Stability and Hopf bifurcation of a delayed predator-prey system with nonlocal competition and herd behaviour

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Abstract

In this paper, we investigate the stability and Hopf bifurcation of a diffusive predator-prey system with herd behaviour. The model is described by introducing both time delay and nonlocal prey intraspecific competition. Compared to the model without time delay, or without nonlocal competition, thanks to the together action of time delay and nonlocal competition, we prove that the first critical value of Hopf bifurcation may be homogenous or non-homogeneous. We also show that a double-Hopf bifurcation occurs at the intersection point of the homogenous and non-homogeneous Hopf bifurcation curves. Furthermore, by the computation of normal forms for the system near equilibria, we investigate the stability and direction of Hopf bifurcation. Numerical simulations also show that the spatially homogeneous and non-homogeneous periodic patters.

Keywords: Predator-prey model; time delay; nonlocal prey competition; Hopf bifurcation

1 Introduction

Predator-prey models have been frequently used to model ecological system. It is an important area to study the dynamics of biological population and attracts many researchers to establish mathematic models for research. Recently, a predator-prey model modeling herd behaviour in population system was considered by Ajraldi et al. [1]. The simplified model is written as

$$\begin{cases} \dfrac{du}{dt} = u(1-u) - \sqrt{u}v, \\ \dfrac{dv}{dt} = rv(-\beta + \sqrt{u}), \end{cases} \tag{1.1}$$

where $u,\ v$ stand for prey and predator densities respectively, $r\beta$ is the death rate of predator in the absence of prey. r is the conversion or consumption rate of prey to predator. In this model, the interaction term is proportional to the square toot of the prey population, which

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appropriately simulates the system in which the prey exhibits a strong herd structure. This means that the predator typically interacts with the prey along the outer corridors of the herd of prey. For the establishment and simplification of the model, please refer to the literatures [1, 2].

When $0 < \beta < 1$, the system (1.1) has a unique positive equilibrium $E_* = (u_*, v_*)$ with

$$u_* = \beta^2, v_* = \beta(1 - \beta^2),$$

which is local asymptotically stable when $\beta > \frac{\sqrt{3}}{3}$.

Many species can move freely. Spatial diffusion is everywhere and reaction-diffusion models play an important role in the study of biological invasions. Consequently, the predator-prey models involving spatial diffusion have been concerned by more and more researchers [3–10]. Adding diffusion term into system (1.1) and supplementing with the Neumann boundary condition and initial condition, then the model in one-dimensional bounded domain reads

$$\begin{cases} u_t = u(1-u) - \sqrt{u}v + d_1 u_{xx}, \quad x \in (0, l\pi), t > 0, \\ v_t = rv(-\beta + \sqrt{u}) + d_2 v_{xx}, \quad x \in (0, l\pi), t > 0, \\ u_x(0,t) = u_x(l\pi, t) = v_x(0,t) = v_x(l\pi, t) = 0, t > 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, \; x \in (0, l\pi), \end{cases} \tag{1.2}$$

where $l > 0$, d_1 and d_2 are the diffusion coefficients for the prey and predator, respectively. Here we choose homogeneous Neumann boundary condition. Biologically speaking, the homogeneous Neumann boundary condition indicates that this system is a closed one (for example, islands and lakes/ponds are such system), and thus there is no population flux on the boundary. Furthermore, in this paper, we are only interested in the bifurcations from the positive constant steady state, corresponding to the homogeneous Neumann boundary condition.

Yuan et al. [9] chose the quadratic mortality for predator population in the model (1.2), i.e., they used $-r\beta v^2$ to represent the quadratic mortality for predator population. Their research presented the Turing pattern selection in a spatial predator-prey model. They also derived that the Turing pattern is induced by quadratic mortality.

