$\bar{\partial}$-dressing method for three-component coupled nonlinear Schrödinger Equations

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Abstract

The dressing method based on 4×4 matrix $\bar{\partial}$-problem is extended to study the three-component coupled nonlinear Schrödinger (3CNLS) equations. The spatial and time spectral problems related to the 3CNLS equations are derived via two linear constraint equations. A 3CNLS hierarchy with source is proposed by using recursive operator. The N-solitions of the 3CNLS equations are given based on the $\bar{\partial}$-equation by selecting a spectral transformation matrix.

Keywords: Three-component coupled nonlinear Schrödinger Equations, $\bar{\partial}$-dressing method, Lax pair, Soliton solution

1. Introduction

The study of multi-component nonlinear systems has attracted more and more attention, since they can describe a variety of complex physical phenomena and have richer dynamical behaviors than low-component systems. Among the various solutions of these models, the soliton plays a crucial role in illustrating some related phenomena. In recent years, many methods for solving soliton solutions have been proposed, including inverse scattering transformation (IST) [1], Darboux transformation (DT) [2], Hirota bilinear method [3–6], $\bar{\partial}$-dressing method, etc. The $\bar{\partial}$-dressing method, as an extension of IST, is based on the inverse scattering theory and Lax framework. It is a powerful tool for constructing and solving integrable nonlinear equations and describing their transformations and reductions. It was first proposed by Zakharov and Shabat [7], and further developed by Beals, Coifman, Ablowitz, ManBakov, Fokas et al. [8–12]. So far, a large number of integrable equations have been successfully studied by the $\bar{\partial}$-dressing method [13–27].

The coupled nonlinear Schrödinger (NLS) equations have been widely used in nonlinear optics, deep ocean, Bose-Einstein (BE) condensation and other fields[28–34]. Therefore, this paper mainly considers

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the three-component coupled nonlinear Schrödinger (3CNLS) equations[35]:

$$
\begin{aligned}
&iu_{1t} + \frac{1}{2}u_{1xx} + \sigma(|u_1|^2 + |u_2|^2 + |u_3|^2)u_1 = 0,\\
&iu_{2t} + \frac{1}{2}u_{2xx} + \sigma(|u_1|^2 + |u_2|^2 + |u_3|^2)u_2 = 0,\\
&iu_{3t} + \frac{1}{2}u_{3xx} + \sigma(|u_1|^2 + |u_2|^2 + |u_3|^2)u_3 = 0.
\end{aligned}
\tag{1.1}
$$

Here, $u_j = u_j(x,t) j = 1,2,3$ are the complex functions with the temporal variable t and spatial variable x, and $\sigma > 0 (< 0)$ stands for the attractive (repulsive) interactions. There has been increasing interest in the study of the dynamic properties of system (1.1). For example, the vector soliton solution has been derived through the Horita bilinear method [36, 37], the bright-bright solitons have been obtained by Darboux transformation (DT) method from a trivial seed solution with $u_3 = 0$ [38], the initial-boundary value problem has been investigated by extending the Fokas unified approach [39]. However, to our knowledge, there is still no research work on system (1.1) by using $\bar{\partial}$-dressing method. For convenience, we take $\sigma = 1$ for the following analysis.

The layout of this paper is organized as follows. In Section 2, starting from the $\bar{\partial}$-equation, we propose a new Lax pair with singular dispersion relation for system (1.1) using the $\bar{\partial}$-dressing method. In Section 3, based on the relationship between $\bar{\partial}$-dressing transformation matrix and potential matrix, we derive a 3CNLS hierarchy with source. In Section 4, the N-soliton solutions formula of system (1.1) are constructed. Finally, the conclusions will be drawn based on the above sections.

2. Spectral problem and Lax pair

2.1. The spatial spectra problem

We consider the 4×4 matrix $\bar{\partial}$-problem in the complex k-plane,

$$\bar{\partial}\psi = \psi R, \tag{2.1}$$

with a boundary condition $\psi(x,t,k) \to I, k \to \infty$, then a solution of the equation (2.1) can be written as

$$\psi(k) = I + \frac{1}{2\pi i}\int\int \frac{\psi(z)R(z)}{z-k} dz \wedge d\bar{z} \equiv I + \psi R C_k, \tag{2.2}$$

where C_k denotes the Cauchy-Green integral operator acting on the left. The formal solution of $\bar{\partial}$-problem (2.1) will be given from (2.2) as

