Attractors for the nonclassical diffusion equation with time-dependent memory kernel and critical nonlinearity*

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Abstract

In this paper, we consider the long-time dynamics of solutions for the nonclassical diffusion equation with time-dependent memory kernel when nonlinear term adheres to critical growth, where the time-dependent memory kernel is used to describe the aging process of viscoelastic conductive medium. Under the new theory framework, we first establish the well-posedness and regularity of the solutions, and then we prove the existence and regularity of the time-dependent global attractors in the time-dependent space $H_0^1(\Omega)\times L^2_{\mu_t}(\mathbb{R}^+;H_0^1(\Omega))$ by use of the delicate integral estimation method and decomposition technique.

Keywords: Nonclassical diffusion equation; Time-dependent memory kernel; Time-dependent global attractors; Regularity

MSC: 35B40; 35B41

1. Introduction

Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. The asymptotic dynamics of the following nonclassical diffusion model with time-dependent memory kernel

$$\partial_t u-\Delta\partial_t u-\Delta u-\int_0^\infty k_t(s)\Delta u(t-s)\mathrm{d}s+f(u)=g,\quad (x,t)\in\Omega\times(\tau,+\infty), \tag{1.1}$$

$$u(x,t)|_{\partial\Omega}=0,\qquad t\in(\tau,+\infty),\qquad u(x,\tau)=u_\tau(x,t),\qquad x\in\Omega,\ t\in(-\infty,\tau], \tag{1.2}$$

are investigated in this article.

Suppose that the time-dependent memory kernel function $k_t(s)$ is nonnegative, convex and summable. And let

$$k_t(s)=\int_s^\infty \mu_t(y)\mathrm{d}y,\ \forall s\in\mathbb{R}^+,\ t\in\mathbb{R}.$$

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Evidently,

$$\mu_t(s) = -\partial_s k_t(s).$$

Remark 1.1. *We can construct the the memory kernel function* $\mu_t(s)$ *through the function* μ *and* ε. *Let* $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ *be a nonnegative and nonincreasing function and so* $\int_0^\infty \mu(s)\mathrm{d}s = m$. *And suppose that* $\varepsilon \in C^1(\mathbb{R}; \mathbb{R}^+)$ *satisfying*

$$\varepsilon'(t) \leqslant 0, \quad \forall\, t \in \mathbb{R},$$

and there exist a positive constant M, *such that*

$$\sup_{t\in\mathbb{R}} (\varepsilon(t) + |\varepsilon'(t)|) \leqslant M.$$

So, we can define

$$\mu_t(s) = \frac{1}{\varepsilon^2(t)} \mu(\frac{s}{\varepsilon(t)}).$$

Furthermore, assume that the map

$$(t, s) \mapsto \mu_t(s) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$$

satisfies the following conditions:
(H$_1$) For every fixed $t \in \mathbb{R}$, the map $s \mapsto \mu_t(s)$ is nonnegative, nonincreasing, absolutely continuous and summable. The total mass of μ_t is defined by the formula

$$\kappa(t) = \int_0^\infty \mu_t(s)\mathrm{d}s,$$

and satisfies

$$\inf_{t\in\mathbb{R}} \kappa(t) > 0.$$

(H$_2$) For every $\tau \in \mathbb{R}$, there exists a function $K_\tau : [\tau, +\infty) \to \mathbb{R}^+$ which is continuous and summable on any interval $[\tau, T]$, such that

$$\mu_t(s) \leqslant K_\tau(t)\mu_\tau(s), \text{ for every } t \geqslant \tau \text{ and every } s > 0.$$

(H$_3$) For almost every fixed $s > 0$, the map $t \mapsto \mu_t(s)$ is differentiable for all $t \in \mathbb{R}$. Besides,

$$(t, s) \mapsto \mu_t(s) \in L^\infty(\mathcal{K}) \text{ and } (t, s) \mapsto \partial_t\mu_t(s) \in L^\infty(\mathcal{K})$$

for every compact set $\mathcal{K} \subset \mathbb{R} \times \mathbb{R}^+$.
(H$_4$) There exists a constant $\delta \leqslant \frac{2M}{m}$, such that

$$\partial_t\mu_t(s) + \partial_s\mu_t(s) + \delta\kappa(t)\mu_t(s) \leqslant 0, \text{ for every } t \in \mathbb{R}^+ \text{ and almost every } s > 0.$$

Remark 1.2. *Obviously, the memory kernel function* $\mu_t(s)$ *in Remark 1.1 satisfies* (H$_1$)-(H$_4$), *with* $K_\tau(t) = \frac{\varepsilon^2(\tau)}{\varepsilon^2(t)}$.

Remark 1.3. *Hereafter, we also give a classical and physically relevant example that be gained by setting*

$$\mu(s) = \mathrm{e}^{-\rho t} \text{ and } k(s) = \frac{1}{\rho}\mathrm{e}^{-\rho s}, \forall\, s \in \mathbb{R}^+,$$

where ρ *is a positive constant. In such the case,*

$$\varepsilon(t) = \frac{\pi}{4} - \frac{1}{2}\arctan t,\ \ \mu_t(s) = \frac{1}{\varepsilon^2}\mathrm{e}^{-\frac{\rho}{\varepsilon(t)}s} \text{ and } k_t(s) = \frac{1}{\rho\varepsilon(t)}\mathrm{e}^{-\frac{\rho}{\varepsilon(t)}s}, \forall\, s \in \mathbb{R}^+, \forall\, t \in \mathbb{R}.$$

Then, the memory kernel function $\mu_t(s)$ *satisfies* (H_1)-(H_4), *with* $K_\tau(t) = \frac{\varepsilon^2(\tau)}{\varepsilon^2(t)}$.

About that forcing term, assume that $g \in L^2(\Omega)$. And let the nonlinearity $f \in C^1(\mathbb{R})$ with $f(0) = 0$ and satisfy

$$|f'(u)| \leqslant C(1 + |u|^{p-1}), \quad f'(u) \geqslant -C_1,\ \forall u \in \mathbb{R}, \tag{1.3}$$

where $1 \leqslant p \leqslant 5$, $C_1 \geqslant 0$, C is a positive constant. Besides, assume that f satisfies the dissipation condition

$$\liminf_{|u|\to\infty} f'(u) > -\lambda_1, \tag{1.4}$$

here, $\lambda_1 > 0$ is the first eigenvalue of the strictly positive Dirichlet operator $A = -\Delta$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ on $(L^2(\Omega), \langle\cdot,\cdot\rangle, \|\cdot\|)$. Obviously, (1.4) implies the following relations: for some $0 < \theta < 1$ and a positive constant c_f,

$$\langle F(u), 1\rangle \geqslant -\frac{1}{2}(1-\theta)\|u\|_1^2 - c_f, \tag{1.5}$$

$$\langle f(u), u\rangle \geqslant \langle F(u), 1\rangle - \frac{1}{2}(1-\theta)\|u\|_1^2 - c_f, \tag{1.6}$$

where $F(u) = \int_0^u f(s)\mathrm{d}s$.

The nonclassical diffusion model

$$\partial_t u - \Delta\partial_t u - \Delta u + f(u) = g \tag{1.7}$$

is widely used in the fields of heat conduction theory and fluid mechanics, which describes a heat conduction process or a fluid diffusion process (see [1, 7, 18]). When the conductive medium or fluid is a viscoelastic material, the corresponding model is

$$\partial_t u - \Delta\partial_t u - \Delta u - \int_0^\infty k(s)\Delta u(t-s)\mathrm{d}s + f(u) = g. \tag{1.8}$$

Furthermore, if the viscoelastic conductive medium or fluid also has aging characteristics, then the corresponding model is (1.1). Because of the model of viscoelasticity with memory (Especially the viscoelastic model with time-dependent memory kernel) has profound application background and research prospect, the asymptotic behavior of solutions for the equation has attracted extensive attention and research interest of many scholars (see [4–6, 9–11, 23, 27, 29–31] and relevant references).

For the usual nonclassical diffusion equation (1.7), there are many research results on this model (see [25, 26, 28, 32, 33] and relevant references). In [28], the existence and continuity of the global attractor which are independent of the parameter μ of $\mu\Delta\partial_t u$ are considered and discussed. In [26], for both autonomous and non-autonomous case, the existence and regularity of the attractor is proved when the nonlinearity satisfies the critical exponent growth. Besides, for the nonclassical diffusion equation with fading memory (1.8), we also have achieved a series of research results on this model. In [30, 31], for both autonomous and non-autonomous case, the asymptotic regularity of solutions and the existence of attractor is obtained when the nonlinearity satisfies the critical exponent growth by use of asymptotic prior estimation and decomposition technique.

Compared with the above mentioned model (1.8), when the memory kernel function is dependent on the time variable t, the problem will become more complex and interesting. From the perspective of application, viscoelastic heat conductive medium or viscoelastic fluid has aging characteristics. This has aroused our strong research interest, so the long-term dynamical behaviors of solutions for the viscoelasticity model (1.1) will be studied in our paper. In fact, time-dependent memory kernel can lead to some essential difficulties in analysis. Firstly, there will be difficult in defining auxiliary variable η^t and its derivative function with respect to time. Secondly, the classical estimation methods and differential inequalities for the nonclassical diffusion equation with fading memory (when the memory kernel function is independent of time variable t) are no longer applicable to the study of (1.1). This will add many difficulties to the dissipative estimation and the compactness verification of the solution process. Inspired by the idea in [4, 5, 19, 27], under the new technical framework, we successfully overcome these essential difficulties in the estimation and proof by use of the delicate integral estimation method and decomposition technique. Finally, we establish the well-posedness and regularity of solutions, and then prove the existence and regularity of the time-dependent global attractor.

The structure of this paper is as follows. In Section 2, we introduce some concepts and preliminary results; in Section 3, we achieve the well-posedness and regularity of the solution; in Section 4, we prove the existence and regularity of the time-dependent global attractor corresponding to the problem (1.1), (1.2).

In the following, for brevity C denotes a positive constant.

2. Notations and preliminaries

As in [14], a new variable which denotes the past history of Eq. (1.1) is defined by

$$\eta^t(s) = \begin{cases} \int_0^s u(t-r)\mathrm{d}r, & 0 < s \leqslant t-\tau, \\ \eta_\tau(s-t+\tau) + \int_0^{t-\tau} u(t-r)\mathrm{d}r, & s > t-\tau. \end{cases} \tag{2.1}$$

Using $\mu_t(s) = -\partial_s k_t(s)$ and $k_t(\infty) = 0$, it is easy to see that the system (1.1), (1.2) is equivalent to the system

$$\partial_t u - \Delta\partial_t u - \Delta u - \int_0^\infty \mu_t(s)\Delta\eta^t(s)\mathrm{d}s + f(u) = g \tag{2.2}$$

with the initial-boundary conditions

$$\begin{cases} u(x,t) = 0, & x \in \partial\Omega,\ t > \tau, \\ \eta^t(x,s) = 0, & (x,s) \in \partial\Omega \times \mathbb{R}^+,\ t > \tau, \\ u(x,t) = u_\tau(x,t), & x \in \Omega,\ t \leqslant \tau, \\ \eta^\tau(x,s) = \eta_\tau(x,s), & (x,s) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{2.3}$$

where $u(\cdot)$ satisfies the following condition: there exist two positive constants $\mathcal{R}$ and $\varrho \leqslant \delta$, such that

$$\int_0^\infty \mathrm{e}^{-\varrho s}\|\nabla u(-s)\|^2 \mathrm{d}s \leqslant \mathcal{R},$$

here, the constant δ is defined in the assumption (H_4), and $\|\cdot\|$ denotes the norm of $L^2(\Omega)$.

The following notations as those in Pata and Squassina [21] will be used. Let $A = -\Delta$ with domain $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. For the family of compact nested Hilbert spaces $V_s = D(A^{\frac{s}{2}})$, the inner products and norms in this family of spaces are defined by the formula

$$\langle u, v\rangle_s = \langle A^{\frac{s}{2}} u, A^{\frac{s}{2}} v\rangle, \ \|u\|_s = \|A^{\frac{s}{2}} u\|, \quad \forall s \in \mathbb{R}, \ \ \forall u, v \in D(A^{\frac{s}{2}}),$$

where $\langle \cdot, \ \cdot\rangle$ and $\|\cdot\|$ mean $L^2(\Omega)$ inner product and norm, respectively. Then, $H = L^2(\Omega)$, $V_1 = H_0^1(\Omega)$, $V_2 = H_0^1(\Omega) \cap H^2(\Omega)$.

Obviously, we have compact embedding $D(A^{\frac{s_1}{2}}) \hookrightarrow\hookrightarrow D(A^{\frac{s_2}{2}})$ for any $s_1 > s_2$ and continuous embedding $D(A^{\frac{s}{2}}) \hookrightarrow L^{\frac{2n}{n-2s}}(\Omega)$ for all $s \in [0, \frac{n}{2})$.

For every fixed time t and every $\sigma \in \mathbb{R}$, according to the assumptions about memory kernel $\mu_t(\cdot)$, we denote the family of Hilbert spaces of functions by $L^2_{\mu_t}(\mathbb{R}^+; V_\sigma)$, which are said to be time-dependent memory space

$$M_t^\sigma = L^2_{\mu_t}(\mathbb{R}^+; V_\sigma) = \{\xi^t : \mathbb{R}^+ \to V_\sigma \mid \int_0^\infty \mu_t(s)\|\xi^t(s)\|_\sigma^2 \mathrm{d}s < +\infty\},$$

endowed with the inner products and norms respectively,

$$\langle \eta^t, \xi^t\rangle_{M_t^\sigma} = \int_0^\infty \mu_t(s)\langle \eta^t(s), \xi^t(s)\rangle_\sigma \mathrm{d}s,$$
$$\|\xi^t\|_{M_t^\sigma}^2 = \int_0^\infty \mu_t(s)\|\xi^t(s)\|_\sigma^2 \mathrm{d}s.$$

Now let us introduce the family of Hilbert spaces

$$\mathcal{H}_t^\sigma = V_\sigma \times M_t^\sigma,$$

with the endowed norms

$$\|z\|_{\mathcal{H}_t^\sigma}^2 = \|(u, \eta^t)\|_{\mathcal{H}_t^\sigma}^2 = \|u\|_\sigma^2 + \|\eta^t\|_{M_t^\sigma}^2.$$

In particular, $\mathcal{H}_t = \mathcal{H}_t^0$.