And since the number of predators does not increase immediately after consuming prey. For example, the pregnancy of some populations takes a certain time. Therefore, Tang and Song

[11] incorporated time delay into the system (1.2) and focused on the following system

$$\begin{cases} u_t = u(1-u) - \sqrt{u}v + d_1 u_{xx}, \quad x \in (0,\pi), t > 0, \\ v_t = rv(-\beta + \sqrt{u_\tau}) + d_2 v_{xx}, \quad x \in (0,\pi), t > 0, \\ u_x(0,t) = u_x(\pi,t) = v_x(0,t) = v_x(\pi,t) = 0, t \geq 0, \\ u(x,t) = \phi(x,t) \geq 0, v(x,t) = \psi(x,t) \geq 0, \ (x,t) \in [0,\pi] \times [-\tau, 0], \end{cases} \tag{1.3}$$

where $u_\tau = u(x, t-\tau)$, τ represents the time delay, which indicates the influence of past consumption of prey on the density of current predators. They investigated the stability of the positive equilibrium, delay-induced Hopf bifurcation of the system (1.3). They also found that the instability of Hopf bifurcation caused by diffusion and time delay respectively can lead to the emergence of spatial patterns.

In [11–15], the effect of time delay is investigated in diffusive predator-prey system with delay. Su et al.[16] considered a reaction-diffusion population model with a general time-delayed growth rate per capita and determined the long time dynamical behavior of the system. Zhao [17] established the global attractivity of the positive steady state for a class of nonmonotone time-delayed reaction-diffusion equations.

In addition, due to the uneven distribution of resources and other reasons, prey and other prey or predators are connected not only in the same place, but also in different places, even in the whole space. Therefore, nonlocal competition exists and many scholars concentrate on the nonlocal interactions in reaction-diffusion equations[18–25].

Recently, Peng and Zhang [26] introduced nonlocal prey competition into the system (1.2):

$$\begin{cases} u_t = u(1 - \int_0^{l\pi} K(x,y)u(y,t)dy) - \sqrt{u}v + d_1 u_{xx}, \quad x \in (0, l\pi), t > 0, \\ v_t = rv(-\beta + \sqrt{u}) + d_2 v_{xx}, \quad x \in (0, l\pi), t > 0, \\ u_x(0,t) = u_x(l\pi,t) = v_x(0,t) = v_x(l\pi,t) = 0, t > 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, \ x \in (0, l\pi), \end{cases} \tag{1.4}$$

where the kernel function $K(x,y) = \frac{1}{l\pi}$. The idea of spatial average of density function was first proposed by Furter and Grinfeld [18]. The effects of nonlocal competition on dynamics of the system (1.4) in the bounded region was investigated by Peng and Zhang. But in unbounded domain $(-\infty, +\infty)$, they took a step function as the kernel function and investigated the influence of nonlocal competition on the stability of the positive equilibrium.

Motivated by literatures [11] and [26], we introduce nonlocal prey competition and time delay into the system (1.2), which is written

$$\begin{cases} u_t = u(1-\tilde{u}) - \sqrt{u}v + d_1 u_{xx}, & x \in (0, l\pi), t > 0, \\ v_t = rv(-\beta + \sqrt{u_\tau}) + d_2 v_{xx}, & x \in (0, l\pi), t > 0, \\ u_x(0,t) = u_x(l\pi, t) = v_x(0,t) = v_x(l\pi, t) = 0, t \geq 0, \\ u(x,t) = \phi(x,t) \geq 0, v(x,t) = \psi(x,t) \geq 0, \ (x,t) \in [0, l\pi] \times [-\tau, 0], \end{cases} \tag{1.5}$$

where $\tilde{u} = \dfrac{1}{l\pi}\displaystyle\int_0^{l\pi} u(x,t)dx$ is the spatial average of prey u and $u_\tau = u(x, t-\tau)$ is the population density of u at time $t - \tau$.

In this paper, we will study the stability of positive equilibrium, Hopf bifurcation induced by delay and nonlocal prey competition and the properties of Hopf bifurcation. The organization of this paper is as follows. The stability of positive equilibrium and Hopf bifurcation are studied by analyzing the characteristic equation in Section 2. In Section 3, we determine the direction of Hopf bifurcation. In Section 4, Numerical simulations verifies the theoretical results. Finally, we conclude this paper by a simple discussion.

