Numerical technique based on generalized Laguerre and shifted Chebyshev polynomials

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Abstract

In this study, we present a numerical scheme for solving a class of fractional-order partial differential equations. First, we introduce ψ-Laguerre polynomials, then, we employ ψ-shifted Chebyshev and ψ-Laguerre polynomials for the solution of space-time fractional-order differential equations. In our approach, we project ψ-shifted Chebyshev and ψ-Laguerre polynomials to develop operational matrices of fractional-order integration. The use of these orthogonal polynomials converts the problem under consideration into a system of algebraic equations. The solution of this system provides the unknown matrix which is then used to obtain the approximated numerical solution. Finally, some illustrative examples are included to observe the validity and applicability of the proposed method.

Keywords: Fractional derivatives ; ψ-shifted Chebyshev polynomials ; ψ-Laguerre polynomials ; Operational matrices of integration.

Mathematics Subject Classification:
35R11; 65M70.

1. Introduction

Fractional calculus is becoming a rapidly growing field in theory and applications. Mathematical modeling using fractional calculus is the best tool to demonstrate many real-world phenomenons successfully in engineering and physical sciences. In the past few decades, fractional order differential equations have gained wide-ranging applications in different fields like diffusion process [1], electrical activity in the heart [2], oscillation theory [3], thermal

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conductivity [4] and many more. For further understanding and application of fractional differential equation in the practical life, the development of different numerical techniques is a need of time now. Fractional-order partial differential equations can be used in the modification of various numerical techniques in natural sciences, physical sciences, optical and engineering technology, fluid mechanics and mechanics. Researchers have developed the analytical solutions of some mathematical models using Laplace transform method [5], Green's function method [6], Fourier method [7] etc, but generally, it is not possible to solve complex models involving fractional order ordinary and partial differential equations for analytic solution. So, various efficient numerical techniques are developed to avoid complexities in computation and difficulty in obtaining explicit analytical solutions.

Recently, researchers are working on numerical methods involving operational matrix of fractional-order differentiation and integration using different types of orthogonal polynomials. I. Talib and F. Ozger in [8] formulated generalized operational matrices in the sense of Riemann-Liouville and Hilfer fractional-order integral and differential operators using Hermite polynomials to develop a numerical scheme for the solution of Bagley-Torvik differential equation. The authors in [9] formulated numerical solutions of multiple non-linear fractional differential equations by developing operational matrices of the Caputo-Fabrizio derivative with the aid of shifted Chebyshev polynomials. In [10], Jafar Biazar and Khadijeh Sadri solved temporal fractional-order partial differential equations using two-variable Jacobi polynomials constructing operational matrices of the integration. S. Kumbinarasaiah et al. presented the solution of the non-linear Rosenau-Hyman equation using Hermite polynomials in [11]. The authors in [12] developed an analytical technique for the investigation of the new soliton configurations of 3D non-linear fractional model.

Fractional calculus deals with numerous definition of fractional integrals and derivatives with different kernels. Kilbas et al. [13] introduced the idea of fractional derivative and integral of a function with respect to some other function. O. P. Agarwal presented some generalized fractional operators of integration and differentiation in [14]. Recently, Ricardo Almeida in [15], introduced ψ-Caputo fractional operator generalizing a class of fractional derivatives. The authors in [16] presented definition and some properties of ψ-Hilfer fractional derivative. In application, a particular problem must be modeled using the given collection of data choosing specific type of operators, which one fits best. On the other hand, mathematically, we try to make the context as general as possible to use the results on different models. The knowledge of some specific desired properties in modeling a particular problem is useful for the choice of the most appropriate model. Thus, the proper approach of classes may be helpful in understanding a variety of emerging behaviors [17, 18].

D. Vivek et al. in [19] studied the existence and stability of solutions for partial differential equations using ψ-Caputo fractional derivative. In [20], the authors presented a numerical method for the solution of fractional differential equations containing generalized Caputo-type fractional derivatives developing generalized derivatives and integral operational matrices. A certain type of non-linear fractional pantograph differential equation for the existence and uniqueness of periodic solutions is investigated in [21]. The authors in [22] presented a numerical scheme to solve space-time fractional partial differential equations utilizing ψ-Caputo fractional derivative to compute differentiation matrices. H. Dehestani et al. in [23], developed operational and pseudo operational matrices of integer and fractional order to solve a class of fractional differential equation using Legendre and Laguerre polynomials. The authors in [24] formulated numerical solution of fractional differential equations of a certain class using hybrid Bernoulli polynomials and block pulse functions developing operational matrices of fractional-order integration.

Taking motivation by above cited work, we formulate a numerical technique for the solution of a class of fractional-order partial differential equations using these generalized orthogonal polynomials. We construct the operational matrices of fractional-order integration using modified polynomials named ψ-shifted Chebyshev and ψ-Laguerre polynomials. The use of these orthogonal polynomials help in reducing the problem into a system of algebraic equations. The unknown matrix is evaluated from this system of equations which is used to obtain the desired solution. Consider fractional-order partial differential equation

$$\mu(x,t)\frac{\partial^{\gamma,\psi}u(x,t)}{\partial x^{\gamma}}+\lambda(x,t)\frac{\partial^{\zeta,\psi}u(x,t)}{\partial t^{\zeta}}=h(x,t), \tag{1.1}$$

on the domain $\Delta=[0,L]\times[0,T]$ for initial and boundary conditions

$$\begin{aligned} u(x,0)&=q_0(x), && 0\le x\le L,\\ u(0,t)=p_0(t)\ \text{ and }\ u(L,t)&=p_1(t), && 0\le t\le T, \end{aligned}$$

where $0<\gamma,\ \zeta\le 2$, $\mu(x,t)$ and $\lambda(x,t)$ may be constants or variables, $h(x,t)$, $q_0(x)$, $p_0(x)$ and $p_1(x)$ are known functions.

The presented numerical scheme involves the idea of computing fractional-order differentiation and integration of one function with respect to some other function ψ. The structure of these specific integral and differential operators is kept in mind and the modification of classical polynomials like shifted Chebyshev and Laguerre is carried out in such a way that these involve the same function with respect to which fractional-order integration and differentiation is performed. This helps in the analytic and numerical assessment of modified polynomials. Generalized polynomials with suitable exponents are also helpful in error minimization and obtaining a better convergence rate. The choice of an appropriate basis for the generalized polynomials often reduces the computational cost and memory requirements of the resulting algorithm in certain classes of differential equations [25].

This article is organized as follows. Some preliminaries and lemmas from fractional calculus are recalled in Section 2. Section 3 is devoted to the introduction of ψ-shifted Chebyshev and ψ-Laguerre polynomials. The approximation of single and two variable functions is presented in Section 4. Operational matrices of fractional-order integration are obtained in Section 5. Section 6 deals with the formulation of the proposed numerical scheme. Error and convergence analysis is discussed in Section 7. In Section 8, some numerical examples are solved to show the applicability of the presented method.

2. Preliminaries

In this section, some basic notations and preliminaries are discussed that will be helpful in the presentation of our work.

Definition 2.1. [26] Consider $\vartheta > 0$ and a finite or infinite interval $\mathcal{I}$ on the real line $\mathbb{R}$. Let $\psi(x)$ be an increasing function with continuous derivative $\psi'(x)$ on (a, b). Then, the fractional integral of the integrable function u with respect to another function ψ is defined as

$$I_a^{\vartheta,\psi} u(x) := \frac{1}{\Gamma(\vartheta)} \int_a^x (\psi(x) - \psi(\rho))^{\vartheta-1} u(\rho)\psi'(\rho) d\rho,$$

and for function of two variables $u(x, t)$ with $u : \mathcal{I} \times \mathbb{R}^+ \to \mathbb{R}$, we define

$$I_{a,x}^{\vartheta,\psi} u(x, t) := \frac{1}{\Gamma(\vartheta)} \int_a^x (\psi(x) - \psi(\rho))^{\vartheta-1} u(\rho, t)\psi'(\rho) d\rho.$$

Definition 2.2. [15] Let $\vartheta > 0$, $\psi \in C^n(\mathcal{I}; \mathbb{R})$ where $n \in \mathbb{N}$, $\mathcal{I} = [a, b]$, and ψ is an increasing function with $\psi'(x) \neq 0\ \forall\ x \in \mathcal{I}$, then, we define ψ-Caputo fractional derivative as:
For $u \in C^n(\mathcal{I}; \mathbb{R})$

$$D_a^{\vartheta,\psi} u(x) := I_a^{n-\vartheta,\psi} u_\psi^{[n]}(x),$$

so that

$$D_a^{\vartheta,\psi}u(x) := \begin{cases} u_\psi^{[k]}(x), & if\ \vartheta = k \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\vartheta)}\int_a^x (\psi(x)-\psi(\rho))^{n-\vartheta-1}\psi'(\rho)u_\psi^{[n]}(\rho)d\rho, & if\ \vartheta \notin \mathbb{N}, \end{cases}$$

where Γ is the famous Gamma function and

$$u_\psi^{[n]}(x) := \Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n u(x). \tag{2.1}$$

For $u \in C^{n-1}(\mathcal{I};\mathbb{R})$

$$D_a^{\vartheta,\psi}u(x) := D_a^{\vartheta,\psi}\Big[u(x) - \sum_{l=0}^{n-1}\frac{u_\psi^{[l]}(a)}{l!}(\psi(x)-\psi(a))^l\Big],$$

with $n = \vartheta$ if $\vartheta \in \mathbb{N}$ and $n = [\vartheta]+1$ if $\vartheta \notin \mathbb{N}$, where $[\vartheta]$ represents integer part of ϑ. In case of $u(x,t)$ and $u : \mathcal{I}\times\mathbb{R}^+ \to \mathbb{R}$, we define
For $u \in C^n(\mathcal{J};\mathbb{R})$

$$D_{a,x}^{\vartheta,\psi}u(x,t) := I_{a,x}^{n-\vartheta,\psi}u_\psi^{[n]}(x,t),$$

so that

$$D_{a,x}^{\vartheta,\psi}u(x,t) := \begin{cases} u_\psi^{[k]}(x,t), & if\ \vartheta = k \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\vartheta)}\int_a^x (\psi(x)-\psi(\rho))^{n-\vartheta-1}\psi'(\rho)u_\psi^{[n]}(\rho,t)d\rho, & if\ \vartheta \notin \mathbb{N}, \end{cases}$$

For $u \in C^{n-1}(\mathcal{J};\mathbb{R})$

$$D_{a,x}^{\vartheta,\psi}u(x,t) := D_{a,x}^{\vartheta,\psi}\Big[u(x,t) - \sum_{l=0}^{n-1}\frac{u_\psi^{[l]}(a,t)}{l!}(\psi(x)-\psi(a))^l\Big],$$

where $\mathcal{J} = \mathcal{I}\times\mathbb{R}^+$ and $u_\psi^{[n]}(x,t) = \Big(\frac{1}{\psi'(x)}\frac{\partial}{\partial x}\Big)^n u(x,t)$.

