On large s-simulation functions and large $\mathcal{Z}_s$-contractions with the link to Picard mappings

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Abstract

The main aim of this paper is to introduce new ideas, called large s-simulation functions and large $\mathcal{Z}_s$-contractions, which are inspired by the broad utility of applications of fixed point results for the enlarged class of nonlinear mappings. Illustrative examples supporting the new idea of large s-simulation functions are presented. Moreover, fixed point results for large $\mathcal{Z}_s$-contractions are investigated.

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1. Introduction and Preliminaries

From the past until now, one of the great power tools for solving many real-world problems and mathematical problems is the theorem concerning fixed points, including the existence theorem for fixed points of nonlinear mappings and the convergence theorem for fixed point algorithms. One cornerstone in metric fixed point theory combining both mentioned parts is the Banach contraction mapping principle in [1]. This principle originates from many more metric fixed point results in this era. In addition to the theoretical aspects mentioned above, the Banach contraction mapping principle has greatly benefited many other fields, such as science, engineering, economics, chaos theory, artificial intelligence studies, big data studies, etc.

Nowadays, there are several ways to improve the Banach contraction mapping principle, such as the investigations on spaces having more structure than metric spaces, the invention of new nonlinear mappings, and proving fixed point results for these new mappings, etc. To lead to the inspiration of this paper, needed details in interesting research articles concerning these mentioned ways are given in the next paragraph.

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In 2015, Khojasteh *et al.* [9] introduced the concept of a simulation function significantly impacting the fixed point theory because it can be used to invent a broad class of new nonlinear mappings covering several nonlinear contraction mappings. They use simulation functions to establish the incredible contraction mapping named $\mathcal{Z}$-contraction mapping. In addition, the existence and uniqueness of fixed point for $\mathcal{Z}$-contraction mappings in metric spaces are presented in such an article. These results also show that any $\mathcal{Z}$-contraction mapping is a Picard mapping. Afterward, Roldán López de Hierro et al. [10] showed that simulation functions require symmetry in their arguments, which is not necessary for the proofs. Then, they slightly modified the original definition to highlight this difference and enlarged the family of all simulation functions.

Although the idea of simulation functions seems helpful, we still need to apply it in more expansive spaces such as in b-metric spaces, formally introduced by Czerwik [6] in 1993. An intelligent solution to the problem, as mentioned earlier, was suggested by Yamaod and Sintunavarat [14] in 2017. They introduced the fantastic class of s-simulation functions, where $s \geq 1$. They also showed that this class could be used to invent the new generalized contraction mappings in b-metric spaces. This contraction is called a $\mathcal{Z}_s$-contraction mapping. Moreover, many results in fixed point theory involving the ideas of s-simulation functions and $\mathcal{Z}_s$-contraction mappings are proved.

On the other hand, it is well-known that many mathematical definitions and theorems originated from the goal that the users want to solve some problems, such as integral equations, differential equations, matrix equations, etc. For instance, the Banach contraction mapping and its fixed point results are presented to solve some integral equations. In 1996, Burton [4] established the concept of large contraction mappings, weaker than Banach contraction mappings, and applied its fixed point results to solve the specific integral equation. Next, we give its definition, which is one of the inspirations for inventing the main idea in this paper.

Definition 1.1 ([4])**.** Let (X,d) be a metric space. A mapping $T : X \to X$ is called a large contraction mapping if the following conditions hold:

(B_1) $d(Tx,Ty) < d(x,y)$ for all $x,y \in X$ with $x \neq y$;
(B_2) for all $\varepsilon > 0$, there is $\delta \in [0,1)$ such that

$$[\forall x,y \in X \text{ with } d(x,y) \geq \varepsilon] \Rightarrow d(Tx,Ty) \leq \delta d(x,y). \tag{1.1}$$

We observe that every Banach contraction mapping is a large contraction, but the converse does not true as in the following example:

Example 1.2. Let $(X,d) = (\mathbb{R}, |\cdot|)$ be a usual metric space and let $T : X \to X$ be defined by $Tx = x - x^3$ for all $x \in X$. By applying the mean value theorem, we get that T is a large contraction mapping, but is not a Banach contraction mapping (the reader can see more details in [4], and also [8]).

Surprisingly, no researchers have combined two ideas of s-simulation functions and large contraction mappings. The main goal of this paper is to fill this gap in the research on this trend. First, the new idea of a large s-simulation function with phenomenal examples is presented. Second, we use the concept of large s-simulation functions to construct the definition of new generalized contraction mappings named large $\mathcal{Z}_s$-contraction mappings and prove that any large $\mathcal{Z}_s$-contraction mapping is a Picard mapping. The example to illustrate our main results and the numerical method are given.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the set of positive integers, the set of non-negative real numbers and the set of real numbers, respectively. In 1993, Czerwik [6] formally introduced the idea of b-metric spaces, which is an extension of metric spaces, and presented the Banach contraction mapping in the framework of b-metric spaces. After the appearance of this research, there are a lot of mathematicians who investigated many results in b-metric spaces. Here, we give some basic ideas related to b-metric spaces as follows:

Definition 2.1 ([6]). Let X be a nonempty set and $s \geq 1$ be a fixed real number. Suppose that the mapping $d : X \times X \to \mathbb{R}_+$ satisfies the following conditions for all $x, y, z \in X$:

1. $d(x,y) = 0$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$;
3. $d(x,y) \leq s[d(x,z) + d(z,y)]$.