$$\psi(k) = I \cdot (I - RC_k)^{-1}. \tag{2.3}$$

For convenience, we define a pairing [14]

$$\langle f, g\rangle = \frac{1}{2\pi i}\int\int f(k) g^T(k) dk \wedge d\bar{k}, \quad \langle f\rangle = \langle f, I\rangle = \frac{1}{2\pi i}\int\int f(k) dk \wedge d\bar{k},$$

which can be shown to possess the following properties

$$\langle f, g\rangle^T = \langle g, f\rangle, \;\; \langle fR, g\rangle = \langle f, gR^T\rangle, \;\; \langle fC_k, g\rangle = -\langle f, gC_k\rangle. \tag{2.4}$$

It is easy to prove that for some matrix functions $f(k)$ and $g(k)$, the operator C_k satisfies

$$g(k)[f(k)C_k]C_k + [g(k)C_k]f(k)C_k = [g(k)C_k][f(k)C_k]. \tag{2.5}$$

It is well known that the Lax pairs of nonlinear equations play an important role in the study of integrable systems. Such as Darboux transformation, inverse scattering transformation, Riemann-Hilbert method, algebro-geometric all depend on on their Lax pairs. Here we prove that if the transform matrix $R(x,t,k)$ satisfies a simple linear equation, the spatial-time spectral problems of system (1.1) can be established from (2.1).

Proposition 1: *Let the transform matrix R satisfies*

$$R_x = ik[J, R], \tag{2.6}$$

where $J = diag(1,-1,-1,-1)$, then the solution ψ of the $\bar{\partial}$-equation (2.1) satisfies the following spatial spectral problem

$$\psi_x - ik[J,\psi] = Q\psi, \tag{2.7}$$

where

$$Q = \begin{pmatrix} 0 & -u_1^* & -u_2^* & -u_3^* \\ u_1 & 0 & 0 & 0 \\ u_2 & 0 & 0 & 0 \\ u_3 & 0 & 0 & 0 \end{pmatrix} = i[J, \langle\psi R\rangle]. \tag{2.8}$$

Proof. Using (2.3) and (2.6), we get

$$\psi_x = ik\psi R\sigma_3 C_k(I - RC_k)^{-1} - ik\psi\sigma_3 RC_k(I - RC_k)^{-1}. \tag{2.9}$$

According to the definition of C_k, we can obtain

$$ik\psi RC_k = i\langle\psi R\rangle + ik(\psi - I). \tag{2.10}$$

Since $RC_k = I - I\cdot(I - RC_K)$, then we have

$$RC_k(I - RC_k)^{-1} = (I - RC_k)^{-1} - I. \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9), we obtain

$$\psi_x = -i\langle\psi R\rangle J\psi + iJk(I - RC_k)^{-1} - ik\psi J. \tag{2.12}$$

From (2.10), we can get

$$k(I-RC_k)^{-1}=\langle\psi R\rangle\psi+k\psi, \tag{2.13}$$

Substituting (2.13) into (2.12), we have Eq.(2.7).

2.2. The time spectral problem

Proposition 2: *Suppose that R satisfies the linear equation*

$$R_t=[R,\Omega], \tag{2.14}$$

where

$$\Omega=\Omega_p+\Omega_s=-ik^2J+\frac{1}{2\pi i}\int\int\frac{\omega(\xi)J}{\xi-k}d\xi\wedge d\bar{\xi}, \tag{2.15}$$

which comprises both a polynomial part $\Omega_p(k)$ and a singular part $\Omega_s(k)$ and $\omega(\xi)$ is a scalar function. Then the solution ψ of the $\bar{\partial}$-equation (2.1) leads to time spectral problem

$$\psi_t-ik^2[J,\psi]=\frac{i}{2}J(Q^2-Q_x)\psi+kQ. \tag{2.16}$$

Proof. We first use the polynomial dispersion relation only$\Omega=\Omega_p=-ik^2J$. From equations (2.2), (2.3) and (2.15), we find that

$$\psi_t=-i[k^2\psi RC_kJ(I-RC_k)^{-1}-k^2\psi J(I-RC_k)^{-1}]-ik^2\psi J. \tag{2.17}$$