In view of (H_2), for every $\eta^t \in M_\tau^\sigma$ and every $t \geqslant \tau$,

$$\|\eta^t\|_{M_t^\sigma}^2 \leqslant K_\tau(t)\|\eta^t\|_{M_\tau^\sigma}^2, \tag{2.4}$$

therefore, we have continuous embedding

$$M_\tau^\sigma \hookrightarrow M_t^\sigma.$$

The linear operator $\mathbb{T}_t$ acting on M_t^σ is defined by

$$\mathbb{T}_t \eta^t = -\partial_s \eta^t \text{ with domain } D(\mathbb{T}_t) = \{\eta^t \in M_t^\sigma | \partial_s \eta^t \in M_t^\sigma, \eta^t(0) = 0\}.$$

Due to the assumption (H_1), for every fixed t the function $s \mapsto \mu_t(s)$ is differential almost everywhere with $\partial_s \mu_t(s) \leqslant 0$. Similar to [17], we have

$$\langle \mathbb{T}_t \eta^t, \eta^t\rangle_{M_t^\sigma} = \frac{1}{2}\int_0^\infty \partial_s \mu_t(s)\|\eta^t(s)\|_\sigma^2 \mathrm{d}s \leqslant 0, \quad \forall \eta^t \in D(\mathbb{T}_t). \tag{2.5}$$

Evidently, $\mathbb{T}_t$ is a dissipative operator. In fact, it is easy to see that $\mathbb{T}_t$ is the infinitesimal generator of the right-translation semigroup on M_t^σ. Especially,

$$\mathbb{T}_\tau \subset \mathbb{T}_t \tag{2.6}$$

and $\{\mathbb{T}_t\}_{t\geqslant\tau}$ are increasingly nested extensions of each other.

Owing to (2.1), we have

$$\partial_t \eta^t(s) = -\partial_s \eta^t(s) + u(t) = \mathbb{T}_t \eta^t + u(t). \tag{2.7}$$

Details of the proof can be found in Lemma 3.2.

The following abstract results will be used to testify the compactness and dissipativity of the solution corresponding to the problem (2.2), (2.3).

Lemma 2.1. *([13, 24]) Let X, B and Y be three Banach spaces. For $T > 0$, if $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$, and*

$$\mathrm{W} = \{u \in L^p([0,T]; X) | \partial_t u \in L^r([0,T]; Y)\}, \ \textit{with} \ \ r > 1, \ 1 \leqslant p < \infty,$$

$$\mathrm{W}_1 = \{u \in L^\infty([0,T]; X) | \partial_t u \in L^r([0,T]; Y)\}, \ \textit{with} \ \ r > 1.$$

Then,

$$W \hookrightarrow\hookrightarrow L^p([0,T]; B), \ W_1 \hookrightarrow\hookrightarrow C([0,T]; B).$$

Lemma 2.2. *([3, 16, 22]) Assume that $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ is a nonnegative function, and satisfies: if there exists $s_0 \in \mathbb{R}^+$ such that $\mu(s_0) = 0$, then $\mu(s) = 0$ for all $s \geqslant s_0$ holds. Moreover, let B_0, B_1, B_2 be Banach spaces, here B_0, B_1 are reflexive and satisfy*

$$B_0 \hookrightarrow\hookrightarrow B_1 \hookrightarrow B_2.$$

If $\mathcal{C} \subset L^2_\mu(\mathbb{R}^+; B_1)$ satisfies

(i) *$\mathcal{C}$ is bounded in $L^2_\mu(\mathbb{R}; B_0) \cap H^1_\mu(\mathbb{R}^+; B_2)$;*

(ii) *$\sup_{\eta\in\mathcal{C}} \|\eta(s)\|^2_{B_1} \leqslant h(s)$, $\forall s \in \mathbb{R}^+$, $h(s) \in L^1_\mu(\mathbb{R}^+)$,*

then $\mathcal{C}$ is relatively compact in $L^2_\mu(\mathbb{R}^+; B_1)$.

By $\operatorname{dist}_{X_t}(A, B)$ we denote the Hausdorff semidistance from a set $A \subset X_t$ to a set $B \subset X_t$:

$$\operatorname{dist}_{X_t}(A, B) = \sup_{x\in A} \operatorname{dist}_{X_t}(x, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|_{X_t}.$$

Lemma 2.3. *[34] Let (M, d) be a metric space and $U(t, \tau)$ be a Lipschitz continuous dynamical process in M, i.e.,*

$$\mathrm{d}(U(t,\tau)m_1, U(t,\tau)m_2) \leqslant C\mathrm{e}^{K(t-\tau)}\mathrm{d}(m_1, m_2),$$

for appropriate constants C and K which are independent of m_i, τ and t. Assume further that there exist three subsets $M_1, M_2, M_3 \subset M$ such that

$$\mathrm{dist}_M(U(t,\tau)M_1, U(t,\tau)M_2) \leqslant L_1 \mathrm{e}^{-\nu_1(t-\tau)},$$

$$\mathrm{dist}_M(U(t,\tau)M_2, U(t,\tau)M_3) \leqslant L_2 \mathrm{e}^{-\nu_2(t-\tau)},$$

for some $\nu_1, \nu_2 > 0$ and $L_1, L_2 > 0$. Then it follows that

$$\mathrm{dist}_M(U(t,\tau)M_1, U(t,\tau)M_3) \leqslant L \mathrm{e}^{-\nu(t-\tau)},$$

where $\nu = \frac{\nu_1\nu_2}{K+\nu_1+\nu_2}$ and $L = CL_1 + L_2$.

Lemma 2.4. *([5])(Gronwall-type lemma in integral form) Let $\tau \in \mathbb{R}$ be fixed, and also let $\Lambda : [\tau, +\infty) \to \mathbb{R}$ be a continuous function. Suppose that for some $\varepsilon > 0$ and every $\tau \leqslant a < b$, the integral inequality*

$$\Lambda(b) + 2\varepsilon \int_a^b \Lambda(y)\mathrm{d}y \leqslant \Lambda(a) + \int_a^b q_1(y)\Lambda(y)\mathrm{d}y + \int_a^b q_2(y)\mathrm{d}y,$$

holds, where $q_1,\ q_2$ are locally nonnegative functions on $[\tau, +\infty)$ satisfying

$$\int_a^b q_1(y)\mathrm{d}y \leqslant \varepsilon(b-a) + c_1 \quad \textit{and} \ \sup_{t\geqslant\tau} \int_t^{t+1} q_2(y)\mathrm{d}y \leqslant c_2,$$

for some $c_1,\ c_2 \geqslant 0$. Then,

$$\Lambda(t) \leqslant \mathrm{e}^{c_1}(|\Lambda(\tau)|\mathrm{e}^{-\varepsilon(t-\tau)} + \frac{c_2\mathrm{e}^{\varepsilon}}{1-\mathrm{e}^{-\varepsilon}}), \quad \forall t \geqslant \tau.$$

As described in [8, 12, 15, 20], we introduce the following concepts and abstract results about time-dependent dynamical system, which are used to investigate the long-time dynamics of solutions.

Definition 2.5. *Let X_t be a family of normed spaces. A two-parameter family of operators $\{U(t,\tau) : X_\tau \to X_t, \tau \leqslant t, \tau \in \mathbb{R}\}$ is said to be a process, if for any $\tau \in \mathbb{R}$,*

(i) *$U(\tau,\tau) = \mathrm{Id}$ is the identity operator on X_τ;*

(ii) *$U(t,s)U(s,\tau) = U(t,\tau)$, $\forall \tau \leqslant s \leqslant t$.*

Assume that X_t is a family of normed spaces. For every $t \in \mathbb{R}$, the R-ball of X_t is defined by:

$$\mathbb{B}_t(R) = \{z \in X_t | \|z\|_{X_t} \leqslant R\}.$$

Definition 2.6. *A family $\mathfrak{C} = \{C_t\}_{t\in\mathbb{R}}$ of bounded sets $C_t \subset X_t$ is called uniformly bounded, if there exists a constant $R > 0$ such that $C_t \subset \mathbb{B}_t(R)$, $\forall t \in \mathbb{R}$.*

Definition 2.7. *A uniformly bounded family* $\mathfrak{B}_t = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$ *is called a time-dependent absorbing set for the process* $U(t,\tau)$*, if for every* $R > 0$*, there exist a* $t_0 = t_0(R) \leqslant t$ *and* $R_0 > 0$ *such that*

$$\tau \leqslant t - t_0 \Rightarrow U(t,\tau)\mathbb{B}_\tau(R) \subset \mathbb{B}_t(R_0).$$

The process $U(t,\tau)$ is said to be dissipative as it possesses a a time-dependent absorbing set.

Definition 2.8. *The smallest family* $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ *is called a time-dependent attractor for the process* $U(t,\tau)$*, if* $\mathfrak{A}$ *satisfies the following properties:*

(i) *Each* A_t *is compact in* X_t *;*

(ii) $\mathfrak{A}$ *is pullback attracting, that is, it is uniformly bounded, and the limit*

$$\lim_{\tau\to-\infty} \mathrm{dist}_{X_t}(U(t,\tau)C_\tau, A_t) = 0$$

holds for every uniformly bounded family $\mathfrak{C} = \{C_t\}_{t\in\mathbb{R}}$ *and every* $t \in \mathbb{R}$.

Theorem 2.9. *([12, 20]) If* $U(t,\tau)$ *is asymptotically compact, that is, the set*

$$\mathbb{K} = \{\mathfrak{K} = \{K_t\}_{t\in\mathbb{R}} |\ \textit{Each } K_t \textit{ is compact in } X_t,\ \mathfrak{K} \textit{ is pullback attracting }\}$$

is not empty, then the time-dependent attractor $\mathfrak{A}$ *exists and coincides with* $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$*. In particular, it is unique.*

Definition 2.10. *A function* $t \to Z(t)$ *and* $Z(t) \in X_t$ *is a complete bounded trajectories* (CBT) *of the process* $U(t,\tau)$*, if and only if*

(i) $\sup_{t\in\mathbb{R}} \|Z(t)\|_{X_t} < \infty$*;*

(ii) $Z(t) = U(t,\tau)Z(\tau), \forall\, \tau \leqslant t,\ \tau \in \mathbb{R}$.

Definition 2.11. *A time-dependent attractor* $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ *is invariant, if for all* $\tau \leqslant t$*,*

$$U(t,\tau)A_\tau = A_t.$$

Theorem 2.12. *([8, 12, 15]) If the time-dependent attractor* $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ *of the process* $U(t,\tau)$ *is invariant, then it coincides with the set of all* CBT *of the process* $U(t,\tau)$*, that is,*

$$\mathfrak{A} = \{Z | t \to Z(t) \in X_t \textit{ and } Z(t) \textit{ is } \mathrm{CBT} \textit{ of the process } U(t,\tau)\}.$$

3. Well-posedness and regularity of solutions

In order to obtain the dissipative estimation and well posedness of the solution, we need to prove the following preliminary results.

Lemma 3.1. *Let $u \in L^{\infty}([\tau,T];V_\sigma)$ and also let*

$$\Gamma(u,\eta_\tau) = 3(t-\tau)^2\kappa(\tau)\|u\|^2_{L^{\infty}([\tau,T];V_\sigma)} + 2\|\eta_\tau\|^2_{M^\sigma_\tau}.$$

Then, we have that $\eta^t \in M^\sigma_\tau \subset M^\sigma_t$ with

$$\|\eta^t\|^2_{M^\sigma_\tau} \leqslant \Gamma(u,\eta_\tau), \quad \forall\ t \in [\tau,T],$$

and

$$\|\eta^t\|^2_{M^\sigma_t} \leqslant \Gamma(u,\eta_\tau)K_\tau(t) \in L^1([\tau,T]).$$

Proof. Since $\mu_\tau(\cdot)$ is nonincreasing, we can obtain from (2.1)

$$\begin{aligned}
&\|\eta^t\|^2_{M^\sigma_\tau}\\
&= \int_0^{t-\tau}\mu_\tau(s)\|\int_0^s u(t-r)\mathrm{d}r\|^2_\sigma\mathrm{d}s + \int_{t-\tau}^{\infty}\mu_\tau(s)\|\eta_\tau(s-t+\tau) + \int_0^{t-\tau}u(t-r)\mathrm{d}r\|^2_\sigma\mathrm{d}s\\
&\leqslant \int_0^{t-\tau}\mu_\tau(s)s\int_0^s\|u(t-r)\|^2_\sigma\mathrm{d}r\mathrm{d}s + \int_{t-\tau}^{\infty}\mu_\tau(s)\|\eta_\tau(s-t+\tau) + \int_0^{t-\tau}u(t-r)\mathrm{d}r\|^2_\sigma\mathrm{d}s\\
&\leqslant \int_0^{t-\tau}\mu_\tau(s)s^2\|u\|^2_{L^{\infty}([\tau,T];V_\sigma)}\mathrm{d}s + 2\int_0^{\infty}\mu_\tau(s+t-\tau)\|\eta_\tau(s)\|^2_\sigma\mathrm{d}s\\
&\quad + 2(t-\tau)^2\|u\|^2_{L^{\infty}([\tau,T];V_\sigma)}\int_0^{\infty}\mu_\tau(s+t-\tau)\mathrm{d}s\\
&\leqslant 3(t-\tau)^2\|u\|^2_{L^{\infty}([\tau,T];V_\sigma)}\int_0^{\infty}\mu_\tau(s)\mathrm{d}s + 2\int_0^{\infty}\mu_\tau(s+t-\tau)\|\eta_\tau(s)\|^2_\sigma\mathrm{d}s\\
&\leqslant 3(t-\tau)^2\kappa(\tau)\|u\|^2_{L^{\infty}([\tau,T];V_\sigma)} + 2\int_0^{\infty}\mu_\tau(s)\|\eta_\tau(s)\|^2_\sigma\mathrm{d}s\\
&= \Gamma(u,\eta_\tau).
\end{aligned}$$

It follows from (H_2) and (2.4) that the latter inequality also holds. The proof is complete. □

Lemma 3.2. *Let $u \in L^{\infty}([\tau,T];V_\sigma)$. If $\eta_\tau \in D(\mathbb{T}_\tau)$, then $\eta^t \in D(\mathbb{T}_\tau)$, for every $t \in [\tau,T]$, $\eta^t \in W^{1,\infty}([\tau,T];M^\sigma_\tau)$ and the equality*

$$\partial_t\eta^t = \mathbb{T}_\tau\eta^t + u(t)$$

holds in M^σ_τ.

