Stability of Traveling Wave Fronts for Nonlocal Diffusive Systems

Shengqiang Zhang*, Zhixian Yu† and Yanling Meng‡

Abstract

The paper is concerned with stability of traveling wave fronts for nonlocal diffusive systems. We adopt L^1–weighted, L^1– and L^2–energy estimates for the perturbation systems, and show that all solutions of the Cauchy problem for the considered systems converge exponentially to traveling wave fronts provided that the initial perturbations around the traveling wave fronts belong to a suitable weighted Sobolev space.

Keywords. Exponential stability, nonlocal dispersals, upper and lower solutions, traveling wave fronts, comparison principle, weighted energy.

AMS subject classifications. 35K57, 35C07, 35R20, 92D25.

*College of Sciences, Shanghai Institute of Technology, 100 Haiquan Road, Shanghai 201418, People's Republic of China. Email: sqzhang@sit.edu.cn

†Department of Mathematics, Shanghai Normal University, Shanghai 200234, People's Republic of China. Email: zxyu0902@163.com.

‡The corresponding author. College of Sciences, Shanghai Institute of Technology, 100 Haiquan Road, Shanghai 201418, People's Republic of China. Email: ylmeng0321@163.com.

1 Introduction

Incorporating spatial variation into models is clearly central to understanding many biological and physical systems. Diffusion has been frequently used to model movement in spatially deterministic models [32, 38]. Diffusion is a local process in which particles move infinitesimal distances in infinitesimal units of time. Thus, Lee et al. [15] argued that, for processes where the spatial scale for movement is large in comparison with its temporal scale, nonlocal models using integro-differential may allow for better estimation of parameters from data and provide more insight into the biological system. Usually, the classic nonlocal model can be described by the following single equation

$$\frac{\partial u(t,x)}{\partial t} = D\left[\int_{\mathbb{R}} J(x-y)u(t,y)dy - u(t,x)\right] + f(u(t,x)), \tag{1.1}$$

where the kernel $J(x)$ is a probability density. The nonlocal model (1.1) with monostable nonlinearity has been widely investigated by authors (see [2, 3, 4, 5, 6, 41]).

The purpose of this work is to investigate the exponential stability of the epidemic model with nonlocal dispersals

$$\begin{cases} u_t(x,t) = d_1[(J_1 * u)(x,t) - u(x,t)] + h(u(x,t), v(x,t-\tau_1)), \\ v_t(x,t) = d_2[(J_2 * v)(x,t) - v(x,t)] + g(u(x,t-\tau_2), v(x,t)), \end{cases} \quad x \in \mathbb{R},\ t > 0, \tag{1.2}$$

where $\tau_1 > 0$ and $\tau_2 > 0$ represent the time delays, $u(x,t)$ and $v(x,t)$ represent the spatial concentration of the bacteria and the infective population at a point $x \in \mathbb{R}$ and time $t \geq 0$, respectively. $d_1 \geq 0$ and $d_2 \geq 0$ are diffusion coefficients. $(J_1 * u)(x,t)$ $(J_2 * v)(x,t)$ represent the total number of the bacteria and the infective population arriving at x from all possible locations y at time t, respectively.

If the diffusion kernel

$$J_i(x) = \delta(x) + \delta''(x)$$

with δ being the Dirac delta function (see [24]), $h(u,v) = -\alpha_1 u + h(v)$ and $g(u,v) = -\alpha_2 v + g(u)$, then (1.2) reduces to the traditional reaction diffusion systems

$$\begin{cases} u_t(x,t) = d_1 u_{xx}(x,t) - \alpha_1 u(x,t) + h(v(x,t)), \\ v_t(x,t) = d_2 v_{xx}(x,t) - \alpha_2 v(x,t) + g(u(x,t)), \end{cases} \quad x \in \mathbb{R},\ t > 0. \tag{1.3}$$

Hsu and Yang [12] investigated the existence, uniqueness and asymptotic behavior of traveling waves for (1.3). See also [8, 10, 49, 50] for some special cases. More recently, Hsu et al. [13] extended (1.3) to more general systems and obtained the existence and stability of traveling waves.

If $d_1 = d_2 = 0$, $h(u,v) = -\alpha_1 u + av$ and $g(u,v) = -\alpha_2 v + g(u)$ for some constant $a > 0$, (1.2) reduces to the classic ODE epidemic model

$$\begin{cases} u_t(x,t) = -\alpha_1 u(x,t) + av(x,t), \\ v_t(x,t) = -\alpha_2 v(x,t) + g(u(x,t)), \end{cases} \quad x \in \mathbb{R},\ t > 0, \tag{1.4}$$

which was proposed in [9] to model the cholera epidemic spread.

From the view of mathematics, letting $u = v$, $d_1 = d_2$, $J_1 = J_2$ and $h(u, v) = g(u, v) = -\alpha u+h(u)$, (1.2) is equivalent to the following single equation with the nonlocal dispersal

$$u_t(x, t) = d[(J * u)(x, t) - u(x, t)] - \alpha u(x, t) + h(u(x, t)), \tag{1.5}$$

which can usually be used to describe the growth and spatial spread of single species population. Yu and Yuan [55] investigated the existence of traveling waves for (1.5). Especially, when $h(u) = pue^{-qu}$, Pan [33] showed the existence of traveling waves of (1.5). We refer to some references about the more general nonlocal monostable equation with delays or without delays, see [33, 35, 34, 39, 46, 48, 54, 57] and some references cited therein.

In this article, we are mainly concerned with the existence and exponential stability of traveling wave solutions for (1.2). More precisely, following the ideas from [12, 13], we can also construct a pair of suitable upper and lower solutions relying on careful local analysis near the stationary solutions. By using the theory in [34], the existence of traveling wave solutions connecting two equilibria is admitted. On the other hand, the stability on traveling waves is an important and interesting project. The stability problems of traveling waves for some specific reaction–diffusion have been widely studied, by using the spectral analysis method [37, 43], a squeezing technique via the upper and lower solutions comparison [7, 30, 40, 47] and the weighted–energy method [13, 14, 17, 18, 19, 29, 25, 36, 44, 49, 50, 52, 53] and many references cited therein. By using weighted–energy method, authors [19, 35] also investigated the stability of traveling wave fronts for single equation with nonlocal diffusion. Recently, there have been many studies on the stability of other types of equations, see [1, 11, 16, 23, 45, 51, 56, 58, 59] and many references cited therein. However, the stability of traveling wave solutions for multi–component systems with nonlocal dispersals is less reported, see [53]. Motivated by the work of [13, 27, 28], we will use the weighted energy method to establish the L^1–weighted, L^1– and L^2–energy estimates for the perturbations between solutions of (1.2) and the traveling wave solutions, and show that all solutions of the Cauchy problem for the considered systems converge exponentially to traveling wave fronts provided that the initial perturbations around the traveling wave fronts belong to a suitable weighted Sobolev spaces.

The rest of our paper is organized as follows. In Section 2, we introduce some notations and main results. In Section 3, by using the the weighted energy method and the comparison principle, we study the asymptotic stability of traveling wave fronts of (1.2). In Section 4, we give an application.

2 Main Results

A traveling wave solution of (1.2) is a pair of solutions with the form $u_1(x, t) = \phi_1(x+ct)$ and $u_2(x, t) = \phi_2(x + ct)$ for some functions $\phi_i(\cdot) \in C^2(\mathbb{R}, \mathbb{R})$, $i = 1, 2$, where $c > 0$ is a constant corresponding to the wave speed and $\xi := x + ct$ is the moving coordinate.

Substituting $(u_1(x,t), u_2(x,t)) = (\phi_1(\xi), \phi_2(\xi))$ into the system (1.2), we can derive the following wave profile equations

$$\begin{cases} c\phi_1'(\xi) = d_1\big(\int_{\mathbb{R}} J_1(\xi-y)\phi_1(y)dy - \phi_1(\xi)\big) + h(\phi_1(\xi), \phi_2(\xi - c\tau_1)), \\ c\phi_2'(\xi) = d_2\big(\int_{\mathbb{R}} J_2(\xi-y)\phi_2(y)dy - \phi_2(\xi)\big) + g(\phi_1(\xi - c\tau_2), \phi_2(\xi)). \end{cases} \tag{2.1}$$

Our goal is to prove the stability of monotone solutions of (2.1) satisfying the following conditions:

$$\lim_{\xi\to-\infty}(\phi_1(\xi), \phi_2(\xi)) = \mathbf{0} \ \text{ and } \lim_{\xi\to+\infty}(\phi_1(\xi), \phi_2(\xi)) = \mathbf{K}. \tag{2.2}$$

For convenience, let us denote the coefficients of the linear parts of $h(u,v)$ and $g(u,v)$ at the equilibrium $\mathbf{0} = (0,0)$ and $\mathbf{K} = (k_1, k_2)$, respectively, by

$$\alpha_1 = \partial_u h(0,0), \ \ \alpha_2 = \partial_v g(0,0), \ \ \beta_1 = \partial_v h(0,0), \ \ \beta_2 = \partial_u g(0,0),$$

$$\bar{\alpha}_1 = \partial_u h(k_1,k_2), \ \ \bar{\alpha}_2 = \partial_v g(k_1,k_2), \ \ \bar{\beta}_1 = \partial_v h(k_1,k_2), \ \ \bar{\beta}_2 = \partial_u g(k_1,k_2).$$

Two vectors $(u_1, \cdots, u_n) \le (v_1, \cdots, v_n)$ in $\mathbb{R}^n$ means $u_i \le v_i$ for $i = 1, 2, \cdots, n$. An interval of $\mathbb{R}^n$ is defined according to this order. For convenience, denote by ∂_i the first differential operator with respect to the i-th variables, and ∂_{ij} the second differential operator with respect to the i-th and j-th variables.