2 Stability and Hopf bifurcations

In this section, we study the stability and Hopf bifurcation of the system (1.5). We know that

$$\phi_k(x) = \begin{cases} \frac{1}{\sqrt{l}}, & k = 0, \\ \sqrt{\frac{2}{l}} cos(\frac{kx}{l}), & k \in \mathbb{N}, \end{cases}$$

are the normalized eigenfunctions of the following eigenvalue problem

$$\begin{cases} \varphi'' + \lambda\varphi = 0, & x \in (0, l\pi), \\ \varphi' = 0, & x = 0, l\pi, \end{cases}$$

whose corresponding eigenvalues are

$$\lambda_k = (\frac{k}{l})^2, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{2.1}$$

with $\mathbb{N} = \{1, 2, 3 \cdots\}$.

It is easy to see that

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_i < \lambda_{i+1} < \ldots < +\infty.$$

When $\tau > 0$, we linearize the equation (1.5) at the positive equilibrium (u_*, v_*)

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = d\frac{\partial^2}{\partial x^2}\begin{pmatrix} u \\ v \end{pmatrix} + A_0 \begin{pmatrix} u \\ v \end{pmatrix} + A_1 \begin{pmatrix} u(t-\tau) \\ v(t-\tau) \end{pmatrix} + A_2 \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \tag{2.2}$$

where

$$d\frac{\partial^2}{\partial x^2} = \begin{pmatrix} d_1\frac{\partial^2}{\partial x^2} & 0 \\ 0 & d_2\frac{\partial^2}{\partial x^2} \end{pmatrix},\ A_0 = \begin{pmatrix} \frac{1}{2}(1-\beta^2) & -\beta \\ 0 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}r(1-\beta^2) & 0 \end{pmatrix}, A_2 = \begin{pmatrix} -\beta^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

The characteristic equation of (2.2) is

$$\det(\mu I - M_k - A_0 - A_1 e^{-\mu\tau} - \chi_k A_2) = 0, \tag{2.3}$$

where I is the 2×2 identity matrix and $M_k = -\lambda_k diag(d_1, d_2)$, $k \in \mathbb{N}_0$. λ_k are given by (2.1) and

$$\chi_k = \begin{cases} 1, & k = 0, \\ 0, & k \in \mathbb{N}. \end{cases} \tag{2.4}$$

It follows from (2.3) that the characteristic equations for the positive constant equilibrium (u_*, v_*) are the following sequence of quadratic transcendental equations

$$\mu^2 - \left(\frac{1}{2}(1-\beta^2) - \chi_k\beta^2 - \lambda_k(d_1+d_2)\right)\mu + d_1d_2\lambda_k^2 - \left(\frac{1}{2}(1-\beta^2)d_2 - \chi_k\beta^2 d_2\right)\lambda_k + \frac{1}{2}r\beta(1-\beta^2)e^{-\mu\tau} = 0, \tag{2.5}$$

where $k \in \mathbb{N}_0$ and λ_k, χ_k are given by (2.1) and (2.4) respectively.

For the distribution of purely imaginary roots of equation (2.5), we have the following results.

Lemma 2.1. *Suppose that* $d_1 > 0,\ d_2 \geq 0,\ r > 0,\ \frac{\sqrt{3}}{3} < \beta < 1$ *and* $l^2 < min\{\frac{2(d_1+d_2)}{1-\beta^2}, \frac{4d_1}{1-\beta^2}\}$ *hold. Let*

$$\begin{cases} \tau_{0i} = \dfrac{1}{\omega_0}\left[\arccos\left(\dfrac{2\omega_0^2}{r\beta(1-\beta^2)}\right) + 2i\pi\right], & i = 0, 1, 2, \cdots, \\ \tau_{ki} = \dfrac{1}{\omega_k}\left[\arccos\left(\dfrac{2\omega_k^2 - 2d_1d_2\lambda_k^2 + (1-\beta^2)d_2\lambda_k}{r\beta(1-\beta^2)}\right) + 2i\pi\right], & k \in \mathbb{N},\ i = 0, 1, 2, \cdots, \end{cases} \tag{2.6}$$

where λ_k *are given by* (2.1), ω_0 *and* ω_k *are the only positive root of equation* (2.10) *and* (2.11) *respectively. Then for the existence of pure imaginary roots of* (2.5), *we have the following results:*

(*i*) *if* $d_1 > 0, d_2 \geq 0$, *the characteristic equation* (2.5) *has a pair of pure imaginary roots* $\pm i\omega_0$ *at* $\tau = \tau_{0i}$, $i \in \mathbb{N}_0$;