Lemma 2.3. *[21] Suppose $\vartheta, \theta > 0$, then, semigroup property satisfied by ψ-fractional integral operator is*

$$I_a^{\theta,\psi}I_a^{\vartheta,\psi}u(x) = I_a^{\theta+\vartheta,\psi}u(x).$$

Lemma 2.4. *[21] Let $\vartheta > 0$, $\theta > 0$, then, fractional integral of $\mathrm{g}(x) = (\psi(x)-\psi(a))^{(\theta-1)}$ is*

$$I_a^{\vartheta,\psi}\mathrm{g}(x) = \frac{\Gamma(\theta)}{\Gamma(\theta+\vartheta)}(\psi(x)-\psi(a))^{\vartheta+\theta-1}.$$

Lemma 2.5. *[15] Let $\vartheta > 0$, $\theta > 0$, then,* $\mathrm{g}(x) = (\psi(x) - \psi(a))^{(\theta-1)}$ *has fractional derivative*

$$D_a^{\vartheta,\psi}\mathrm{g}(x) = \frac{\Gamma(\theta)}{\Gamma(\theta-\vartheta)}(\psi(x) - \psi(a))^{(\theta-\vartheta-1)}.$$

Theorem 2.6. *[15] For $u : [a, b] \to \mathbb{R}$, $n - 1 < \vartheta < n$ where $n \in \mathbb{N}$*
(i) if $u \in C^1[a, b]$, then, $D_a^{\vartheta,\psi} I_a^{\vartheta,\psi} u(x) = u(x)$,
(ii) if $u \in C^{n-1}[a, b]$, then, $I_a^{\vartheta,\psi} D_a^{\vartheta,\psi} u(x) = u(x) - \sum_{s=0}^{n-1} \frac{u_\psi^{[s]}(a)}{s!}(\psi(x) - \psi(a))^s$,
where $u_\psi^{[s]}(a)$ is defined in (2.1).

Definition 2.7. [26] Let ψ be an increasing function such that $\psi' \neq 0$ on the interval $\mathcal{I} = [a, b]$, then the space H_ψ^2 defined as

$$H_\psi^2(\mathcal{I}; \mathbb{R}) = \left\{\tilde{\mathfrak{g}} : \mathcal{I} \to \mathbb{R} \ : \tilde{\mathfrak{g}} \text{ is measurable and } \int_{\mathcal{I}} |\tilde{\mathfrak{g}}(x)|^2 \tau(x)\psi'(x)dx < \infty\right\},$$

where $\tau(x)$ is the weight function, is a Hilbert space. The inner product and the norm are defined as

$$(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})_{H_\psi^2(\mathcal{I};\mathbb{R})} = \int_{\mathcal{I}} \tilde{\mathfrak{g}}(x)\tilde{\mathfrak{h}}(x)\tau(x)\psi'(x)dx, \qquad \tilde{\mathfrak{g}}, \tilde{\mathfrak{h}} \in H_\psi^2(\mathcal{I}; \mathbb{R}),$$

$$\|\tilde{\mathfrak{g}}\|_{H_\psi^2(\mathcal{I};\mathbb{R})} = \left(\int_{\mathcal{I}} |\tilde{\mathfrak{g}}(x)|^2 \tau(x)\psi'(x)dx\right)^{\frac{1}{2}}, \quad \tilde{\mathfrak{g}} \in H_\psi^2(\mathcal{I}; \mathbb{R}).$$

3. Modified polynomials

Approximation by orthogonal family of functions is a topic of interest these days due to its wide application in science, engineering and many other fields. The most common orthogonal polynomials are Chebyshev, Hermite, Legendre, Jacobi, Laguerre, Genochhi etc. Original and modified forms of these orthogonal polynomials are used in different mathematical techniques to solve fractional differential equations. A. Baseri et al. in [27] formulated the numerical solution of diffusion equation by using rational Chebyshev functions. In [28], O. Postavaru and A. Toma presented a numerical method for two-dimensional fractional differential equations by developing fractional-order hybrid functions of block-pulse functions and Bernoulli polynomials. The authors in [22] presented a numerical scheme for the solution of a class of fractional partial differential equations by constructing the operational matrices of fractional differentiation using ψ-shifted Chebyshev polynomials.

3.1. ψ-shifted Chebyshev polynomials

Chebyshev polynomials have a vast range of applications in different fields. These orthogonal polynomials are defined as $T_k(x) = \cos(k \arccos(x))$ on the interval $[-1, 1]$. The orthogonality relation for these polynomials is

$$\int_{-1}^{1} T_q(x)\ T_l(x)\ \mathrm{W}(x)dx = \varepsilon_q \delta_{ql},$$

where $\mathrm{W}(x) = (1 - x^2)^{-\frac{1}{2}}$ is the weight function, $\varepsilon_0 = \pi$, $\varepsilon_q = \frac{\pi}{2}$ when $q \neq 0$ and δ_{ql} is the Kronecker delta. In analysis and computation, we use shifted Chebyshev polynomials that are defined on the interval $[0, 1]$ by introducing change of variable $t = 2x - 1$ as

$$T_k^*(x) = \cos(k \arccos(2x - 1)) = T_k(2x - 1), \qquad x \in [0, 1],$$

and orthogonality of shifted Chebyshev polynomial is

$$\int_{0}^{1} T_q^*(x)\ T_l^*(x)\ \mathrm{W}^*(x)dx = \varepsilon_q \delta_{ql},$$

where $\mathrm{W}^*(x) = (x - x^2)^{-\frac{1}{2}}$ is the weight function. The analytical form of shifted Chebyshev polynomials is [29]

$$T_m^*(x) = \sum_{i=0}^{m} \frac{m(-1)^{(m-i)}(2)^{2i}(m+i-1)!}{(2i)!(m-i)!}\ x^i, \qquad m > 0, \tag{3.1}$$

ψ-shifted Chebyshev polynomials which are modified form of shifted Chebyshev polynomials are defined as [29]

$$\mathfrak{T}_m^{*\psi}(x) = \sum_{i=0}^{m} \frac{m(-1)^{(m-i)}(2)^{2i}(m+i-1)!}{(2i)!(m-i)!}\ (\psi(x))^i, \qquad m > 0, \tag{3.2}$$

where $\mathfrak{T}_m^{*\psi}(x) = T_m^*(\psi(x))$. The first few ψ-shifted Chebyshev polynomials are $\mathfrak{T}_0^{*\psi}(x) = 1$, $\mathfrak{T}_1^{*\psi}(x) = 2\psi(x) - 1$, $\mathfrak{T}_2^{*\psi}(x) = 8(\psi(x))^2 - 8\psi(x) + 1$.

$$\int_{0}^{1} \mathfrak{T}_q^{*\psi}(x)\ \mathfrak{T}_l^{*\psi}(x)\ \mathrm{W}^{*\psi}(x)\psi'(x)dx = \varepsilon_q \delta_{ql},$$

is the orthogonality relation satisfied by ψ-shifted Chebyshev polynomials with weight function $\mathrm{W}^{*\psi}(x) = \frac{1}{\sqrt{\psi(x) - (\psi(x))^2}}$.

Lemma 3.1. *[29] The set $\{\mathfrak{T}_m^{*\psi}(x) : m \in \mathbb{N}_0\}$ is an orthonormal basis of the Hilbert space $H_\psi^2(\mathcal{I}; \mathbb{R})$.*

Fractional derivative of ψ-shifted Chebyshev polynomials is

$$ {}^{c}D_{a}^{\vartheta,\psi}\ \mathfrak{T}_{m}^{*\psi}(x) = \begin{cases} 0, & m < \lceil \vartheta \rceil, \\ \sum_{i=\lceil \vartheta \rceil}^{m} \frac{m(-1)^{(m-i)}(2)^{2i}(m+i-1)!i!}{(2i)!(m-i)!(i-\vartheta)!} (\psi(x))^{i-\vartheta}, & m \geq \lceil \vartheta \rceil, \end{cases} $$

where $\lceil \vartheta \rceil$ represents the smallest integer greater than or equal to ϑ and is called the ceiling function.