In order to state our main results, throughout this article, we assume the nonlinearities $h(\cdot)$ and $g(\cdot)$ satisfy the following assumptions.

(J) $J_i \in C(\mathbb{R})$, $J_i(x) = J_i(-x) \ge 0$, $\int_{\mathbb{R}} J_i(y)dy = 1$, $i = 1, 2$, and $\int_{\mathbb{R}} |x|^j J_i(x) e^{-\lambda x} dx < \infty$ for every $\lambda > 0$, $j = 0, 1, 2$, $i = 1, 2$.

(H1) Assume $h_2 := \partial_2 h \ge 0$, $g_1 := \partial_1 g \ge 0$ on the interval $[(0,0), (k_1, k_2)]$.

(H2) Assume $\alpha_i < 0$, $\bar{\alpha}_i < 0$,

$$\alpha_1\alpha_2 < \beta_1\beta_2 \ \text{ and } \ \bar{\alpha}_1\bar{\alpha}_2 > \bar{\beta}_1\bar{\beta}_2.$$

Since $h(\cdot)$, $g(\cdot)$ are C^2, the assumption (H1) is equivalent to the following *quasimonotonicity* assumption:

there exist constants $m_i > 0, i = 1, 2$ such that the functions $\hat{h}(u,v) := m_1 u + h(u,v)$ and $\hat{g}(u,v) := m_2 v + g(u,v)$ satisfy

$$\hat{h}(\phi_1(\xi), \phi_2(\xi - c\tau_1)) \ge \hat{h}(\psi_1(\xi), \psi_2(\xi - c\tau_1)), \tag{2.3}$$

$$\hat{g}(\phi_1(\xi - c\tau_2), \phi_2(\xi)) \ge \hat{g}(\psi_1(\xi - c\tau_2), \psi_2(\xi)), \tag{2.4}$$

for any $\phi(\xi) := (\phi_1(\xi), \phi_2(\xi))$, $\psi(\xi) := (\psi_1(\xi), \psi_2(\xi)) \in C(\mathbb{R}, \mathbb{R}^2)$ satisfying $0 \le \psi_i(\xi) \le \phi_i(\xi) \le K_i$ for all $\xi \in \mathbb{R}$ and $i = 1, 2$.

The assumption (H2) can help us to investigate the characteristic roots of the linearized equations for the profile equations (2.1) at the equilibria $\mathbf{0}$ and $\mathbf{K}$, respectively.

The c_* is actually the threshold speed such that the linearized equation of (2.1) at $\mathbf{0}$ has positive eigenvalues. Given a fixed $c > 0$, let $\lambda_1(c)$ be the smallest positive eigenvalue of the linearized equation of (2.1) at $\mathbf{0}$, and $\lambda_2(c)$ be the largest negative eigenvalue of the linearized equation of (2.1) at $\mathbf{K}$.

Now we recall the known result on the existence of traveling wave fronts (see [21, 31]).

Theorem 2.1 (Existence)**.** *Assume* (J) *and* (H1)–(H2) *hold. There exists a positive constant* $c_* > 0$ *such that* (1.2) *admits a positive traveling wave front* $(\phi_1(x+ct), \phi_2(x+ct))$ *with the wave speed* $c \geq c_*$ *and satisfying* (2.2)*. For* $0 < c < c_*$*, the system* (1.2) *has no positive monotone traveling wave solution satisfying* (1.2)*.*

Next, we state the stability result of traveling wave fronts derived in Theorem 4.1. Before that, let us introduce the following notations.

- Let I be an interval, especially $I = \mathbb{R}$, then we denote $L^2(I)$ by the space of the square integrable functions on I.
- The space $H^k(I)$ $(k \geq 0)$ means the Sobolev space of the L^2–functions $f(x)$ defined on I whose derivatives $\frac{d^i}{dx^i} f (i = 1, \cdots, k)$ also belong to $L^2(I)$.
- Let us write $L^2_\omega(I)$ and $W^{k,p}_\omega(I)$ by the weight L^2–space and weight Sobolev space with positive weighted function $\omega(x) : \mathbb{R} \to \mathbb{R}$, respectively. For any $f \in L^2_\omega(I)$ or $W^{k,p}_\omega(I)$, its norm is given (resp.) by

$$\|f\|_{L^2_w(I)} = \Big(\int_I w(x)|f(x)|^2 dx \Big)^{1/2} \text{ or } \|f\|_{W^{k,p}_\omega(I)} = \Big(\sum_{i=0}^{k} \int_I \omega(x) |\frac{d^i}{dx^i} f(x)|^p dx \Big)^{1/p}.$$

  Furthermore, we set $H^k_w(I) := W^{k,2}_\omega(I)$.
- Letting $T > 0$ and $\mathcal{B}$ be a Banach space, we denote by $C^0([0,T];\mathcal{B})$ the space of the $\mathcal{B}$–valued continuous functions on $[0,T]$ and $L^2([0,T];\mathcal{B})$ as the space of the $\mathcal{B}$–valued L^2–function on $[0,T]$. The corresponding spaces of the $\mathcal{B}$–valued functions on $[0,\infty)$ are defined similarly.

Define the weight function $\omega(\cdot)$ by

$$\omega(\xi) = \begin{cases} \omega_1(\xi), & \text{for } \xi \leq \xi_0, \\ 1, & \text{for } \xi > \xi_0, \end{cases} \quad \text{with } \omega_1(\xi) := e^{-\gamma(\xi - \xi_0)}, \tag{2.5}$$

where γ and ξ_0 is large enough, which will be determined later.

In order to obtain the stability, we assume the nonlinearities $h(\cdot)$ and $g(\cdot)$ satisfy the following assumptions.

(H3) $\partial_{ij} h \leq 0$ and $\partial_{ij} g \leq 0$ for $i, j = 1, 2$ on the interval $[(0,0), (k_1, k_2)]$.

(H4) Assume $\bar{\alpha}_1 + \bar{\beta}_2 < 0$, $\bar{\alpha}_2 + \bar{\beta}_1 < 0$,

$$2\bar{\alpha}_1 + \bar{\beta}_1 + \bar{\beta}_2 < 0 \quad \text{and} \quad 2\bar{\alpha}_2 + \bar{\beta}_1 + \bar{\beta}_2 < 0.$$

Let $(\phi_1(\xi), \phi_2(\xi))$ be a traveling wave solution of (1.2) satisfying (2.2) with the wave speed $c > c_*$. Motivated by the work of [13, 27, 28], we will adopt the weighted energy method to establish the L^1–weighted, L^1– and L^2–energy estimates (see Section 4) for the perturbations between solutions of (1.2) and $(\phi_1(\xi), \phi_2(\xi))$.

Moreover, we recall the following lemmas from [12, 20], which will play an important role in establishing the L^1–weighted, L^1– and L^2–energy estimates.

Lemma 2.1. (1) *If $c > c_*$, $\Delta(c, \lambda) = 0$ has two positive roots $\lambda_1(c) < \lambda_2(c)$ in $(0, \lambda_m^c)$. Moreover, $f_i(c, \lambda_1(c) + \varepsilon) < 0$ for $i = 1, 2$ and $\Delta(c, \lambda_1(c) + \varepsilon) > 0$ when $\varepsilon > 0$ is small enough, where*

$$f_i(c, \lambda) = d_i \int_{\mathbb{R}} J_i(y) e^{-\lambda y} dy - c\lambda - d_i + \alpha_i, \; i = 1, 2.$$

(2) *Let $A = (a_{ij})$ be a two by two matrix such that $a_{ii} < 0$, $i = 1, 2$ and $a_{ij} > 0$ for $i \neq j$. Then the system of the following equalities*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 < 0 \;\; (> 0, resp.), \\ a_{21}x_1 + a_{22}x_2 < 0 \;\; (> 0, resp.), \end{cases} \tag{2.6}$$

has a solution (x_1, x_2) with $x_i > 0$, $i = 1, 2$, if and only if $\det A > 0$ $(< 0$, *resp.*).

Then, by the comparison principle and Hölder inequality, we can obtain the following stability result.

Theorem 2.2. *(Asymptotic stability) Assume that* (J) *and* (H1)–(H4) *hold. Let $\tau := \max\{\tau_1, \tau_2\}$. For a given traveling wave front $(\phi_1(x + ct), \phi_2(x + ct))$ of* (1.2) *satisfying* (2.2) *with the wave speed $c > c_*$. If the Cauchy problem* (1.2) *with the initial data $(u_0(x, s), v_0(x, s))$ satisfying the following conditions*

$$\begin{array}{l} u_0(x, s) - \phi_1(x + cs), \; v_0(x, s) - \phi_2(x + cs) \in C(L^1_\omega(\mathbb{R}) \cap H^1(\mathbb{R})), \\ \mathbf{0} \leq (u_0(x, s), v_0(x, s)) \leq \mathbf{K} \; \text{for} \; (x, s) \in \mathbb{R} \times [-\tau, 0], \end{array} \tag{2.7}$$

then the solution of (1.2) *with initial data $(u_0(x, s), v_0(x, s))$ uniquely exists and satisfies*

$$\sup_{x \in \mathbb{R}} |u(x, t) - \phi_1(x + ct)| \leq Ce^{-\mu t}, \quad \sup_{x \in \mathbb{R}} |v(x, t) - \phi_2(x + ct)| \leq Ce^{-\mu t}, \quad t \geq 0$$

for some positive constants μ and C.