(*ii*) *if* $d_2 = 0,\ d_1 > 0$, *the characteristic equation* (2.5) *has a pair of pure imaginary roots* $\pm i\omega_k$ *at* $\tau = \tau_{ki}$, $k \in \mathbb{N}$, $i \in \mathbb{N}_0$;

(*iii*) *if* $d_2 > 0,\ 0 < d_1 < \dfrac{d_2(1-\beta^2)}{8r\beta}$, *the characteristic equation* (2.5) *has a pair of pure imaginary roots* $\pm i\omega_k$ *at* $\tau = \tau_{ki}$ *for* $N_1 < k \le N_2$ $(N_1 < k < N_2)$ *and has no purely imaginary roots for* $k \le N_1$ $(k < N_1)$ *or* $k > N_2,\ k \in \mathbb{N},\ i \in \mathbb{N}_0$, *where*

$$N_1 = [\sqrt{x_2}l], \quad N_2 = [\sqrt{x_4}l],$$

with

$$x_2 = \frac{\frac{1}{2}d_2(1-\beta^2) + \sqrt{\Delta_1}}{2d_1d_2},\ x_4 = \frac{\frac{1}{2}d_2(1-\beta^2) + \sqrt{\Delta_2}}{2d_1d_2}, \tag{2.7}$$

$$\Delta_1 = \frac{1}{4}d_2^2(1-\beta^2)^2 - 2d_1d_2r\beta(1-\beta^2),\ \Delta_2 = \frac{1}{4}d_2^2(1-\beta^2)^2 + 2d_1d_2r\beta(1-\beta^2). \tag{2.8}$$

(*iv*) *if* $d_2 > 0,\ d_1 \ge \dfrac{d_2(1-\beta^2)}{8r\beta}$, *the characteristic equation* (2.5) *has a pair of pure imaginary roots* $\pm i\omega_k$ *at* $\tau = \tau_{ki}$ *for* $k \le N_2(k < N_2)$ *and has no purely imaginary roots for* $k > N_2,\ k \in \mathbb{N},\ i \in \mathbb{N}_0$. N_2 *is given in* (*iii*).

Remark 2.2. For the conclusion (*iii*), if $\sqrt{x_2}l$ is not a positive integer, then $N_1 < k \le N_2$ and if $\sqrt{x_2}l$ is a positive integer, then $N_1 < k < N_2$; For the conclusion (*iv*), if $\sqrt{x_4}l$ is not a positive integer, then $k \le N_2$ and if $\sqrt{x_4}l$ is a positive integer, then $k < N_2$.

Next, we give the proof of Lemma 2.1.

Proof. Suppose that $\mu = i\omega_k(\omega_k > 0)$ is a root of characteristic equation (2.5). Substituting $\mu = i\omega_k$ into equation (2.5), we obtain

$$-\omega_k^2 + i\omega_k[\lambda_k(d_1+d_2) - \frac{1}{2}(1-\beta^2) + \chi_k\beta^2] + d_1d_2\lambda_k^2 - d_2[\frac{1}{2}(1-\beta^2) - \chi_k\beta^2]\lambda_k + \frac{1}{2}r\beta(1-\beta^2)e^{-i\omega_k\tau} = 0,$$

which leads to

$$\begin{cases} -\omega_0^2 + i\omega_0(\frac{3}{2}\beta^2 - \frac{1}{2}) + \frac{1}{2}r\beta(1-\beta^2)e^{-i\omega_0\tau} = 0, \\ -\omega_k^2 + i\omega_k[\lambda_k(d_1+d_2) - \frac{1}{2}(1-\beta^2)] + d_1d_2\lambda_k^2 - \frac{1}{2}d_2(1-\beta^2)\lambda_k + \frac{1}{2}r\beta(1-\beta^2)e^{-i\omega_k\tau} = 0, \quad k \in \mathbb{N}. \end{cases} \tag{2.9}$$