Then d is called a b-metric, and (X,d) is called a b-metric space with the coefficient s.

It is easy to see that each metric is a b-metric with $s = 1$, but the converse is not true. The mapping $\mathbb{R} \times \mathbb{R} \ni (x,y) \mapsto |x-y|^p$, where $p \geq 1$, is a known b-metric on $\mathbb{R}$ with the coefficient $s = 2^{p-1}$. The reader can see more examples of b-metrics from many research articles in fixed point theory (see [3, 5, 7, 11, 12, 13] and references therein).

Definition 2.2 ([3]). Let (X,d) be a b-metric space and $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ is b-convergent if there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n,x) = 0$. In this case, we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is called a b-Cauchy sequence if $\lim_{n,m\to\infty} d(x_n,x_m) = 0$.

(iii) (X,d) is called b-complete if every b-Cauchy sequence in X b-converges.

From the above definition, it is well-known that for each b-metric space (X,d), a b-convergent sequence in X is a b-Cauchy sequence and it has a unique limit. In general, a b-metric is not continuous (see more details in [3]). Recently, a good survey about a brief history and survey of b-metric spaces with some important related aspects and the early developments in fixed point theory on b-metric spaces is presented in [2].

Next, we review some basic knowledge about simulation functions and $\mathcal{Z}$-contraction mappings which are needed for our investigation.

Definition 2.3 ([9]). A function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called a simulation function if it satisfies the following conditions:

(S_1) $\zeta(0,0) = 0$;

(S_2) $\zeta(t,s) < s - t$ for all $t,s > 0$;

(S_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

We denote by $\mathcal{Z}$ the class of all simulation functions.

Example 2.4 ([9]). Let $\zeta_1, \zeta_2, \zeta_3 : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by

(a) $\zeta_1(t,s) = ks - t$ for all $t,s \in [0,\infty)$, where $k \in [0,1)$;

(b) $\zeta_2(t,s) = \psi(s) - \phi(t)$ for all $t,s \in [0,\infty)$, where $\psi, \phi : [0,\infty) \to [0,\infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$ for all $t > 0$;

(c) $\zeta_3(t,s) = s - \varphi(s) - t$ for all $t,s \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$;

(d) $\zeta_4(t,s) = s - \frac{f(t,s)}{g(t,s)} t$ for all $t,s \in [0,\infty)$, where $f, g : [0,\infty) \to [0,\infty)$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t,s > 0$.

Then $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathcal{Z}$.

Definition 2.5 ([9]). Let (X,d) be a metric space. A mapping $T : X \to X$ is called a $\mathcal{Z}$-contraction mapping with respect to $\zeta \in \mathcal{Z}$ if the following condition is satisfied:

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0$$

for all $x,y \in X$.

It is easy to see that a Banach contraction mapping is a $\mathcal{Z}$-contraction mapping concerning $\zeta_1 \in \mathcal{Z}$ defined in Definition 2.4. Furthermore, by utilization of simulation functions, we can show that many contraction mappings in the past are $\mathcal{Z}$-contraction mappings.

In 2017, the concept of simulation functions was extended to the following idea:

Definition 2.6 ([14]). Let $s \geq 1$ be a given real number. A function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called an s-simulation function if it satisfies (S_2) and the following condition:

(S_4) if $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,\infty)$ such that

$$0 < \liminf_{n\to\infty} \alpha_n \leq s \left(\limsup_{n\to\infty} \beta_n \right) \leq s^2 \left(\liminf_{n\to\infty} \alpha_n \right)$$

and

$$0 < \liminf_{n\to\infty} \beta_n \leq s \left(\limsup_{n\to\infty} \alpha_n \right) \leq s^2 \left(\liminf_{n\to\infty} \beta_n \right),$$

then

$$\limsup_{n\to\infty} \zeta(\alpha_n, \beta_n) < 0.$$

3. Large s-simulation functions

The main aim of this section is to present a new type of simulation function, which is named a large s-simulation function. Using this simulation function to define the new contractive condition in fixed point theory has the advantage of being used more than other simulations in the past. First, we start with defining the mentioned simulation function type.

Definition 3.1. Let $s \geq 1$ be a given real number. A function $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is called a large s-simulation function if it satisfies (S_4). Throughout this paper, we denote by $\mathcal{C}_s$ the collection of all large s-simulation functions.

Directly from Definition 3.1, each s-simulation function is a large s-simulation function and so it is also a generalization of a simulation function. The reader can see the relation of various types of simulations from Figure 1.