3 Exponential stability of traveling wave fronts

This section is devoted to proving the exponential stability of noncritical traveling wave fronts of (1.2) and (2.2) with an exponential convergence rate. Throughout this section, it is assumed that (J) and (H1)–(H4) hold. We first give some auxiliary statements about the global solutions of the Cauchy problem (1.2) and the comparison principle. Via the standard energy method and continuity extension method (see, [25, 26]), we have the following result.

Proposition 3.1. *Assume that* (J) *and* (H1)–(H4) *hold. If the initial data* $(u_0(x,s), v_0(x,s))$ *satisfies* (2.7)*, then* (1.2) *admits a unique solution* $(u(x,t), v(x,t))$ *such that*

$$u(\cdot,t) - \phi_1(\cdot + ct),\quad v(\cdot,t) - \phi_2(\cdot + ct) \in C(L^1_\omega(\mathbb{R}) \cap H^1(\mathbb{R})) \text{ for } t \in [0,\infty) \text{ and}$$
$$\mathbf{0} \le (u(x,t), v(x,t)) \le \mathbf{K} \text{ for } (x,t) \in \mathbb{R} \times [0,\infty).$$

Similar to the proofs of Proposition 3 in [29], Lemma 3.2 in [42] and Lemma 3 in [22], we easily obtain the following comparison principle.

Proposition 3.2. *Assume that* (J) *and* (H1)–(H2) *hold. Let* $(u^-(x,t), v^-(x,t))$ *and* $(u^+(x,t), v^+(x,t))$ *be the solutions of* (1.2) *with the initial data* $(u_0^-(x,s), v_0^-(x,s))$ *and* $(u_0^+(x,s), v_0^+(x,s))$*, respectively. If*

$$(u_0^-(x,s), v_0^-(x,s)) \le (u_0^+(x,s), v_0^+(x,s)) \text{ for } (x,s) \in \mathbb{R} \times [-\tau, 0].$$

Then

$$(u^-(x,t), v^-(x,t)) \le (u^+(x,t), v^+(x,t)) \text{ for } (x,t) \in \mathbb{R} \times \mathbb{R}_+.$$

Assume the initial data $(u_0(x,s), v_0(x,s))$ satisfies the assumptions of Theorem 2.2. Let $u_0^-(x,s) \triangleq \min\{u_0(x,s), \phi_1(x+cs)\}$, $v_0^-(x,s) \triangleq \min\{v_0(x,s), \phi_2(x+cs)\}$, $u_0^+(x,s) \triangleq \max\{u_0(x,s), \phi_1(x+cs)\}$, $v_0^+(x,s) \triangleq \max\{v_0(x,s), \phi_2(x+cs)\}$ and $(u^\pm(x,t), v^\pm(x,t))$ be the nonnegative solutions of system (1.2) with the initial data $(u_0^\pm(x,s), v_0^\pm(x,s))$. Then it follows from Proposition 3.2 (the comparison principle) that

$$0 \le u^-(x,t) \le u(x,t),\ \phi_1(x+ct) \le u^+(x,t) \le k_1,$$
$$0 \le v^-(x,t) \le v(x,t),\ \phi_2(x+ct) \le v^+(x,t) \le k_2$$

for $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

Denote

$$U^+(\xi,t) \triangleq u^+(\xi - ct, t) - \phi_1(\xi),\ \ V^+(\xi,t) \triangleq v^+(\xi - ct, t) - \phi_2(\xi)$$

and

$$U^-(\xi,t) \triangleq \phi_1(\xi) - u^-(\xi - ct, t),\ \ V^-(\xi,t) \triangleq \phi_2(\xi) - v^-(\xi - ct, t),$$

where $\xi = x + ct$. Furthermore, since

$$(U_0^-(\xi,s), V_0^-(\xi,s)) \le (U_0(\xi,s), V_0(\xi,s)) \le (U_0^+(\xi,s), V_0^+(\xi,s)),\ (\xi,s) \in \mathbb{R} \times [-\tau, 0],$$

by the Comparison Theorem, we have

$$(U^-(x,t),V^-(x,t)) \le (U(x,t),V(x,t)) \le (U^+(x,t),V^+(x,t)),\ (x,t)\in\mathbb{R}\times[0,\infty).$$

Therefore, our goal is to show that there exist positive constants C and μ such that

$$\sup_{\xi\in\mathbb{R}}|U^\pm(\xi,t)|,\quad \sup_{\xi\in\mathbb{R}}|V^\pm(\xi,t)| \le Ce^{-\mu t},\quad t\ge 0. \tag{3.1}$$

For convenience, we denote the column vectors

$$\begin{aligned} X(\xi,t) :=&\big(U(\xi,t),V(\xi-c\tau_1,t-\tau_1)\big)^T, \quad Y(\xi,t) := \big(U(\xi-c\tau_2,t-\tau_2),V(\xi,t)\big)^T,\\ \Phi(\xi) :=&(\phi_1(\xi),\phi_2(\xi-c\tau_1))^T, \qquad\qquad\quad \Psi(\xi) := (\phi_1(\xi-c\tau_2),\phi_2(\xi))^T. \end{aligned}$$

For the sake of convenience, let us simply denote $U^+(\xi,t),\ V^+(\xi,t)$ by $U(\xi,t),\ V(\xi,t)$. Hence, $U(\xi,t)$ and $V(\xi,t)$ satisfy

$$\begin{cases} \partial_t U(\xi,t)+c\partial_\xi U(\xi,t)-d_1[(J_1*U)(\xi,t)-U(\xi,t)] \\ = h(U(\xi,t)+\phi_1(\xi),V(\xi-c\tau_1,t-\tau_1)+\phi_2(\xi-c\tau_1))-h(\phi_1(\xi),\phi_2(\xi-c\tau_1)),\\ \partial_t V(\xi,t)+c\partial_\xi V(\xi,t)-d_2[(J_2*V)(\xi,t)-V(\xi,t)] \\ = g(U(\xi-c\tau_2,t-\tau_2)+\phi_1(\xi-c\tau_2),V(\xi,t)+\phi_2(\xi))-g(\phi_1(\xi-c\tau_2),\phi_2(\xi)), \end{cases} \tag{3.2}$$

with the initial data

$$\begin{cases} U_0(\xi,s)\triangleq u_0^+(\xi,s)-\phi_1(\xi+cs),\\ V_0(\xi,s)\triangleq u_0^+(\xi,s)-\phi_2(\xi+cs), \end{cases}$$

where $(\xi,s)\in\mathbb{R}\times[-\tau,0]$. Obviously, $U_0(x,s),V_0(x,s)\in\mathcal{X}_\omega$ and Proposition 3.1 implies that the solution $U(\xi,t),V(\xi,t)\in\mathcal{N}_{\omega_1}$ for each $t\in[0,+\infty)$.

According to (H3), it is easy to see that (3.2) is equivalent to the following system:

$$\begin{cases} \partial_t U(\xi,t)+c\partial_\xi U(\xi,t)-d_1[(J_1*U)(\xi,t)-U(\xi,t)]-\nabla h(\Phi(\xi))X(\xi,t)\\ =\frac{1}{2}X(\xi,t)^T A\big(\bar\Phi(\xi)\big)X(\xi,t)\le 0,\\ \partial_t V(\xi,t)+c\partial_\xi V(\xi,t)-d_2[(J_2*V)(\xi,t)-V(\xi,t)]-\nabla g(\Psi(\xi))Y(\xi,t)\\ =\frac{1}{2}Y(\xi,t)^T B\big(\bar\Psi(\xi)\big)Y(\xi,t)\le 0, \end{cases} \tag{3.3}$$

where $\Phi(\xi)\le\bar\Phi(\xi)\le\Phi(\xi)+X(\xi,t)$ and $\Psi(\xi)\le\bar\Psi(\xi)\le\Psi(\xi)+Y(\xi,t)$. To obtain the estimations of (3.1), we first establish the $L^1_{w_1}$-energy, L^1-energy and L^2-energy estimates for $(U(\xi,t),V(\xi,t))$ in the following subsections.

3.1 $L^1_{\omega_1}$–energy and L^1–energy estimates

Then we have the following results.

Lemma 3.1. *Assume that* (J) *and* (H1)–(H4) *hold. For any $c>c_*$ and $\gamma=\lambda_1(c)+\varepsilon$ ($\varepsilon>0$ small enough), there exist positive constants μ and C such that*

$$e^{\mu t}(\|U(\cdot,t)\|_{L^1_{\omega_1}(\mathbb{R})}+\|V(\cdot,t)\|_{L^1_{\omega_1}(\mathbb{R})})+\int_0^t e^{\mu s}(\|U(s)\|_{L^1_{\omega_1}(\mathbb{R})}+\|V(s)\|_{L^1_{\omega_1}(\mathbb{R})})ds\le C$$

for each $t\ge0$, where $\omega_1(\xi)=e^{-\gamma(\xi-\xi_0)}$.