3.2. ψ-Laguerre polynomials

The classical Laguerre equation is

$$ t\frac{d^2u}{dt^2} + (1-t)\frac{du}{dt} + nu = 0, \qquad t > 0. \tag{3.3} $$

In application, we try to find a solution of Equation (3.3) which is finite for all finite values of t. So, we obtain a solution in series form which is an analytic expression for Laguerre polynomials [30].

$$ L_n(t) = \sum_{j=0}^{n} \frac{(-1)^j n!}{(n-j)!(j!)^2}\ t^j, \qquad n > 0, \tag{3.4} $$

with $L_0(t) = 1$. For Laguerre polynomials, the orthogonality relation with weight function $\mathcal{W}(t) = e^{-t}$ is

$$ \int_0^\infty L_p(t)\ L_s(t)\ \mathcal{W}(t)dt = \delta_{ps}. $$

Next, we define ψ-Laguerre polynomials by replacing t with $\psi(t)$ in Equation (3.4)

$$ \mathfrak{L}_n^{*\psi}(t) = \sum_{j=0}^{n} \frac{(-1)^j n!}{(n-j)!(j!)^2}\ (\psi(t))^j, \qquad n > 0, \tag{3.5} $$

where $\mathfrak{L}_n^{*\psi}(t) = L_n(\psi(t))$. A few ψ-Laguerre polynomials are $\mathfrak{L}_0^{*\psi}(t) = 1$, $\mathfrak{L}_1^{*\psi}(t) = 1 - \psi(t)$, $\mathfrak{L}_2^{*\psi}(t) = \frac{1}{2}(\psi(t))^2 - 2\psi(t) + 1$. It is easy to verify that ψ-Laguerre polynomials satisfy the orthogonality relation

$$ \int_0^\infty \mathfrak{L}_p^{*\psi}(t)\ \mathfrak{L}_s^{*\psi}(t)\ \mathcal{W}^{*\psi}(t)\psi'(t)dt = \delta_{ps}, $$

where $\mathcal{W}^{*\psi}(t) = e^{-\psi(t)}$ is the weight function for ψ-Laguerre polynomials. Fractional derivative of ψ-Laguerre polynomials is

$$ {}^{c}D_{a}^{\vartheta,\psi}\ \mathfrak{L}_{n}^{*\psi}(t) = \begin{cases} 0, & n < \lceil \vartheta \rceil, \\ \sum_{j=\lceil \vartheta \rceil}^{n} \frac{(-1)^j n!}{(n-j)!(j-\vartheta)!j!} (\psi(t))^{j-\vartheta}, & n \geq \lceil \vartheta \rceil. \end{cases} $$

4. Approximation of functions

In analysis and computation, we usually adopt the technique of series representation of a function using orthogonal polynomials. This technique mostly reduces the problem into a system of algebraic equations which is then solved to obtain the numerical solution of fractional differential equations.

4.1. Approximation of one variable functions

We represent any function $f(x)$ over $[0,1]$ in terms of ψ-shifted Chebyshev polynomials as

$$f(x) = \sum_{m=0}^{\infty} \mathfrak{b}_m \mathfrak{T}_m^{*\psi}(x),$$

where the expansion coefficients $\mathfrak{b}_m$ are calculated as

$$\mathfrak{b}_m = \frac{1}{\varepsilon_m} \int_0^1 f(x)\mathfrak{T}_m^{*\psi}(x)\mathrm{W}^{*\psi}(x)\psi'(x)dx, \qquad m = 0, 1, 2, \ldots.$$

For practical purposes, we use truncated series for $f(x)$ as

$$f(x) \simeq \sum_{m=0}^{M} \mathfrak{b}_m \mathfrak{T}_m^{*\psi}(x) = \mathfrak{B}^T \mathfrak{T}^{*\psi}(x), \tag{4.1}$$

where $\mathfrak{B} = [\mathfrak{b}_0, \mathfrak{b}_1, ..., \mathfrak{b}_M]^T$ is ψ-shifted Chebyshev coefficient vector and $\mathfrak{T}^{*\psi}(x) = [\mathfrak{T}_0^{*\psi}(x), \mathfrak{T}_1^{*\psi}(x)$ $, ..., \mathfrak{T}_M^{*\psi}(x)]^T$ is ψ-shifted Chebyshev vector.

Similarly, a function $g(t)$ defined over $[0,\infty)$ can be represented in terms of ψ-Laguerre polynomials as

$$g(t) = \sum_{n=0}^{\infty} \mathfrak{c}_n \mathfrak{L}_n^{*\psi}(t),$$

where the coefficients $\mathfrak{c}_n$ are given as

$$\mathfrak{c}_n = \int_0^{\infty} g(t)\mathfrak{L}_n^{*\psi}(t)\mathcal{W}^{*\psi}(t)\psi'(t)dt, \qquad n = 0, 1, 2, \ldots.$$

For practical applications, we will approximate $g(t)$ as

$$g(t) \simeq \sum_{n=0}^{N} \mathfrak{c}_n \mathfrak{L}_n^{*\psi}(t) = \mathfrak{C}^T \mathfrak{L}^{*\psi}(t), \tag{4.2}$$

where $\mathfrak{C} = [\mathfrak{c}_0, \mathfrak{c}_1, ..., \mathfrak{c}_N]^T$ and $\mathfrak{L}^{*\psi}(t) = [\mathfrak{L}_0^{*\psi}(t), \mathfrak{L}_1^{*\psi}(t), ..., \mathfrak{L}_N^{*\psi}(t)]^T$ are ψ-Laguerre coefficient vector and ψ-Laguerre vector respectively.

4.2. Approximation of two variable functions

In order to find the solution of two-dimensional fractional differential equations, we need to introduce two variable functions. These functions play an important role in the construction of numerical techniques for the solution of space-time fractional-order partial differential equations. The authors in [31] developed a numerical method to solve coupled system of fractional order partial differential equations constructing operational matrices of fractional integration and differentiation of two-variable functions. S. Sabermahani et al. in [32] presented a numerical technique to solve fractional differential equations by constructing one and two dimensional Muntz-Legendre functions. Thus, for computational purposes, we use ψ-shifted Chebyshev polynomials $\mathfrak{T}_m^{*\psi}(x)$ and ψ-Laguerre polynomials $\mathfrak{L}_n^{*\psi}(t)$ to define two-variable functions $\mathrm{P}_{mn}^{*\psi}(x,t)$ on the interval $(x,t) \in \nabla = [0,1] \times [0,\infty)$ as

$$\mathrm{P}_{mn}^{*\psi}(x,t) = \mathfrak{T}_m^{*\psi}(x)\mathfrak{L}_n^{*\psi}(t), \quad m = 0,1,\ldots,M, \quad n = 0,1,\ldots,N. \tag{4.3}$$

In Theorem 4.1, we see that $\mathrm{P}_{mn}^{*\psi}(x,t)$ forms orthogonal basis.

Theorem 4.1. *The two variable functions $\mathrm{P}_{mn}^{*\psi}(x,t)$ are orthogonal on $\nabla = [0,1] \times [0,\infty)$ with weight function $\mathbf{W}^{*\psi}(x,t) = \mathrm{W}^{*\psi}(x)\mathcal{W}^{*\psi}(t)$ and the orthogonality relation is*

$$\int_0^\infty \int_0^1 \mathrm{P}_{mn}^{*\psi}(x,t)\mathrm{P}_{ij}^{*\psi}(x,t)\mathbf{W}^{*\psi}(x,t)\psi'(x)\psi'(t)dxdt = \varepsilon_m \delta_{mi}\delta nj.$$

For proof see Theorem 4.1 in [29].

The function $k(x,t)$ is approximated by using ψ-shifted Chebyshev and ψ-Laguerre polynomials as

$$k(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathfrak{a}_{mn}\mathrm{P}_{mn}^{*\psi}(x,t),$$

where

$$\mathfrak{a}_{mn} = \frac{1}{\varepsilon_m}\int_0^\infty \int_0^1 k(x,t)\mathrm{P}_{mn}^{*\psi}(x,t)\mathbf{W}^{*\psi}(x,t)\psi'(x)\psi'(t)dxdt.$$

Thus, the truncated series for $k(x,t)$ is

$$k(x,t) \simeq \sum_{m=0}^{M}\sum_{n=0}^{N} \mathfrak{a}_{mn}\mathrm{P}_{mn}^{*\psi}(x,t) = (\mathfrak{T}^{*\psi}(x))^T A_{mn}\mathfrak{L}^{*\psi}(t),$$

where A_{mn} is the matrix of expansion coefficients.

5. Operational integration matrices

In this section, we will formulate operational matrices of fractional-order integration for ψ-shifted Chebyshev vector $\mathfrak{T}^{*\psi}(x)$ and ψ-Laguerre vector $\mathfrak{L}^{*\psi}(t)$ using properties of Riemann-Liouville fractional integration.

5.1. Operational matrices of fractional-order integration for ψ-shifted Chebyshev functions.

Theorem 5.1. *For ψ-shifted Chebyshev functions $\mathfrak{T}^{*\psi}(x)$, the fractional-order integration of order ξ for $\xi > 0$ is given by*

$$I_{a,x}^{\xi,\psi}(\mathfrak{T}^{*\psi}(x)) \simeq \mathcal{R}^{\xi,*\psi}\mathfrak{T}^{*\psi}(x), \tag{5.1}$$

where $(M+1)\times(M+1)$ dimensional matrix $\mathcal{R}^{\xi,*\psi}$ is the operational matrix of fractional-order integration of order ξ given as

$$\mathcal{R}^{\xi,*\psi} = \begin{bmatrix} \Theta_{0,0,0}^{\xi,*\psi} & \Theta_{0,1,0}^{\xi,*\psi} & \cdots & \Theta_{0,M,0}^{\xi,*\psi} \\ \sum_{i=0}^{1}\Theta_{1,0,i}^{\xi,*\psi} & \sum_{i=0}^{1}\Theta_{1,1,i}^{\xi,*\psi} & \cdots & \sum_{i=0}^{1}\Theta_{1,M,i}^{\xi,*\psi} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{M}\Theta_{M,0,i}^{\xi,*\psi} & \sum_{i=0}^{M}\Theta_{M,1,i}^{\xi,*\psi} & \cdots & \sum_{i=0}^{M}\Theta_{M,M,i}^{\xi,*\psi} \end{bmatrix}.$$

Proof. Applying $I_{a,x}^{\xi,\psi}$ to both sides of Equation (3.2)

$$I_{a,x}^{\xi,\psi}(\mathfrak{T}_m^{*\psi}(x)) = I_{a,x}^{\xi,\psi}\Big(\sum_{i=0}^{m}\frac{m(-1)^{(m-i)}(2)^{2i}(m+i-1)!}{(2i)!(m-i)!}\,(\psi(x))^i\Big).$$

Using linearity of fractional integrals and Lemma 2.4

$$I_{a,x}^{\xi,\psi}(\mathfrak{T}_m^{*\psi}(x)) = \sum_{i=0}^{m}\eta_{m,i}^{\xi,*\psi}(\psi(x))^{i+\xi}, \tag{5.2}$$

where

$$\eta_{m,i}^{\xi,*\psi} = \frac{m(-1)^{(m-i)}(2)^{2i}(m+i-1)!}{(2i)!(m-i)!}\times\frac{i!}{(i+\xi)!}.$$