Proof. Differentiating (2.1) with respect s and t in the weak sense, we have

$$\partial_s\eta^t(s) = \begin{cases} u(t-s), & s \leqslant t-\tau,\\ \partial_s\eta_\tau(s-t+\tau), & s > t-\tau, \end{cases} \tag{3.1}$$

$$\partial_t\eta^t(s) = \begin{cases} u(t) - u(t-s), & s \leqslant t-\tau, \\ u(t) - \partial_s\eta_\tau(s-t+\tau), & s > t-\tau. \end{cases} \tag{3.2}$$

And by (2.1), we find that

$$\eta^t(0) = 0.$$

Moreover, since $\mu_\tau(\cdot)$ is nonincreasing and $\eta_\tau \in D(\mathbb{T}_\tau) \subset M_\tau^\sigma$, we obtain

$$\begin{aligned} \|\partial_s\eta^t\|_{M_\tau^\sigma}^2 &= \int_0^{t-\tau} \mu_\tau(s)\|u(t-s)\|_\sigma^2 \mathrm{d}s + \int_{t-\tau}^\infty \mu_\tau(s)\|\partial_s\eta_\tau(s-t+\tau)\|_\sigma^2 \mathrm{d}s \\ &\leqslant \kappa(\tau)\|u\|_{L^\infty([\tau,T];V_\sigma)}^2 + \|\partial_s\eta_\tau\|_{M_\tau^\sigma}^2, \end{aligned} \tag{3.3}$$

thus, $\partial_s\eta^t \in M_\tau^\sigma$, namely, $\eta^t \in D(\mathbb{T}_\tau)$.

Be similar to the above estimation, we have

$$\text{ess} \sup_{t\in[\tau,T]} \|\partial_t\eta^t\|_{M_\tau^\sigma} < \infty.$$

Applying Lemma 3.1, we obtain that $\eta^t \in W^{1,\infty}([\tau,T]; M_\tau^\sigma)$.

By (3.1) and (3.2), we have the equality

$$\partial_t\eta^t = \mathbb{T}_\tau\eta^t + u(t)$$

holds in M_τ^σ. □

Remark 3.3. *Due to $M_\tau^\sigma \hookrightarrow M_t^\sigma$ and (2.6), for any fixed t, the differential equation*

$$\partial_t\eta^t = \mathbb{T}_t\eta^t + u(t) \tag{3.4}$$

holds in M_t^σ.

Remark 3.4. *When $\eta_\tau \in D(\mathbb{T}_\tau)$, we can obtain from (2.4) and (3.3)*

$$\|\partial_s\eta^t\|_{M_t^\sigma}^2 \leqslant \Xi(u,\eta_\tau)K_\tau(t), \quad \forall t \in [\tau,T], \tag{3.5}$$

where $\Xi(u,\eta_\tau) = \kappa(\tau)\|u\|_{L^\infty([\tau,T];V_\sigma)}^2 + \|\partial_s\eta_\tau\|_{M_\tau^\sigma}^2$.

Lemma 3.5. *Suppose that $u \in C([\tau,T];V_\sigma)$ and $\eta_\tau \in C^1(\mathbb{R}^+,V_\sigma) \cap D(\mathbb{T}_\tau)$. Then, the following inequality*

$$\|\eta^b\|_{M_b^\sigma}^2 - \int_a^b\int_0^\infty (\partial_t\mu_t(s) + \partial_s\mu_t(s))\|\eta^t(s)\|_\sigma^2 \mathrm{d}s\mathrm{d}t \leqslant \|\eta^a\|_{M_a^\sigma}^2 + 2\int_a^b \langle u(t),\eta^t\rangle_{M_t^\sigma}\mathrm{d}t \tag{3.6}$$

holds for all $\tau \leqslant a \leqslant b \leqslant T$.

Proof. For every $\varepsilon > 0$ small, let the cut-off function

$$\phi_\varepsilon(s) = \begin{cases} 0, & 0 \leqslant s < \varepsilon, \\ \frac{s}{\varepsilon} - 1, & \varepsilon \leqslant s < 2\varepsilon, \\ 1, & 2\varepsilon \leqslant s < \frac{1}{\varepsilon}, \\ 2 - \varepsilon s, & \frac{1}{\varepsilon} \leqslant s < \frac{2}{\varepsilon}, \\ 0, & \frac{2}{\varepsilon} \leqslant s. \end{cases}$$

We denote

$$\mu_t^\varepsilon(s)=\phi_\varepsilon(s)\mu_t(s),\;\; y_\varepsilon(t,s)=\mu_t^\varepsilon(s)\|\eta^t(s)\|_\sigma^2,$$

hence,

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}y_\varepsilon(t,s)\mathrm{d}s=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty y_\varepsilon(t,s)\mathrm{d}s. \tag{3.7}$$

For every fixed t and for every s, we deduce that from Lemma 3.1

$$s\mapsto y_\varepsilon(t,s)\in L^1(\mathbb{R}^+),$$

$$t\mapsto \|\eta^t(s)\|_\sigma^2\in C^1([\tau,T]).$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}y_\varepsilon(t,s)=\partial_t\mu_t^\varepsilon(s)\|\eta^t(s)\|_\sigma^2+2\mu_t^\varepsilon(s)\langle\partial_t\eta^t(s),\eta^t(s)\rangle_\sigma.$$

From (2.1) and (3.2), we can get that

$$\sup_{t\in[\tau,T]}\sup_{s\in[\varepsilon,\frac{2}{\varepsilon}]}(\|\eta^t\|_\sigma+\|\partial_t\eta^t\|_\sigma)<\infty.$$

Bearing in mind (H_3) on the compact set $\mathcal{K}=[\tau,T]\times[\varepsilon,\frac{2}{\varepsilon}]$, we know that there exists $C_\varepsilon>0$, such that

$$|\frac{\mathrm{d}}{\mathrm{d}t}y_\varepsilon(t,s)|\leqslant C_\varepsilon\phi_\varepsilon(s)\leqslant C_\varepsilon\chi_{[\varepsilon,\frac{2}{\varepsilon}]}(s). \tag{3.8}$$

Hence, it follows from (3.8) that

$$\int_0^\infty \sup_{t\in[\tau,T]}|\frac{\mathrm{d}}{\mathrm{d}t}y_\varepsilon(t,s)|\mathrm{d}s<\infty. \tag{3.9}$$

We define

$$M_t^{\sigma,\varepsilon}=L^2_{\mu_t^\varepsilon}(\mathbb{R}^+;V_\sigma).$$

Multiplying (3.4) by $2\eta^t$ in $M_t^{\sigma,\varepsilon}$, we can obtain that

$$2\langle\partial_t\eta^t,\eta^t\rangle_{M_t^{\sigma,\varepsilon}}=2\langle\mathbb{T}_t\eta^t,\eta^t\rangle_{M_t^{\sigma,\varepsilon}}+2\langle u(t),\eta^t\rangle_{M_t^{\sigma,\varepsilon}}.$$

Owing to (3.7), we deduce that

$$\begin{aligned}2\langle\partial_t\eta^t,\eta^t\rangle_{M_t^{\sigma,\varepsilon}}&=\int_0^\infty \mu_t^\varepsilon(s)\frac{\mathrm{d}}{\mathrm{d}t}\|\eta^t(s)\|_\sigma^2\mathrm{d}s\\&=\int_0^\infty(\frac{\mathrm{d}}{\mathrm{d}t}(\mu_t^\varepsilon(s)\|\eta^t(s)\|_\sigma^2)-\partial_t\mu_t^\varepsilon(s)\|\eta^t(s)\|_\sigma^2)\mathrm{d}s\\&=\frac{\mathrm{d}}{\mathrm{d}t}\|\eta^t\|^2_{M_t^{\sigma,\varepsilon}}-\int_0^\infty\partial_t\mu_t^\varepsilon(s)\|\eta^t(s)\|_\sigma^2\mathrm{d}s.\end{aligned}$$

By (2.5), we find

$$2\langle\mathbb{T}_t\eta^t,\eta^t\rangle_{M_t^{\sigma,\varepsilon}}=\int_0^\infty\partial_s\mu_t^\varepsilon(s)\|\eta^t(s)\|_\sigma^2\mathrm{d}s.$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\eta^t\|^2_{M_t^{\sigma,\varepsilon}} = \int_0^\infty (\partial_t\mu_t^\varepsilon(s) + \partial_s\mu_t^\varepsilon(s))\|\eta^t(s)\|^2_\sigma \mathrm{d}s + 2\langle u(t), \eta^t\rangle_{M_t^{\sigma,\varepsilon}}. \tag{3.10}$$

By virtue of (3.7) and (3.8), it can be seen that the map $t \mapsto \|\eta^t\|^2_{M_t^{\sigma,\varepsilon}}$ is absolutely continuous. Thus, integrating (3.10) over $[a, b]$, we get

$$\begin{aligned}
&\|\eta^b\|^2_{M_b^{\sigma,\varepsilon}} - \|\eta^a\|^2_{M_a^{\sigma,\varepsilon}} - \int_a^b \int_0^\infty (\partial_t\mu_t^\varepsilon(s) + \partial_s\mu_t^\varepsilon(s))\|\eta^t(s)\|^2_\sigma \mathrm{d}s\mathrm{d}t \\
&\quad = 2\int_a^b \langle u(t), \eta^t\rangle_{M_t^{\sigma,\varepsilon}}\mathrm{d}t.
\end{aligned} \tag{3.11}$$

Next, let us show (3.11) pass to the limit (3.6) as $\varepsilon \to 0$.
For any fixed t, we have

$$0 \leqslant \|\eta^t\|^2_{M_t^\sigma} - \|\eta^t\|^2_{M_t^{\sigma,\varepsilon}} \leqslant \int_0^{2\varepsilon} \mu_t(s)\|\eta^t(s)\|^2_\sigma \mathrm{d}s + \int_{\frac{1}{\varepsilon}}^\infty \mu_t(s)\|\eta^t(s)\|^2_\sigma \mathrm{d}s \to 0.$$

Be similar to the above estimate, we obtain

$$\langle u(t), \eta^t\rangle_{M_t^{\sigma,\varepsilon}} \to \langle u(t), \eta^t\rangle_{M_t^\sigma}.$$

Using $(\mathrm{H}_1), (\mathrm{H}_2)$ and applying Lemma 3.1, we can obtain that

$$\begin{aligned}
|\langle u(t), \eta^t\rangle_{M_t^{\sigma,\varepsilon}}| &\leqslant \sqrt{\kappa(t)}\|u(t)\|_\sigma\|\eta^t\|_{M_t^\sigma} \\
&\leqslant \sqrt{\kappa(\tau)}\|u(t)\|_\sigma\sqrt{K_\tau(t)}\|\eta^t\|_{M_\tau^\sigma} \in L^1([a, b]).
\end{aligned}$$

Thanks to Lebesgue dominated convergence theorem, we get

$$\int_a^b \langle u(t), \eta^t\rangle_{M_t^{\sigma,\varepsilon}}\mathrm{d}t \to \int_a^b \langle u(t), \eta^t\rangle_{M_t^\sigma}\mathrm{d}t.$$

We set

$$\begin{aligned}
q_\varepsilon(t, s) &= -(\partial_t\mu_t^\varepsilon(s) + \partial_s\mu_t^\varepsilon(s))\|\eta^t(s)\|^2_\sigma, \\
q(t, s) &= -(\partial_t\mu_t(s) + \partial_s\mu_t(s))\|\eta^t(s)\|^2_\sigma.
\end{aligned}$$

By use of (H_4), we have

$$\begin{aligned}
q_\varepsilon(t, s) &= -(\phi_\varepsilon(s)\partial_t\mu_t(s) + \phi_\varepsilon(s)\partial_s\mu_t(s) + \phi'_\varepsilon(s)\mu_t(s))\|\eta^t(s)\|^2_\sigma \\
&\geqslant \delta\kappa(t)\mu_t(s)\|\eta^t(s)\|^2_\sigma - \frac{1}{\varepsilon}\chi_{[\varepsilon,2\varepsilon]}(s)\mu_t(s)\|\eta^t(s)\|^2_\sigma \\
&\geqslant -\delta\kappa(t)\mu_t(s)\|\eta^t(s)\|^2_\sigma - \frac{1}{\varepsilon}\chi_{[\varepsilon,2\varepsilon]}(s)\mu_t(s)\|\eta^t(s)\|^2_\sigma \\
&\in L^1([a, b] \times \mathbb{R}^+).
\end{aligned}$$

In addition,

$$\|\eta^t(s)\|^2_\sigma \leqslant (\int_0^s \|\partial_s\eta^t(y)\|_\sigma \mathrm{d}y)^2 \leqslant s\int_0^s \|\partial_s\eta^t(y)\|^2_\sigma \mathrm{d}y.$$

Since $\mu_t(\cdot)$ is nonincreasing, we obtain

$$\mu_t(s)\|\eta^t(s)\|_\sigma^2 \leqslant s\int_0^s \mu_t(y)\|\partial_s\eta^t(y)\|_\sigma^2 \mathrm{d}y \leqslant s\|\partial_s\eta^t\|^2_{M_t^\sigma} \leqslant \Xi(u,\eta_\tau)sK_\tau(t).$$

And we can presume that $\varepsilon \leqslant 1$, so

$$\frac{s}{\varepsilon}\chi_{[\varepsilon,2\varepsilon]}(s) \leqslant 2\chi_{[0,2]}(s).$$

Combining with the two estimate, we have

$$\frac{1}{\varepsilon}\chi_{[\varepsilon,2\varepsilon]}(s)\mu_t(s)\|\eta^t(s)\|_\sigma^2 \leqslant 2\Xi(u,\eta_\tau)\chi_{[0,2]}(s)K_\tau(t) \in L^1([a,b]\times\mathbb{R}^+).$$

Consequently, we find a positive function

$$\psi(t,s) = \delta\kappa(t)\mu_t(s)\|\eta^t(s)\|_\sigma^2 + 2\Xi(u,\eta_\tau)\chi_{[0,2]}(s)K_\tau(t) \in L^1([a,b]\times\mathbb{R}^+),$$

satisfying

$$q_\varepsilon(t,s) \geqslant -\psi(t,s).$$

According to Fatou Lemma and using $q_\epsilon(t,s)\to q(t,s)$ almost everywhere, we deduce that

$$\int_a^b\int_0^\infty q(t,s)\mathrm{d}s\mathrm{d}t \leqslant \liminf_{\varepsilon\to 0}\int_a^b\int_0^\infty q_\varepsilon(t,s)\mathrm{d}s\mathrm{d}t.$$

Finally, we conclude that (3.6) holds. □

Theorem 3.6. *For all $\tau \leqslant a \leqslant b \leqslant T$, the following estimate*

$$\begin{aligned}\|\eta^b\|^2_{M_b^\sigma} + \delta\int_a^b \kappa(t)\|\eta^t(s)\|^2_{M_t^\sigma}\mathrm{d}s\mathrm{d}t &\leqslant \|\eta^b\|^2_{M_b^\sigma} - \int_a^b\int_0^\infty(\partial_t\mu_t(s)+\partial_s\mu_t(s))\|\eta^t(s)\|_\sigma^2\mathrm{d}s\mathrm{d}t \\ &\leqslant \|\eta^a\|^2_{M_a^\sigma} + 2\int_a^b\langle u(t),\eta^t\rangle_{M_t^\sigma}\mathrm{d}t \end{aligned}\tag{3.12}$$

holds.