Proof. Multiplying the equation (3.3) by $e^{\mu t}\omega_1(\xi)$ for some $\mu > 0$, respectively, and integrating it over $\mathbb{R}\times[0,t]$, we can obtain

$$\begin{aligned}
0 \geq & \int_0^t \int_{-\infty}^{\infty} e^{\mu s}\omega_1 \partial_s U(\xi,s) + e^{\mu s}\omega_1 \Big(c\partial_\xi U(\xi,s) - d_1 (J_1 * U)(\xi,s) + d_1 U(\xi,s) \\
& \qquad - a_1 U(\xi,s) - \beta_1 V(\xi - c\tau_1, s-\tau_1) \Big) d\xi ds \\
= & \int_0^t \int_{-\infty}^{\infty} (e^{\mu s}\omega_1 U(\xi,s))_s - \mu e^{\mu s}\omega_1 U(\xi,s) + (c e^{\mu s}\omega_1 U(\xi,s))_\xi + c\gamma e^{\mu s}\omega_1 U(\xi,s) \\
& \qquad - \omega_1 \Big(d_1 (J_1 * U)(\xi,s) - (d_1-\alpha_1) U(\xi,s) + \beta_1 V(\xi - c\tau_1, s-\tau_1) \Big) d\xi ds \\
= & e^{\mu t}\|U(\cdot,t)\|_{L^1_{\omega_1}(\mathbb{R})} - \|U_0(0)\|_{L^1_{\omega_1}(\mathbb{R})} - \int_0^t \int_{-\infty}^{\infty} e^{\mu s}\omega_1 \beta_1 V(\xi - c\tau_1, s-\tau_1) d\xi ds \\
& + \int_0^t \int_{-\infty}^{\infty} e^{\mu s}\omega_1 U(\xi,s) \Big(-\mu + c\gamma - d_1 \int_{\mathbb{R}} J_1(y) e^{-\gamma y} dy + (d_1 - \alpha_1) \Big) d\xi ds \\
\geq & e^{\mu t}\|U(\cdot,t)\|_{L^1_{\omega_1}(\mathbb{R})} - \|U_0(0)\|_{L^1_{\omega_1}(\mathbb{R})} \\
& + \int_0^t \int_{-\infty}^{\infty} e^{\mu s}\omega_1 U(\xi,s) \Big(-\mu + c\gamma - d_1 \int_{\mathbb{R}} J_1(y) e^{-\gamma y} dy + (d_1 - \alpha_1) \Big) d\xi ds \\
& - \int_0^t \int_{-\infty}^{\infty} \beta_1 e^{\mu(s+\tau_1)} e^{-\gamma c\tau_1} \omega_1 V(\xi,s) d\xi ds - \int_{-\tau_1}^0 \int_{-\infty}^{\infty} \beta_1 e^{\mu(s+\tau_1)} e^{-\gamma c\tau_1} \omega_1 V(\xi,s) d\xi ds.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& e^{\mu t}\|U(\cdot,t)\|_{L^1_{\omega_1}(\mathbb{R})} + \int_0^t \int_{-\infty}^{\infty} e^{\mu s}\omega_1 U(\xi,s) \Big(-\mu + c\gamma - d_1 \int_{\mathbb{R}} J_1(y) e^{-\gamma y} dy + (d_1 - \alpha_1) \Big) d\xi ds \\
& - \int_0^t \int_{-\infty}^{\infty} \beta_1 e^{\mu(s+\tau_1)} e^{-\gamma c\tau_1} \omega_1 V(\xi,s) d\xi ds \leq C_1 \qquad (3.4)
\end{aligned}$$

for some constant $C_1 > 0$. Similarly, it follows from the second equation of (3.3) that

$$\begin{aligned}
0 \geq & \int_0^t \int_{-\infty}^{\infty} e^{\mu s}\omega_1 \partial_s V(\xi,s) + e^{\mu s}\omega_1 \Big(c\partial_\xi V(\xi,s) - d_2 (J_2 * V)(\xi,s) + d_2 V(\xi,s) \\
& \qquad - \alpha_2 V(\xi,s) - \beta_2 U(\xi - c\tau_2, s-\tau_2) \Big) d\xi ds \\
= & \int_0^t \int_{-\infty}^{\infty} (e^{\mu s}\omega_1 V(\xi,s))_s - \mu e^{\mu s}\omega_1 V(\xi,s) + (c e^{\mu s}\omega_1 V(\xi,s))_\xi + c\gamma e^{\mu s}\omega_1 V(\xi,s) \\
& \qquad - \omega_1 \Big(d_2 (J_2 * V)(\xi,s) - (d_2-\alpha_2) V(\xi,s) + \beta_2 U(\xi - c\tau_2, s-\tau_2) \Big) d\xi ds \\
= & e^{\mu t}\|V(\cdot,t)\|_{L^1_{\omega_1}(\mathbb{R})} - \|V_0(0)\|_{L^1_{\omega_1}(\mathbb{R})} - \int_0^t \int_{-\infty}^{\infty} e^{\mu s}\omega_1 \beta_1 U(\xi - c\tau_2, s-\tau_2) d\xi ds \\
& + \int_0^t \int_{-\infty}^{\infty} e^{\mu s}\omega_1 V(\xi,s) \Big(-\mu + c\gamma - d_2 \int_{\mathbb{R}} J_2(y) e^{-\gamma y} dy + (d_2 - \alpha_2) \Big) d\xi ds
\end{aligned}$$

$$\geq e^{\mu t}\|V(\cdot,t)\|_{L^1_{\omega_1}(\mathbb{R})} - \|V_0(0)\|_{L^1_{\omega_1}(\mathbb{R})}$$
$$+\int_0^t\int_{-\infty}^{\infty} e^{\mu s}\omega_1 V(\xi,s)\Big(-\mu+c\gamma-d_2\int_{\mathbb{R}}J_2(y)e^{-\gamma y}dy+(d_2-\alpha_2)\Big)d\xi ds$$
$$-\int_0^t\int_{-\infty}^{\infty}\beta_2 e^{\mu(s+\tau_2)}e^{-\gamma c\tau_2}\omega_1 U(\xi,s)d\xi ds-\int_{-\tau_2}^{0}\int_{-\infty}^{\infty}\beta_2 e^{\mu(s+\tau_2)}e^{-\gamma c\tau_2}\omega_1 U(\xi,s)d\xi ds.$$

Thus, it holds

$$e^{\mu t}\|V(\cdot,t)\|_{L^1_{\omega_1}(\mathbb{R})}+\int_0^t\int_{-\infty}^{\infty} e^{\mu s}\omega_1 V(\xi,s)\Big(-\mu+c\gamma-d_2\int_{\mathbb{R}}J_2(y)e^{-\gamma y}dy+(d_2-\alpha_2)\Big)d\xi ds$$
$$-\int_0^t\int_{-\infty}^{\infty}\beta_2 e^{\mu(s+\tau_2)}e^{-\gamma c\tau_2}\omega_1 U(\xi,s)d\xi ds\leq C_2 \tag{3.5}$$

for some constant $C_2>0$. Let $\gamma=\lambda_1+\varepsilon$, where $\varepsilon>0$ is small enough such that $f_i(\lambda_1+\varepsilon)<0$ for $i=1,2$. By Lemma 2.1, there are two positive constants p and q such that

$$pf_1(\gamma)+q\beta_2 e^{-\gamma c\tau_2}=p\Big(-c\gamma+d_1\int_{\mathbb{R}}J_1(y)e^{-\gamma y}dy-(d_1-\alpha_1)\Big)+q\beta_2 e^{-\gamma c\tau_2}<0$$

and

$$p\beta_1 e^{-\gamma c\tau_1}+qf_2(\gamma)+=p\beta_1 e^{-\gamma c\tau_1}+q\Big(-c\gamma+d_2\int_{\mathbb{R}}J_2(y)e^{-\gamma y}dy-(d_2-\alpha_2)\Big)<0.$$

Multiplying (3.4)–(3.5) by p and q, respectively, and adding them, we can obtain

$$e^{\mu t}\Big(p\|U(\cdot,t)\|_{L^1_{\omega_1}(\mathbb{R})}+q\|V(\cdot,t)\|_{L^1_{\omega_1}(\mathbb{R})}\Big)-\Big(p\mu+pf_1(\gamma)+q\beta_2 e^{-\gamma c\tau_2}\Big)\int_0^t\|U(\cdot,s)\|_{L^1_{\omega_1}(\mathbb{R})}ds$$
$$-\Big(q\mu+qf_2(\gamma)+p\beta_1 e^{-\gamma c\tau_1}\Big)\int_0^t\|V(\cdot,s)\|_{L^1_{\omega_1}(\mathbb{R})}ds$$
$$\leq pC_1+qC_2, \tag{3.6}$$

where $f_i(\gamma)=-c\gamma+d_i\int_{\mathbb{R}}J_i(y)e^{-\gamma y}dy-(d_i-\alpha_i)$, $i=1,2$. By taking $\mu>0$ small enough, it follows that

$$-\Big(p\mu+pf_1(\gamma)+q\beta_2 e^{-\gamma c\tau_2}\Big)>0 \text{ and } -\Big(q\mu+qf_2(\gamma)+p\beta_1 e^{-\gamma c\tau_1}\Big)>0.$$

Then we establish the key energy estimate

$$\|U(\cdot,t)\|_{L^1_{\omega_1}(\mathbb{R})}+\|V(\cdot,t)\|_{L^1_{\omega_1}(\mathbb{R})}+\int_0^t e^{\mu(s-t)}\Big(\|U(s)\|_{L^1_{\omega_1}(\mathbb{R})}+\|V(s)\|_{L^1_{\omega_1}(\mathbb{R})}\Big)ds\leq Ce^{-\mu t}.$$

This completes the proof. □

Using the $L^1_{\omega_1}$–estimate of Lemma 3.1, we further have the following L^1–estimate.