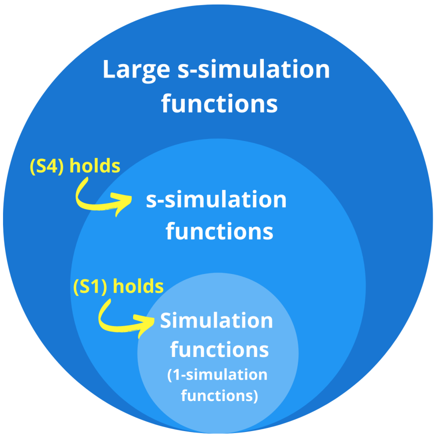

Figure 1: The relation of various types of simulations.

To claim the accurate proper generalization of a class of large s-simulation functions, we must give the example of a large s-simulation function, which is not an s-simulation function.

Example 3.2. Let $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ be defined by

$$\zeta(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha = 0; \\ 2\beta - 4\alpha - 1 & \text{if } \alpha \neq 0. \end{cases}$$

First, we will claim that for any $s \geq 1$, we obtain ζ is not an s-simulation function. To claim this, we will show that (S_2) does not hold. Choosing $\alpha,\beta \in (0,\infty)$ with $\beta \geq 3\alpha + 1$, we have

$$\begin{aligned} \zeta(\alpha,\beta) - (\beta - \alpha) &= (2\beta - 4\alpha - 1) - (\beta - \alpha) \\ &= \beta - 3\alpha - 1 \\ &\geq 0. \end{aligned}$$

It follows that $\zeta(\alpha,\beta) \geq \beta - \alpha$ and then (S_2) does not hold. Therefore, ζ is not an s-simulation function for all $s \geq 1$.

Next, we will show that $\zeta \in \mathcal{C}_2$, that is, ζ satisfies (S_4) with $s = 2$. To show this, we suppose that $\{\alpha_n\}, \{\beta_n\} \subseteq (0,\infty)$ are two sequences such that

$$0 < \liminf_{n\to\infty} \alpha_n \leq 2\left(\limsup_{n\to\infty} \beta_n\right) \leq 4\left(\liminf_{n\to\infty} \alpha_n\right)$$

and

$$0 < \liminf_{n\to\infty} \beta_n \leq 2\left(\limsup_{n\to\infty} \alpha_n\right) \leq 4\left(\liminf_{n\to\infty} \beta_n\right).$$

From all the above relations, we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \zeta(\alpha_n,\beta_n) &= \limsup_{n\to\infty}\ (2\beta_n - 4\alpha_n - 1) \\
&\leq 2\left(\limsup_{n\to\infty} \beta_n\right) - 4\left(\liminf_{n\to\infty} \alpha_n\right) - 1 \\
&\leq 4\left(\liminf_{n\to\infty} \alpha_n\right) - 4\left(\liminf_{n\to\infty} \alpha_n\right) - 1 \\
&< 0.
\end{aligned}$$

It yields that (S_4) holds. Therefore, $\zeta \in \mathcal{C}_2$, that is, ζ is a large 2-simulation function.

The above example can be extended to the following example:

Example 3.3. Let $A \in \mathbb{R}$, $B > 0$, $s \geq 2$ and $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by

$$\zeta(\alpha,\beta) = \begin{cases} A & \text{if } \alpha = 0; \\ s\beta - s^2\alpha - B & \text{if } \alpha \neq 0. \end{cases}$$

First, we will claim that ζ is not an s-simulation function by showing that (S_2) does not hold. Choosing $\alpha,\beta \in (0,\infty)$ with $\beta \geq \frac{1}{s-1}\left[(s^2-1)\alpha + B\right]$, we have

$$\begin{aligned}
\zeta(\alpha,\beta) - (\beta - \alpha) &= (s\beta - s^2\alpha - \beta) - (\beta - \alpha) \\
&= (s-1)\beta - (s^2-1)\alpha - B \\
&\geq 0.
\end{aligned}$$

This implies that $\zeta(\alpha,\beta) \geq \beta - \alpha$. Therefore, (S_2) does not hold and then ζ is not an s-simulation function.

Here, we will show that $\zeta \in \mathcal{C}_s$, that is, ζ satisfies (S_4) with arbitrary $s \geq 2$. Assume that $\{\alpha_n\}, \{\beta_n\} \subseteq (0,\infty)$ are two sequences such that

$$0 < \liminf_{n\to\infty} \alpha_n \leq s\left(\limsup_{n\to\infty} \beta_n\right) \leq s^2\left(\liminf_{n\to\infty} \alpha_n\right)$$

and

$$0 < \liminf_{n\to\infty} \beta_n \leq s \left(\limsup_{n\to\infty} \alpha_n \right) \leq s^2 \left(\liminf_{n\to\infty} \beta_n \right).$$

Then we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \zeta(\alpha_n, \beta_n) &= \limsup_{n\to\infty} \left(s\beta_n - s^2\alpha_n - B \right) \\
&\leq s \left(\limsup_{n\to\infty} \beta_n \right) - s^2 \left(\liminf_{n\to\infty} \alpha_n \right) - B \\
&\leq s^2 \left(\liminf_{n\to\infty} \alpha_n \right) - s^2 \left(\liminf_{n\to\infty} \alpha_n \right) - B \\
&< 0.
\end{aligned}$$

It follows that (S_4) holds. Therefore, $\zeta \in \mathcal{C}_s$, that is, ζ is a large s-simulation function.