Through the following direct computation,

$$\begin{aligned}k^2\psi RC_k&=\langle\zeta\psi R\rangle+k\langle\psi R\rangle+k^2(\psi-I),\\ k^2(I-RC_k)^{-1}&=(\langle\zeta\psi R\rangle+\langle\psi R\rangle^2+k\langle\psi R\rangle+k^2)\psi,\end{aligned}$$

then (2.17) is changed to

$$\begin{aligned}\psi_t&=-i[\langle\zeta\psi R\rangle,J]\psi-i[\langle\psi R\rangle,J]\langle\psi R\rangle\psi-ik[\langle\psi R\rangle,J]\psi+ik^2[J,\psi]\\&=2iJ\langle\zeta\psi R\rangle^{off}\psi+Q\langle\psi R\rangle\psi+kQ\psi+ik^2[J,\psi].\end{aligned} \tag{2.18}$$

By virtue of (2.6), (2.7) and (2.8), we have

$$\begin{aligned}\langle\psi R\rangle_x&=Q\langle\psi R\rangle+i[J,\langle k\psi R\rangle],\\ \langle\psi R\rangle_x^{off}&=Q\langle\psi R\rangle^{diag}+2iJ\langle k\psi R\rangle^{off},\\ \langle k\psi R\rangle^{off}&=\frac{i}{2}JQ\langle\psi R\rangle^{diag}-\frac{1}{4}Q_x,\\ \langle\psi R\rangle-\langle\psi R\rangle^{diag}&=J(J\langle\psi R\rangle-J\langle\psi R\rangle^{diag})=\frac{J}{2}[J,\langle\psi R\rangle]=-\frac{i}{2}JQ.\end{aligned} \tag{2.19}$$

Substituting (2.19) into (2.18) leads to the time-dependent linear equation

$$\begin{aligned}\psi_t &= 2iJ(\frac{i}{2}JQ\langle\psi R\rangle^{diag} - \frac{1}{4}Q_x)\psi + Q\langle\psi R\rangle\psi + kQ\psi + ik^2[J,\psi] \\ &= Q[\langle\psi R\rangle - \langle\psi R\rangle^{diag}]\psi + (kQ - \frac{i}{2}JQ_x)\psi + ik^2[J,\psi] \\ &= \frac{i}{2}J(Q^2 - Q_x)\psi + kQ + ik^2[J,\psi].\end{aligned} \tag{2.20}$$

3. Recursive operators and equation hierarchy

In this section, we derive 3CNLS equations with a source. First, we define the 4×4 matrix $M = \psi J\psi^{-1}$. By using the Eq.(2.8) and definition of M, we can prove the following proposition.

Proposition 4: *Q defined by (2.8) satisfies a coupled hierarchy with a source M*

$$\begin{aligned}&Q_t + 2\alpha_n J\Lambda^n Q = i[J,\langle\omega(k)M(k)\rangle], \quad n = 1,2,\ldots \\ &M_x - ik[J,M] = [Q,M].\end{aligned} \tag{3.1}$$

Proof. Differentiating Q with respect to t gives

$$Q_t = i[J,\langle\psi R\rangle_t]. \tag{3.2}$$

Because of $\bar{\partial}f(k)C_k = f(k)$, then we have

$$\begin{aligned}(\psi R)_t &= \bar{\partial}\psi_t(k) = \bar{\partial}[I\cdot(I - RC_k)_t^{-1}] \\ &= \bar{\partial}[\psi R_t(I - RC_k)^{-1}]C_k \\ &= \psi R_t(I - RC_k)^{-1}.\end{aligned} \tag{3.3}$$

By using the properties (3.3), we can obtain

$$Q_t = i[J,\langle\psi R_t(I - RC_k)^{-1}, I\rangle] = i[J,\langle\psi R_t, I\cdot(I + R^TC_k)^{-1}\rangle]. \tag{3.4}$$

From the $\bar{\partial}$-equation (2.1), we have

$$\bar{\partial}\psi^{-1} = -R\psi^{-1},$$

which leads to

$$(\psi^{-1})^T = I\cdot(I + R^TC_k)^{-1}.$$

Therefore, using (2.4) and (2.14), Eq. (3.4) can be simplified to

$$Q_t = i[J,\langle(\bar{\partial}\psi)\Omega\psi^{-1} + \psi\Omega\bar{\partial}\psi^{-1}\rangle]. \tag{3.5}$$