Lemma 3.2. *Assume that* (J) *and* (A1)–(A2) *hold, in addition,* (A3) *holds. For any $c>c_*$, there exist positive constants μ, ξ_0 and C such that*

$$e^{\mu t}(\|U(\cdot,t)\|_{L^1(\mathbb{R})}+\|V(\cdot,t)\|_{L^1(\mathbb{R})})\leq C \ \textit{ for all } \ t\geq 0.$$

Proof. Multiplying the inequalities (3.3) by $e^{\mu t}$ and integrating it over $\mathbb{R}\times[0,t]$, we can obtain

$$
\begin{aligned}
0 \geq & \int_0^t \int_{-\infty}^{\infty} e^{\mu s}\partial_s U(\xi,s) + ce^{\mu s}\partial_\xi U(\xi,s) + e^{\mu s}\Big(- d_1(J_1 * U)(\xi,s) + d_1 U(\xi,s)\Big) \\
& \qquad - e^{\mu s}\Big(h_1(\Phi(\xi))U(\xi,s) + h_2(\Phi(\xi))V(\xi - c\tau_1, s-\tau_1)\Big) d\xi ds \\
= & \int_0^t \int_{-\infty}^{\infty} (e^{\mu s}U(\xi,s))_s - \mu e^{\mu s}U(\xi,s) + (ce^{\mu s}U(\xi,s))_\xi + e^{\mu s}\Big(- d_1(J_1 * U)(\xi,s) \\
& \qquad + d_1 U(\xi,s) - h_1(\Phi(\xi))U(\xi,s) - h_2(\Phi(\xi))V(\xi - c\tau_1, s-\tau_1)\Big) d\xi ds \\
= & \, e^{\mu t}\|U(\cdot,t)\|_{L^1(\mathbb{R})} - \|U(0)\|_{L^1(\mathbb{R})} + \int_0^t \int_{-\infty}^{\infty} e^{\mu s}\Big(- \mu - h_1(\Phi(\xi))\Big)U(\xi,s) d\xi ds \\
& - \int_0^t \int_{-\infty}^{\infty} e^{\mu s} h_2(\Phi(\xi))V(\xi - c\tau_1, s-\tau_1) d\xi ds \\
\geq & \, e^{\mu t}\|U(\cdot,t)\|_{L^1(\mathbb{R})} - \|U(0)\|_{L^1(\mathbb{R})} + \int_0^t \int_{-\infty}^{\infty} e^{\mu s}\Big(- \mu - h_1(\Phi(\xi))\Big)U(\xi,s) d\xi ds \\
& - \int_0^t \int_{-\infty}^{\infty} e^{\mu(s+\tau_1)} h_2(\Phi(\xi + c\tau_1))V(\xi,s) d\xi ds - \int_{-\tau_1}^0 \int_{-\infty}^{\infty} e^{\mu(s+\tau_1)} h_2(\Phi(\xi + c\tau_1))V(\xi,s) d\xi ds \\
= & \, e^{\mu t}\|U(\cdot,t)\|_{L^1(\mathbb{R})} - \|U(0)\|_{L^1(\mathbb{R})} - \int_{-\tau_1}^0 \int_{-\infty}^{\infty} e^{\mu(s+\tau_1)} h_2(\Phi(\xi + c\tau_1))V(\xi,s) d\xi ds \\
& + \int_0^t \Big(\int_{-\infty}^{\xi_0} + \int_{\xi_0}^{\infty} \Big) e^{\mu s}\Big(\mathcal{Q}_1(\xi)U(\xi,s) + \mathcal{Q}_2(\xi)V(\xi,s)\Big) d\xi ds, \qquad (3.7)
\end{aligned}
$$

where $\mathcal{Q}_1(\xi) := -\mu - h_1(\Phi(\xi))$ and $\mathcal{Q}_2(\xi) := -e^{\mu\tau_1}h_2(\Phi(\xi + c\tau_1))$.

Since $\omega_1(\xi) \geq 1$ for $\xi \leq \xi_0$, by Lemma 3.1, we have

$$
\begin{aligned}
& \Big| \int_0^t \int_{-\infty}^{\xi_0} e^{\mu s}\Big(\mathcal{Q}_1(\xi)U(\xi,s) + \mathcal{Q}_2(\xi)V(\xi,s)\Big) d\xi ds \Big| \\
& \leq C_4 \int_0^t e^{\mu s}\Big(\|U(\cdot,s)\|_{L^1_{\omega_1}(-\infty,\xi_0]} + \|V(\cdot,s)\|_{L^1_{\omega_1}(-\infty,\xi_0]}\Big) ds \\
& \leq C_5 \qquad (3.8)
\end{aligned}
$$

for some positive constants C_4 and C_5. Then it follows from (3.7) and (3.8), we have

$$
e^{\mu t}\|U(\cdot,t)\|_{L^1(\mathbb{R})} + \int_0^t \int_{\xi_0}^{\infty} e^{\mu s}\Big(\mathcal{Q}_1(\xi)U(\xi,s) + \mathcal{Q}_2(\xi)V(\xi,s)\Big) d\xi ds \leq C_6 \qquad (3.9)
$$

for some positive constant C_6. Similarly, there exists a constant $C_7 > 0$ such that

$$
e^{\mu t}\|V(\cdot,t)\|_{L^1(\mathbb{R})} + \int_0^t \int_{\xi_0}^{\infty} e^{\mu s}\Big(\mathcal{U}_1(\xi)U(\xi,s) + \mathcal{U}_2(\xi)V(\xi,s)\Big) d\xi ds \leq C_7, \qquad (3.10)
$$

where $\mathcal{U}_1(\xi) := -e^{\mu\tau_2} g_1(\Psi(\xi + c\tau_2))$ and $\mathcal{U}_2(\xi) := -\mu - g_2(\Psi(\xi))$.

Summing (3.9) and (3.10), there exists a constant $C > 0$ such that

$$\begin{aligned} & e^{\mu t}(\|U(\cdot,t)\|_{L^1(\mathbb{R})} + \|V(\cdot,t)\|_{L^1(\mathbb{R})}) \\ & + \int_0^t \int_{\xi_0}^{+\infty} e^{\mu s}\Big(\Big[\mathcal{Q}_1(\xi) + \mathcal{U}_1(\xi)\Big]U(\xi,s) + \Big[\mathcal{Q}_2(\xi) + \mathcal{U}_2(\xi)\Big]V(\xi,s)\Big)d\xi ds \\ & \le C. \end{aligned} \tag{3.11}$$

Taking $\mu = 0$, according to the assumption (H4), we see that

$$\lim_{\xi\to+\infty}(\mathcal{Q}_1(\xi) + \mathcal{U}_1(\xi)) = -h_1(k_1,k_2) - g_1(k_1,k_2) = -\bar{\alpha}_1 - \bar{\beta}_2 > 0$$

and

$$\lim_{\xi\to+\infty}(\mathcal{Q}_2(\xi) + \mathcal{U}_2(\xi)) = -h_2(k_1,k_2) - g_2(k_1,k_2) = -\bar{\alpha}_2 - \bar{\beta}_1 > 0.$$

Then choosing $\xi_0 > 0$ large enough and $\mu > 0$ small enough, for $\xi > \xi_0$, we have

$$e^{\mu t}\Big(\|U(\cdot,t)\|_{L^1(\mathbb{R})} + \|V(\cdot,t)\|_{L^1(\mathbb{R})}\Big) \le C \ \text{ for all } \ t \ge 0.$$

This completes the proof. □

3.2 L^2 –energy estimate

Now we begin to establish the following L^2–energy estimate.

Lemma 3.3. *Assume that* (J) *and* (H1)–(H4) *hold. For any $c > c_*$, there exist positive constants ξ_0 and C such that for $t \ge 0$, we have*

$$\|U(\cdot,t)\|^2_{L^2(\mathbb{R})} + \|V(\cdot,t)\|^2_{L^2(\mathbb{R})} \le C.$$

Proof. Multiplying the inequalities (3.3) by $U(\xi,t)$ and $V(\xi,s)$, respectively, and integrating them over $\mathbb{R}\times[0,t]$, we can obtain

$$\begin{aligned} 0 \ge & \int_0^t \int_{-\infty}^{+\infty} \Big\{(U^2(\xi,s))_s + c(U^2(\xi,s))_\xi - 2d_1\int_{\mathbb{R}} J_1(y)U(\xi-y,s)U(\xi,s)dy + 2d_1U^2(\xi,s) \\ & - 2U(\xi,s)\Big(h_1(\Phi(\xi))U(\xi,s) + h_2(\Phi(\xi))V(\xi-c\tau_1,s-\tau_1)\Big)\Big\}d\xi ds \\ \ge & \int_0^t \int_{-\infty}^{+\infty} \Big\{(U^2(\xi,s))_s + c(U^2(\xi,s))_\xi + \int_{\mathbb{R}} -d_1J_1(y)U^2(\xi-y,s)dy - d_1\int_{\mathbb{R}} J_1(y)U^2(\xi,s)dy \\ & + 2d_1U^2(\xi,s) - 2h_1(\Phi(\xi))U^2(\xi,s) - h_2(\Phi(\xi))U^2(\xi,s) - h_2(\Phi(\xi))V^2(\xi-c\tau_1,s-\tau_1)\Big\}d\xi ds \\ \ge & \|U(\cdot,t)\|^2_{L^2(\mathbb{R})} - \|U(0)\|^2_{L^2(\mathbb{R})} - \int_{-\tau_1}^0 \int_{-\infty}^{+\infty} h_2(\Phi(\xi+c\tau_1))V^2(\xi,s)d\xi ds \end{aligned}$$

$$+\int_0^t\int_{-\infty}^{+\infty}\Big((-2h_1(\Phi(\xi))-h_2(\Phi(\xi)))U^2(\xi,s)+(-h_2(\Phi(\xi+c\tau_1)))V^2(\xi,s)\Big)d\xi ds$$

$$=\|U(\cdot,t)\|^2_{L^2(\mathbb{R})}-\|U(0)\|^2_{L^2(\mathbb{R})}-\int_{-\tau_1}^0\int_{-\infty}^{+\infty}h_2(\Phi(\xi+c\tau_1))V^2(\xi,s)d\xi ds$$

$$+\int_0^t\Big(\int_{-\infty}^{\xi_0}+\int_{\xi_0}^{+\infty}\Big)\Big(\mathcal{F}_1(\xi)U^2(\xi,s)+\mathcal{F}_2(\xi)V^2(\xi,s)\Big)d\xi ds, \tag{3.12}$$

where $\mathcal{F}_1(\xi):=-2h_1(\Phi(\xi))-h_2(\Phi(\xi))$ and $\mathcal{F}_2(\xi):=-h_2(\Phi(\xi+c\tau_1))$.