4. The connection between large $\mathcal{Z}_s$-contraction mappings and Picard mappings

Based on the help of large s-simulation functions in the previous section, the new idea of a $\mathcal{Z}_s$-contraction mapping is defined. The class of new contractions covers classes of generalized contraction mappings related to simulation functions and classes of some famous generalized contraction mappings in fixed point theory from the past until the present (see Figure 2). Moreover, the connection between large $\mathcal{Z}_s$-contraction mappings and Picard mappings is investigated in this section.

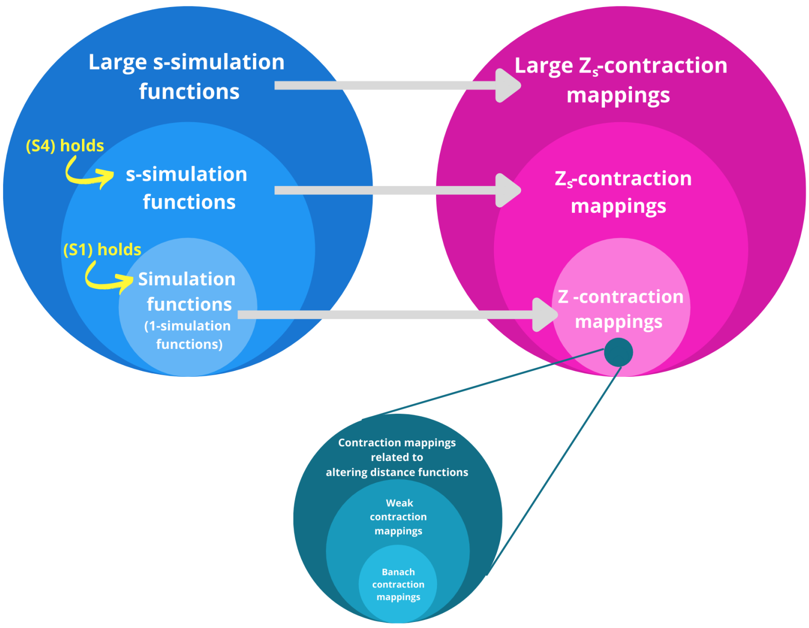

Figure 2: The relation of classes of important generalized contraction mappings.

Definition 4.1. Let (X,d) be a b-metric space with the coefficient $s \geq 1$. A mapping $T : X \to X$ is called a large $\mathcal{Z}_s$-contraction mapping if the following conditions hold:

(L_1) $d(Tx,Ty) < d(x,y)$ for all $x,y \in X$ with $x \neq y$;
(L_2) for all $\varepsilon > 0$, there is $\zeta \in \mathcal{C}_s$ such that

$$[\forall x,y \in X \text{ with } d(x,y) \geq \varepsilon] \Longrightarrow \zeta(d(Tx,Ty),d(x,y)) \geq 0. \tag{4.1}$$

Remark 4.2. For each b-metric space (X,d), we observe that if $T : X \to X$ is a $\mathcal{Z}_s$-contraction mapping with respect to an s-simulation function ζ_s, then T is also a large $\mathcal{Z}_s$-contraction mapping because (S_2) implies (L_1), and the $\mathcal{Z}_s$-contractive condition implies (4.1). But, the converse is not true in general.

Theorem 4.3. *Let (X,d) be a b-complete metric space with the coefficient $s \geq 1$ and $T : X \to X$ be a large s-simulation contraction mapping. Then T is a Picard mapping, that is, T has a unique fixed point $x_\star \in X$, and the Picard sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, where $x_0 \in X$, converges to the fixed point $x_\star$.*

Proof. It is easy to see that if T has a fixed point, then the uniqueness of its fixed point follows from (L_1). Hence, in the remaining proof of this theorem, we will show only the existence of fixed points of T. By fixing $x_0 \in X$, the proof is finish if $T^{n_0}x_0 = T^{n_0-1}x_0$ for some $n_0 \in \mathbb{N}$. This implies that $T^n x_0 \neq T^{n-1}x_0$ for all $n \in \mathbb{N}$, where T^0 is an identity mapping. It follows from (L_1) that for each $n \in \mathbb{N}$, we obtain

$$d(T^{n+1}x_0,T^n x_0) < d(T^n x_0,T^{n-1}x_0) < \cdots < d(Tx_0,x_0).$$

Thus, the sequence $\{\gamma_n := d(T^n x_0,T^{n-1}x_0)\}$ is strictly decreasing. It is easy to see that $\{\gamma_n\}$ is also bounded below. Then there exists $\inf\limits_{n\in\mathbb{N}} \gamma_n =: \gamma \geq 0$ such that $\lim\limits_{n\to\infty} \gamma_n = \gamma$. We will show that $\gamma = 0$. Assume that $\gamma > 0$. By using (L_2), there is $\zeta \in \mathcal{C}_s$ such that

$$\zeta(d(T^{n+1}x_0,T^n x_0),d(T^n x_0,T^{n-1}x_0)) \geq 0 \tag{4.2}$$

for all $n \in \mathbb{N}$ because $d(T^n x_0,T^{n-1}x_0) \geq \gamma$ for all $n \in \mathbb{N}$. Since $\zeta \in \mathcal{C}_s$, we obtain ζ satisfies (S_4) and then

$$\limsup_{n\to\infty} \zeta(d(T^{n+1}x_0,T^n x_0),d(T^n x_0,T^{n-1}x_0)) < 0. \tag{4.3}$$