Here we shall consider $\Omega_p = \alpha_n k^n J, \alpha_n = const$ and the fact that $\Omega_s \to 0$ as $k \to \infty$, then the above equation can be further simplified like

$$
\begin{aligned}
Q_t &= i[J, \langle \psi \Omega \bar{\partial} \psi^{-1} \rangle] + i[J, \langle (\bar{\partial}\psi) \Omega \psi^{-1} \rangle] \\
&= i\alpha_n [J, \langle \bar{\partial}(k^n M(k)) \rangle] + i[J, \langle \omega(k) M(k) \rangle] \\
&= 2i\alpha_n J \langle \bar{\partial}(k^n M(k)^{off}) \rangle + i[J, \langle \omega(k) M(k) \rangle].
\end{aligned}
\tag{3.6}
$$

By using (2.7), it can be checked that M(k) satisfies the equation

$$
M_x - ik[J, M] - [Q, M] = 0. \tag{3.7}
$$

From Eq.(3.7), they satisfy the following equations

$$
\begin{aligned}
M_x^{diag} &= [Q, M^{off}], \\
M_x^{off} &= 2ikJM^{off} + [Q, M^{diag}],
\end{aligned}
\tag{3.8}
$$

which lead to

$$
\begin{aligned}
M^{diag} &= J + \partial_x^{-1}[Q, M^{off}], \\
M^{off} &= i(\Lambda - k)^{-1} Q,
\end{aligned}
\tag{3.9}
$$

where

$$
\Lambda \cdot = -\frac{i}{2} J(\partial_x \cdot - [Q, \partial_x^{-1}[Q, \cdot]]).
$$

The operator Λ usually be called as recursion operator. We expand $(\Lambda - k)^{-1}$ in the series

$$
(\Lambda - k)^{-1} = -\sum_{j=1}^{\infty} k^{-j} \Lambda^{j-1}.
$$

By using $\bar{\partial} k^{n-j} = \pi \delta(k) \delta_{j,n+1}$, $j = 1, 2, \ldots$, we can derive that

$$
\sum_{j=1}^{\infty} \langle \bar{\partial} k^{n-j} \rangle \Lambda^{j-1} Q = -\Lambda^n Q.
$$

Substituting it into (3.6) leads to the Eq.(3.1).

4. *N*-Soliton solutions of cmKdV equation

In this section, we will construct the N-soliton solutions of the syetem (1.1) still based on the $\bar{\partial}$-equation (2.1),we first introduce a spectral matrix R as

$$
R = \pi \sum_{j=1}^{N} \begin{pmatrix} 0 & -a_j e^{2i\theta(k)} \delta(k - \bar{k}_j) & -b_j e^{2i\theta(k)} \delta(k - \bar{\tilde{k}}_j) & -c_j e^{2i\theta(k)} \delta(k - \bar{\hat{k}}_j) \\ a_j e^{2i\theta(k)} \delta(k - k_j) & 0 & 0 & 0 \\ b_j e^{2i\theta(k)} \delta(k - \tilde{k}_j) & 0 & 0 & 0 \\ c_j e^{2i\theta(k)} \delta(k - \hat{k}_j) & 0 & 0 & 0 \end{pmatrix} \tag{4.1}
$$

where $k_j, \tilde{k}_j, \hat{k}_j$, $j = 1, 2, \ldots$ are constants distinct from each other, $\theta(k) = -kx - k^2 t$. Substituting (4.1) into (2.8) leads to

$$\begin{aligned} u_1 &= -2i\langle \psi R\rangle_{21} \\ &= -\frac{1}{\pi}\int\int(\psi_{22}(\zeta)R_{21}(\zeta) + \psi_{23}(\zeta)R_{31}(\zeta) + \psi_{24}(\zeta)R_{41}(\zeta))d\zeta\wedge d\bar{\zeta} \\ &= -\sum_{j=1}^{N}(a_j e^{2i\theta(k_j)}\psi_{22}(k_j) + b_j e^{2i\theta(\tilde{k}_j)}\psi_{23}(\tilde{k}_j) + c_j e^{2i\theta(\hat{k}_j)}\psi_{24}(\hat{k}_j)). \end{aligned} \tag{4.2}$$