From (2.9), we obtain

$$\omega_0^4 + (\frac{3}{2}\beta^2 - \frac{1}{2})^2\omega_0^2 - \frac{1}{4}r^2\beta^2(1-\beta^2)^2 = 0, \tag{2.10}$$

and

$$\omega_k^4 + B_k\omega_k^2 + C_k = 0, \quad k \in \mathbb{N}, \tag{2.11}$$

where

$$B_k = \lambda_k^2(d_1^2 + d_2^2) - \lambda_k d_1(1-\beta^2) + \frac{1}{4}(1-\beta^2)^2 = \left(d_1\lambda_k - \frac{1}{2}(1-\beta^2)\right)^2 + d_2^2\lambda_k^2,$$

$$C_k = \left(-d_1 d_2 \lambda_k^2 + \frac{1}{2} d_2 (1-\beta^2)\lambda_k\right)^2 - \frac{1}{4} r^2 \beta^2 (1-\beta^2)^2.$$

Separating real and imaginary parts from (2.9), we have

$$\begin{cases} \cos(\omega_0 \tau) = \dfrac{2\omega_0^2}{r\beta(1-\beta^2)}, \\ \sin(\omega_0 \tau) = \dfrac{\omega_0(3\beta^2-1)}{r\beta(1-\beta^2)}, \end{cases} \tag{2.12}$$

and when $k \in \mathbb{N}$,

$$\begin{cases} \cos(\omega_k \tau) = \dfrac{2\omega_k^2 - 2d_1 d_2 \lambda_k^2 + d_2(1-\beta^2)\lambda_k}{r\beta(1-\beta^2)}, \\ \sin(\omega_k \tau) = \dfrac{\omega_k[2\lambda_k(d_1+d_2) - (1-\beta^2)]}{r\beta(1-\beta^2)}. \end{cases} \tag{2.13}$$

Thanks to $\frac{\sqrt{3}}{3} < \beta < 1$ and $l^2 < \frac{2(d_1+d_2)}{1-\beta^2}$, it follows from (2.12) and (2.13) that $\sin(\omega_0\tau) > 0$ and $\sin(\omega_k \tau) > 0$. Because $\frac{1}{4} r^2 \beta^2 (1-\beta^2)^2 > 0$, according to (2.10), it is easy to prove the conclusion (i).

If $d_2 = 0$, $d_1 > 0$, then $B_k \geq 0$ and $C_k < 0$. It follows from (2.11) and (2.13) that the conclusion (ii) is obviously true.

If $d_2 > 0$, $d_1 > 0$, we have $B_k > 0$. Then when $C_k < 0$, the equation (2.11) has a unique positive root .

According to the expression of C_k, it can be rewritten as

$$C_k = (-d_1 d_2 \lambda_k^2 + \frac{1}{2} d_2(1-\beta^2)\lambda_k - \frac{1}{2} r\beta(1-\beta^2))(-d_1 d_2 \lambda_k^2 + \frac{1}{2} d_2(1-\beta^2)\lambda_k + \frac{1}{2} r\beta(1-\beta^2)). \tag{2.14}$$

Let

$$f(x) = -d_1 d_2 x^2 + \frac{1}{2} d_2 (1-\beta^2) x - \frac{1}{2} r\beta(1-\beta^2), \quad x \geq \frac{1}{l^2}.$$

Then

$$f(x) = -d_1 d_2 (x - x_1)(x - x_2), \tag{2.15}$$

where $x_1 = \frac{\frac{1}{2} d_2(1-\beta^2) - \sqrt{\Delta_1}}{2 d_1 d_2}$, x_2 and Δ_1 are defined by (2.7) and (2.8) respectively.

*Case*1 If $d_2 > 0$ and $0 < d_1 < \frac{d_2(1-\beta^2)}{8r\beta}$, then we have $\Delta_1 > 0$, $x_1 > 0$ and $x_2 > 0$. It follows from $l^2 < \frac{4d_1}{1-\beta^2}$ that

$$\frac{1}{l^2} > \frac{1-\beta^2}{4d_1} = \frac{d_2(1-\beta^2)}{4d_1 d_2} > \frac{d_2(1-\beta^2) - 2\sqrt{\Delta_1}}{4 d_1 d_2} = x_1,$$

i.e., $\lambda_1 > x_1$. On the other hand, when $\lambda_k < x_2$, i.e., $k < \sqrt{x_2} l$, it follows from (2.15) that $f(\lambda_k) > 0$.