In the next step, we will expand $(\psi(x))^{i+\xi}$ in terms of ψ-shifted Chebyshev polynomials.

$$(\psi(x))^{i+\xi} \simeq \sum_{p=0}^{M}\varphi_{i,p}^{\xi,*\psi}\mathfrak{T}_p^{*\psi}(x). \tag{5.3}$$

where the coefficient $\varphi_{i,p}^{\xi,*\psi}$ are calculated by

$$\varphi_{i,p}^{\xi,*\psi} = \frac{1}{\varepsilon_i}\int_0^1 (\psi(x))^{i+\xi}\mathfrak{T}_p^{*\psi}(x)\mathrm{W}^{*\psi}(x)\psi'(x)dx.$$

Combining (5.2) and (5.3)

$$I_{a,x}^{\xi,\psi}(\mathfrak{T}_m^{*\psi}(x)) = \sum_{p=0}^{M}\Big(\sum_{i=0}^{m}\Theta_{m,p,i}^{\xi,*\psi}\Big)\mathfrak{T}_p^{*\psi}(x), \tag{5.4}$$

where

$$\Theta_{m,p,i}^{\xi,*\psi} = \eta_{m,i}^{\xi,*\psi}\ \varphi_{i,p}^{\xi,*\psi}.$$

Equation (5.4) can be rewritten as

$$I_{a,x}^{\xi,\psi}(\mathfrak{T}_m^{*\psi}(x)) \simeq \Big[\sum_{i=0}^{m}\Theta_{m,0,i}^{\xi,*\psi}, \sum_{i=0}^{m}\Theta_{m,1,i}^{\xi,*\psi}, \cdots, \sum_{i=0}^{m}\Theta_{m,M,i}^{\xi,*\psi}\Big]\mathfrak{T}^{*\psi}(x).$$

Thus, we get the desired matrix of integration. □

5.2. Operational matrices of fractional-order integration for ψ-Laguerre functions.

Theorem 5.2. *For ψ-Laguerre functions $\mathfrak{L}^{*\psi}(t)$, the fractional-order integration of order β for $\beta > 0$ is given by*

$$I_{a,t}^{\beta,\psi}(\mathfrak{L}^{*\psi}(t)) \simeq \mathcal{P}^{\beta,*\psi}\mathfrak{L}^{*\psi}(t), \tag{5.5}$$

where $\mathcal{P}^{\beta,*\psi}$ is the operational matrix of fractional-order integration of order β with $(N+1)\times(N+1)$ dimension and

$$\mathcal{P}^{\beta,*\psi} = \begin{bmatrix} \Omega_{0,0,0}^{\beta,*\psi} & \Omega_{0,1,0}^{\beta,*\psi} & \cdots & \Omega_{0,N,0}^{\beta,*\psi} \\ \sum_{j=0}^{1}\Omega_{1,0,j}^{\beta,*\psi} & \sum_{j=0}^{1}\Omega_{1,1,j}^{\beta,*\psi} & \cdots & \sum_{j=0}^{1}\Omega_{1,N,j}^{\beta,*\psi} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=0}^{N}\Omega_{N,0,j}^{\beta,*\psi} & \sum_{j=0}^{N}\Omega_{N,1,j}^{\beta,*\psi} & \cdots & \sum_{j=0}^{N}\Omega_{N,N,j}^{\beta,*\psi} \end{bmatrix}.$$

Proof. Applying $I_{a,t}^{\beta,\psi}$ to both sides of Equation (3.2)

$$I_{a,t}^{\beta,\psi}(\mathfrak{L}_n^{*\psi}(t)) = I_{a,t}^{\beta,\psi}\Big(\sum_{j=0}^{n}\frac{(-1)^j n!}{(n-j)!(j!)^2}\ (\psi(t))^j\Big).$$

Using Lemma 2.4 and the linear property of fractional integrals

$$I_{a,t}^{\beta,\psi}(\mathfrak{L}_n^{*\psi}(t)) = \sum_{j=0}^{n} \Im_{n,j}^{\beta,*\psi}(\psi(t))^{j+\beta}, \tag{5.6}$$

where

$$\Im_{n,j}^{\beta,*\psi} = \frac{(-1)^j n!}{(n-j)!(j!)(j+\beta)!}.$$

In the next step, we will expand $(\psi(t))^{j+\beta}$ in terms of ψ-Laguerre polynomials.

$$(\psi(t))^{j+\beta} \simeq \sum_{q=0}^{N} \varrho_{j,q}^{\beta,*\psi} \mathfrak{L}_q^{*\psi}(t). \tag{5.7}$$

Combining (5.6) and (5.7)

$$I_{a,t}^{\beta,\psi}(\mathfrak{L}_n^{*\psi}(t)) = \sum_{q=0}^{N} \Big(\sum_{j=0}^{n} \Omega_{n,q,j}^{\beta,*\psi} \Big) \mathfrak{L}_q^{*\psi}(t), \tag{5.8}$$

where

$$\Omega_{n,q,j}^{\beta,*\psi} = \Im_{n,j}^{\beta,*\psi} \ \varrho_{j,q}^{\beta,*\psi}.$$

Equation (5.8) can be rewritten as

$$I_{a,t}^{\beta,\psi}(\mathfrak{L}_n^{*\psi}(t)) \simeq \Big[\sum_{j=0}^{n} \Omega_{n,0,j}^{\beta,*\psi}, \sum_{j=0}^{n} \Omega_{n,1,j}^{\beta,*\psi}, \cdots, \sum_{j=0}^{n} \Omega_{n,N,j}^{\beta,*\psi} \Big] \mathfrak{L}^{*\psi}(t).$$

So, the matrix of integration is achieved. □

6. Numerical formulation

This section is specified for the development of numerical scheme using operational matrices of fractional-order integration of ψ-shifted Chebyshev and ψ-Laguerre polynomials for the solution of space-time fractional differential equations of integer and non-integer order. For this purpose, let us take

$$\frac{\partial^{\gamma,\psi} u(x,t)}{\partial x^{\gamma}} = v(x,t). \tag{6.1}$$

Applying $I_{a,x}^{\gamma,\psi}$ to both sides and using Theorem 2.6

$$u(x,t) = I_{a,x}^{\gamma,\psi} v(x,t) + \Psi_1(x,t), \tag{6.2}$$

where

$$\Psi_1(x,t) = \sum_{s=0}^{k-1} \frac{D_{a,x}^{s,\psi} u(a,t)}{s!} (\psi(x) - \psi(a))^s,$$

with

$$D_{a,x}^{s,\psi} u(a,t) = \Big(\frac{1}{\psi'(x)} \frac{\partial}{\partial x}\Big)^s u(a,t). \tag{6.3}$$

Taking the derivative with respect to t of order ζ to both sides of Equation (6.2)

$$\frac{\partial^{\zeta,\psi} u(x,t)}{\partial t^\zeta} = I_{a,x}^{\gamma,\psi} D_{a,t}^{\zeta,\psi} v(x,t) + \omega_1(x,t), \tag{6.4}$$

where $\omega_1(x,t) = D_{a,t}^{\zeta,\psi}(\Psi_1(x,t))$.
Using Equation (6.1) and (6.4) in Equation (1.1)

$$\mu(x,t)v(x,t) + \lambda(x,t) I_{a,x}^{\gamma,\psi} D_{a,t}^{\zeta,\psi} v(x,t) = \omega_2(x,t), \tag{6.5}$$

where

$$\omega_2(x,t) = h(x,t) - \lambda(x,t)\omega_1(x,t).$$

Next, suppose

$$D_{a,t}^{\zeta,\psi} v(x,t) = (\mathfrak{T}^{*\psi}(x))^T N(\mathfrak{L}^{*\psi}(t)), \tag{6.6}$$

$\mathfrak{T}^{*\psi}(x)$ and $\mathfrak{L}^{*\psi}(t)$ are ψ-shifted Chebyshev and ψ-Laguerre vectors and N is a square matrix. Applying $I_{a,t}^{\zeta,\psi}$ to both sides and using Theorem 2.6

$$v(x,t) = (\mathfrak{T}^{*\psi}(x))^T N \mathcal{P}^{\zeta,*\psi} \mathfrak{L}^{*\psi}(t) + \Psi_2(x,t), \tag{6.7}$$

where

$$\Psi_2(x,t) = \sum_{j=0}^{l-1} \frac{D_{a,t}^{j,\psi} v(x,a)}{j!} (\psi(t) - \psi(a))^j,$$

and

$$D_{a,t}^{j,\psi} v(x,a) = \Big(\frac{1}{\psi'(t)} \frac{\partial}{\partial t}\Big)^j v(x,a). \tag{6.8}$$

In addition, we have

$$I_{a,x}^{\gamma,\psi}D_{a,t}^{\zeta,\psi}v(x,t) = I_{a,x}^{\gamma,\psi}\big((\mathfrak{T}^{*\psi}(x))^T N(\mathfrak{L}^{*\psi}(t))\big) = (\mathcal{R}^{\gamma,*\psi}\mathfrak{T}^{*\psi}(x))^T N\mathfrak{L}^{*\psi}(t). \tag{6.9}$$

Using Equation (6.7) and (6.9) in Equation (6.5), we get

$$\mu(x,t)\Big((\mathfrak{T}^{*\psi}(x))^T N\mathcal{P}^{\zeta,*\psi}\mathfrak{L}^{*\psi}(t)\Big) + \lambda(x,t)\Big((\mathcal{R}^{\gamma,*\psi}\mathfrak{T}^{*\psi}(x))^T N\mathfrak{L}^{*\psi}(t)\Big) = \omega_3(x,t), \tag{6.10}$$

where $\omega_3(x,t) = \omega_2(x,t) - \mu(x,t)\Psi_2(x,t)$. Next, we reduce Equation (6.10) to Sylvester equation and obtain square matrix N. Finally, using Equation (6.7) in Equation (6.2), we have

$$u(x,t) = (\mathcal{R}^{\gamma,*\psi}\mathfrak{T}^{*\psi}(x))^T N(\mathcal{P}^{\zeta,*\psi}\mathfrak{L}^{*\psi}(t)) + \Psi_3(x,t),$$

where

$$\Psi_3(x,t) = \Psi_1(x,t) + I_{a,x}^{\gamma,\psi}(\Psi_2(x,t)).$$

So, we have a system of algebraic equations that is numerically solved to obtain the proposed solution $u(x,t)$.