Two inequlities (4.2) and (4.3) imply

$$0 \leq \limsup_{n\to\infty} \zeta(d(T^{n+1}x_0,T^n x_0),d(T^n x_0,T^{n-1}x_0)) < 0,$$

which is a contradiction. Hence, $\gamma = 0$ and so

$$\lim_{n\to\infty} d(T^n x_0,T^{n-1}x_0) = 0. \tag{4.4}$$

Consider the Picard sequence $\{x_n\}$ defined by

$$x_n = Tx_{n-1}$$

for all $n \in \mathbb{N}$. Let us prove that $x_n \neq x_m$ for all $n,m \in \mathbb{N}$. Indeed, assume that there are $n_0, m_0 \in \mathbb{N}$ such that $n_0 < m_0$ and $x_{n_0} = x_{m_0}$. Let $p_0 = m_0 - n_0$. Clearly, $p_0 \in \mathbb{N}$ and $p_0 \geq 2$. In this case, $x_{n_0+p_0} = x_{m_0} = x_{n_0}$. Furthermore,

$$x_{n_0+2p_0} = x_{n_0+p_0+p_0} = x_{m_0+p_0} = T^{p_0}x_{m_0} = T^{p_0}x_{n_0} = x_{n_0+p_0} = x_{m_0} = x_{n_0}.$$

By induction, it can be proved that $x_{n_0+k\cdot p_0} = x_{n_0}$ for all $k \in \mathbb{N}$. Therefore,

$$\{d\left(x_{n_0+k\cdot p_0}, x_{n_0+k\cdot p_0+1}\right)\}_{k\in\mathbb{N}} = \{d\left(x_{n_0}, x_{n_0+1}\right)\}_{k\in\mathbb{N}},$$

which is a posive constant sequence. However, this is a contradiction because $\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0$. As a consequence, we have proved that

$$x_n \neq x_m \quad \text{for all } n,m \in \mathbb{N}. \tag{4.5}$$

Next, we will show that the Picard sequence $\{x_n\}$ is a b-Cauchy sequence in X. Assume that $\{x_n\}$ is not a b-Cauchy sequence in (X,d). Then there exists $\varepsilon > 0$ and two subsequence $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that for each $k \in \mathbb{N}$, n_k is the smallest number such that

$$d(x_{n_k}, x_{m_k-1}) \leq \varepsilon < d(x_{n_k}, x_{m_k})$$

and $k \leq n_k < m_k$. By the triangular inequality, we have

$$\varepsilon < d(x_{n_k}, x_{m_k}) \leq s[d(x_{n_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k})] \leq s\varepsilon + sd(x_{m_k-1}, x_{m_k}).$$

Taking the limit superior as $k \to \infty$ in the above inequality and using (4.4), we get

$$\varepsilon \leq \limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq s\varepsilon. \tag{4.6}$$

In the same way, we have

$$\varepsilon \leq \liminf_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq s\varepsilon. \tag{4.7}$$

From the triangular inequality, we obtain

$$\begin{aligned} \varepsilon &< d(x_{n_k}, x_{m_k}) \\ &< d(x_{n_k-1}, x_{m_k-1}) \\ &\leq s[d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k-1})] \\ &\leq sd(x_{n_k-1}, x_{n_k}) + s\varepsilon. \end{aligned} \tag{4.8}$$

Taking the limit superior as $k \to \infty$ in the above inequality and using (4.4), we have

$$\varepsilon \leq \limsup_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) \leq s\varepsilon. \tag{4.9}$$

Similarly, we obtain

$$\varepsilon \leq \liminf_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) \leq s\varepsilon. \tag{4.10}$$

From (4.6), (4.7), (4.9) and (4.10), we have

$$\begin{aligned}
\varepsilon &< \liminf_{k\to\infty} d(x_{n_k}, x_{m_k}) \\
&\leq s\left(\limsup_{k\to\infty} d(x_{n_k-1}, x_{m_k-1})\right) \\
&\leq s^2\left(\liminf_{k\to\infty} d(x_{n_k}, x_{m_k})\right) \\
&\leq s^3\varepsilon \qquad (4.11)
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon &< \liminf_{n\to\infty} d(x_{n_k-1}, x_{m_k-1}) \\
&\leq s\left(\limsup_{n\to\infty} d(x_{n_k}, x_{m_k})\right) \\
&\leq s^2\left(\liminf_{n\to\infty} d(x_{n_k-1}, x_{m_k-1})\right) \\
&\leq s^3\varepsilon. \qquad (4.12)
\end{aligned}$$