Choose $N_1 = [\sqrt{x_2} l]$, then we obtain

$(a1)$ if $\sqrt{x_2}l$ is not a positive integer, $f(\lambda_k) > 0$ for $k \leq N_1$ and $f(\lambda_k) < 0$ for $k > N_1$;

$(a2)$ if $\sqrt{x_2}l$ is a positive integer, $f(\lambda_k) > 0$ for $k < N_1$ and $f(\lambda_k) < 0$ for $k > N_1$.

*Case*2 If $d_2 > 0$ and $d_1 = \dfrac{d_2(1-\beta^2)}{8r\beta}$, then we have $\Delta_1 = 0$, $x_1 = x_2 = \dfrac{1-\beta^2}{4d_1} > 0$.

From $l^2 < \dfrac{4d_1}{1-\beta^2}$, we have

$$\frac{1}{l^2} > \frac{1-\beta^2}{4d_1} = x_1 = x_2.$$

It follows from (2.15) that $f(\lambda_k) < 0$ for any $k \in \mathbb{N}$.

*Case*3 If $d_2 > 0$ and $d_1 > \dfrac{d_2(1-\beta^2)}{8r\beta}$, then we have $\Delta_1 < 0$, which implies that $f(\lambda_k) < 0$ for any $k \in \mathbb{N}$.

Similarly, let

$$g(x) = -d_1 d_2 x^2 + \frac{1}{2} d_2(1-\beta^2)x + \frac{1}{2} r\beta(1-\beta^2), \quad x \geq \frac{1}{l^2}.$$

Then

$$g(x) = -d_1 d_2 (x - x_3)(x - x_4), \tag{2.16}$$

where $x_3 = \dfrac{\frac{1}{2} d_2(1-\beta^2) - \sqrt{\Delta_2}}{2d_1 d_2}$, x_4 and Δ_2 are defined by (2.7) and (2.8) respectively.

It is obvious that $\Delta_2 > 0$, which implies that $x_3 < 0$ and $x_4 > 0$ hold. When $\lambda_k < x_4$, i.e., $k < \sqrt{x_4}l$, it follows from (2.16) that $g(\lambda_k) > 0$.

Choose $N_2 = [\sqrt{x_4}l]$, we obtain

$(b1)$ if $\sqrt{x_4}l$ is not a positive integer, then $g(\lambda_k) > 0$ for $k \leq N_2$ and $g(\lambda_k) < 0$ for $k > N_2$;

$(b2)$ if $\sqrt{x_4}l$ is a positive integer, then $g(\lambda_k) > 0$ for $k < N_2$ and $g(\lambda_k) < 0$ for $k > N_2$.

In addition, due to $x_2 < x_4$, it follows from the definition of N_1 and N_2 that $N_1 \leq N_2$ hold.

Based on the above analysis, we prove that

(iii) if $d_2 > 0$ and $0 < d_1 < \dfrac{d_2(1-\beta^2)}{8r\beta}$, then when $k \leq N_1 (k < N_1)$ or $k \geq N_2$, we have $C_k > 0$ and (2.11) has no positive root; when $N_1 < k \leq N_2 (N_1 < k < N_2)$, we have $C_k < 0$ and (2.11) has a positive root. This implies that the conclusion (iii) of Lemma 2.1 is proved;

(iv) if $d_2 > 0$ and $d_1 \geq \dfrac{d_2(1-\beta^2)}{8r\beta}$, then when $k > N_2$, we have $C_k > 0$ and (2.11) has no positive root; when $k \leq N_2 (k < N_2)$, we have $C_k < 0$ and (2.11) has a positive root. This implies that the conclusion (iv) of Lemma 2.1 is proved. □

To prove the existence of Hopf bifurcation of the system (1.5), we need to verify the following transversality condition.