Since $\omega_1(\xi)\geq 1$ for $\xi\leq\xi_0$ and $0\leq U(\xi,t)\leq k_1$, Lemma 3.1 can guarantee that

$$\int_{-\infty}^{\xi_0}U^2(\xi,t)d\xi\leq k_1\int_{-\infty}^{\xi_0}\omega_1U(\xi,t)d\xi\leq k_1\|U(\cdot,t)\|_{L^1_{\omega_1}(\mathbb{R})}\leq Ce^{-\mu t}\quad\text{for } t>0.$$

Similarly, it yields

$$\int_{-\infty}^{\xi_0}V^2(\xi,t)d\xi\leq Ce^{-\mu t}\quad\text{for } t>0.$$

Then,

$$\begin{aligned}&\Big|\int_0^t\int_{-\infty}^{\xi_0}\Big(\mathcal{F}_1(\xi)U^2(\xi,s)+\mathcal{F}_2(\xi)V^2(\xi,s)\Big)d\xi ds\Big|\\&\leq C_8\int_0^t\int_{-\infty}^{\xi_0}\Big(U^2(\xi,s)+V^2(\xi,s)\Big)d\xi ds\leq C_9,\end{aligned}$$

where C_8 and C_9 are positive constants. Thus, it holds

$$\|U(\cdot,t)\|^2_{L^2(\mathbb{R})}+\int_0^t\int_{\xi_0}^{+\infty}\Big(\mathcal{F}_1(\xi)U^2(\xi,s)+\mathcal{F}_2(\xi)V^2(\xi,s)\Big)d\xi ds\leq C_{10} \tag{3.13}$$

for some positive constant C_{10}.

Similarly, there exists a constant $C_{11}>0$ such that

$$\|V(\cdot,t)\|^2_{L^2(\mathbb{R})}+\int_0^t\int_{\xi_0}^{+\infty}\Big(\mathcal{R}_1(\xi)U^2(\xi,s)+\mathcal{R}_2(\xi)V^2(\xi,s)\Big)d\xi ds\leq C_{11}, \tag{3.14}$$

where $\mathcal{R}_1(\xi):=-g_1(\Psi(\xi+c\tau_2))$ and $\mathcal{R}_2(\xi):=-g_1(\Psi(\xi))-2g_2(\Psi(\xi))$.

Summing (3.13) and (3.14), it follows

$$\begin{aligned}&\|U(\cdot,t)\|^2_{L^2(\mathbb{R})}+\|V(\cdot,t)\|^2_{L^2(\mathbb{R})}\\&+\int_0^t\int_{\xi_0}^{+\infty}\Big\{\Big(\mathcal{F}_1(\xi)+\mathcal{R}_1(\xi)\Big)U^2(\xi,s)+\Big(\mathcal{F}_2(\xi)+\mathcal{R}_2(\xi)\Big)V^2(\xi,s)\Big\}d\xi ds\leq C.\end{aligned} \tag{3.15}$$

According to (H4), it yields

$$\lim_{\xi\to+\infty}\Big(\mathcal{F}_1(\xi)+\mathcal{R}_1(\xi)\Big)=-2h_1(k_1,k_2)-h_2(k_1,k_2)-g_1(k_1,k_2)=-2\bar{\alpha}_1-\bar{\beta}_1-\bar{\beta}_2>0$$

and

$$\lim_{\xi\to+\infty}\Big(\mathcal{F}_2(\xi)+\mathcal{R}_2(\xi)\Big) = -h_2(k_1,k_2)-g_1(k_1,k_2)-2g_2(k_1,k_2) = -2\bar{\alpha}_2-\bar{\beta}_1-\bar{\beta}_2>0.$$

Thus for $\xi_0>0$ large enough and $\xi\ge\xi_0$, it holds

$$\mathcal{F}_1(\xi)+\mathcal{R}_1(\xi)>0 \text{ and } \mathcal{F}_2(\xi)+\mathcal{R}_2(\xi)>0.$$

By (3.15), it holds

$$\|U(\cdot,t)\|^2_{L^2(\mathbb{R})}+\|V(\cdot,t)\|^2_{L^2(\mathbb{R})}\le C.$$

Then the assertion of this lemma follows. This completes the proof. □

In order to derive a L^2–energy estimate for $(U_\xi(\xi,t),V_\xi(\xi,t))$, we differentiate the system (3.2) with respect to ξ, we can obtain

$$\begin{cases}\partial_{t\xi}U(\xi,t)+c\partial_{\xi\xi}U(\xi,t)-d_1\Big((J_1*U_\xi)(\xi,t)-U_\xi(\xi,t)\Big)-\nabla h(X(\xi,t)+\Phi(\xi))X_\xi(\xi,t)\\ =\Big(\nabla h(X(\xi,t)+\Phi(\xi))-\nabla h(\Phi(\xi))\Big)\Phi'(\xi):=\mathcal{H}_h(\xi,t)\le 0,\\ \partial_{t\xi}V(\xi,t)+c\partial_{\xi\xi}V(\xi,t)-d_2\Big((J_2*V_\xi)(\xi,t)-V_\xi(\xi,t)\Big)-\nabla g(Y(\xi,t)+\Psi(\xi))Y_\xi(\xi,t)\\ =\Big(\nabla g(Y(\xi,t)+\Psi(\xi))-\nabla g(\Psi(\xi))\Big)\Psi'(\xi):=\mathcal{H}_g(\xi,t)\le 0.\end{cases} \tag{3.16}$$

Similar to the process of Lemmas 3.1–3.3, we can obtain the following result.

Lemma 3.4. *Assume that* (J) *and* (H1)–(H4) *hold. For any $c>c_*$, there exist positive constants ξ_0 and C such that for $t\ge0$, it holds*

$$\|U_\xi(\cdot,t)\|^2_{L^2(\mathbb{R})}+\|V_\xi(\cdot,t)\|^2_{L^2(\mathbb{R})}\le C.$$

3.3 Proof of Theorem 2.2

Lemma 3.5. *Assume that* (J) *and* (H1)–(H4) *hold. For any $c>c_*$, it holds*

$$\|U(\cdot,t)\|_{L^\infty(\mathbb{R})}\le M_1e^{-\frac{1}{3}\mu_1 t},$$

$$\|V(\cdot,t)\|_{L^\infty(\mathbb{R})}\le M_2e^{-\frac{1}{3}\mu_2 t}$$

for some positive constants μ_1, μ_2, M_1, M_2 and $t>0$.

Proof. It is easily checked that

$$\begin{aligned}\|U(\cdot,t)\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} U^2(\cdot,t)d\xi \le \sup_{\xi\in\mathbb{R}} |U(\xi,t)| \int_{\mathbb{R}} |U(\cdot,t)|d\xi \\ &= \|U(\cdot,t)\|_{L^\infty(\mathbb{R})} \cdot \|U(\cdot,t)\|_{L^1(\mathbb{R})} \end{aligned} \tag{3.17}$$

for any $t \ge 0$. Since $U(\cdot,t) \in H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$, by Hölder inequality, we have

$$\begin{aligned} U^2(\cdot,t) &= 2\int_{-\infty}^{\xi} U_\xi(\cdot,t)U(\cdot,t)d\xi \le 2\Big(\int_{-\infty}^{\xi} |U_\xi(\cdot,t)|^2 d\xi\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{\xi} |U(\cdot,t)|^2 d\xi\Big)^{\frac{1}{2}} \\ &\le 2\|U(\cdot,t)\|_{L^2(-\infty,\xi)} \cdot \|U_\xi(\cdot,t)\|_{L^2(-\infty,\xi)} \end{aligned}$$

for any $\xi \in \mathbb{R}$ and $t > 0$. Then it follows that

$$\|U(\cdot,t)\|^2_{L^\infty(\mathbb{R})} \le 2\|U(\cdot,t)\|_{L^2(\mathbb{R})} \cdot \|U_\xi(\cdot,t)\|_{L^2(\mathbb{R})},\ \forall t > 0. \tag{3.18}$$

Combining (3.17) and (3.18), we have

$$\|U(\cdot,t)\|_{L^\infty(\mathbb{R})} \le 2^{\frac{2}{3}} \|U(\cdot,t)\|^{\frac{1}{3}}_{L^1(\mathbb{R})} \cdot \|U_\xi(\cdot,t)\|^{\frac{2}{3}}_{L^2(\mathbb{R})},\ \forall t > 0.$$