Since $\varepsilon > 0$, from (4.8) and (L_2), there is $\zeta' \in \mathcal{C}_s$ such that

$$\zeta'(d(Tx_{n_k-1}, Tx_{m_k-1}), d(x_{n_k-1}, x_{m_k-1})) \geq 0 \qquad (4.13)$$

for all $k \in \mathbb{N}$. Since ζ' satisfies (S_4), from (4.11) and (4.12), we get

$$\limsup_{k\to\infty} \zeta'(d(x_{n_k}, x_{m_k}), d(x_{n_k-1}, x_{m_k-1})) < 0. \qquad (4.14)$$

It follows from (4.13) and (4.14) that

$$\begin{aligned}
0 &\leq \limsup_{k\to\infty} \zeta'(d(Tx_{n_k-1}, Tx_{m_k-1}), d(x_{n_k-1}, x_{m_k-1})) \\
&= \limsup_{k\to\infty} \zeta'(d(x_{n_k}, x_{m_k}), d(x_{n_k-1}, x_{m_k-1})) \\
&< 0,
\end{aligned}$$

which is a contradiction. Therefore, $\{x_n\}$ is a b-Cauchy sequence in X. Since (X,d) is a complete b-metric space, there exists $x_\star \in X$ such that $\lim_{n\to\infty} x_n = x_\star$, that is,

$$\lim_{n\to\infty} d(x_n, x_\star) = 0. \qquad (4.15)$$

Next, we are going to claim that $x_\star$ is a fixed point of T reasoning by contradiction. Assume to contrary that $x_\star$ is not a fixed point of T, that is, $Tx_\star \neq x_\star$ and hence

$$d(x_\star, Tx_\star) > 0.$$

From (4.15), there is $n_1 \in \mathbb{N}$ such that $d(x_n, x_\star) < d(x_\star, Tx_\star)$ for all $n \geq n_1$. In particular, $x_n \neq Tx_\star$ for all $n \geq n_1$, that is,

$$d(Tx_n, Tx_\star) = d(x_{n+1}, Tx_\star) > 0 \tag{4.16}$$

for all $n \geq n_1$.

On the other hand, it is impossible that there exists $n_2 \in \mathbb{N}$ such that $x_n = x_\star$ for all $n \geq n_2$. Hence, there exists a subsequence $\{x_{\sigma(n)}\}$ of $\{x_n\}$ such that

$$x_{\sigma(n)} \neq x_\star \tag{4.17}$$

for all $n \in \mathbb{N}$. Let $n_3 \in \mathbb{N}$ be such that $\sigma(n_3) \geq n_1$. Then, by (4.16) and (4.17), we have $d(x_{\sigma(n)}, x_\star) > 0$ and $d(Tx_{\sigma(n)}, Tx_\star) > 0$ for all $n \geq n_3$. By using the contractive of mapping T with $x_{\sigma(n)} \neq x_\star$, we have

$$0 \leq d(Tx_{\sigma(n)}, Tx_\star) < d(x_{\sigma(n)}, x_\star)$$

for all $n \geq n_3$. In particular, by (4.15), we obtain

$$x_{\sigma(n)+1} = Tx_{\sigma(n)} \to Tx_\star.$$

By the unicity of the limit, we get $x_\star = Tx_\star$, which is a contradiction with the fact that we have supposed that $Tx_\star \neq x_\star$. Therefore, $x_\star$ is a fixed point of T. This completes the proof. □

Now, we give an illustrative example supporting Theorem 4.3.

Example 4.4. Let $X = [0, \infty)$ and $d : X \times X \to [0, \infty)$ be defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y; \\ (x+y)^2 & \text{if } x \neq y, \end{cases}$$

for all $x, y \in X$. Therefore, (X, d) is a complete b-metric space with $s = 2$. Define $T : X \to X$ by

$$Tx = \begin{cases} \dfrac{x^2}{2} & \text{if } x \in [0,1); \\ \dfrac{x}{2} & \text{if } x \in [1,\infty). \end{cases}$$

Here, we will show that T is a large $\mathcal{Z}_s$-contraction mapping with $s = 2$. From the definition of T, it is easy to see that (L_1) holds, and then we will only show (L_2). Suppose that $\varepsilon > 0$. We will show that (4.1) holds with $\zeta \in \mathcal{C}_2$ defined by

$$\zeta(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha = 0; \\ 2\beta - 4\alpha - \varepsilon & \text{if } \alpha \neq 0 \end{cases}$$

(see the claim for $\zeta \in \mathcal{C}_2$ in Example 3.2). Assume that $x, y \in X$ with $d(x,y) \geq \varepsilon$. It implies that $x \neq y$. Next, we will divide into 4 cases.