Lemma 2.3. *For $k \in \mathbb{N}_0$ and $i \in \mathbb{N}_0$, we have* $\left.\dfrac{d\mathcal{R}e(\mu)}{d\tau}\right|_{\tau=\tau_{ki}} > 0.$

Proof. Taking the derivative on both sides of the equation (2.5) with respect to τ, we get

$$\begin{aligned}\left.\frac{d\mu}{d\tau}\right|_{\tau=\tau_{ki}} &= \left.\frac{\frac{1}{2}\mu r\beta(1-\beta^2)e^{-\mu\tau}}{2\mu+[\lambda_k(d_1+d_2)-\frac{1}{2}(1-\beta^2)+\chi_k\beta^2]-\frac{1}{2}r\beta\tau(1-\beta^2)e^{-\mu\tau}}\right|_{\tau=\tau_{ki}}\\ &= \frac{\frac{1}{2}i\omega_k r\beta(1-\beta^2)e^{-i\omega_k\tau_{ki}}}{2i\omega_k+[\lambda_k(d_1+d_2)-\frac{1}{2}(1-\beta^2)+\chi_k\beta^2]-\frac{1}{2}r\beta\tau_{ki}(1-\beta^2)e^{-i\omega_k\tau_{ki}}},\end{aligned}$$

which implies that

$$\left.\frac{d\mathcal{R}e(\mu)}{d\tau}\right|_{\tau=\tau_{ki}} = \mathcal{R}e\left(\left.\frac{d\mu}{d\tau}\right|_{\tau=\tau_{ki}}\right) = \frac{Q_k}{P_k},$$

where

$$Q_k = \omega_k^2 r\beta(1-\beta^2)\cos(\omega_k\tau_{ki}) + \frac{1}{2}\omega_k r\beta(1-\beta^2)\sin(\omega_k\tau_{ki})[\lambda_k(d_1+d_2)-\frac{1}{2}(1-\beta^2)+\chi_k\beta^2],$$

$$P_k = [\lambda_k(d_1+d_2)-\frac{1}{2}(1-\beta^2)+\chi_k\beta^2-\frac{1}{2}r\beta\tau_{ki}(1-\beta^2)\cos(\omega_k\tau_{ki})]^2+[2\omega_k+\frac{1}{2}r\beta\tau_{ki}(1-\beta^2)\sin(\omega_k\tau_{ki})]^2.$$

It is obvious that $P_k > 0$ for any $k \in \mathbb{N}_0$. Next, we prove that $Q_k > 0$ is also true. Substituting (2.12) and (2.13) into the expression of Q_k respectively, we have

$$\begin{aligned}Q_0 &= \omega_0^2 r\beta(1-\beta^2)\cos(\omega_0\tau_{0i}) + \frac{1}{2}\omega_0 r\beta(1-\beta^2)\sin(\omega_0\tau_{0i})(\frac{3}{2}\beta^2-\frac{1}{2})\\ &= 2\omega_0^4 + \frac{1}{4}\omega_0^2(3\beta^2-1)^2 > 0,\end{aligned}$$

and when $k \in \mathbb{N}$,

$$\begin{aligned}Q_k &= \omega_k^2 r\beta(1-\beta^2)\cos(\omega_k\tau_{ki}) + \frac{1}{2}\omega_k r\beta(1-\beta^2)\sin(\omega_k\tau_{ki})[\lambda_k(d_1+d_2)-\frac{1}{2}(1-\beta^2)]\\ &= \omega_k^2[(d_1^2+d_2^2)\lambda_k^2 - d_1(1-\beta^2)\lambda_k + \frac{1}{4}(1-\beta^2)^2 + 2\omega_k^2]\\ &= \omega_k^2[(d_1\lambda_k - \frac{1}{2}(1-\beta^2))^2 + d_2^2\lambda_k^2 + 2\omega_k^2] > 0,\end{aligned}$$

which implies that $\left.\frac{d\mathcal{R}e(\mu)}{d\tau}\right|_{\tau=\tau_{ki}} > 0$. This completes the proof. □

From (2.6), we know that $\tau_{k0} = \min_{i\in\mathbb{N}_0}\{\tau_{ki}\}$, and

$$\tau_{k0} = \frac{1}{\omega_k}\left[\arccos\left(\frac{2\omega_k^2 - 2d_1d_2\lambda_k^2 + (1-\beta^2)d_2\lambda_k}{r\beta(1-\beta^2)}\right)\right], k \in \mathbb{N}.$$

Lemma 2.4. *Suppose that* $\frac{\sqrt{3}}{3} < \beta < 1,\ r > 0$ *and* $l^2 < \frac{2d_1}{1-\beta^2}$ *hold, then we have*