Proof. Take two sequences

$$\{\eta_\tau^n\}\subset C^1(\mathbb{R}^+;V_\sigma)\cap D(\mathbb{T}_\tau) \text{ and } \{u^n\}\subset C([\tau,T];V_\sigma),$$

such that

$$\eta_\tau^n \to \eta_\tau, \quad u^n \to u.$$

We set

$$\eta^{tn}(s) = \begin{cases} \int_0^s u^n(t-r)\mathrm{d}r, & 0 < s \leqslant t-\tau, \\ \eta_\tau^n(s-t+\tau) + \int_0^{t-\tau} u^n(t-r)\mathrm{d}r, & s > t-\tau. \end{cases}$$

Thanks to Lemma 3.5 and the assumption (H_4), we deduce that

$$\|\eta_b^n\|^2_{M_b^\sigma} + \delta\int_a^b\kappa(t)\|\eta^{tn}(s)\|^2_{M_t^\sigma}\mathrm{d}t \leqslant \|\eta_a^n\|^2_{M_a^\sigma} + 2\int_a^b\langle u^n(t),\eta^{tn}\rangle_{M_t^\sigma}\mathrm{d}t. \tag{3.13}$$

We will show that the sequences is passing to the limit in the above inequality.

Bearing in mind Lemma 3.1, we obtain

$$\|\eta^{tn} - \eta^t\|^2_{\mathcal{M}^\sigma_t} \leqslant \Gamma(u^n - u, \eta^n_\tau - \eta_\tau) K_\tau(t).$$

Hence, the pointwise convergence

$$\eta^{tn} \to \eta^t \text{ in } M^\sigma_t, \ \forall t \in [a,b]$$

holds. And

$$\|\eta^{tn}\|^2_{M^\sigma_t} \to \|\eta^t\|^2_{M^\sigma_t}, \quad \kappa(t)\|\eta^{tn}\|^2_{M^\sigma_t} \to \kappa(t)\|\eta^t\|^2_{M^\sigma_t}, \quad \forall t \in [a,b].$$

By Lemma 3.1, we know that

$$\kappa(t)\|\eta^{tn}\|^2_{M^\sigma_t} \leqslant \kappa(\tau)\Gamma(u^n, \eta^n_\tau)(K_\tau(t))^2 \in L^1([a,b]),$$

where we have used (H$_2$) and $\kappa(t) \leqslant K_\tau(t)\kappa(\tau)$. According to the dominated convergence theorem, we have

$$\int_a^b \kappa(t)\|\eta^{tn}(s)\|^2_{M^\sigma_t} \mathrm{d}t \to \int_a^b \kappa(t)\|\eta^t(s)\|^2_{M^\sigma_t} \mathrm{d}t.$$

It can be easily shown that

$$\langle u^n(t), \eta^{tn}\rangle_{M^\sigma_t} \to \langle u(t), \eta^t\rangle_{M^\sigma_t}, \text{ for almost every } t \in [a,b].$$

Using $\kappa(t) \leqslant K_\tau(t)\kappa(\tau)$, we find

$$|\langle u^n(t), \eta^{tn}\rangle_{M^\sigma_t}| \leqslant \sqrt{\kappa(t)}\|u^n(t)\|_\sigma \|\eta^{tn}\|_{M^\sigma_t} \leqslant C K_\tau(t) \in L^1([a,b]),$$

here, $C = \sup_n (\sqrt{\kappa(\tau)\Gamma(u^n, \eta^{tn})}\|u^n\|_{C([\tau,T];V_\sigma)})$. By the dominated convergence theorem, we have

$$\int_a^b \langle u^n(t), \eta^{tn}\rangle_{M^\sigma_t} \mathrm{d}t \to \int_a^b \langle u(t), \eta^t\rangle_{M^\sigma_t} \mathrm{d}t.$$

This completes proof. □

Definition 3.7. *Let $g \in L^2(\Omega)$ and also let $T > \tau \in \mathbb{R}$. A binary $z(t) = (u(t), \eta^t)$ is said to be a*

- *strong solution of the problem* (2.2), (2.3) *on the interval* $[\tau, T]$, *if*
  - (i) $(u, \eta^t) \in L^\infty([\tau,T]; \mathcal{H}^2_t)$,
  - (ii) *The function* η^t *satisfies the formula* (2.1),
  - (iii) *For every* $\phi \in V_1$ *and almost every* $t \in [\tau, T]$,

$$\langle \partial_t u, \phi\rangle + \langle \partial_t u, \phi\rangle_1 + \langle u, \phi\rangle_1 + \int_0^\infty \mu_t(s)\langle \eta^t(s), \phi\rangle_1 \mathrm{d}s + \langle f(u), \phi\rangle = \langle g, \phi\rangle; \quad (3.14)$$

- *weak solution of the problem* (2.2), (2.3) *on an interval* $[\tau, T]$,
  - (i) *if there exists a sequence of regular data* $(u^n_\tau, \eta^n_\tau) \in \mathcal{H}^2_\tau$ *such that*

$$(u^n_\tau, \eta^n_\tau) \to (u_\tau, \eta_\tau) \text{ in } \mathcal{H}^1_\tau, \ (u, \eta^t) \in L^\infty([\tau,T]; \mathcal{H}^1_t), \text{ and } u^n \to u \text{ in } C([\tau,T]; V_1),$$

where, (u^n_τ, η^n_τ) *is the sequence of the strong solution of the problem* (2.2), (2.3) *with initial data* $z^n_\tau = (u^n_\tau, \eta^n_\tau) \in \mathcal{H}^2_\tau$,

  - (ii) *The function* η^t *satisfies the formula* (2.1),
  - (iii) *For every* $\phi \in V_1$ *and almost every* $t \in [\tau, T]$, *Eq.* (2.2) *satisfies* (3.14).

Theorem 3.8. *(Well-posedness and regularity) Let $T > \tau$ be arbitrary and* (1.3), (1.4) *hold. If $g \in L^2(\Omega)$ and the assumptions* (H$_1$)-(H$_4$) *are valid. Then,*

(i) *for any $(u_\tau, \eta_\tau) \in \mathcal{H}^1_\tau$, the problem* (2.2), (2.3) *admits a weak solution (u, η^t) satisfying*

$$\sup_{t\geqslant\tau} \|z(t)\|^2_{\mathcal{H}^1_t} + \int_\tau^t \|u(r)\|_1^2 dr + \int_\tau^t \kappa(r)\|\eta^r\|^2_{M^1_r}\mathrm{d}r + \int_\tau^t \|\partial_t u(r)\|_1^2\mathrm{d}r \leqslant Q,$$

here, $Q = \max\{Q_1, Q_2\}$. In addition, if there exists a sequence of regular data $(u^n_\tau, \eta^n_\tau) \in \mathcal{H}^2_\tau$ such that

$$(u^n_\tau, \eta^n_\tau) \to (u_\tau, \eta_\tau) \text{ in } \mathcal{H}^1_\tau,$$

then $u^n \to u$ in $C([\tau, T]; V_1)$;

(ii) *for any $(u_\tau, \eta_\tau) \in \mathcal{H}^2_\tau$, the problem* (2.2), (2.3) *admits a strong solution (u, η^t) satisfying*

$$\sup_{t\geqslant\tau} \|z(t)\|^2_{\mathcal{H}^2_t} + \int_\tau^t \|u(r)\|_2^2 dr + \int_\tau^t \kappa(r)\|\eta^r\|^2_{M^2_r}\mathrm{d}r \leqslant Q_3;$$

(iii) *moreover, the solutions of the problem* (2.2), (2.3) *depend on initial data continuously. That is*

$$\|z_1(t) - z_2(t)\|^2_{\mathcal{H}^1_t} \leqslant C\mathrm{e}^{C(R,\lambda_1)(t-\tau)}\|z_1(\tau) - z_2(\tau)\|^2_{\mathcal{H}^1_\tau}, \quad z_{1_\tau}, z_{2_\tau} \in \mathcal{H}^1_\tau, \quad t \in [\tau, T],$$

or

$$\|z_1(t) - z_2(t)\|^2_{\mathcal{H}^2_t} \leqslant C\mathrm{e}^{C(R,\lambda_1)(t-\tau)}\|z_1(\tau) - z_2(\tau)\|^2_{\mathcal{H}^2_\tau}, , \quad z_{1_\tau}, z_{2_\tau} \in \mathcal{H}^2_\tau, \quad t \in [\tau, T],$$

where $z_1(t)$, $z_2(t)$ are two weak solutions of the problem (2.2), (2.3) *with initial data $z_{1_\tau} = (u_{1_\tau}, \eta_{1_\tau})$, $z_{2_\tau} = (u_{2_\tau}, \eta_{2_\tau})$, respectively.*

Proof. Multiplying (2.2) by u, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + \|u\|_1^2) + 2\|u\|_1^2 + 2\langle u, \eta^t\rangle_{M^1_t} + 2\langle f(u), u\rangle - 2\langle g, u\rangle = 0. \tag{3.15}$$

In view of (1.4), we obtain

$$-2\langle f(u), u\rangle \leqslant 2(1-\theta)\|u\|_1^2 + 4c_f,$$

here, $\theta \in (0, 1)$. And it is easy to see that

$$2\langle g, u\rangle \leqslant \theta\|u\|_1^2 + \frac{1}{\lambda_1\theta}\|g\|^2.$$

We define

$$N(t) = \|u\|^2 + \|u\|_1^2.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) + \theta\|u\|_1^2 + 2\langle u, \eta^t\rangle_{M^1_t} \leqslant \frac{1}{\lambda_1\theta}\|g\|^2 + 4c_f := Q_0. \tag{3.16}$$

Integrating (3.16) over $[\tau, t]$, we get

$$N(t)+\theta\int_\tau^t\|u(r)\|_1^2\mathrm{d}r+2\int_\tau^t\langle u,\eta^r\rangle_{M_r^1}\mathrm{d}r\leqslant N(\tau)+Q_0(t-\tau),\quad\forall t\geqslant\tau. \tag{3.17}$$

Applying Theorem 3.6, we know

$$\begin{aligned}N(t)+\|\eta^t\|_{M_t^1}^2+\theta\int_\tau^t\|u(r)\|_1^2\mathrm{d}r-\int_\tau^t\int_0^\infty(\partial_t\mu_t(s)+\partial_s\mu_t(s))\|\eta^r(s)\|_1^2\mathrm{d}s\mathrm{d}r\\ \leqslant N(\tau)+\|\eta_\tau\|_{M_\tau^1}^2+Q_0(t-\tau),\quad\forall t\geqslant\tau.\end{aligned}$$

We set

$$\mathcal{N}(t)=N(t)+\|\eta^t\|_{M_t^1}^2.$$

Then

$$\|z(t)\|_{\mathcal{H}_t^1}^2\leqslant\mathcal{N}(t)\leqslant(1+\frac{1}{\lambda_1})\|z(t)\|_{\mathcal{H}_t^1}^2. \tag{3.18}$$

Therefore,

$$\mathcal{N}(t)+\theta\int_\tau^t\|u(r)\|_1^2\mathrm{d}r-\int_\tau^t\int_0^\infty(\partial_t\mu_t(s)+\partial_s\mu_t(s))\|\eta^r(s)\|_1^2\mathrm{d}s\mathrm{d}r\leqslant\mathcal{N}(\tau)+Q_0(t-\tau). \tag{3.19}$$

Namely,

$$\sup_{t\geqslant\tau}\|z(t)\|_{\mathcal{H}_t^1}^2+\int_\tau^t\|u(r)\|_1^2\mathrm{d}r+\int_\tau^t\kappa(r)\|\eta^r\|_{M_r^1}^2\mathrm{d}r\leqslant C(R,T,\|g\|,\theta,\delta,\lambda_1,c_f):=Q_1. \tag{3.20}$$

Taking the multiplier $\partial_t u$ in (2.2) yields

$$\|\partial_t u\|^2+\|\partial_t u\|_1^2=-\langle u,\partial_t u\rangle_1-\int_0^\infty\mu_t(s)\langle\eta^t(s),\partial_t u\rangle_1\mathrm{d}s-\langle f(u),\partial_t u\rangle+\langle g,\partial_t u\rangle.$$

In view of (1.3), we have

$$|\langle f(u),\partial_t u\rangle|\leqslant\|f(u)\|_{L^{1+\frac{1}{p}}}\|\partial_t u\|_{L^{p+1}}\leqslant C(1+\|u(t)\|_1^p)\|\partial_t u\|_1.$$