Substituting (4.1) into $\bar{\partial}$-equation (2.1) and resorting the properties of function, we can obtain

$$\begin{aligned} \psi_{22}(k) &= 1 + \frac{1}{2\pi i}\int\int\frac{\psi_{21}(\zeta)R_{12}(\zeta)}{\zeta - k}d\zeta\wedge d\bar{\zeta} = 1 - \sum_{p=1}^{N}\frac{\bar{a}_p e^{-2i\theta(\bar{k}_p)}}{k - \bar{k}_p}\psi_{21}(\bar{k}_p), \\ \psi_{23}(k) &= \frac{1}{2\pi i}\int\int\frac{\psi_{21}(\zeta)R_{13}(\zeta)}{\zeta - k}d\zeta\wedge d\bar{\zeta} = -\sum_{l=1}^{N}\frac{\bar{b}_j e^{-2i\theta(\bar{\tilde{k}}_l)}}{k - \bar{\tilde{k}}_l}\psi_{21}(\bar{\tilde{k}}_l), \\ \psi_{24}(k) &= \frac{1}{2\pi i}\int\int\frac{\psi_{21}(\zeta)R_{14}(\zeta)}{\zeta - k}d\zeta\wedge d\bar{\zeta} = -\sum_{m=1}^{N}\frac{\bar{c}_m e^{-2i\theta(\bar{\hat{k}}_m)}}{k - \bar{\hat{k}}_m}\psi_{21}(\bar{\hat{k}}_m). \end{aligned} \tag{4.3}$$

then introducing notation A_p, B_l, C_m written as

$$A_p(k) = \frac{a_p}{k - k_p}e^{-2i\theta(k_p)},\ \ B_l(k) = \frac{b_l}{k - \tilde{k}_l}e^{-2i\theta(\tilde{k}_l)},\ \ C_m(k) = \frac{c_m}{k - \hat{k}_m}e^{-2i\theta(\hat{k}_m)}. \tag{4.4}$$

From(4.3), we have

$$\begin{aligned} \psi_{22}(k) &= 1 - \sum_{p,j=1}^{N}\overline{A_p(\bar{k})}[A_j(\bar{k}_p)\psi_{22}(k_j) + B_j(\bar{k}_p)\psi_{23}(\tilde{k}_j) + C_j(\bar{k}_p)\psi_{22}(\hat{k}_j)], \\ \psi_{23}(k) &= -\sum_{j,l=1}^{N}\overline{B_l(\bar{k})}[A_j(\bar{\tilde{k}}_l)\psi_{22}(k_j) + B_j(\bar{\tilde{k}}_l)\psi_{23}(\tilde{k}_j) + C_j(\bar{\tilde{k}}_l)\psi_{24}(\hat{k}_j)], \\ \psi_{24}(k) &= -\sum_{j,m=1}^{N}\overline{C_m(\bar{k})}[A_j(\bar{\hat{k}}_m)\psi_{22}(k_j) + B_j(\bar{\hat{k}}_m)\psi_{23}(\tilde{k}_j) + C_j(\bar{\hat{k}}_m)\psi_{24}(\hat{k}_j)], \end{aligned} \tag{4.5}$$

taking $z = z_j$, $z = \tilde{z}_j$ and $z = \hat{z}_j$ respectively, we have

$$\begin{aligned} &(I + M)\psi_{22}(k) + N\psi_{23}(\tilde{k}) + P\psi_{24}(\hat{k}) = E, \\ &\widetilde{M}\psi_{22}(k) + (I + \widetilde{N})\psi_{23}(\tilde{k}) + \widetilde{P}\psi_{24}(\hat{k}) = 0, \\ &\hat{M}\psi_{22}(k) + \hat{N}\psi_{23}(\tilde{k}) + (I + \hat{P})\psi_{24}(\hat{k}) = 0, \end{aligned} \tag{4.6}$$

where $E = (1, \cdots, 1)^T$, and M, N, P are $N \times N$ matrix

$$M_{n,p} = \sum_{j=1}^{N}\overline{A_j(\bar{k}_n)}A_p(\bar{k}_j),\ \ N_{n,p} = \sum_{j=1}^{N}\overline{A_j(\bar{k}_n)}B_p(\bar{k}_j),\ \ P_{n,p} = \sum_{j=1}^{N}\overline{A_j(\bar{k}_n)}C_p(\bar{k}_j),$$

$$\widetilde{M}_{n,p} = \sum_{j=1}^{N} \overline{B_j(\bar{k}_n)} A_p(\bar{\tilde{k}}_j), \quad \widetilde{N}_{n,p} = \sum_{j=1}^{N} \overline{B_j(\bar{k}_n)} B_p(\bar{\tilde{k}}_j), \quad \widetilde{P}_{n,p} = \sum_{j=1}^{N} \overline{B_j(\bar{k}_n)} C_p(\bar{\hat{k}}_j),$$