7. Error analysis

In this section, we will calculate the error bounds for the presented numerical approximation of fractional integral by the shifted Chebyshev polynomials.

Lemma 7.1. *[33] Suppose $\widetilde{H}$ with $dim\widetilde{H} < \infty$ is a closed subspace of Hilbert space H. Let $\{\psi_1, \psi_2, ..., \psi_m\}$ is any basis of $\widetilde{H}$ and ψ be an arbitrary element in H such that its unique best approximation is ψ^* out of $\widetilde{H}$, then*

$$\|\psi - \psi^*\| = \left(\frac{G(\psi, \psi_1, ..., \psi_m)}{G(\psi_1, \psi_2, ..., \psi_m)}\right)^{1/2},$$

$$G(\psi, \psi_1, ..., \psi_m) = \begin{bmatrix} \langle\psi,\psi\rangle & \langle\psi,\psi_1\rangle & \cdots & \langle\psi,\psi_m\rangle \\ \langle\psi_1,\psi\rangle & \langle\psi_1,\psi_1\rangle & \cdots & \langle\psi_1,\psi_m\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_m,\psi\rangle & \langle\psi_m,\psi_1\rangle & \cdots & \langle\psi_m,\psi_m\rangle \end{bmatrix}.$$

Lemma 7.2. *[23] Let us approximate $\widehat{f} \in L^2[0,1]$ by ψ-shifted Chebyshev polynomials as*

$$\widehat{f} \simeq \widehat{f}_M = \sum_{m=0}^{M} \widehat{b}_m \mathfrak{T}_m^{*\psi}(x), \text{ then, we have}$$

$$\lim_{M\to\infty} \mathcal{L}_M(\widehat{f}) = 0,$$

where

$$\mathcal{L}_M(\widehat{f}) = \int_0^1 \left[\widehat{f}(x) - \widehat{f}_M(x)\right]^2 dx.$$

Lemma 7.3. *[23] Let us approximate $\widehat{g} \in L^2[0,T]$ using ψ-Laguerre polynomials as*

$$\widehat{g} \simeq \widehat{g}_N = \sum_{n=0}^{N} \widehat{c}_n \mathfrak{L}_n^{*\psi}(t), \text{ then, we have}$$

$$\lim_{N\to\infty} \widetilde{\mathcal{L}}_N(\widehat{g}) = 0,$$

where

$$\widetilde{\mathcal{L}}_N(\widehat{g}) = \int_0^T \left[\widehat{g}(t) - \widehat{g}_N(t)\right]^2 dt.$$

Suppose $E_x^{\xi,*\psi}$ is the error vector of fractional-order integration, then, for ψ-shifted Chebyshev vector $\mathfrak{T}^{*\psi}(x)$, we have

$$E_x^{\xi,*\psi} = I_{a,x}^{\xi,\psi}(\mathfrak{T}^{*\psi}(x)) - \mathcal{R}^{\xi,*\psi}\mathfrak{T}^{*\psi}(x), \quad E_x^{\xi,*\psi} = [\mathbf{e}_{x,m}^{\xi,*\psi}], \quad m = 0,1,\ldots,M.$$

Using Equation (5.3) and Lemma 7.1

$$\|(\psi(x))^{i+\xi} - \sum_{p=0}^{M} \varphi_{i,p}^{\xi,*\psi}\mathfrak{T}_p^{*\psi}(x)\| = \left(\frac{G\big((\psi(x))^{i+\xi}, \mathfrak{T}_0^{*\psi}(x), \cdots, \mathfrak{T}_M^{*\psi}(x)\big)}{G\big(\mathfrak{T}_0^{*\psi}(x), \mathfrak{T}_1^{*\psi}(x), \cdots, \mathfrak{T}_M^{*\psi}(x)\big)}\right)^{1/2}.$$

Using Equations (5.1-5.4), we have

$$\begin{aligned} \left\|\mathbf{e}_{x,m}^{\xi,*\psi}\right\| &\leq \Big|\sum_{i=0}^{m} \eta_{m,i}^{\xi,*\psi}\Big| \, \Big\|(\psi(x))^{i+\xi} - \sum_{p=0}^{M} \varphi_{i,p}^{\xi,*\psi}\mathfrak{T}_p^{*\psi}(x)\Big\| \\ &\leq \sum_{i=0}^{m} \eta_{m,i}^{\xi,*\psi}\mathfrak{G}_m^{*\psi}(x), \qquad m = 0,1,\ldots,M, \end{aligned} \tag{7.1}$$

where

$$\mathfrak{G}_m^{*\psi}(x) = \left(\frac{G\big((\psi(x))^{i+\xi}, \mathfrak{T}_0^{*\psi}(x), \cdots, \mathfrak{T}_M^{*\psi}(x)\big)}{G\big(\mathfrak{T}_0^{*\psi}(x), \mathfrak{T}_1^{*\psi}(x), \cdots, \mathfrak{T}_M^{*\psi}(x)\big)} \right)^{1/2}.$$

Let $\overline{E}_t^{\beta,*\psi}$ is the error vector of fractional-order integration of ψ-Laguerre functions, then

$$\overline{E}_t^{\beta,*\psi} = I_{a,t}^{\beta,\psi}(\mathfrak{L}^{*\psi}(t)) - \mathcal{P}^{\beta,*\psi}\mathfrak{L}^{*\psi}(t), \quad \overline{E}_t^{\beta,*\psi} = [\overline{\mathbf{e}}_{t,n}^{\beta,*\psi}], \quad n = 0, 1, \ldots, N.$$

Using Equation (5.7) and Lemma 7.1

$$\|(\psi(t))^{j+\beta} - \sum_{q=0}^{N} \varrho_{j,q}^{\beta,*\psi}\mathfrak{L}_q^{*\psi}(t)\| = \left(\frac{G\big((\psi(t))^{j+\beta}, \mathfrak{L}_0^{*\psi}(t), \cdots, \mathfrak{L}_N^{*\psi}(t)\big)}{G\big(\mathfrak{L}_0^{*\psi}(t), \mathfrak{L}_1^{*\psi}(t), \cdots, \mathfrak{L}_N^{*\psi}(t)\big)} \right)^{1/2}.$$

Using Equations (5.5-5.8), we have

$$\begin{aligned} \big\|\overline{\mathbf{e}}_{t,n}^{\beta,*\psi}\big\| &\leq \Big| \sum_{j=0}^{n} \Im_{n,j}^{\beta,*\psi} \Big| \, \Big\|(\psi(t))^{j+\beta} - \sum_{q=0}^{N} \varrho_{j,q}^{\beta,*\psi}\mathfrak{L}_q^{*\psi}(t)\Big\| \\ &\leq \sum_{j=0}^{n} \Im_{n,j}^{\beta,*\psi}\widehat{\mathfrak{G}}_n^{*\psi}(t), \qquad n = 0, 1, \ldots, N, \end{aligned} \tag{7.2}$$

where

$$\widehat{\mathfrak{G}}_n^{*\psi}(t) = \left(\frac{G\big((\psi(t))^{j+\beta}, \mathfrak{L}_0^{*\psi}(t), \cdots, \mathfrak{L}_N^{*\psi}(t)\big)}{G\big(\mathfrak{L}_0^{*\psi}(t), \mathfrak{L}_1^{*\psi}(t), \cdots, \mathfrak{L}_N^{*\psi}(t)\big)} \right)^{1/2}.$$

Next, suppose that $\widetilde{\mathrm{P}}^{*\psi}$ is rearrangement of $\mathrm{P}^{*\psi}$ with respect to x, then the error vector of fractional-order integration for $\widetilde{\mathrm{P}}^{*\psi}$ is

$$\widetilde{\mathcal{E}}_x^{\xi,*\psi} = I_{a,x}^{\xi,\psi}(\widetilde{\mathrm{P}}^{*\psi}) - \widetilde{\mathcal{R}}^{\xi,*\psi}\widetilde{\mathrm{P}}^{*\psi},$$

where $\widetilde{\mathrm{P}}^{*\psi} = [\widetilde{\mathrm{P}}_{00}^{*\psi}, \cdots, \widetilde{\mathrm{P}}_{M0}^{*\psi}, \cdots, \widetilde{\mathrm{P}}_{0N}^{*\psi}, \cdots, \widetilde{\mathrm{P}}_{MN}^{*\psi}]^T$ and $\widetilde{\mathcal{R}}^{\xi,*\psi}$ is the diagonal matrix with each diagonal entry $\mathcal{R}^{\xi,*\psi}$ and each non-diagonal entry zero matrix of $(M+1)\times(M+1)$ dimension.
Also, we have
$\widetilde{\mathcal{E}}_x^{\xi,*\psi} = [\widetilde{E}_0^{\xi,*\psi}, \widetilde{E}_1^{\xi,*\psi}, \cdots, \widetilde{E}_N^{\xi,*\psi}]^T$ with $\widetilde{E}_n^{\xi,*\psi} = [\mathfrak{e}_{0n}^{\xi,*\psi}, \mathfrak{e}_{1n}^{\xi,*\psi}, \cdots, \mathfrak{e}_{Mn}^{\xi,*\psi}]^T$.
For $m = 0, 1, \cdots, M$ *and* $n = 0, 1, \cdots, N$, we have

$$\widetilde{E}_n^{\xi,*\psi} = [\mathfrak{e}_{mn}^{\xi,*\psi}] = I_{a,x}^{\xi,\psi}(\mathfrak{T}_m^{*\psi}(x)\mathfrak{L}_n^{*\psi}(t)) - \mathcal{R}^{\xi,*\psi}\mathfrak{T}_m^{*\psi}(x)\mathfrak{L}_n^{*\psi}(t).$$