And we can obtain from (H_1) that

$$\begin{aligned}|-\int_0^\infty\mu_t(s)\langle\eta^t(s),\partial_t u\rangle_1\mathrm{d}s|&\leqslant\|\partial_t u\|_1\int_0^\infty\mu_t(s)\|\eta^t(s)\|_1\mathrm{d}s\\&\leqslant\|\partial_t u\|_1(\int_0^\infty\mu_t(s)\mathrm{d}s)^{\frac12}(\int_0^\infty\mu_t(s)\|\eta^t(s)\|_1^2\mathrm{d}s)^{\frac12}\\&\leqslant\|\partial_t u\|_1\sqrt{\kappa(t)}\|\eta^t\|_{M_t^1}.\end{aligned}$$

Then

$$\begin{aligned}\|\partial_t u\|_1^2&\leqslant C(\|u(t)\|_1+1+\|u(t)\|_1^p+\sqrt{\kappa(t)}\|\eta^t\|_{M_t^1}+\frac{\|g\|}{\lambda_1^{\frac12}})\|\partial_t u\|_1\\&\leqslant C(1+Q_0^{\frac12}+Q_0^{\frac{p}{2}}+\sqrt{\kappa(t)}\|\eta^t\|_{M_t^1}+\frac{\|g\|}{\lambda_1^{\frac12}})\|\partial_t u\|_1\\&\leqslant\frac12\|\partial_t u\|_1^2+C(R,T,\|g\|,\theta,\delta,\lambda_1,c_f)(1+\kappa(t)\|\eta^t\|_{M_t^1}^2)\\&=\frac12\|\partial_t u\|_1^2+Q_1(1+\kappa(t)\|\eta^t\|_{M_t^1}^2),\quad\forall t\in[\tau,T].\end{aligned} \tag{3.21}$$

Therefore,

$$\int_\tau^t \|\partial_t u(s)\|_1^2 \mathrm{d}s \leqslant 2Q_1(1+\int_\tau^t \kappa(s)\|\eta^s\|_{M_s^1}^2 \mathrm{d}s) \leqslant Q_2. \tag{3.22}$$

Multiplying (2.2) by $-\Delta u$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_1^2+\|u\|_2^2)+2\|u\|_2^2+2\langle u,\eta^t\rangle_{M_t^2}+2\langle f(u),-\Delta u\rangle-2\langle g,-\Delta u\rangle=0. \tag{3.23}$$

By virtue of (1.3), we obtain

$$-2\langle f(u),-\Delta u\rangle=-2\int_\Omega f'(u)|\nabla u|^2 \mathrm{d}x \leqslant 2C_1\|u\|_1^2. \tag{3.24}$$

Obviously,

$$2\langle g,-\Delta u\rangle \leqslant \|u\|_2^2+\|g\|^2.$$

Define

$$N_1(t)=\|u\|_1^2+\|u\|_2^2.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}N_1(t)+\|u\|_2^2+2\langle u,\eta^t\rangle_{M_t^2} \leqslant 2C_1\|u\|_1^2+\|g\|^2. \tag{3.25}$$

Integrating (3.25) over $[\tau,t]$, we have

$$N_1(t)+\int_\tau^t \|u\|_2^2 \mathrm{d}r+2\int_\tau^t \langle u,\eta^r\rangle_{M_r^2}\mathrm{d}r \leqslant N_1(\tau)+2C_1\int_\tau^t \|u(r)\|_1^2 \mathrm{d}r+\|g\|^2(t-\tau). \tag{3.26}$$

Thanks to Theorem 3.6, we find

$$\begin{aligned} N_1(t)&+\int_\tau^t \|u\|_2^2 \mathrm{d}r+\|\eta^t\|_{M_t^2}^2+\delta\int_\tau^t \kappa(r)\|\eta^r\|_{M_r^2}^2 \mathrm{d}r \\ &\leqslant N_1(\tau)+\|\eta_\tau\|_{M_\tau^2}^2+2C_1\int_\tau^t \|u(r)\|_1^2 \mathrm{d}r+\|g\|^2(t-\tau), \quad \forall t\geqslant\tau. \end{aligned} \tag{3.27}$$

We set

$$\mathcal{N}_1(t)=N_1(t)+\|\eta^t\|_{M_t^2}^2.$$

Then

$$\|z(t)\|_{\mathcal{H}_t^2}^2 \leqslant \mathcal{N}_1(t) \leqslant (1+\frac{1}{\lambda_1})\|z(t)\|_{\mathcal{H}_t^2}^2.$$

Thus,

$$\mathcal{N}_1(t)+\int_\tau^t \|u\|_2^2 \mathrm{d}r+\delta\int_\tau^t \kappa(r)\|\eta^r\|_{M_r^2}^2 \mathrm{d}r \leqslant \mathcal{N}_1(\tau)+2C_1\int_\tau^t \|u(s)\|_1^2 \mathrm{d}s+\|g\|^2(t-\tau), \quad \forall t\geqslant\tau.$$

Applying Gronwall inequality, we conclude that

$$\begin{aligned} \sup_{t\geqslant\tau}\|z(t)\|_{\mathcal{H}_t^2}^2&+\int_\tau^t \|u\|_2^2 \mathrm{d}r+\int_\tau^t \kappa(r)\|\eta^r\|_{M_r^2}^2 \mathrm{d}r \\ &\leqslant C(\|z(\tau)\|_{\mathcal{H}_\tau^2},T,\|g\|,\theta,\delta,\lambda_1,C_1,c_f):=Q_3. \end{aligned} \tag{3.28}$$

Let $\{w^n\}$ be an orthonormal basis of $L^2(\Omega)$ which is also orthogonal in V_1 and $-\Delta w^j = \lambda_j w^j,\ j=1,2,\cdots$. And let $\{\zeta^n\}$ be an orthonormal basis of $L^2_{\mu_t}(\mathbb{R}^+;V_1)$ which is also orthogonal in $L^2_{\mu_t}(\mathbb{R}^+;V_1)$ and $-\Delta\zeta^j = \lambda_j\zeta^j,\ j=1,2,\cdots$. For every $n\in\mathbb{N}$, the finite-dimensional subspace is defined by:

$$H_n = \operatorname{span}\{w^1,\cdots,w^n\}\subset V_1,\quad M_n = \operatorname{span}\{\zeta^1,\cdots,\zeta^n\}\subset L^2_{\mu_t}(\mathbb{R}^+;V_1).$$

$P_n : V_1 \to H_n$ is denoted by the orthogonal projection onto H_n; $Q_n : L^2_{\mu_t}(\mathbb{R}^+;V_1)\to M_n$ is denoted by the orthogonal projection onto M_n.

The initial datum $z_\tau = (u_\tau,\eta_\tau)$ is approximated with a sequence $\{z^n_\tau = (u^n_\tau,\eta^n_\tau)\}\subset\mathcal{H}^2_\tau$, where

$$u^n_\tau = P_n u_\tau \to u_\tau \text{ in } V_1, \tag{3.29}$$

$$\eta^n_\tau = Q_n\eta_\tau \to \eta_\tau \text{ in } M^1_\tau. \tag{3.30}$$

For every $n\in\mathbb{N}$, let $z^n = (u^n,\eta^{tn})$ be the approximation solutions of the problem (2.2), (2.3). Where, $u^n = \Sigma^n_{j=1}T^n_j(t)w^j$, $T^n_j\in C^1([\tau,T])$ and $\eta^{tn} = \Sigma^n_{j=1}\Lambda^n_j(t)\zeta^j$, $\Lambda^n_j\in C^1([\tau,T])$. So, for every test function $\psi\in H_n$ and every $t\in[\tau,T]$, $z^n=(u^n,\eta^{tn})$ solves the following system:

$$\begin{aligned}&\langle\partial_t u^n,\psi\rangle + \langle\partial_t u^n,\psi\rangle_1 + \langle u^n,\psi\rangle_1 + \int_0^\infty \mu_t(s)\langle\eta^{tn}(s),\psi\rangle_1 \mathrm{d}s + \langle f(u^n),\psi\rangle\\ &\quad = \langle g,\psi\rangle,\end{aligned} \tag{3.31}$$

and

$$\eta^{tn}(s) = \begin{cases}\int_0^s u^n(t-r)\mathrm{d}r, & 0<s\leqslant t-\tau,\\ \eta^n_\tau(s-t+\tau)+\int_0^{t-\tau}u^n(t-r)\mathrm{d}r, & s>t-\tau.\end{cases} \tag{3.32}$$

Assume that $\psi\in H_m$ is fixed. Then for every $n\geqslant m$, we have (3.31) holds. Multiplying (3.31) by an arbitrary $\varphi\in C^\infty_0([\tau,T])$ and integrating (3.31) over $[\tau,T]$, we find

$$\begin{aligned}&\int_\tau^T \varphi\langle\partial_t u^n(r),\psi\rangle\mathrm{d}r + \int_\tau^T\varphi\langle\partial_t u^n(r),\psi\rangle_1\mathrm{d}r + \int_\tau^T\varphi\langle u^n(r),\psi\rangle_1\mathrm{d}r\\ &\quad + \int_\tau^T\varphi\int_0^\infty\mu_r(s)\langle\eta^{rn}(s),\psi\rangle_1\mathrm{d}s\mathrm{d}r + \int_\tau^T\varphi\langle f(u^n),\psi\rangle\mathrm{d}r = \int_\tau^T\varphi\langle g,\psi\rangle\mathrm{d}r.\end{aligned} \tag{3.33}$$

Evidently, for the sequence $\{z^n\}$, the estimates (3.20), (3.22) and (3.28) are valid. Then,

$$\begin{aligned}&\partial_t u^n \text{ is bounded in } L^2([\tau,T];V_1);\\ &u^n \text{ is bounded in } L^\infty([\tau,T];V_2);\\ &u^n \text{ is bounded in } L^2([\tau,T];V_2);\\ &\eta^{tn} \text{ is bounded in } L^\infty([\tau,T];M^2_t).\end{aligned}$$

Since $\|f(u^n)\|_{L^{1+\frac{1}{p}}}\leqslant C(1+\|u^n\|_1^p)\leqslant C$, we deduce that

$$f(u^n) \text{ is bounded in } L^{1+\frac{1}{p}}(\Omega).$$

For the Galerkin approximation solutions $z^n = (u^n, \eta^{tn})$, we know that there exists a binary $z = (u, \eta^t)$ such that (subsequence if necessary)

$$\partial_t u^n \to \partial_t u \text{ weakly in } L^2([\tau, T]; V_1); \tag{3.34}$$

$$u^n \to u \text{ weakly}^* \text{ in } L^\infty([\tau, T]; V_2); \tag{3.35}$$

$$u^n \to u \text{ weakly in } L^2([\tau, T]; V_2); \tag{3.36}$$

$$\eta^{tn} \to q^t \text{ weakly}^* \text{ in } L^\infty([\tau, T]; M_t^2); \tag{3.37}$$

$$f(u^n) \to f(u) \text{ weakly in } L^{1+\frac{1}{p}}(\Omega). \tag{3.38}$$

Applying Lemma 2.1, we can obtain from (3.34) and (3.35)

$$u^n \to u \text{ in } C([\tau, T]; V_1), \tag{3.39}$$

and the pointwise convergence

$$u^n(x, t) \to u(x, t) \text{ a.e. in } \Omega \times [\tau, T].$$

According to the continuity of f,

$$f(u^n(x, t)) \to f(u(x, t)) \text{ a.e. in } \Omega \times [\tau, T]$$

is also valid.

Using (3.34) and (3.36), we easily obtain the convergence of the first term to the third term at the left end of (3.33). We will deal with the remaining two terms.

Due to $\psi \in H_n \subset V_1$, it is easy to see that $\psi \in P_n L^{p+1}(\Omega)$. Thus, (3.38) ensures

$$\langle f(u^n) - f(u), \psi \rangle \mathrm{d}r \to 0$$

holds. Owing to the boundedness of $f(u^n)$ and $f(u)$ in $L^{1+\frac{1}{p}}(\Omega)$, applying dominated convergence theorem, we deduce that

$$\int_\tau^T \varphi \langle f(u^n) - f(u), \psi \rangle \mathrm{d}r \to 0.$$

Let us show the convergence of $\int_\tau^T \varphi \int_0^\infty \mu_r(s) \langle \eta^{rn}(s), \psi \rangle_1 \mathrm{d}s\mathrm{d}r$. To this end, we set

$$\bar{\eta}_\tau = \eta_\tau^n - \eta_\tau, \;\; \bar{u}_\tau = u_\tau^n - u_\tau,$$

and for every $t \in [\tau, T]$,

$$\bar{\eta}^t = \eta^{tn} - \eta^t, \;\; \bar{u}(t) = u^n(t) - u(t).$$

Taking account of (H_2) and using

$$\bar{\eta}^t(s) = \begin{cases} \int_0^s \bar{u}(t - \zeta) \mathrm{d}\zeta, & 0 < s \leqslant t - \tau, \\ \bar{\eta}_\tau(s - t + \tau) + \int_0^{t-\tau} \bar{u}(t - \zeta) \mathrm{d}\zeta, & s > t - \tau, \end{cases}$$

we have

$$
\begin{aligned}
&\|\bar\eta^t\|^2_{M^1_t}\\
&\quad\leqslant K_\tau(t)\|\bar\eta^t\|^2_{M^1_\tau}\\
&\quad= C(T)\Big(\int_0^{t-\tau}\mu_\tau(s)\|\int_0^s \bar u(t-\zeta)\mathrm{d}\zeta\|_1^2\mathrm{d}s\\
&\qquad+\int_{t-\tau}^\infty \mu_\tau(s)\|\bar\eta_\tau(s-t+\tau)+\int_0^{t-\tau}\bar u(t-\zeta)\mathrm{d}\zeta\|_1^2\mathrm{d}s\Big)\\
&\quad\leqslant C(T)\Big(3(T-\tau)^2\|\bar u\|^2_{C([\tau,T];V_1)}\int_0^\infty \mu_\tau(s)\mathrm{d}s+2\int_0^\infty \mu_\tau(s+t-\tau)\|\bar\eta_\tau(s)\|_1^2\mathrm{d}s\Big)\\
&\quad\leqslant C(T)\Big(3(T-\tau)^2\|\bar u\|^2_{C([\tau,T];V_1)}\kappa(\tau)+2\|\bar\eta_\tau\|^2_{M^1_\tau}\Big)\to 0,\quad \forall t\in[\tau,T].
\end{aligned}
$$

Due to the uniqueness of the limit, we obtain that $q^t=\eta^t$.