$$\widehat{M}_{n,p} = \sum_{j=1}^{N} \overline{C_j(\bar{k}_n)} A_p(\bar{\hat{k}}_j), \quad \widehat{N}_{n,p} = \sum_{j=1}^{N} \overline{C_j(\bar{k}_n)} B_p(\bar{\hat{k}}_j), \quad \widehat{P}_{n,p} = \sum_{j=1}^{N} \overline{C_j(\bar{k}_n)} C_p(\bar{\hat{k}}_j),$$

then we can solve $\psi_{24}(\hat{k}), \psi_{22}(k)$ and $\psi_{23}(\tilde{k})$

$$\begin{aligned} \psi_{24}(\hat{k}) &= (I + X_1)^{-1} Y_1, \\ \psi_{22}(k) &= (I + X_2)^{-1} Y_2, \\ \psi_{23}(\tilde{k}) &= (I + X_3)^{-1} Y_3, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
X_1 &= [(I + M)^{-1} P - \widetilde{M}^{-1} P]^{-1} [(I + M)^{-1} N - \widetilde{M}^{-1}(I + \widetilde{N})][\widetilde{M}^{-1}(I + \widetilde{N}) - \widehat{M}^{-1}\widehat{N}]^{-1}[\widetilde{M}^{-1}\widetilde{P} - \widehat{M}^{-1}(I + \widehat{P})], \\
X_2 &= [N^{-1}(I + M) - (I + \widetilde{N})^{-1}\widetilde{M}]^{-1}[N^{-1}P - (I + \widetilde{N})^{-1}\widetilde{P}][(I + \widetilde{N})^{-1}\widetilde{P} - \widehat{N}^{-1}(I + \widehat{P})]^{-1}[(I + \widetilde{N})^{-1}\widetilde{M} - \widehat{N}^{-1}\widehat{M}], \\
X_3 &= [P^{-1}N - \widetilde{P}^{-1}(I + \widetilde{N})]^{-1}[P^{-1}(I + M) - \widetilde{P}^{-1}\widetilde{M}][(I + \widehat{P})^{-1}\widehat{M} - \widetilde{P}^{-1}\widetilde{M}]^{-1}[(I + \widehat{P})^{-1}\widehat{N} - \widetilde{P}^{-1}(I + \widetilde{N})], \\
Y_1 &= [(I + M)^{-1} P - \widetilde{M}^{-1} P]^{-1}(I + M^{-1})E, \\
Y_2 &= [N^{-1}(I + M) - (I + \widetilde{N})^{-1}\widetilde{M}]^{-1} N^{-1} E, \\
Y_3 &= [P^{-1}N - \widetilde{P}^{-1}(I + \widetilde{N})] P^{-1} E.
\end{aligned}$$

Hence, the N-soliton solutions of the system (1.1) take the form

$$\begin{aligned}
u_1 &= 2i(h_1\psi_{22}(k) + h_2\psi_{23}(\tilde{k}) + h_3\psi_{24}(\hat{k})) \\
&= 2i[h_1(I + X_1)^{-1}Y_1 + h_2(I + X_2)^{-1}Y_2 + h_3(I + X_3)^{-1}Y_3] \\
&= 2i\mathrm{tr}[(I + X_1)^{-1}Y_1h_1 + (I + X_2)^{-1}Y_2h_2 + (I + X_3)^{-1}Y_3h_3] \\
&= 2i\left[\frac{\det(I + X_1 + Y_1h_1)}{\det(I + X_1)} + \frac{\det(I + X_2 + Y_2h_2)}{\det(I + X_2)} + \frac{\det(I + X_3 + Y_3h_3)}{\det(I + X_3)} - 3\right],
\end{aligned} \tag{4.8}$$

$$u_2 = 2i\left[\frac{\det(I + X_1 + Z_1h_1)}{\det(I + X_1)} + \frac{\det(I + X_2 + Z_2h_2)}{\det(I + X_2)} + \frac{\det(I + X_3 + Z_3h_3)}{\det(I + X_3)} - 3\right], \tag{4.9}$$