Case I: For each $x, y \in [0,1)$, we obtain

$$\begin{aligned}
\zeta(d(Tx,Ty),d(x,y)) &= \zeta\left(\left(\frac{x^2}{2}+\frac{y^2}{2}\right)^2,(x+y)^2\right) \\
&= 2(x+y)^2-4\left(\frac{x^2}{2}+\frac{y^2}{2}\right)^2-\varepsilon \\
&\geq 2(x+y)^2-(x+y)^2-\varepsilon \\
&= (x+y)^2-\varepsilon \\
&= d(x,y)-\varepsilon \\
&\geq 0.
\end{aligned}$$

Case II: For each $(x,y) \in [0,1) \times [1,\infty)$, we obtain

$$\begin{aligned}
\zeta(d(Tx,Ty),d(x,y)) &= \zeta\left(\left(\frac{x^2}{2}+\frac{y}{2}\right)^2,(x+y)^2\right) \\
&= 2(x+y)^2-4\left(\frac{x^2}{2}+\frac{y}{2}\right)^2-\varepsilon \\
&\geq 2(x+y)^2-\left(x^2+y\right)^2-\varepsilon \\
&\geq 2(x+y)^2-(x+y)^2-\varepsilon \\
&= (x+y)^2-\varepsilon \\
&= d(x,y)-\varepsilon \\
&\geq 0.
\end{aligned}$$

Case III: For each $(x,y) \in [1,\infty) \times [0,1)$, we obtain

$$\begin{aligned}
\zeta(d(Tx,Ty),d(x,y)) &= \zeta\left(\left(\frac{x}{2}+\frac{y^2}{2}\right)^2,(x+y)^2\right) \\
&= 2(x+y)^2-4\left(\frac{x}{2}+\frac{y^2}{2}\right)^2-\varepsilon \\
&\geq 2(x+y)^2-\left(x+y^2\right)^2-\varepsilon \\
&\geq 2(x+y)^2-(x+y)^2-\varepsilon \\
&= (x+y)^2-\varepsilon \\
&= d(x,y)-\varepsilon \\
&\geq 0.
\end{aligned}$$

Case IV: For each $x,y \in [1,\infty)$, we obtain

$$\begin{aligned}
\zeta_s(d(Tx,Ty),d(x,y)) &= \zeta_s\left(\left(\frac{x}{2}+\frac{y}{2}\right)^2,(x+y)^2\right) \\
&= 2(x+y)^2-4\left(\frac{x}{2}+\frac{y}{2}\right)^2-\varepsilon \\
&= 2(x+y)^2-(x+y)^2-\varepsilon \\
&= (x+y)^2-\varepsilon \\
&= d(x,y)-\varepsilon \\
&\geq 0.
\end{aligned}$$

From all cases, we get (4.1) is satisfied. Hence, all conditions of Theorem 4.3 hold and so T is a Picard mapping. In this case, 0 is a fixed point of T. Figure 3 presents comparative results of Picard iterations with initial points $x_0 = 0.5, 5, 15, 25$. Figure 4 shows convergence behaviors of Picard iterations with initial points $x_0 = 0.5, 5, 15, 25$.

Step	$x_0 = 0.5$	$x_0 = 5$	$x_0 = 15$	$x_0 = 25$
1	0.1250000000	2.5000000000	7.5000000000	12.5000000000
2	0.0078125000	1.2500000000	3.7500000000	6.2500000000
3	0.0000305176	0.6250000000	1.8750000000	3.1250000000
4	0.0000000005	0.1953125000	0.9375000000	1.5625000000
5	0.0000000000	0.0190734863	0.4394531250	0.7812500000
6	0.0000000000	0.0001818989	0.0965595245	0.3051757813
7	0.0000000000	0.0000000165	0.0046618709	0.0465661287
8	0.0000000000	0.0000000000	0.0000108665	0.0010842022
9	0.0000000000	0.0000000000	0.0000000001	0.0000005877
10	0.0000000000	0.0000000000	0.0000000000	0.0000000000
11	0.0000000000	0.0000000000	0.0000000000	0.0000000000
12	0.0000000000	0.0000000000	0.0000000000	0.0000000000
⋮	⋮	⋮	⋮	⋮

Figure 3: Comparative results of Picard iterations with initial points $x_0 = 0.5, 5, 15, 25$ in Example 4.4

From the fact in Figure 2, we obtain the the following result covering many fixed point results in the literature.

Corollary 4.5 ([14])**.** *Let (X,d) be a b-complete metric space with the coefficient $s \geq 1$ and $T : X \to X$ be a $\mathcal{Z}_s$-simulation contraction mapping. Then T is a Picard mapping.*

Based on the variety of mappings in class $\mathcal{C}_s$, where $s \geq 1$, we get the following results.