Thus

$$\begin{aligned}
\|\widetilde{E}_n^{\xi,*\psi}\| &= \|[I_{a,x}^{\xi,\psi}\mathfrak{T}_m^{*\psi}(x) - \mathcal{R}^{\xi,*\psi}\mathfrak{T}_m^{*\psi}(x)]\mathfrak{L}_n^{*\psi}(t)\| \\
&= \Big(\int_0^\infty\int_0^1 \big|[I_{a,x}^{\xi,\psi}\mathfrak{T}_m^{*\psi}(x) - \mathcal{R}^{\xi,*\psi}\mathfrak{T}_m^{*\psi}(x)]\mathfrak{L}_n^{*\psi}(t)\big|^2 \mathcal{W}^{*\psi}(t)\psi'(x)\psi'(t)dxdt\Big)^{1/2} \\
&\leq \Big(\int_0^1 \big|I_{a,x}^{\xi,\psi}\mathfrak{T}_m^{*\psi}(x) - \mathcal{R}^{\xi,*\psi}\mathfrak{T}_m^{*\psi}(x)\big|^2\psi'(x)dx\Big)^{1/2}\Big(\int_0^\infty \big|\mathfrak{L}_n^{*\psi}(t)\big|^2\mathcal{W}^{*\psi}(t)\psi'(t)dt\Big)^{1/2} \\
&= \|I_{a,x}^{\xi,\psi}\mathfrak{T}_m^{*\psi}(x) - \mathcal{R}^{\xi,*\psi}\mathfrak{T}_m^{*\psi}(x)\|\|\mathfrak{L}_n^{*\psi}(t)\| \\
&\leq \sum_{i=0}^m \eta_{m,i}^{\xi,*\psi}\mathfrak{G}_m^{*\psi}(x). \qquad (7.3)
\end{aligned}$$

Further, consider $\widehat{\mathrm{P}}^{*\psi}$ is rearrangement of $\mathrm{P}^{*\psi}$ with respect to t, then, we define the error vector $\widehat{\mathrm{P}}^{*\psi}$ as

$$\widehat{\mathcal{E}}_t^{\beta,*\psi} = I_{a,t}^{\beta,\psi}(\widehat{\mathrm{P}}^{*\psi}) - \widehat{\mathcal{P}}^{\beta,*\psi}\widehat{\mathrm{P}}^{*\psi},$$

where $\widehat{\mathrm{P}}^{*\psi} = [\widehat{\mathrm{P}}_{00}^{*\psi}, \cdots, \widehat{\mathrm{P}}_{0N}^{*\psi}, \cdots, \widehat{\mathrm{P}}_{M0}^{*\psi}, \cdots, \widehat{\mathrm{P}}_{MN}^{*\psi}]^T$ and $\widehat{\mathcal{P}}^{\beta,*\psi}$ is the diagonal matrix with each diagonal entry $\mathcal{P}^{\beta,*\psi}$ and each non-diagonal entry zero matrix of $(N+1)\times(N+1)$ dimension. We have
$\widehat{\mathcal{E}}_t^{\beta,*\psi} = [\widehat{E}_0^{\beta,*\psi}, \widehat{E}_1^{\beta,*\psi}, \cdots, \widehat{E}_M^{\beta,*\psi}]^T$ with $\widehat{E}_m^{\beta,*\psi} = [\widehat{\mathfrak{e}}_{m0}^{\beta,*\psi}, \widehat{\mathfrak{e}}_{m1}^{\beta,*\psi}, \cdots, \widehat{\mathfrak{e}}_{mN}^{\beta,*\psi}]^T$,
and

$$\widehat{E}_m^{\beta,*\psi} = [\widehat{\mathfrak{e}}_{mn}^{\beta,*\psi}] = I_{a,t}^{\beta,\psi}(\mathfrak{T}_m^{*\psi}(x)\mathfrak{L}_n^{*\psi}(t)) - \mathcal{P}^{\beta,*\psi}\mathfrak{T}_m^{*\psi}(x)\mathfrak{L}_n^{*\psi}(t).$$

Thus

$$\begin{aligned}
\|\widehat{E}_m^{\beta,*\psi}\| &= \|\mathfrak{T}_m^{*\psi}(x)[I_{a,t}^{\beta,\psi}\mathfrak{L}_n^{*\psi}(t) - \mathcal{P}^{\beta,*\psi}\mathfrak{L}_n^{*\psi}(t)]\| \\
&= \Big(\int_0^\infty\int_0^1 \big|\mathfrak{T}_m^{*\psi}(x)[I_{a,t}^{\beta,\psi}\mathfrak{L}_n^{*\psi}(t) - \mathcal{P}^{\beta,*\psi}\mathfrak{L}_n^{*\psi}(t)]\big|^2 W^{*\psi}(x)\psi'(x)\psi'(t)dxdt\Big)^{1/2} \\
&\leq \Big(\int_0^1 \big|\mathfrak{T}_m^{*\psi}(x)\big|^2 W^{*\psi}(x)\psi'(x)dx\Big)^{1/2}\Big(\int_0^\infty \big|I_{a,t}^{\beta,\psi}\mathfrak{L}_n^{*\psi}(t) - \mathcal{P}^{\beta,*\psi}\mathfrak{L}_n^{*\psi}(t)\big|^2\psi'(t)dt\Big)^{1/2} \\
&= \|\mathfrak{T}_m^{*\psi}(x)\|\|I_{a,t}^{\beta,\psi}\mathfrak{L}_n^{*\psi}(t) - \mathcal{P}^{\beta,*\psi}\mathfrak{L}_n^{*\psi}(t)\| \\
&\leq \varepsilon_m \sum_{j=0}^n \Im_{n,j}^{\beta,*\psi}\widehat{\mathfrak{G}}_n^{*\psi}(t). \qquad (7.4)
\end{aligned}$$

From above discussion, it can be concluded that the error vectors tend to be zero as we increase the number of ψ-shifted Chebyshev and ψ-Laguerre functions bases.

8. Numerical results

Example 8.1. Consider (1.1) with $\mu(x,t) = \lambda(x,t) = 1$, $0 < \zeta, \gamma \leq 2$, initial-boundary conditions $u(x,0) = 0 = u(0,t)$ and $u(1,t) = (\psi(t))^2$. The exact solution is $u(x,t) = (\psi(x))^2(\psi(t))^2$ and the function $h(x,t)$ is

$$h(x,t) = \frac{2}{\Gamma(3-\gamma)}(\psi(x))^{(2-\gamma)}(\psi(t))^2 + \frac{2}{\Gamma(3-\zeta)}(\psi(x))^2(\psi(t))^{(2-\zeta)}.$$

We will solve problem 8.1 by applying the proposed technique for three different chosen functions ψ. ψ is required to be increasing and $\psi' \neq 0$ for all $x \in [0,L]$, $t \in [0,T]$. We will test the method for the following choices of ψ.

- $\psi_1(x) = x,\ \psi_1(t) = t$;
- $\psi_2(x) = \frac{1}{4}x(x^3 + x^2 + x + 1),\ \psi_2(t) = \frac{1}{4}t(t^3 + t^2 + t + 1)$;
- $\psi_3(x) = \frac{1}{\log 3}x\log(x+2),\ \psi_3(t) = \frac{1}{\log 3}t\log(t+2)$.

Table 1 shows absolute errors for fractional and integral values of γ and ζ, applying the proposed technique for three different functions ψ. From the Table 1, we can observe that error is mostly increasing with the increase in the values of x and t except for a few points. In Table 2, we have presented exact and proposed numerical solutions at $\gamma = 1.8,\ \zeta = 1.3$ for $u(x,1)$ taking different ψ. From Table 2, we can observe that both exact and proposed solutions are very close. The exact and proposed solution for ψ_1 at $\gamma = 1.8,\ \zeta = 1.3$ is plotted in Figure 1. Figure 2-4 represents the absolute error analysis for multiple ψ.

	$\gamma = 2,\ \zeta = 1.5$			$\gamma = 1.8,\ \zeta = 1.3$		
(x,t)	ψ_1	ψ_2	ψ_3	ψ_1	ψ_2	ψ_3
(0.2,0.2)	9.73E-07	4.38E-06	9.46E-08	7.91E-07	3.45E-07	6.48E-07
(0.4,0.4)	5.41E-05	2.42E-08	1.59E-05	5.40E-05	1.97E-07	1.68E-05
(0.6,0.6)	2.85E-04	9.99E-06	1.62E-04	2.93E-04	1.42E-05	1.88E-04
(0.8,0.8)	3.59E-04	7.01E-05	3.11E-04	2.93E-04	1.79E-04	3.34E-04
(1.0,1.0)	3.09E-04	4.10E-03	5.29E-04	6.77E-04	2.60E-03	6.57E-04

Table 1: Absolute error when $N = 5$

	ψ_1	ψ_1	ψ_2	ψ_2	ψ_3	ψ_3
x	Exact	Proposed	Exact	Proposed	Exact	Proposed
0.2	0.0400	0.0400	0.0039	0.0039	0.0206	0.0206
0.4	0.1600	0.1600	0.0264	0.0264	0.1016	0.1016
0.6	0.3600	0.3601	0.1065	0.1066	0.2723	0.2724
0.8	0.6400	0.6403	0.3486	0.3487	0.5621	0.5624

Table 2: Exact and proposed solution at $\gamma = 1.8,\ \zeta = 1.3$ for $u(x, 1)$

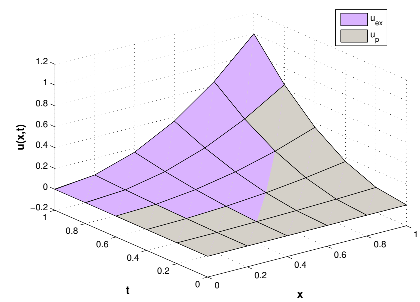

Figure 1: Exact and proposed solution for ψ_1 at $\gamma = 1.8,\ \zeta = 1.3$ when $N = 5$.

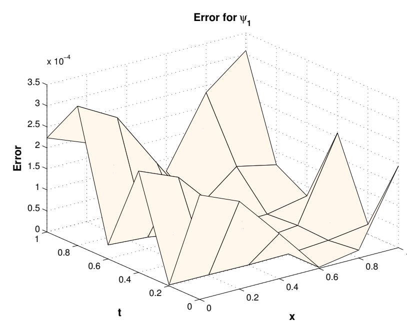

Figure 2: Absolute error for ψ_1 at $\gamma = 1.8,\ \zeta = 1.3$ when $N = 5$.