According to Lemmas 3.2 and 3.4, there exist positive constants μ_1 and M_1 such that

$$\|U(\cdot,t)\|_{L^\infty(\mathbb{R})} \le M_1 e^{-\frac{1}{3}\mu_1 t},\ \forall t > 0.$$

Similarly, there exist $\mu_2 > 0$ and $M_2 > 0$ such that

$$\|V(\cdot,t)\|_{L^\infty(\mathbb{R})} \le M_2 e^{-\frac{1}{3}\mu_2 t},\ \forall t > 0.$$

This completes the proof . □

Proof of Theorem 2.2 By Lemma 3.5, it is easily see that

$$\sup_{x\in\mathbb{R}} |u_i^+(x,t) - \phi_i(x+ct)| \le Ce^{-\mu t}\ (i=1,2),\ \forall\ t \ge 0.$$

Similarly, we can verify that for any $c > c_*$, it holds

$$\sup_{x\in\mathbb{R}} |u_i^-(x,t) - \phi_i(x+ct)| \le Ce^{-\mu t}\ (i=1,2),\ \forall\ t \ge 0.$$

Since

$$0 \le u_i^-(x,t) \le u_i(x,t), \phi_i(x+ct) \le u_i^+(x,t) \le k_i\ (i=1,2),$$

the squeezing argument implies that

$$\sup_{x\in\mathbb{R}} |u_i(x,t) - \phi_i(x+ct)| \le Ce^{-\mu t}\ (i=1,2),\ \forall\ t \ge 0.$$

This completes the proof of Theorem 2.2. □

4 An application

In this section, we give an application as follow. If $h(u,v) = -\alpha_1 u + h(v)$ and $g(u,v) = -\alpha_2 v + g(u)$, then (1.2) reduces to the traditional reaction diffusion systems:

$$\begin{cases} u_t(x,t) = d_1[(J_1 * u)(x,t) - u(x,t)] - \alpha_1 u(x,t) + h(v(x,t-\tau_1)), \\ v_t(x,t) = d_2[(J_2 * v)(x,t) - v(x,t)] - \alpha_2 v(x,t) + g(u(x,t-\tau_2)), \end{cases} \quad x \in \mathbb{R},\ t > 0, \tag{4.1}$$

where $u(x,t)$ and $v(x,t)$ represent the spatial concentration of the bacteria and the infective population at a point $x \in \mathbb{R}$ and time $t \geq 0$, respectively. $d_1 \geq 0$ and $d_2 \geq 0$ are diffusion coefficients. $(J_1 * u)(x,t)$ and $(J_2 * v)(x,t)$ represent the total number of the bacteria and the infective population arriving at x from all possible locations y at time t, respectively. $-\alpha_1 u$ is the natural death rate of the bacterial population and the nonlinearity $h(v)$ is the contribution of the infective humans to the growth rate of the bacterial. $-\alpha_2 v$ is the natural diminishing rate of the infective population due to the finite mean duration of the infectious population and $g(u)$ is the infection rate of the human population under the assumption that the total susceptible human population is constant during the evolution of the epidemic.

Before applying the results of stability, we give the following assumptions.

(J) $J_i \in C(\mathbb{R})$, $J_i(x) = J_i(-x) \geq 0$, $\int_{\mathbb{R}} J_i(y)dy = 1$, $i = 1, 2$, and $\int_{\mathbb{R}} |x|^j J_i(x) e^{-\lambda x} dx < \infty$ for every $\lambda > 0$, $j = 0, 1, 2$, $i = 1, 2$. Furthermore, J_1 and J_2 are compactly supported.

(A1) $h(u)$ and $g(u)$ are nondecreasing on $(0, +\infty)$ and $h''(u) < 0$ and $g''(u) < 0$ for all $u \in (0, +\infty)$.

(A2) $h, g \in C^2(\mathbb{R}^+, \mathbb{R}^+)$, $h(0) = g(0) = 0$, $k_2 = g(k_1)/\alpha_2$, $h(g(k_1)/\alpha_2) = \alpha_1 k_1$ and $h(g(u)/\alpha_2) > \alpha_1 u$ for $u \in (0, k_1)$, where k_1 is a positive constant.

(A3) $\min\{\alpha_1,\ \alpha_2\} > \max\{\bar{\beta}_1,\ \bar{\beta}_2\}$.

The existence result can be found in [21] without delays, the existence of system 4.1 can be obtained similarly, here we just review the result of existence.

Theorem 4.1 (Existence)**.** *Assume* (J) *and* (A1)–(A2) *hold. There exists a positive constant* $c_* > 0$ *such that* (4.1) *admits a positive traveling wave front* $(\phi_1(x+ct), \phi_2(x+ct))$ *with the wave speed* $c \geq c_*$ *and satisfying*

$$\lim_{\xi \to -\infty} (\phi_1(\xi), \phi_2(\xi)) = \mathbf{0} \ \ \textit{and} \ \ \lim_{\xi \to +\infty} (\phi_1(\xi), \phi_2(\xi)) = \mathbf{K}. \tag{4.2}$$

For $0 < c < c_*$, *the system* (4.1) *has no positive monotone traveling wave solution satisfying* (4.2).

In order to apply the stability result, we first give a description of the hypothetical conditions.

It is obvious that the assumption (A1) is equivalent to the assumptions (H1) and (H3). From (A2), it is easy to know that $\beta_1\beta_2 > \alpha_1\alpha_2$ holds. And assumption (A3) implies that $-\alpha_1 + \bar{\beta}_2 < 0$, $-\alpha_2 + \bar{\beta}_1 < 0$, $-2\alpha_1 + \bar{\beta}_1 + \bar{\beta}_2 < 0$ and $-2\alpha_2 + \bar{\beta}_1 + \bar{\beta}_2 < 0$ hold, in other words, the assumptions (A2) and (A3) are equivalent to the assumptions (H2) and (H4).

Similar to the process of Lemmas 3.1–3.3, we can also obtain the following results.

Lemma 4.1. *Assume that* (J) *and* (A1)–(A3) *hold. For any $c > c_*$, there exist positive constants ξ_0 and C such that for $t \geq 0$, it holds*

$$\|U(\cdot,t)\|^2_{L^2(\mathbb{R})} + \|V(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq C$$

and

$$\|U_\xi(\cdot,t)\|^2_{L^2(\mathbb{R})} + \|V_\xi(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq C.$$

Then, by applying the techniques of weighted energy method, comparison principle and the squeezing argument, we can get

$$\sup_{x\in\mathbb{R}} |u(x,t) - \phi_1(x+ct)| \leq Ce^{-\mu t} \quad \text{and} \quad \sup_{x\in\mathbb{R}} |v(x,t) - \phi_2(x+ct)| \leq Ce^{-\mu t}, \quad \forall\, t \geq 0.$$

This means that all solutions of the Cauchy problem for the considered systems 4.1 converge exponentially to traveling wave solutions provided that the initial perturbations around the traveling wave fronts belong to a suitable weighted Sobolev space. This completes the explain of application. □

ACKNOWLEDGMENTS

Shengqiang Zhang is partially supported by Natural Science Foundation of China (No. 12301635) and the Introduction of Talent Research Initiation Project (No. YJ2023-23). Yanling Meng is partially supported by the Introduction of Talent Research Initiation Project (No. YJ2022-64). Zhixian Yu is partially supported by Natural Science Foundation of China (No. 12071297).

References

[1] J. Anderson, S. Zbarsky, Stability and instability of traveling wave solutions to nonlinear wave equations, *Int. Math. Res. Notices*, **2023(1)** (2023), 95-184.

[2] P. W. Bates, P. C. Fife, X. Ren and X. Wang, Traveling waves in a convolution model for phase transition, *Arch. Rational Mech. Anal.*, **138** (1997), 105-136.

[3] J. Carr and A. Chmaj, Uniqueness of travelling waves for nonlocal monostable equations, *Proc. Amer. Math. Soc.*, **132** (2004), 2433-2439.

[4] J. Coville, On uniqueness and monotonicity of solutions of nonlocal reaction diffusion equation, *Ann. Mat. Pura Appl.*, **185** (2006), 461-485.

[5] J. Coville and L. Dupaigne, Propagation speed of travelling fronts in nonlocal reaction-diffusion equation, *Nonlinear Anal. TMA*, **60** (2005), 797-819.

[6] J. Coville and L. Dupaigne, On a non-local eqution arising in population dynamics, *Proc. Roy. Soc. Edinburgh,* **137A** (2007), 727-755.

[7] X. Chen, Existence, uniqueness and asymptotic stability of traveling waves in non–local evolution equations, *Adv. Differ. Equ.,* **2** (1997), 125-160.

[8] V. Capasso, K. Kunisch, A reaction-diffusion system arising in modelling man-environment diseases, *J. Quart. Appl. Math.*, **46** (1988), 431-450.

[9] V. Capasso, S. L. Paveri-Fontana, A mathematical model for the 1973 cholera epidemic in the European Mediterranean region, *Rev. dEpidemiol. Sante Publique*, **27** (1979), 121-132.

[10] V. Capasso, R. E. Wilson, Analysis of reaction-diffusion system modeling man-environment-man epidemics, *M. Society for Industrial and Applied Mathematics*, **57** (1997), 327-346.

[11] Y. Guo, S.S. Ge, A. Arbi, Stability of traveling waves solutions for nonlinear cellular neural networks with distributed delays, *J. Syst. Sci. Complex.*, **35(1)** (2022), 18-31.