$$u_3 = 2i\left[\frac{\det(I + X_1 + L_1h_1)}{\det(I + X_1)} + \frac{\det(I + X_2 + L_2h_2)}{\det(I + X_2)} + \frac{\det(I + X_3 + L_3h_3)}{\det(I + X_3)} - 3\right], \tag{4.10}$$

where

$$h_1 = (h_{1,1}, h_{1,2}, \cdots, h_{1,N}), \quad h_{1,j} = a_j e^{2i\theta(k_j)},$$

$$h_2 = (h_{2,1}, h_{2,2}, \cdots, h_{2,N}), \quad h_{2,j} = b_j e^{2i\theta(\tilde{k}_j)},$$

$$h_3 = (h_{3,1}, h_{3,2}, \cdots, h_{3,N}), \quad h_{3,j} = c_j e^{2i\theta(\hat{k}_j)}.$$

5. Conclusion

In this paper, we have presented the dressing method based on the $\bar{\partial}$-problem to study the three-component coupled nonlinear Schrödinger (3CNLS) equations (1.1). By means of the $\bar{\partial}$-dressing method, we have obtained the spatial and time spectral problems associated with the 3CNLS equations. Then we proposed a 3CNLS hierarchy with source by using recursive operator. Finally, the N-soliton solutions of the 3CNLS equations have been constructed based on the $\bar{\partial}$-equation by selecting a special spectral transformation matrix. It is hoped that our results can help enrich the nonlinear dynamics of the NLS-type equations.

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Reference

[1] M.J. Ablowitz, P.A. Clarkson, Solitons, nonlinear evolution equations and inverse scattering, Cambridge University Press, Cambridge, 1991.

[2] V.B. Matveev, M.A. Salle, Darboux Transformation and Solitons, Springer, Berlin, 1991.

[3] R. Hirota, Direct Methods in Soliton Theory, Springer, Berlin, 2004.

[4] W.X. Ma, Soliton solutions by means of Hirota bilinear forms, Partial Diff. Equ. Appl. Math., 2022, 5, 100220.

[5] D.S. Wang, X.D. Zhu, Direct and inverse scattering problems of the modified Sawada-Kotera equation: Riemann-Hilbert approach, Proc. R. Soc. A, 2022, 478, 20220541.

[6] D.S. Wang, The "good" Boussinesq equation: long-time asymptotics, Analysis and PDE, 2023, 16 (6), 1351-1388.

[7] V.E. Zakharov, S. V. Manakov, Construction of multidimensional nonlinear integrable systems and their solutions, Funkc. Anal. Prilozh, 1985, 19 (2), 11-25.

[8] M.J. Ablowitz, D. Bar Yaacov, A.S. Fokas, On the inverse scattering transform for the Kadomtsev-Petviashvili equation, Stud. Appl. Math, 1983, 69, 135-142.

[9] R. Beals , R.R. Coifman, The D-bar approach to inverse scattering and nonlinear evolutions, Physica D, 1986, 18, 242-249.

[10] R. Beals , R.R. Coifman, Scattering, spectral transformations and nonlinear evolution equations, Goulaouic-Meyer-Schwartz, 1981, 22.

[11] A.S. Fokas , P.M. Santini, Dromions and a boundary value problem for the Davey-Stewartson I equation, Physica D, 1990, 44, 99-130.

[12] S.V. Manakov, The inverse scattering transform for the time-dependent Schrödinger equation and Kadomtsev-Petviashvili equation, Physica D, 1981, 3 (1-2), 420-427.

[13] L.V. Bogdanov, S.V. Manakov, The nonlocal $\bar{\partial}$-problem and (2+1)-dimensional soliton equations, J. Phys. A, 1988, 21, 537.

[14] E.V. Doktorov, S.B. Leble, A Dressing Method in Mathematical Physics, Springer, 2007.

[15] A.S. Fokas , V.E. Zakharov, The dressing method and nonlocal Riemann-Hilbert problem, J. Nonlinear Sci.,1992, 2, 109-134.

[16] B.G. Konopelchenko, B.T. Matkarimov, Inverse spectral transform for the nonlinear evolution equation generating the Davey-Stewartson and Ishimori equations, Stud. Appl. Math., 1990, 82, 319-359.

[17] Y.K. Kuang , J.Y.Zhu, A three-wave interaction model with self-consistent sources: the $\bar{\partial}$-dressing method and solutions, J. Math. Anal. Appl., 2015, 426, 783-793.