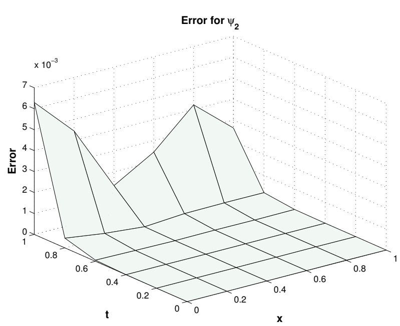

Figure 3: Absolute error for ψ_2 at $\gamma = 1.8,\ \zeta = 1.3$ when $N = 5$.

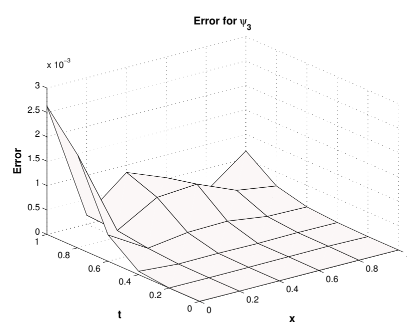

Figure 4: Absolute error for ψ_3 at $\gamma = 1.8,\ \zeta = 1.3$ when $N = 5$.

Next, we solve this example for $\Delta_1 = [0, 1] \times [0, 1]$ and $\Delta_2 = [0, 1] \times [0, 5]$ for ψ_1. Table 3 depicts the absolute error analysis for $u(x, 1)$ and $u(x, 5)$ for multiple values of γ and ζ. In Table 3, it can be seen that absolute error is high for $u(x, 5)$ when both derivatives are of fractional order. We can reduce the error by taking one derivative fractional and the other of integer-order. In the Table 3, we observe that N is small which is another reason of high error. The increase or decrease in N affect the absolute error, so, increase in N and change in the value of derivative help in reducing the error. Exact and proposed solutions for $u(x, 1)$ and $u(x, 5)$ at $\gamma = 2,\ \zeta = 1.5$ are plotted in Figure 5 where Figure 6 gives absolute error analysis of exact and proposed solutions.

	$u(x,1)$	$u(x,5)$	$u(x,1)$	$u(x,5)$
	$\gamma = 2.0$	$\gamma = 2.0$	$\gamma = 1.5$	$\gamma = 1.5$
x	$\zeta = 1.5$	$\zeta = 1.5$	$\zeta = 1.2$	$\zeta = 1.2$
1/5	5.67E-07	2.27E-06	1.43E-05	2.01E-01
2/5	2.80E-06	3.58E-05	6.80E-05	3.19E-01
3/5	2.36E-05	2.13E-04	1.51E-04	3.61E-01
4/5	1.09E-04	6.07E-04	2.92E-04	3.80E-01
5/5	3.09E-04	9.82E-04	5.98E-04	4.10E-01

Table 3: Absolute error for $u(x, 1)$ and $u(x, 5)$ when $N = 5$

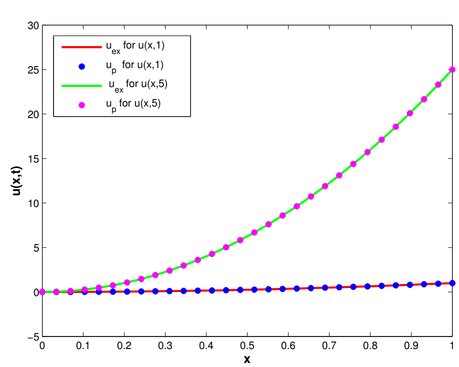

Figure 5: Exact and proposed solution for $u(x, 1)$ and $u(x, 5)$ at $\gamma = 2,\ \zeta = 1.5$

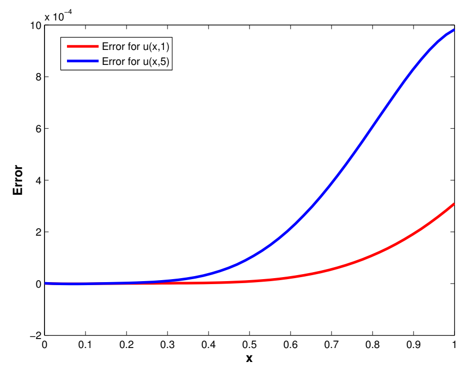

Figure 6: Absolute error for $u(x, 1)$ and $u(x, 5)$ at $\gamma = 2,\ \zeta = 1.5$

Example 8.2. Consider fractional-order partial differential equation (1.1) with $\gamma = \zeta = 1$, $\nu(x, t) = \lambda(x, t) = 1$, initial condition $u(x, 0) = 0$ and boundary conditions $u(0, t) = 0$, $u(1, t) = \sin(1) * \sin(\psi(t))$ with $h(x, t) = \sin(\psi(x) + \psi(t))$. The exact solution is $u(x, t) = \sin(\psi(x)) * \sin(\psi(t))$.

(x,t)	Exact	Ref [34]	Proposed
(1/8,1/8)	0.0155	0.0155	0.0155
(3/8,3/8)	0.1342	0.1342	0.1342
(5/8,5/8)	0.3423	0.3423	0.3423
(7/8,7/8)	0.5891	0.5891	0.5891

Table 4: Comparison of exact, proposed and reference solution for $N = 8$

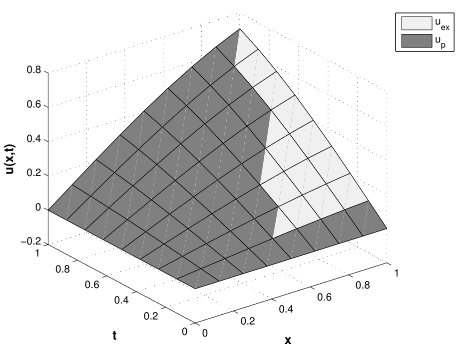

Figure 7: Exact and proposed solution for $\gamma = \zeta = 1$ and $N = 8$.

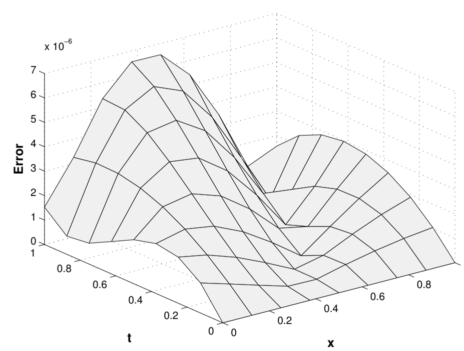

Figure 8: Absolute error at $\gamma = \zeta = 1$ and $N = 8$.

To compare our results with the existing numerical approach, we take $\psi(x) = x$ and $\psi(t) = t$. Harendra Singh and C.S. Singh solved this problem in [34] by operational matrices approach taking Legendre scaling functions as a basis. The numerical results of exact, proposed and Legendre scaling approach are presented in Table 4. From Table 4, we see that proposed results reasonably match with the solution in [34] and the exact solution. Figure 7 and Figure 8 shows the exact, proposed solution and their absolute error at $\gamma = \zeta = 1$ and $N = 8$.

Example 8.3. Consider Equation (1.1) with $\mu(x,t) = \lambda(x,t) = 1$, $\gamma = 0.5$, $\zeta = 2/3$, $u(0,t) = u(x,0) = 0$, $u(1,t) = (\psi(t))^s$, the exact solution $u_{ex} = (\psi(x))^r(\psi(t))^s$ and

$$h(x,t) = \frac{\Gamma(r+1)}{\Gamma(r+1-\gamma)}(\psi(x))^{r-\gamma}(\psi(t))^s + \frac{\Gamma(s+1)}{\Gamma(s+1-\zeta)}(\psi(x))^r(\psi(t))^{s-\zeta}.$$

N	Ref [35] $r=5,s=1$	Ref [35] $r=s=12$	Proposed $r=5,s=1$	Proposed $r=s=12$
1	3.27E-01	9.16E-01	0.0	0.0
2	1.17E-01	4.92E-01	1.93E-02	1.93E-02
3	2.18E-02	1.96E-01	8.85E-02	8.58E-02
4	1.95E-03	7.68E-02	8.47E-02	8.47E-02

Table 5: Maximum absolute error for different parameters.

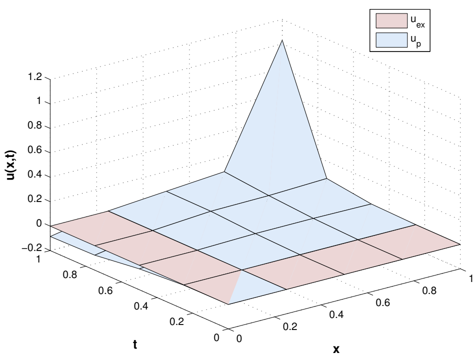

Figure 9: Exact and proposed solution for $\gamma = 0.5,\ \zeta = 2/3$ when $N = 4$.

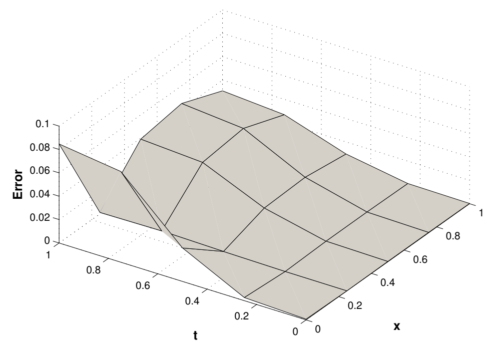

Figure 10: Absolute error at $\gamma = 0.5,\ \zeta = 2/3$ when $N = 4$.

S. Mockary et al. solved this problem in [35] for $\psi(x) = x$ and $\psi(t) = t$ by using matrices of fractional-order integration for Chebyshev polynomials. Table 5 reports maximum absolute error for different values of r and s at $\gamma = 0.5$, $\zeta = 2/3$. From Table 5, we can observe that the method provides good results. Figure 9 describes comparison of the exact and proposed solution for $\gamma = 0.5$, $\zeta = 2/3$ when $N = 4$ and Figure 10 presents graphical analysis of absolute error for these parameters.