[12] C.H. Hsu, T.S. Yang, Existence, uniqueness, monotonicity and asymptotic behaviour of travelling waves for epidemic models, *Nonlinearity*, **26** (2013), 121-139.

[13] C.H. Hsu, T.S. Yang, Z.X. Yu, Existence and exponential stability of traveling waves for delayed reaction-diffusion systems, *Nonlinearity*, **31** (2018), 838-863.

[14] R. Huang, M. Mei, Y. Wang, Planar traveling waves for nonlocal dispersal equation with monostable nonlinearity, *Discrete Contin. Dyn. Syst.-Series A*, **32** (2012), 3621-3649.

[15] C.T. Lee, M.F. Hoopes, J. Diehl, W. Gilliland, G. Huxel, E.V. Leaver, K. Mc-Cann, J. Umbanhowar, A. Mogilner, Non-local concepts in models in biology, *J. Theor. Biol.*, **210** (2001), 201-219.

[16] K.P. Leisman, J.C. Bronski, M.A. Johnson, et al. Stability of Traveling Wave Solutions of Nonlinear Dispersive Equations of NLS Type, *Arch. Rational Mech. Anal.*, **240** (2021), 927-969.

[17] C.K. Lin, M. Mei, On traveling wavefronts of the Nicholsons blowflies equation with diffusion, *Proc. R. Soc. Edinburgh A*, **140** (2010), 135-152.

[18] G.Y. Lv, M.X. Wang, Nonlinear stability of traveling wave fronts for delayed reaction diffusion systems, *Nonlinear Anal. RWA*, **13** (2012), 1854-1865.

[19] G.Y. Lv, M.X. Wang, Nonlinear stability of traveling wave fronts for nonlocal delayed reaction-diffusion equations, *J. Math. Anal. Appl.*, **385** (2012), 1094-1106.

[20] W.T. Li, W.B. Xu and L. Zhang, Traveling waves and entire solutions for an epidemic model with asymmetric dispersal, *Discrete Contin. Dyn. Syst.-Series A,* **37** (2017), 2483-2512.

[21] W.T. Li, W.B. Xu, L. Zhang, Traveling waves and entire solutions for an epidemic model with asymmetric dispersal, *Discrete Contin. Dyn. Syst.,* **37** (2017), 2483-2512.

[22] W.T. Li, L. Zhang and G. Zhang, Invasion entire solutions in a competition system with nonlocal dispersal, *Nonlinear Anal.,* **35** (2015), 1531-1560.

[23] M. Ma, W. Meng, C. Ou, Impact of nonlocal dispersal and time periodicity on the global exponential stability of bistable traveling waves, *Stud. Appl. Math.*, **150(3)** (2023), 818-840.

[24] J. Medlock, M. Kot, Spreading disease: integro-differential equations old and new, *Math. Biosci.*, **184** (2003), 201-222.

[25] M. Mei, J.W.H. So, Stability of strong traveling waves for a nonlocal time-delayed reaction-diffusion equation, *Proc. R. Soc. Edinb.*, **138** (2008), 551-568.

[26] M. Mei, J. So, M. Li, S. Shen, Asymptotic stability of travelling waves for Nicholson's blowflies equation with diffusion, *Proc. R. Soc. Edinb.*, **134** (2004), 579-594.

[27] M. Mei, C.H. Ou, X.Q. Zhao, Global stability of monostable traveling waves for nonlocal time-delayed reation-diffusion equations, *SIAM J. Math. Anal.*, **42** (2010), 2762-2790.

[28] M. Mei, Y. Wang, Remark on stability of traveling waves for nonlocal Fisher-KPP equations, *Int. J. Numer. Anal. Model. B*, **2** (2011), 379-401.

[29] R.H. Martin, H.L. Smith, Abstract functional-differential equations and reaction d-iffusion systems, *Trans. Amer. Math. Soc.*, **321** (1990), 1-44.

[30] S.W. Ma, X.F. Zou, Existence, uniqueness and stability of traveling waves in a discrete reaction diffusion monostable equation with delay, *J. Differ. Equ.*, **217** (2005), 54-87.

[31] Y.L. Meng, Z.X. Yu and C.H. Hsu, Entire solutions for a delayed nonlocal dispersal system with monostable nonlinearities, *Nonlinearity*, **32** (2019), 1206-1236.

[32] A. Okubo, Diffusion and Ecological Problems: Mathematical Models, *Springer-Verlag*, New York, 1980.

[33] S. Pan, Traveling wave fronts of delayed nonlocal diffusion systems without quasi-monotonicity, *J. Math. Anal. Appl.,* **346** (2008), 415-424.

[34] S. Pan, W. Li, G. Lin, Traveling wave fronts in nonlocal delayed reaction-diffusion systems and applications, *Z. Angew. Math. Phys.*, **60** (2009), 377-392.

[35] S. Pan, W. Li, G. Lin, Existence and stability of traveling wavefronts in a nonlocal diffusion equation with delay, *Nonlinear Anal. TMA*, **72** (2009), 3150-3158.

[36] S.X. Pan, G. Lin, Invasion traveling wave solutions of a competitive systems with dispersal, *Bound. Value Probl.* **2012** (2012), 120-130.

[37] D.H. Sattinger, On the stability of waves of nonlinear parabolic systems, *Adv. Math.*, **22** (1976), 312-355.

[38] N. Shigesada, K. Kawasaki, Biological Invasions: Theory and Practice, *Oxford University Press*, New York, 1997.

[39] Y. Sun, W.T. Li, Z.C. Wang, Traveling waves for a nonlocal anisotropic dispersal equation with monostable nonlinearity, *Nonlinear Anal. TMA*, **74** (2011), 814-826.

[40] H.L. Smith, X.Q. Zhao, Global asymptotic stability of travelling waves in delayed reaction-diffusion equations, *SIAM J. Math. Anal.*, **31** (2000), 514-534.

[41] K. Schumacher, Travelling-front solutions for integro-differential equations. I. *J. Reine Angew. Math.* **316** (1980), 54-70.

[42] H.R. Thieme, Asymptotic estimates of the solutions of nonlinear integral equations and asymptotic speeds for the spread of populations, *J. Reine Angew. Math.*, **306** (1979), 94-121.

[43] A.I. Volpert, V.A. Volpert, V.A. Volpert, Travelling wave solutions of parabolic systems, Translations ofmathematical monographs, Vol.**104**. Providence (RI): *American Mathematical Society*; (1994).

[44] S.L. Wu, W.T. Li, S.Y. Liu, Asymptotic stability of traveling wave fronts in nonlocal reaction-diffusion equations with delay. *J. Math. Anal. Appl.*, **360** (2009), 439-458.

[45] X. Wu, Z. Ma, Exponential stability of traveling waves for a nonlocal dispersal SIR model with delay, *Open Math.*, **20(1)** (2022), 1451-1469.

[46] Z.Q. Xu, D.M. Xiao, Regular traveling waves for a nonlocal diffusion equation, *J. Differ. Equ.*, **258** (2015), 191-223.

[47] Z.C. Wang, W.T. Li, S.G. Ruan, Traveling fronts in monostable equations with nonlocal delayed effects, *J. Dyn. Diff. Equ.*, **20** (2008), 573-607.

[48] H. Yagisita, Existence and nonexistence of traveling waves for a nonlocal monostable equation, *Publ. Res. Inst. Math. Sci.*, **45** (2009), 925-953.

[49] Y.R. Yang, W.T. Li, S.L. Wu, Stability of traveling waves in a monostable delayed system without quasi-monotonicity, *Nonlinear Anal. RWA.*, **14** (2013), 1511-1526.

[50] Y.R. Yang, W.T. Li, S.L. Wu, Exponential stability of traveling fronts in a diffusion epidemic system with delay, *Nonlinear Anal. RWA.,* **12** (2011), 1223-1234.

[51] C. Yang, N. Rodriguez, Existence and stability traveling wave solutions for a system of social outbursts, **494(1)** (2021), *J. Math. Anal. Appl.*, 124583.

[52] Z.X. Yu, M. Mei, Uniqueness and stability of traveling waves for cellular neural networks with multiple delays, *J. Differ. Equ.*, **260** (2016), 241-267.

[53] Z.X. Yu, F. Xu, W.G. Zhang, Stability of invasion traveling waves for a competition system with nonlocal dispersals. *Appl. Anal.*, **96** (2017), 1107-1125.

[54] Z.X. Yu, R. Yuan, Existence and asymptotics of traveling waves for nonlocal diffusion systems, *Chaos, Solitons and Fractals*, **45** (2012), 1361-1367.

[55] Z.X. Yu, R. Yuan, Traveling waves of a nonlocal dispersal delayed age-structured population model, *Japan J. Indust. Appl. Math.*, **30** (2013), 165-184.

[56] Z.X. Yu, Y.J. Wan, C.H. Hsu, Wave propagation and its stability for a class of discrete diffusion systems, *Z. Angew. Math. Phys.*, **194** (2020).

[57] G.B. Zhang, Traveling waves in a nonlocal dispersal population model with age-structure, *Nonlinear Anal. TMA*, **74** (2011), 5030-5047.

[58] H. Zhang, H. Izuhara, Y. Wu, Asymptotic stability of two types of traveling waves for some predator-prey models, *Discrete Cont. Dyn.-B*, **26(4)** (2021).

[59] T. Zhang, W. Li, Y Han, et al. Global exponential stability of bistable traveling waves in a reaction-diffusion system with cubic nonlinearity, *Commun. Pur. Appl. Anal.*, **22(7)** (2023), 2215-2232.