omega g(x)\ |u|^{1-\beta}\ dx - \eta r\,(2r-1) \int_{\Omega\times\Omega} \frac{H(y,u)\,H(x, u+ts)}{|x-y|^\mu}\ dxdy.\end{aligned}$$

Since $u^+ \in \mathcal{N}_\eta^+$, we have $\zeta_s(0) > 0$. Thanks to [18, Proposition 3.5], there exists $\varphi(t) > 0$ for all $t \in [0, c_2]$ such that

$$\varphi(t)\left(u^+ + ts\right) \in \mathcal{N}_\eta^+ \quad \text{and} \quad \varphi(t) \to 1 \ \text{ as } \ t \to 0^+. \tag{3.23}$$

By Proposition 3.1, we have

$$\mathcal{J}_\eta^+ = J_\eta\left(u^+\right) \le J_\eta\left(\varphi(t)\left(u^+ + ts\right)\right) \quad \forall t \in [0, c_2].$$

By (3.23) and the previous fact, there exists $\kappa \in [0, c_2]$ sufficiently small such that

$$\mathcal{J}_\eta^+ = J_\eta\left(u^+\right) \le J_\eta\left(u^+ + ts\right) \quad \text{for all} \ \ t \in [0, \kappa].$$

Since the continuity of Φ'' in t, and $\Phi''_{u^+}(1) > 0$ we obtain $\Phi''_{u^+ + ts}(1) > 0$ for all $t \in [0, \kappa]$ with $\kappa \in (0, c_2]$. which achieve the proof. □

Proof of Theorem 3.1 : To demonstrate that the minimizers found in Propositions 3.1 and 3.2 are non-trivial solutions to the problem (1.1). We use similar arguments to [18, propositions 3.6 and 3.9], which complete the proof of Theorem 3.1.

ACKNOWLEDGEMENT

The authors would like to thank the reviewers for their insightful remarks and constructive criticism that helped polish the final version of the work.

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