Example 8.4. Take Equation (1.1) with $\mu(x,t) = \lambda(x,t) = 1$, $\gamma = 2$, $\zeta = 0.5$ and zero initial and boundary conditions. The exact solution is $u_{ex} = (\psi(t))^2 \sin(2\pi\psi(x))$ and

$$h(x,t) = \frac{2}{\Gamma(3-\zeta)}(\psi(t))^{(2-\zeta)} \sin(2\pi\psi(x)) + 4\pi^2(\psi(t))^2 \sin(2\pi\psi(x)).$$

	t=0.25			t=0.50			t=0.75	
x	Ref [36]	Proposed	x	Ref [36]	Proposed	x	Ref [36]	Proposed
0.3	0.0007	0.000098	0.3	0.009	0.0039	0.3	0.06	0.0089
0.6	0.0020	0.000048	0.6	0.01	0.0017	0.6	0.01	0.0034
0.9	0.0020	0.001300	0.9	0.01	0.0072	0.9	0.03	0.0191

Table 6: Maximum absolute error for $\gamma = 2$ and $\zeta = 0.5$ when $N = 6$

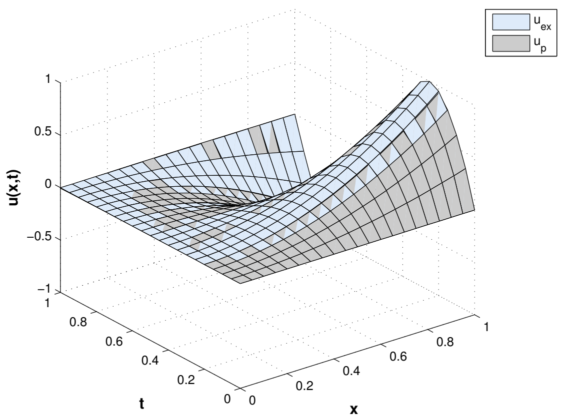

Figure 11: Exact and proposed solution at $\gamma = 2,\ \zeta = 0.5$ when $N = 6$.

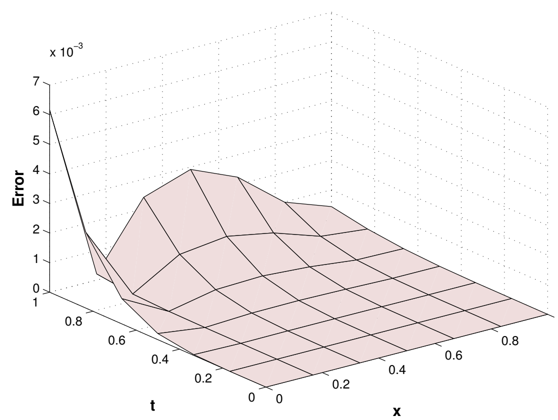

Figure 12: Absolute error at $\gamma = 2,\ \zeta = 0.5$ when $N = 6$.

For $\psi(x) = x$ and $\psi(t) = t$, this is time-fractional diffusion equation and this problem is solved in [36]. Table 6 gives the comparison of the maximum absolute error at $\gamma = 2$, $\zeta = 0.5$ and $N = 6$ for two techniques. From the Table 6, it can be observed that our method provides appropriate results which proves the suitability of the method for certain types of fractional differential equations. The graphical analysis of the exact and proposed numerical solution is presented in Figure 11 and the absolute error is in Figure 12. Next, we solve this problem for $\gamma = 2$, $\zeta = 0.5$ and $N = 10$ on the interval $[0, 1] \times [0, 5]$. Table 7 presents the absolute error of the problem 8.4 for $x \in [0, 1]$ and $t \in [0, 5]$. In Table 7, we observe that there is a rise and fall in the absolute error for different values of x and t. We can minimize the error by the suitable adjustment of the parameters γ, ζ, N, x and t. Figure 13 shows the graphical analysis of the exact and proposed solution at $\gamma = 2$ and $\zeta = 0.5$ for $N = 10$ and Figure 14 gives the picture of absolute error.

x	t	Error	x	t	Error	x	t	Error
0.25	0.25	2.21E-04	0.15	0.75	1.60E-03	0.60	3.00	1.26E-05
0.50	0.50	3.58E-06	0.30	1.50	1.70E-07	0.75	3.75	4.95E-02
0.75	0.75	1.97E-03	0.45	2.25	5.52E-03	0.90	4.50	2.87E-04

Table 7: Maximum absolute error for $\gamma = 2$, $\zeta = 0.5$ and $N = 10$ on $[0, 1] \times [0, 5]$.

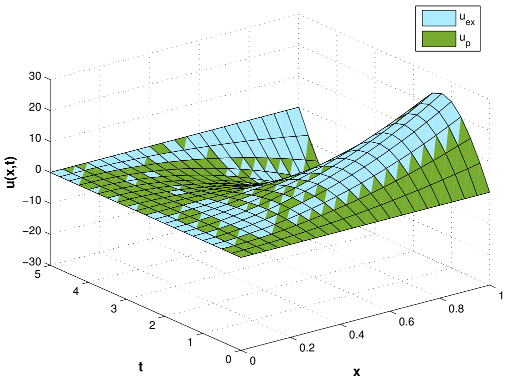

Figure 13: Exact and proposed solution at $\gamma = 2,\ \zeta = 0.5$ when $N = 10$.

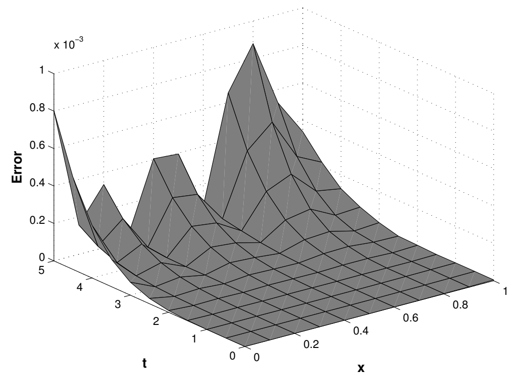

Figure 14: Absolute error at $\gamma = 2,\ \zeta = 0.5$ when $N = 10$.

Example 8.5. Again consider (1.1) with $\lambda(x,t) = 1$, $\mu(x,t) = \Gamma(0.2)x^{1.8}$, $\gamma = 1.8$, $\zeta = 1$, $u(0,t) = u(1,t) = 0$ and $u(x,0) = \psi(x) - \psi(x)^2$. The exact solution is $u_{ex} = (\psi(x) - \psi(x)^2)\ \exp(-\psi(t))$ and $h(x,t) = (11\psi(x)^2 - 2\psi(x))\exp(-\psi(t))$.

(x,t)	Exact	Proposed $N = 4$	Proposed $N = 8$
(0.25,0.25)	0.146025	0.145953	0.146023
(0.50,0.50)	0.151633	0.151277	0.151596
(0.75,0.75)	0.088569	0.089081	0.088572

Table 8: Comparison of exact and proposed solutions for $\gamma = 1.8$ and $\zeta = 1$.

The authors in [37] solved this space-fractional diffusion problem for $\psi(x) = x$ and $\psi(t) = t$. The comparison of the exact and proposed numerical solutions taking different

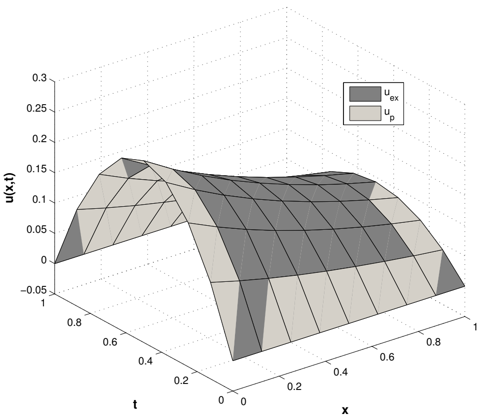

Figure 15: Exact and proposed solution at $\gamma = 1.8,\ \zeta = 1$ when $N = 8$.

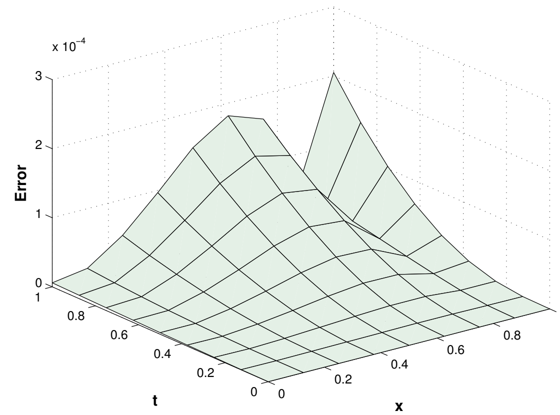

Figure 16: Absolute error at $\gamma = 1.8,\ \zeta = 1$ when $N = 8$.

values of N is presented in Table 8. The analysis of the table shows that the proposed technique provides better results. The exact and proposed solution for $\gamma = 1.8$ and $\zeta = 1$ are plotted in Figure 15 where Figure 16 gives a picture of absolute error analysis.

9. Conclusion

In this paper, a numerical technique dealing with the solution of certain fractional-order partial differential equations is presented. In the first step, shifted Chebyshev and Laguerre polynomials are used to formulate ψ-shifted Chebyshev and ψ-Laguerre polynomials, then, operational matrices of fractional-order integration are developed. In the final step, these operational matrices are used to obtain the solution of fractional-order partial differential equations. Error bounds are calculated for convergence analysis. Some examples are solved by comparing results with existing techniques to show the validity of proposed numerical technique. In Example 1, we compute the approximate solution of fractional-order partial differential equation for different values of function ψ and get good results. In Example 2, 3 and 4 we see that our results are better than the results of other numerical techniques. In Example 4, we also calculate the absolute error for the interval $[0, 1] \times [0, 5]$ and get good approximations. In Example 5, we observe that the proposed solution is close to the exact solution. This proves the applicability of the proposed numerical technique to solve certain classes of fractional-order differential equations. In the future, this technique can be extended to Hadamard fractional differential operators for the solution of a variety of fractional differential equations.

Conflict of interest:

All authors declare that they have no conflict of interest.

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