Obviously,

$$
\begin{aligned}
&\int_0^\infty \mu_t(s)\langle\bar\eta^t(s),\psi\rangle_1\mathrm{d}s\\
&\quad=\int_0^{t-\tau}\mu_t(s)\langle\int_0^s \bar u_n(t-\zeta)\mathrm{d}\zeta,\psi\rangle_1\mathrm{d}s+\int_{t-\tau}^\infty \mu_t(s)\langle\bar\eta_\tau(s-t+\tau),\psi\rangle_1\mathrm{d}s\\
&\qquad+\int_{t-\tau}^\infty \mu_t(s)\langle\int_0^{t-\tau}\bar u(t-\zeta)\mathrm{d}\zeta,\psi\rangle_1\mathrm{d}s\\
&\quad=\int_0^{t-\tau}\mu_t(s)s\int_0^s\langle\bar u(t-\zeta),\psi\rangle_1\mathrm{d}\zeta\mathrm{d}s+\int_0^\infty \mu_t(s+t-\tau)\langle\bar\eta_\tau(s),\psi\rangle_1\mathrm{d}s\\
&\qquad+\int_{t-\tau}^\infty \mu_t(s)s\int_\tau^t\langle\bar u(\zeta),\psi\rangle_1\mathrm{d}\zeta\mathrm{d}s.
\end{aligned}
$$

Using (H$_2$) once again, we get

$$
\begin{aligned}
&\int_0^{t-\tau}\mu_t(s)s\int_0^s\langle\bar u(t-\zeta),\psi\rangle_1\mathrm{d}\zeta\mathrm{d}s\\
&\quad\leqslant\int_0^{t-\tau}\mu_t(s)s\int_0^s\|\bar u(t-\zeta)\|_1\|\psi\|_1\mathrm{d}\zeta\mathrm{d}s\\
&\quad\leqslant\|\bar u\|_{C([\tau,T];V_1)}\|\psi\|_1(T-\tau)^2K_\tau(t)\kappa(\tau)\to 0,\qquad a.e.\ \ t\in[\tau,T],
\end{aligned}
$$

$$
\begin{aligned}
&\int_{t-\tau}^\infty \mu_t(s)s\int_\tau^t\langle\bar u(\zeta),\psi\rangle_1\mathrm{d}\zeta\mathrm{d}s\\
&\quad\leqslant\int_{t-\tau}^\infty \mu_t(s)s\int_\tau^t\|\bar u(\zeta)\|_1\|\psi\|_1\mathrm{d}\zeta\mathrm{d}s\\
&\quad\leqslant\|\bar u\|_{C([\tau,T];V_1)}\|\psi\|_1(T-\tau)^2K_\tau(t)\kappa(\tau)\to 0,\qquad a.e.\ \ t\in[\tau,T],
\end{aligned}
$$

$$
\int_0^\infty \mu_t(s+t-\tau)\langle\bar\eta_\tau(s),\psi\rangle_1\mathrm{d}s\leqslant\|\psi\|_1K_\tau(t)\sqrt{\kappa(\tau)}\|\bar\eta_\tau\|_{M^1_\tau}\to 0,\qquad a.e.\ \ t\in[\tau,T].
$$

Consequently,

$$
\lim_{n\to\infty}\int_0^\infty \mu_t(s)\langle\bar\eta^t(s),\psi\rangle_1\mathrm{d}s=0,\quad a.e.\ \ t\in[\tau,T].
$$

And

$$\left|\int_0^\infty \mu_t(s)\langle\bar{\eta}^t(s),\psi\rangle_1 \mathrm{d}s\right| \leqslant \int_0^\infty \mu_t(s)\|\bar{\eta}^t(s)\|_1\|\psi\|_1\mathrm{d}s$$
$$\leqslant \|\psi\|_1\sqrt{K_\tau(t)\kappa(\tau)}\|\bar{\eta}^t\|_{M_t^1} \in L^1([\tau,T]).$$

Apply the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty}\int_\tau^T \varphi\int_0^\infty \mu_r(s)\langle\bar{\eta}^r(s),\psi\rangle_1\mathrm{d}s\mathrm{d}r = 0.$$

Finally, we obtain that $z=(u,\eta^t)$ is a weak solution of the problem (2.2), (2.3). Similarly, the existence of the strong solution of the problem (2.2), (2.3) can be proved.

Now, let us show the continuous dependence of the solutions on initial values. Assume that

$$z_1(t)=(u_1(t),\eta_1^t),\ \ z_2(t)=(u_2(t),\eta_2^t)$$

are two weak solutions of the problem (2.2), (2.3) on $[\tau,T]$. Then the difference $\tilde{z}(t)=z_1(t)-z_2(t)=(\tilde{u}(t),\tilde{\eta}^t)$ satisfies

$$\partial_t\tilde{u}+A\partial_t\tilde{u}+A\tilde{u}+\int_0^\infty \mu_t(s)A\tilde{\eta}^t(s)\mathrm{d}s=-f(u_1)+f(u_2), \tag{3.40}$$

where

$$\tilde{\eta}^t(s)=\begin{cases}\int_0^s\tilde{u}(t-r)\mathrm{d}r, & s\leqslant t-\tau,\\ \tilde{\eta}_\tau(s-t+\tau)+\int_0^{t-\tau}\tilde{u}(t-r)\mathrm{d}r, & s>t-\tau.\end{cases} \tag{3.41}$$

Multiplying (3.40) by $\tilde{u}$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}F(t)+2\int_0^\infty \mu_t(s)\langle\tilde{\eta}^t(s),\tilde{u}(t)\rangle_1\mathrm{d}s\\ &\quad=-2\|\tilde{u}\|_1^2-2\langle f(u_1)-f(u_2),\tilde{u}(t)\rangle\\ &\quad\leqslant -\frac{2}{\lambda_1}\|\tilde{u}\|^2+C(1+\|u_1\|_{L^{p+1}}^{p-1}+\|u_2\|_{L^{p+1}}^{p-1})\|\tilde{u}\|_{L^{p+1}}^2\\ &\quad\leqslant -\frac{2}{\lambda_1}\|\tilde{u}\|^2+C(1+\|u_1\|_1^{p-1}+\|u_2\|_1^{p-1})\|\tilde{u}\|_1^2\\ &\quad\leqslant C(R,\lambda_1)F(t),\quad t\in[\tau,T],\end{aligned}$$

where $F(t)=(\|\tilde{u}\|^2+\|\tilde{u}\|_1^2)$. Integrating the above estimate over $[\tau,t]$, we find

$$F(t)+2\int_\tau^t\langle\bar{u}(y),\tilde{\eta}^y\rangle_{M_y^1}\mathrm{d}y\leqslant F(\tau)+C(R,\lambda_1)\int_\tau^t F(y)\mathrm{d}y,\quad t\in[\tau,T]. \tag{3.42}$$

According to Theorem 3.6, we know that

$$\|\tilde{\eta}^t\|_{M_t^1}^2+\delta\int_\tau^t\kappa(y)\|\tilde{\eta}^y(s)\|_{M_y^1}^2\mathrm{d}y\leqslant\|\tilde{\eta}^\tau\|_{M_\tau^1}^2+2\int_\tau^t\langle\tilde{u},\tilde{\eta}^y\rangle_{M_y^1}\mathrm{d}y. \tag{3.43}$$

Setting $\mathcal{F}(t)=F(t)+\|\tilde{\eta}^t\|_{M_t^1}^2$, we have

$$\|\tilde{z}(t)\|_{\mathcal{H}_t^1}^2\leqslant\mathcal{F}(t)\leqslant C\|\tilde{z}(t)\|_{\mathcal{H}_t^1}^2.$$

Combining (3.42) with (3.43), we get

$$\mathcal{F}(t) \leqslant \mathcal{F}(\tau) + C(R,\lambda_1)\int_\tau^t \mathcal{F}(y)\mathrm{d}y.$$

Applying Gronwall inequality, we obtain

$$\|\tilde{z}(t)\|^2_{\mathcal{H}^1_t} \leqslant C\mathrm{e}^{C(R,\lambda_1)(t-\tau)}\|\tilde{z}(\tau)\|^2_{\mathcal{H}^1_\tau}, \quad t\in[\tau,T].$$

At the same time, we have proved the uniqueness of the weak solutions of the problem (2.2), (2.3). Besides, similar to the above estimates, we can also show the continuous dependence on initial data (i.e. the uniqueness) of the strong solutions of the problem (2.2), (2.3). □

Thanks to Theorem 3.8, a process $U(t,\tau)$ corresponding to the problem (2.2), (2.3) can be defined by:

$$z(t) = U(t,\tau)z(\tau) : \mathcal{H}^1_\tau \to \mathcal{H}^1_t,$$

which is continuous from $\mathcal{H}^1_\tau$ to $\mathcal{H}^1_t$.

4. Existence and regularity of time-dependent global attractor

4.1. The existence of time-dependent absorbing set in $\mathcal{H}^1_t$

Theorem 4.1. *(Dissipativity) Suppose that* $g\in L^2(\Omega)$. *If* (1.3), (1.4) *and* (H_1)-(H_4) *hold, and there exists a sequence of regular data* $(u^n_\tau,\eta^n_\tau)\in\mathcal{H}^2_\tau$ *such that*

$$(u^n_\tau,\eta^n_\tau)\to(u_\tau,\eta_\tau)\in\mathbb{B}_\tau(R)\subset\mathcal{H}^1_\tau,$$

then there exists $R_0>0$*, such that the process* $U(t,\tau)$ *corresponding to the problem* (2.2), (2.3) *possesses a time-dependent absorbing set in* $\mathcal{H}^1_t$*, namely, the family* $\mathfrak{B}_t = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$.

Proof. Using Poincaré inequality and (H_4), we can obtain from (3.17)

$$\begin{aligned}&\mathcal{N}(t) + \frac{\theta\lambda_1}{2}\int_\tau^t \|u(r)\|^2\mathrm{d}r + \frac{\theta}{2}\int_\tau^t \|u(r)\|_1^2\mathrm{d}r + \delta\int_\tau^t \kappa(r)\|\eta^r(s)\|^2_{M^1_t}\mathrm{d}s\mathrm{d}r\\ &\leqslant \mathcal{N}(\tau) + Q_1(t-\tau).\end{aligned} \tag{4.1}$$

Namely,

$$\mathcal{N}(t) + 2\varepsilon\int_\tau^t \mathcal{N}(r)\mathrm{d}r \leqslant \mathcal{N}(\tau) + \varepsilon\int_\tau^t \mathcal{N}(r)\mathrm{d}r + Q_1(t-\tau),$$

here, $\varepsilon = \min\{\frac{1}{2}\theta\lambda_1, \frac{1}{2}\theta, \delta\inf_{r\in[\tau,t]}\kappa(r)\}$. Applying Lemma 2.4, we deduce that

$$\mathcal{N}(t) \leqslant \mathcal{N}(\tau)\mathrm{e}^{-\varepsilon(t-\tau)} + \frac{Q_1\mathrm{e}^\varepsilon}{1-\mathrm{e}^{-\varepsilon}}.$$

Moreover,

$$\|z(t)\|^2_{\mathcal{H}^1_t} \leqslant \mathcal{N}(t) \leqslant (1+\frac{1}{\lambda_1})\|z(\tau)\|^2_{\mathcal{H}^1_\tau}\mathrm{e}^{-\varepsilon(t-\tau)} + \frac{R_0^2}{2}, \tag{4.2}$$

where $R_0^2 = 2\frac{Q_1\mathrm{e}^\varepsilon}{1-\mathrm{e}^{-\varepsilon}}$. Then for every $R>0$, there exist a $t_0 = t_0(R) = \frac{1}{\varepsilon}\ln\frac{2(1+\frac{1}{\lambda_1})R^2}{R_0^2} \leqslant t$ and $R_0>0$ such that

$$\tau\leqslant t-t_0 \Rightarrow U(t,\tau)\mathbb{B}_\tau(R)\subset\mathbb{B}_t(R_0).$$

The proof is complete. □

4.2. The existence of time-dependent global attractor in $\mathcal{H}_t^1$

Next, we will testify the asymptotic compactness of the solution process $U(t,\tau)$ corresponding to the problem (2.2), (2.3). To this end, we need to make some decompositions about nonlinear term, solution and solution process.

About the nonlinearity f, inspired by [2], we decompose it as follows:

$$f(s) = f_0(s) + f_1(s),$$

where $f_0, f_1 \in C^1(\mathbb{R})$ and satisfy:

$$|f_0'(u)| \leqslant C(1+|u|^{p-1}), \quad \forall u \in \mathbb{R},\ 1 \leqslant p \leqslant 5, \tag{4.3}$$

$$f_0(u)u \geqslant 0, \quad \forall u \in \mathbb{R}, \tag{4.4}$$

$$|f_1'(u)| \leqslant C(1+|u|^{\gamma}), \quad \forall u \in \mathbb{R},\ 1 \leqslant \gamma < 4, \tag{4.5}$$

$$\liminf_{|u|\to\infty} f_1'(u) > -\lambda_1. \tag{4.6}$$

Influenced by the idea in [26], the solution $z(t) = (u(t), \eta^t)$ of the problem (2.2), (2.3) is decomposed as follows:

$$z(t) = z_1(t) + z_2(t), \text{ with } u(t) = v(t) + w(t) \text{ and } \eta^t = \zeta^t + \xi^t,$$

here $z_1(t) = (v(t), \zeta^t)$ and $z_2(t) = (w(t), \xi^t)$ solve the following equations:

$$\begin{cases} \partial_t v + A\partial_t v + Av + \int_0^\infty \mu_t(s) A\zeta^t(s)\mathrm{d}s + f_0(v) = 0, \\ \partial_t \zeta^t + \partial_s \zeta^t = v(t), \\ v(x,t)|_{\partial\Omega} = 0, \quad v(x,\tau) = u_\tau(x,t), \\ \zeta^t(x,s)|_{\partial\Omega} = 0, \quad \zeta^\tau(x,s) = \eta_\tau(x,s), \end{cases} \tag{4.7}$$

where,

$$\zeta^t(s) = \begin{cases} \int_0^s v(t-r)\mathrm{d}r, & 0 < s \leqslant t-\tau, \\ \zeta_\tau(s-t+\tau) + \int_0^{t-\tau} v(t-r)\mathrm{d}r, & s > t-\tau, \end{cases}$$

and

$$\begin{cases} \partial_t w + A\partial_t w + Aw + \int_0^\infty \mu_t(s) A\xi^t(s)\mathrm{d}s + f(u) - f_0(v) = g, \\ \partial_t \xi^t + \partial_s \xi^t = w(t), \\ w(x,t)|_{\partial\Omega} = 0, \quad w(x,\tau) = 0, \\ \xi^t(x,s)|_{\partial\Omega} = 0, \quad \xi^\tau(x,s) = 0, \end{cases} \tag{4.8}$$

where,

$$\xi^t(s) = \begin{cases} \int_0^s w(t-r)\mathrm{d}r, & 0 < s \leqslant t-\tau, \\ \int_0^{t-\tau} w(t-r)\mathrm{d}r, & s > t-\tau. \end{cases}$$

Analogue to the proof of Theorem 3.8, the existence and uniqueness of the solution of Eqs. (4.7) and (4.8) can be obtained. Further, it is easy to know that the solution processes $U_1(t,\tau)$ and $U_2(t,\tau)$ corresponding to Eqs. (4.7) and (4.8) can be defined. For simplicity, we set

$$U(t,\tau)z_\tau = U_1(t,\tau)z_1(\tau) + U_2(t,\tau)z_2(\tau) = z_1(t) + z_2(t),$$

where $z_1(\tau) = z(\tau),\ z_2(\tau) = 0$.