Corollary 4.6. *Let (X,d) be a b-complete metric space with the coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying the following conditions:*

(BS_1) *$d(Tx,Ty) < d(x,y)$ for all $x,y \in X$ with $x \neq y$;*

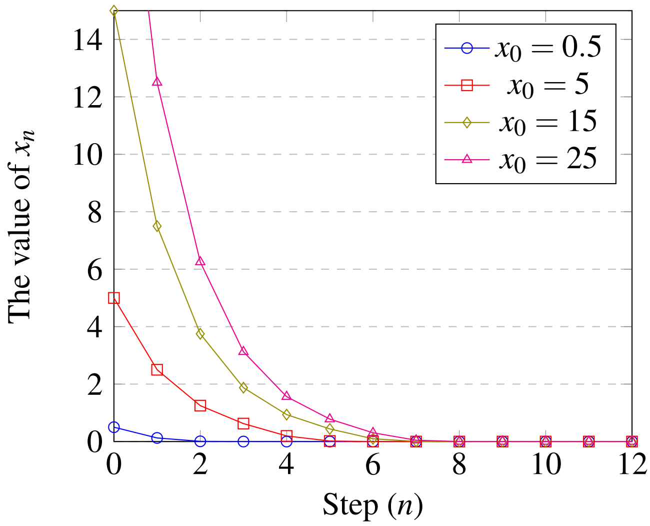

Figure 4: Behavior of the Picard iteration with initial points $x_0 = 0.5, 5, 15, 25$ in Example 4.4.

(BS_2) *for all* $\varepsilon > 0$, *there is* $k \in [0,1)$ *such that*

$$[\forall x, y \in X,\ \textit{with}\ d(x,y) \geq \varepsilon] \Longrightarrow sd(Tx,Ty) \leq kd(x,y). \tag{4.18}$$

Then T is a Picard mapping.

Corollary 4.7. *Let (X,d) be a complete metric space and $T : X \to X$ be a large Banach contraction mapping. Then T is a Picard mapping.*

5. Conclusions

This paper represents a significant advancement in the field of metric fixed point theory, particularly in the context of an evolution of the Banach contraction mapping principle. The introduction of simulation functions by Khojasteh et al. and the subsequent development of $\mathcal{Z}$-contraction mappings have marked a pivotal shift in the study of nonlinear mappings in metric spaces. Furthermore, adapting these concepts to b-metric spaces through the innovative work of Yamaod and Sintunavarat [14] has expanded the applicability of fixed point theory. This paper delves into the intricacies of large contraction mappings, initially presented by Burton [4]. This exploration underscores the versatility and practical relevance of fixed point theory. A novel contribution of this paper is the amalgamation of the concepts of s-simulation functions and large contraction mappings, a fusion that has yet to be explored. The introduction of large s-simulation functions and the establishment of large $\mathcal{Z}_s$-contraction mappings as Picard mappings open new avenues for research and application. The presented examples and numerical methods serve not only to validate our findings but also to offer a practical perspective on their implementation. As we conclude, it's evident that while this paper fills a crucial gap in the existing literature, it also sets the stage for further exploration. The potential for discoveries and applications in the realm of fixed point theory remains vast, and continued research in this field promises to yield even more groundbreaking results and solutions to complex problems.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

References

[1] S. Banach, Sur les operations dans les ensembles abstrait et leur application aux equations integrals, Fund. Math., 3 (1922), 133–181.

[2] V. Berinde, M. Pacurar, The early developments in fixed point theory on b-metric spaces: a brief survey and some important related aspects, Carrpathian J. Math. 38(3) (2022), 523–538.

[3] M. Boriceanu, M. Bota and A. Petrusel, Multivalued fractals in b-metric spaces, Cent. Eur. J. Math., 8(2) (2010), 367–377.

[4] T. A. Burton, Integral equations, implicit relations and fixed points, Proc. Amer. Math. Soc., 124 (1996), 2383-2390.

[5] M. Bota, A. Molnar and V. Csaba, On Ekeland's variational principle in b-metric spaces, Fixed Point Theory, 12 (2011), 21–28.

[6] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav., 1(1993), 5–11.

[7] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Semin. Mat. Fis. Univ. Modena, 46 (1998), 263–276.

[8] A. Dehiciy, M. B. Mesmouliz and E. Karapinar, On The Fixed Points Of Large-Kannan Contraction Mappings And Applications, Applied Mathematics E-Notes, 19 (2019), 535–551

[9] F. Khojasteh, S. Shukla and S. Radenovic, A New Approach to the Study of Fixed Point Theory for Simulation Functions, Filomat, 29:6 (2015), 1189–1194.

[10] A. F. Roldán López de Hierro, E. Karapınar, C. Roldán López de Hierro, J. Martínez-Moreno, Coincidence point theorems on metric spaces via simulation functions, Journal of Computational and Applied Mathematics 275 (2015), 345–355.

[11] K. Sawangsup, W. Sintunavarat, On solving nonlinear matrix equations in terms of b-simulation functions in b-metric spaces with numerical solutions, Computational and Applied Mathematics 37 (2018), 5829–5843.

[12] W. Sintunavarat, Nonlinear integral equations with new admissibility types in b-metric spaces, Journal of Fixed Point Theory and Applications 18 (2) (2016), 397–416.

[13] W. Sintunavarat, Fixed point results in b-metric spaces approach to the existence of a solution for nonlinear integral equations, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas 110 (2)(2016), 585–600.

[14] O. Yamaod and W. Sintunavarat, An approach to the existence and uniqueness of fixed point results in b-metric spaces via s-simulation functions, J. Fixed Point Theory Appl., 19 (2017), 2819–2830.