[18] J.Y. Zhu, X.G. Geng, The AB equations and the $\bar{\partial}$-dressing method in semi-characteristic coordinates, Math. Phys. Anal. Geom., 2014, 17, 49-65.

[19] J.H. Luo, E.G. Fan, $\bar{\partial}$-dressing method for the coupled Gerdjikov-Ivanov equation, Appl. Math. Lett., 2020, 110, 106589.

[20] J.H. Luo, E.G. Fan, Dbar-dressing method for the Gerdjikov-Ivanov equation with nonzero boundary conditions, Appl. Math. Lett., 2021, 120, 107297.

[21] J.H. Luo, E.G. Fan, A $\bar{\partial}$-dressing approach to the Kundu-Eckhaus equation, J. Geom. Phys., 2021, 167, 104291.

[22] Y.Q. Yao, Y.H. Huang, E.G. Fan, The $\bar{\partial}$-dressing method and Cauchy matrix for the defocusing matrix NLS system, Appl. Math. Lett., 2021, 117, 107143.

[23] S.X. Yang, B. Li, $\bar{\partial}$-dressing method for the (2+1)-dimensional Korteweg-de Vries equation, Appl. Math. Lett., 2023, 140, 108589.

[24] P.V. Nabelek, V.E. Zakharov, Solutions to the Kaup-Broer system and its (2+1) dimensional integrable generalization via the dressing method, Physica D, 2020, 409, 132478.

[25] V.G. Dubrovsky, A.V. Topovsky, Multi-lump solutions of KP equation with integrable boundary via $\bar{\partial}$-dressing method, Physica D, 2020, 414, 132740.

[26] V.G. Dubrovsky, A.V. Topovsky, Multi-soliton solutions of KP equation with integrable boundary via $\bar{\partial}$-dressing method, Physica D, 2021, 428, 133025.

[27] X.D. Chai, Y.F. Zhang, The $\bar{\partial}$-dressing method for the (2+1)-dimensional Konopelchenko-Dubrovsky equation, Appl. Math. Lett., 2022, 134, 108378.

[28] B.L. Guo, L.M. Ling, Rogue wave, breathers and bright-dark-rogue solutions for the coupled Schrödinger equations, Chin. Phys. Lett., 2011, 28 (11), 110202.

[29] F. Baronio, et al., Solutions of the vector nonlinear Schrödinger equations: evidence for deterministic rogue waves, Phys. Rev. Lett., 2012, 109 (4).

[30] X. Wang, B. Yang, Y. Chen, et al., Higher-order localized waves in coupled nonlinear Schrödinger equations, Chin. Phys. Lett., 2014, 31 (09), 090201.

[31] X. Wang, Y. Chen, Rogue-wave pair and dark-bright rogue wave solutions of the coupled Hirota equations, Chin. Phys. B., 2014, 23, 070203.

[32] X. Wang, Y. Li, Y. Chen,Generalized Darboux transformation and localized waves in coupled Hirota equations, Wave Motion, 2014, 51, 1149-1160.

[33] S.H. Chen, L.Y. Song, Rogue waves in coupled Hirota systems, Phys. Rev. E., 2013, 87, 032910.

[34] L.C. Zhao, L. Jie, Rogue-wave solutions of a three-component coupled nonlinear Schrödinger equation, Phys. Rev. E., 2013, 87, 013201.

[35] G.Q. Zhang, Z.Y. Yan, Three-component nonlinear Schrödinger equations: modulational instability, Nth-order vector rational and semi-rational rogue waves, and dynamics, Commun. Nonlinear S-ci.,2018, 62, 117-133.

[36] M. Vijayajayanthi, T. Kanna, M. Lakshmanan, Bright-dark solitons and their collisions in mixed N-coupled nonlinear Schrödinger equations, Phys. Rev. A., 2008, 77, 013820.

[37] M. Vijayajayanthi, T. Kanna, M. Lakshmanan, Multisoliton solutions and energy sharing collisions in coupled nonlinear Schrödinger equations with focusing, defocusing and mixed type nonlinearities, Eur. Phys. J-Spec. Top., 2009, 173, 57-80.

[38] L. Zhao, S. He, Matter wave solitons in coupled system with external potentials, Phys. Lett. A., 2011, 375, 3017-3020.

[39] Z.Y. Yan, An initial-boundary value problem of the general three-component nonlinear Schrödinger equation with a 4×4 lax pair on a finite interval, Chaos, 2017, 27, 053117.