Similar to Theorem 4.1, the following result can be gained.

Lemma 4.2. *Assume that f_0 satisfies (4.3), (4.4). If there exists a sequence of regular data $(u^n_\tau, \eta^n_\tau) \in \mathcal{H}^2_\tau$ such that*

$$(u^n_\tau, \eta^n_\tau) \to (u_\tau, \eta_\tau) \in \mathbb{B}_\tau(R) \subset \mathcal{H}^1_\tau,$$

and (H_1)-(H_4) *hold, then the solutions of* (4.7) *satisfy the estimate:*

$$\|z_1(t)\|^2_{\mathcal{H}^1_t} \leqslant C(R)\mathrm{e}^{-\varepsilon_1(t-\tau)}. \tag{4.9}$$

Proof. Multiplying the first equation in (4.7) by v and integrating over Ω, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|v\|^2 + \|v\|_1^2) + 2\|v\|_1^2 + 2\langle v, \zeta^t\rangle_{M^1_t} + 2\langle f_0(v), v\rangle = 0. \tag{4.10}$$

We define

$$F(t) = \|v\|^2 + \|v\|_1^2.$$

In consideration of (4.4), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t) + 2\|v\|_1^2 + 2\langle v, \zeta^t\rangle_{M^1_t} \leqslant 0. \tag{4.11}$$

Integrating (4.11) over $[\tau, t]$, we have

$$F(t) + 2\int_\tau^t \|v(r)\|_1^2 \mathrm{d}r + 2\int_\tau^t \langle v, \zeta^r\rangle_{M^1_r} \mathrm{d}r \leqslant F(\tau), \quad \forall t \geqslant \tau. \tag{4.12}$$

Thanks to Theorem 3.6, we get

$$\begin{aligned} F(t) + \|\zeta^t\|^2_{M^1_t} + 2\int_\tau^t \|v(r)\|_1^2 \mathrm{d}r + \delta\int_\tau^t \kappa(r)\|\zeta^r(s)\|^2_{M^1_t}\mathrm{d}r \\ \leqslant F(\tau) + \|\zeta_\tau\|^2_{M^1_\tau}, \quad \forall t \geqslant \tau. \end{aligned}$$

We set

$$\mathcal{F}(t) = F(t) + \|\zeta^t\|^2_{M^1_t}.$$

Then

$$\|z_1(t)\|^2_{\mathcal{H}^1_t} \leqslant \mathcal{F}(t) \leqslant (1 + \frac{1}{\lambda_1})\|z_1(t)\|^2_{\mathcal{H}^1_t}. \tag{4.13}$$

Consequently,

$$\mathcal{F}(t) + 2\int_\tau^t \|v(r)\|_1^2 \mathrm{d}r + \delta\int_\tau^t \kappa(r)\|\zeta^r(s)\|^2_{M^1_t}\mathrm{d}r \leqslant \mathcal{F}(\tau). \tag{4.14}$$

That is,

$$\mathcal{F}(t) + 2\varepsilon_1\int_\tau^t \mathcal{F}(r)\mathrm{d}r \leqslant \mathcal{F}(\tau) + \varepsilon_1\int_\tau^t \mathcal{F}(r)\mathrm{d}r,$$

here, $\varepsilon_1 = \min\{\lambda_1, 1, \delta \inf_{r\in[\tau,t]} \kappa(r)\}$. Applying Lemma 2.4, we obtain that

$$\mathcal{F}(t) \leqslant \mathcal{F}(\tau)\mathrm{e}^{-\varepsilon_1(t-\tau)}.$$

Furthermore,

$$\|z_1(t)\|^2_{\mathcal{H}^1_t} \leqslant \mathcal{F}(t) \leqslant (1 + \frac{1}{\lambda_1})\|z(\tau)\|^2_{\mathcal{H}^1_\tau}\mathrm{e}^{-\varepsilon_1(t-\tau)} \leqslant C(R, \lambda_1)\mathrm{e}^{-\varepsilon_1(t-\tau)}, \tag{4.15}$$

where $\|z(\tau)\|_{\mathcal{H}^1_\tau} \leqslant R$. This completes the proof. □

Lemma 4.3. *Assume that the nonlinearity f satisfy* (1.3), (1.4) *and* (4.3)-(4.6). *If $g \in L^2(\Omega)$ and* (H$_1$)-(H$_4$) *hold, and there exists a sequence of regular data $(u_\tau^n, \eta_\tau^n) \in \mathcal{H}_\tau^2$ such that*

$$(u_\tau^n, \eta_\tau^n) \to (u_\tau, \eta_\tau) \in \mathbb{B}_\tau(R) \subset \mathcal{H}_\tau^1,$$

then for each time $T > 0$, there exists a positive constant $I = I(\|g\|, \|z_\tau\|_{\mathcal{H}_\tau^1}, T, \lambda_1)$, such that the solutions of (4.8) *satisfy:*

$$\|U_2(T+\tau,\tau)z_2(\tau)\|^2_{\mathcal{H}^{\frac{4}{3}}_{T+\tau}} = \|z_2(T+\tau)\|^2_{\mathcal{H}^{\frac{4}{3}}_{T+\tau}} \leqslant I. \tag{4.16}$$

Proof. Multiplying the first equation in (4.8) by $A^{\frac{1}{3}}w$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}G(t) + 2\|w(t)\|^2_{\frac{4}{3}} + 2\langle \xi^t, w(t)\rangle_{M_t^{\frac{4}{3}}}\\ &= 2\langle g, A^{\frac{1}{3}}w\rangle - 2\langle f_1(v), A^{\frac{1}{3}}w\rangle - 2\langle f(u)-f(v), A^{\frac{1}{3}}w\rangle,\end{aligned} \tag{4.17}$$

where $G(t) = \|w(t)\|^2_{\frac{1}{3}} + \|w(t)\|^2_{\frac{4}{3}}$. It is easy to know that

$$2|\langle g, A^{\frac{1}{3}}w\rangle| \leqslant \frac{1}{4}\|w\|^2_{\frac{4}{3}} + \frac{4\|g\|^2}{\lambda_1^{\frac{2}{3}}}. \tag{4.18}$$

We can obtain from (4.5) and (1.3)

$$\begin{aligned}&-2\langle f_1(v), A^{\frac{1}{3}}w\rangle\\ &\leqslant C\int_\Omega (1+|v|^\gamma)|A^{\frac{1}{3}}w|\mathrm{d}x\\ &\leqslant C\Big(\int_\Omega (1+|v|^{\frac{18\gamma}{13}})\mathrm{d}x\Big)^{\frac{13}{18}}\Big(\int_\Omega |A^{\frac{1}{3}}w|^{\frac{18}{5}}\mathrm{d}x\Big)^{\frac{5}{18}}\\ &\leqslant C(1+\|v\|^\gamma_{L^6})\|A^{\frac{1}{3}}w\|_{L^{\frac{18}{5}}}\\ &\leqslant C(R,\lambda_1)\|w\|_{\frac{4}{3}}\\ &\leqslant \frac{1}{4}\|w\|^2_{\frac{4}{3}} + C\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}&-2\langle f(u)-f(v), A^{\frac{1}{3}}w\rangle\\ &\leqslant C\int_\Omega (1+|u|^{p-1}+|v|^{p-1})|w||A^{\frac{1}{3}}w|\mathrm{d}x\\ &\leqslant C(\|u\|^{p-1}_{L^{\frac{3(p-1)}{2}}} + \|v\|^{p-1}_{L^{\frac{3(p-1)}{2}}})\|w\|_{L^{18}}\|A^{\frac{1}{3}}w\|_{L^{\frac{18}{5}}}\\ &\leqslant C(\|u\|_1^{p-1} + \|v\|_1^{p-1})\|w\|_{L^{18}}\|A^{\frac{1}{3}}w\|_{L^{\frac{18}{5}}}\\ &\leqslant c_0\|w\|^2_{\frac{4}{3}},\end{aligned} \tag{4.20}$$

where $c_0 = c_0(Q,R,\lambda_1)$, and we have used the embedding $V_{\frac{4}{3}} \hookrightarrow L^{18}$, $V_{\frac{2}{3}} \hookrightarrow L^{\frac{18}{5}}$, $V_1 \hookrightarrow L^6$.

Thus, inserting (4.18)-(4.20) into (4.17), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t) + 2\langle \xi^t, w(t)\rangle_{M_t^{\frac{4}{3}}} \leqslant (c_0 - \frac{3}{2})\|w(t)\|^2_{\frac{4}{3}} + C. \tag{4.21}$$

Integrating over $[\tau, T+\tau]$, we have

$$\begin{aligned} &G(T+\tau) + 2\int_\tau^{T+\tau} \langle \xi^r, w(r)\rangle_{M_r^{\frac{4}{3}}} \mathrm{d}r \\ &\quad \leqslant G(\tau) + (c_0 - \frac{3}{2})\int_\tau^{T+\tau} \|w(r)\|^2_{\frac{4}{3}} \mathrm{d}r + CT. \end{aligned} \tag{4.22}$$

Define

$$\mathcal{G}(t) = \|w(t)\|^2_{\frac{1}{3}} + \|w(t)\|^2_{\frac{4}{3}} + \|\xi^t\|^2_{M_t^{\frac{4}{3}}}.$$

Applying Theorem 3.6, we get that

$$\mathcal{G}(T+\tau) + \delta\int_\tau^{T+\tau} \kappa(r)\|\xi^r(s)\|^2_{M_r^{\frac{4}{3}}} \mathrm{d}r \leqslant \mathcal{G}(\tau) + (c_0 - \frac{3}{2})\int_\tau^{T+\tau} \|w(r)\|^2_{\frac{4}{3}} \mathrm{d}r + CT. \tag{4.23}$$

That is

$$\mathcal{G}(T+\tau) \leqslant \mathcal{G}(\tau) + c_1\int_\tau^{T+\tau} \mathcal{G}(r)\mathrm{d}r + CT. \tag{4.24}$$

By Gronwall inequality, we conclude that

$$\mathcal{G}(T+\tau) \leqslant \mathrm{e}^{c_1 T}(\mathcal{G}(\tau) + CT) = CT\mathrm{e}^{c_1 T}.$$

Similarly,

$$\|z_2(T+\tau)\|^2_{\mathcal{H}_{T+\tau}^{\frac{4}{3}}} \leqslant \mathcal{G}(T+\tau) \leqslant CT\mathrm{e}^{c_1 T} = I.$$

We complete the proof. □

Moreover, for any $\xi_\tau \in L^2_{\mu_\tau}(\mathbb{R}^+; V_1)$, Cauchy problem (see [3, 16, 22])

$$\begin{cases} \partial_t \xi^t = -\partial_s \xi^t + w, \quad t > \tau, \\ \xi^\tau = \xi_\tau, \end{cases} \tag{4.25}$$

has a unique solution $\xi^t \in C([\tau, +\infty); L_{\mu_\tau}(\mathbb{R}^+; V_1))$ and explicit expression:

$$\xi^t(s) = \begin{cases} \int_0^s w(t-r)\mathrm{d}r, & 0 < s \leqslant t-\tau, \\ \int_0^{t-\tau} w(t-r)\mathrm{d}r, & s > t-\tau. \end{cases} \tag{4.26}$$

We denote by $\mathfrak{B}_t$ the time-dependent absorbing set obtained by Theorem 4.1. Then, we set

$$\mathcal{K}_T = \Pi U_2(T,\tau)\mathfrak{B}_\tau,$$

here, $\Pi : V_1 \times L_{\mu_t}(\mathbb{R}^+; V_1) \to L_{\mu_t}(\mathbb{R}^+; V_1)$ is a projection operator.

Lemma 4.4. *Let $z_2(t) = (w(t), \xi^t)$ be a solution of the problem (4.8). Suppose that the nonlinearity satisfies (1.3), (1.4) and (4.3)-(4.6). If $g \in L^2(\Omega)$ and the assumptions (H$_1$)-(H$_4$) hold, then for every given $T > \tau$, there exists a positive constant $I_1 = I_1(\|\mathfrak{B}_\tau\|_{\mathcal{H}^1_\tau})$, such that*

(i) $\mathcal{K}_T$ *is bounded in* $L^2_{\mu_\tau}(\mathbb{R}^+; V_{\frac{4}{3}}) \cap H^1_{\mu_\tau}(\mathbb{R}^+; V_1)$;

(ii) $\sup\limits_{\eta^T \in \mathcal{K}_T} \|\xi^T(s)\|_1^2 \leqslant I_1$.

Proof. In view of (4.26), we deduce that

$$\partial_s \xi^t(s) = \begin{cases} w(t-s), & 0 < s \leqslant t-\tau, \\ 0, & s > t-\tau. \end{cases} \tag{4.27}$$

And thanks to Lemma 4.3, it can be shown that (i) holds.

Next, it is easy to know that

$$\|\xi^T(s)\|_1 \leqslant \begin{cases} \int_0^s \|w(T-r)\|_1 \mathrm{d}r \leqslant \int_0^{T-\tau} \|w(T-r)\|_1 \mathrm{d}r, & 0 < s \leqslant T-\tau, \\ \int_0^{T-\tau} \|w(T-r)\|_1 \mathrm{d}r, & s > T-\tau, \end{cases} \tag{4.28}$$

holds. From (4.16), (ii) is proved. □

Lemma 4.5. *Let the assumptions of Lemma 4.4 hold. Then for every given* $T > \tau$, $U_2(T,\tau)\mathfrak{B}_\tau$ *is relatively compact in* $\mathcal{H}^1_T$.

Proof. Indeed, applying Lemma 2.2 we know that $\mathcal{K}_T$ is relatively compact in $L_{\mu_\tau}(\mathbb{R}^+; V_1)$. And using the assumption (H_2) once again, we obtain that $\mathcal{K}_T$ is relatively compact in $L_{\mu_t}(\mathbb{R}^+; V_1)$. Furthermore, from the compact embedding: $V_{\frac{4}{3}} \hookrightarrow\hookrightarrow V_1$, we conclude that

$$U_2(T,\tau)\mathfrak{B}_\tau \text{ is relatively compact in } \mathcal{H}^1_T.$$

The proof is complete. □

Theorem 4.6. *Let* $U(t,\tau)$ *be the solution process of the problem* (2.2), (2.3). *Suppose that the nonlinear term* f *satisfies* (1.3), (1.4) *and* (4.3)-(4.6). *If* $g \in L^2(\Omega)$ *and* (H_1)-(H_4) *hold, then the process* $U(t,\tau)$ *possesses the time-dependent attractor* $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ *in* $\mathcal{H}^1_t$. *In addition, the attractor* $\mathfrak{A}$ *is invariant, namely,*

$$U(t,\tau)A_\tau = A_t, \quad \forall t \geqslant \tau.$$

Proof. Let $\mathfrak{B}_t = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$ be the time-dependent absorbing set obtained from Theorem 4.1. From Lemma 4.2 and Lemma 4.3, for a large enough positive constant R_1, it is easy to know that

$$\text{the family } B_t^{\frac{1}{3}} = \{\mathbb{B}_t^{\frac{1}{3}}(R_1)\}_{t\in\mathbb{R}} \text{ is pullback attracting,}$$

here $\mathbb{B}_t^{\frac{1}{3}}(R_1) = \{\xi \mid \|\xi\|_{\mathcal{H}_t^{\frac{4}{3}}} \leqslant R_1\}$.

In fact, combining (4.9) with (4.16), we deduce that

$$\begin{aligned} \mathrm{dist}_{\mathcal{H}^1_t}(U(t,\tau)\mathfrak{B}_\tau, B_t^{\frac{1}{3}}) &\leqslant \mathrm{dist}_{\mathcal{H}^1_t}(U_1(t,\tau)\mathfrak{B}_\tau + U_2(t,\tau)\mathfrak{B}_\tau, B_t^{\frac{1}{3}}) \\ &= \mathrm{dist}_{\mathcal{H}^1_t}(U_1(t,\tau)\mathfrak{B}_\tau, B_t^{\frac{1}{3}}) \\ &\leqslant C(\|\mathfrak{B}_\tau\|_{\mathcal{H}^1_\tau}) e^{-\varepsilon_1(t-\tau)}, \end{aligned}$$

here, $\varepsilon_1 = \min\{\lambda_1, 1, \delta \inf_{r\in[\tau,t]} \kappa(r)\}$.

If there exists a sequence of regular data $(u_\tau^n, \eta_\tau^n) \in \mathcal{H}_\tau^2$ such that

$$(u_\tau^n, \eta_\tau^n) \to (u_\tau, \eta_\tau) \in \mathbb{B}_\tau(R) \subset \mathcal{H}_\tau^1,$$

then for the bounded (in $\mathcal{H}_\tau^1$) set $B_\tau = \{\mathbb{B}_\tau(R)\}_{\tau\in\mathbb{R}}$ corresponding to initial data (u_τ, η_τ), by Theorem 4.1, there exists a $t_0 = t_0(R)$ such that

$$\tau \leqslant t - t_0 \Rightarrow U(t,\tau)\mathbb{B}_\tau(R) \subset \mathbb{B}_t(R_0).$$

Thus,

$$\mathrm{dist}_{\mathcal{H}_t^1}(U(t,\tau)B_\tau, \mathfrak{B}_t) \leqslant \varpi \mathrm{e}^{\varepsilon_1 t_0}\mathrm{e}^{-\varepsilon_1(t-\tau)},$$

where $\varpi = \sup_{0\leqslant t-\tau\leqslant t_0} \|U(t,\tau)B_\tau\|_{\mathcal{H}_t^1}$.

Applying Lemma 2.3 and Theorem 3.8, we can obtain that

$$\mathrm{dist}_{\mathcal{H}_t^1}(U(t,\tau)B_\tau, B_t^{\frac{1}{3}}) \leqslant C(\|B_\tau\|_{\mathcal{H}_\tau^1})\mathrm{e}^{-\varepsilon(t-\tau)}.$$

Combining with Lemma 4.5, we have the process $U(t,\tau)$ corresponding to the problem (2.2), (2.3) is asymptotically compact in $\mathcal{H}_t^1$. Therefore, applying Theorem 2.9, Theorem 2.12 and Theorem 3.8, we can show the existence and invariance of the time-dependent attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ in $\mathcal{H}_t^1$, that is

$$U(t,\tau)A_\tau = A_t,$$

and

$$\mathfrak{A} = \{Z | t \to Z(t) \in \mathcal{H}_t^1 \text{ and } Z(t) \text{ is CBT of the process } U(t,\tau)\}.$$

We complete the proof. □

4.3. Regularity of the time-dependent attractors

Subsequently, we will prove that the time-dependent attractor $\mathfrak{A}$ is bounded in $\mathcal{H}_t^2$ and the bound is independent of t.

To this end, we make a decomposition of the solution $z(t)$ of the problem (2.2), (2.3):

$$U(t,\tau)z_\tau = z(t) = z_1(t) + z_2(t) = U_3(t,\tau)z_{1_\tau} + U_4(t,\tau)z_{2_\tau},$$

where, $z_1(t)$ and $z_2(t)$ solve the following equations, respectively,

$$\begin{cases} \partial_t v + A\partial_t v + Av + \int_0^\infty \mu_t(s)A\zeta^t(s)\mathrm{d}s = 0, \\ \partial_t\zeta^t + \partial_s\zeta^t = v(t), \\ v(x,t)|_{\partial\Omega} = 0, \quad v(x,\tau) = u_\tau(x,t), \\ \zeta^t(x,s)|_{\partial\Omega} = 0, \quad \zeta^\tau(x,s) = \eta_\tau(x,s), \end{cases} \tag{4.29}$$

where,

$$\zeta^t(s) = \begin{cases} \int_0^s v(t-r)\mathrm{d}r, & 0 < s \leqslant t-\tau, \\ \zeta_\tau(s-t+\tau) + \int_0^{t-\tau} v(t-r)\mathrm{d}r, & s > t-\tau, \end{cases}$$

and

$$
\begin{cases}
\partial_t w + A\partial_t w + Aw + \int_0^\infty \mu_t(s)A\xi^t(s)\mathrm{d}s + f(u) = g, \\
\partial_t \xi^t + \partial_s \xi^t = w(t), \\
w(x,t)|_{\partial\Omega} = 0, \quad w(x,\tau) = 0, \\
\xi^t(x,s)|_{\partial\Omega} = 0, \quad \xi^\tau(x,s) = 0,
\end{cases}
\tag{4.30}
$$

where,

$$
\xi^t(s) = \begin{cases} \int_0^s w(t-r)\mathrm{d}r, & 0 < s \leqslant t-\tau, \\ \int_0^{t-\tau} w(t-r)\mathrm{d}r, & s > t-\tau. \end{cases}
$$

Be similar to the proof of Lemma 4.2, we can obtain easily that

$$
\|U_3(t,\tau)z_{1_\tau}\|_{\mathcal{H}_t^1} \leqslant C\mathrm{e}^{-\varepsilon(t-\tau)}. \tag{4.31}
$$

Theorem 4.7. *Let $z_2(t)$ be the solution of (4.30) with initial data $z_2(\tau) \in A_\tau$ satisfying $\|z_2(\tau)\|_{\mathcal{H}_\tau^1} = 0$. And also let $g \in L^2(\Omega)$. If the presumptions (1.3), (1.4) and (H_1)-(H_4) hold, then $\{A_t\}_{t\in\mathbb{R}}$ is bounded in $\mathcal{H}_t^2$ and the bound is independent of t.*

Proof. Taking the scalar product of (4.30) with $-\Delta w$, we find

$$
\frac{\mathrm{d}}{\mathrm{d}t}G_1(t) + 2\|w(t)\|_2^2 + 2\langle \xi^t, w(t)\rangle_{M_t^2} + 2\langle f(u), -\Delta w\rangle = 2\langle g, -\Delta w\rangle, \tag{4.32}
$$

where $G_1(t) = \|w(t)\|_1^2 + \|w(t)\|_2^2$.

It is easy to know that

$$
2\langle g, -\Delta w\rangle \leqslant 2\|g\|^2 + \frac{1}{2}\|w\|_2^2.
$$

Due to the invariance of $\mathfrak{A}$, we obtain

$$
\|U(t,\tau)z(\tau)\|_{\mathcal{H}_t^{\frac{4}{3}}} \leqslant C.
$$

From (1.3), we can get that

$$
\begin{aligned}
-2\langle f(u), -\Delta w\rangle| &\leqslant C\int_\Omega (1+|u|^{p-1})|\nabla u||\nabla w|\mathrm{d}x \\
&\leqslant C(1+\|u\|^{p-1}_{L^{\frac{9(p-1)}{4}}})\|\nabla u\|_{L^{\frac{18}{7}}}\|\nabla w\|_{L^6} \\
&\leqslant C(1+\|u\|^{p-1}_{\frac{4}{3}})\|u\|_{\frac{4}{3}}\|w\|_2 \\
&\leqslant \frac{1}{2}\|w\|_2^2 + C.
\end{aligned}
$$

where, we have used the embedding $V_{\frac{4}{3}} \hookrightarrow L^{18} \hookrightarrow L^{\frac{9(p-1)}{4}}$, $V_{\frac{1}{3}} \hookrightarrow L^{\frac{18}{7}}$ and $V_1 \hookrightarrow L^6$.

Combining the above estimates, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}G_1(t) + \|w\|_2^2 + 2\langle \xi^t, w(t)\rangle_{M_t^2} \leqslant C. \tag{4.33}
$$

Integrating (4.33) over $[\tau, t]$, we have

$$
G_1(t) + \int_\tau^t \|w(r)\|_2^2\mathrm{d}r + 2\int_\tau^t \langle \xi^r, w(r)\rangle_{M_r^2}\mathrm{d}r \leqslant G_1(\tau) + C(t-\tau).
$$

Thanks to Theorem 3.6, we get

$$\mathcal{G}_1(t)+\int_\tau^t \|w(r)\|_2^2\mathrm{d}r+\delta\int_\tau^t \kappa(r)\|\xi^r(s)\|^2_{M_r^2}\mathrm{d}r\leqslant \mathcal{G}_1(\tau)+C(t-\tau), \tag{4.34}$$

where,

$$\mathcal{G}_1(t)=\|w(t)\|_1^2+\|w(t)\|_2^2+\|\xi^t\|^2_{M_t^2}.$$

Namely,

$$\mathcal{G}_1(t)+2\varepsilon_2\int_\tau^t \mathcal{G}_1(r)\mathrm{d}r\leqslant \mathcal{G}_1(\tau)+\varepsilon_2\int_\tau^t \mathcal{G}_1(r)\mathrm{d}r+C(t-\tau), \tag{4.35}$$

here, $\varepsilon_2=\min\{\frac{\lambda_1}{2},\frac{1}{2},\delta\inf\limits_{r\in[\tau,t]}\kappa(r)\}$.

Applying Lemma 2.4, we deduce that

$$\mathcal{G}_1(t)\leqslant \mathcal{G}_1(\tau)\mathrm{e}^{-\varepsilon_2(t-\tau)}+\frac{C\mathrm{e}^{\varepsilon_2}}{1-\mathrm{e}^{-\varepsilon_2}}.$$

Since

$$\|z_2(t)\|^2_{\mathcal{H}_t^2}\leqslant \mathcal{G}_1(t)\leqslant (1+\frac{1}{\lambda_1})\|z_2(t)\|^2_{\mathcal{H}_t^2}, \tag{4.36}$$

we get that

$$\begin{aligned}\|z_2(t)\|^2_{\mathcal{H}_t^2}&\leqslant (1+\frac{1}{\lambda_1})\|z_2(\tau)\|^2_{\mathcal{H}_\tau^2}\mathrm{e}^{-\varepsilon_2(t-\tau)}+\frac{C\mathrm{e}^{\varepsilon_2}}{1-\mathrm{e}^{-\varepsilon_2}}\\&=\frac{C\mathrm{e}^{\varepsilon_2}}{1-\mathrm{e}^{-\varepsilon_2}}\leqslant I_1.\end{aligned} \tag{4.37}$$

Then, $\|U_4(t,\tau)z_{2_\tau}\|_{\mathcal{H}_t^2}$ is uniformly bounded with respect to t.

We set

$$K_t^2=\{z|\|z(t)\|_{\mathcal{H}_t^2}\leqslant I_1\}.$$

We can obtain from (4.31) and (4.37) that

$$\lim_{\tau\to-\infty}\mathrm{dist}_{\mathcal{H}_t^1}(U(t,\tau)A_\tau,K_t^2)=0,\quad \forall t\in\mathbb{R}.$$

Due to the invariance of the time-dependent attractor $\mathfrak{A}$, we obtain

$$\mathrm{dist}_{\mathcal{H}_t^1}(A_t,K_t^2)=0,\quad \forall t\in\mathbb{R}.$$

Hence, $A_t\subset\overline{K_t^2}=K_t^2$. Finally, we conclude that $\{A_t\}_{t\in\mathbb{R}}$ is bounded in $\mathcal{H}_t^2$ and the bound is independent of t.

This finishes the proof. □

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