(i) τ_{k0} *is a strictly increasing sequence with respect to* k*;*

(ii) τ_{10} *is strictly increasing with respect to* d_2 *for fixed* $d_1 > 0$*.*

Proof. (i) Let $p = \lambda_k$, we rewrite τ_{k0} as follows.

$$\tau_{k0}(p) = \frac{1}{\omega_k(p)}\left(\arccos\left(\frac{2\omega_k^2(p) - 2d_1d_2p^2 + (1-\beta^2)d_2p}{r\beta(1-\beta^2)}\right)\right),$$

then differentiating with respect of p, we obtain

$$\begin{aligned}\frac{d[\tau_{k0}(p)]}{dp} &= -\frac{\omega_k^{'}(p)}{\omega_k^2(p)}\arccos\left(\frac{2\omega_k^2(p) - 2d_1d_2p^2 + (1-\beta^2)d_2p}{r\beta(1-\beta^2)}\right)\\ &-\frac{4\omega_k(p)\omega_k^{'}(p) - 4d_1d_2p + (1-\beta^2)d_2}{\omega_k(p)r\beta(1-\beta^2)\sqrt{1-\left(\frac{2\omega_k^2(p) - 2d_1d_2p^2 + (1-\beta^2)d_2p}{r\beta(1-\beta^2)}\right)^2}}.\end{aligned} \tag{2.17}$$

Next, we judge the sign of $\omega_k^{'}(p)$. Rewrite (2.11) as follows.

$$\omega_k^4(p)+[p^2(d_1^2+d_2^2)-pd_1(1-\beta^2)+\frac{1}{4}(1-\beta^2)^2]\omega_k^2(p)+[-d_1d_2p^2+\frac{1}{2}(1-\beta^2)d_2p]^2-\frac{1}{4}r^2\beta^2(1-\beta^2)^2 = 0.$$

Differentiating with respect of p for the above equation, we obtain

$$\begin{aligned}&4\omega_k^3(p)\omega_k^{'}(p) + 2\omega_k(p)\omega_k^{'}(p)[p^2(d_1^2+d_2^2) - pd_1(1-\beta^2) + \frac{1}{4}(1-\beta^2)^2]\\ &+\omega_k^2(p)[2p(d_1^2+d_2^2) - d_1(1-\beta^2)] + 2[-d_1d_2p^2 + \frac{1}{2}(1-\beta^2)d_2p][-2d_1d_2p + \frac{1}{2}(1-\beta^2)d_2] = 0,\end{aligned}$$

which yields

$$\omega_k^{'}(p) = -\frac{\omega_k^2(p)[2p(d_1^2+d_2^2) - d_1(1-\beta^2)] + 2[-d_1d_2p^2 + \frac{1}{2}(1-\beta^2)d_2p][-2d_1d_2p + \frac{1}{2}(1-\beta^2)d_2]}{4\omega_k^3(p) + 2\omega_k(p)[p^2(d_1^2+d_2^2) - pd_1(1-\beta^2) + \frac{1}{4}(1-\beta^2)^2]}.$$

Obviously,

$$p^2(d_1^2+d_2^2) - pd_1(1-\beta^2) + \frac{1}{4}(1-\beta^2)^2 = \left(d_1p - \frac{1}{2}(1-\beta^2)\right)^2 + d_2^2p^2 \geq 0.$$

Thanks to

$$l^2 < \frac{2d_1}{1-\beta^2},\ p = \lambda_k \geq \lambda_1 = \frac{1}{l^2},$$

we obtain

$$2p(d_1^2+d_2^2) - d_1(1-\beta^2) > 0,$$

and

$$[-d_1d_2p^2 + \frac{1}{2}(1-\beta^2)d_2p][-2d_1d_2p + \frac{1}{2}(1-\beta^2)d_2] > 0.$$

So $\omega_k^{'}(p) < 0$. Combining with (2.17), we get $\frac{d[\tau_{k0}(p)]}{dp} > 0$. Therefore, τ_{k0} is increasing in λ_k. And we know that λ_k is increasing in k, thus τ_{k0} is increasing in k.

(ii) Considering τ_{10} as a function of d_2,

$$\tau_{10}(d_2) = \frac{1}{\omega_1(d_2)}\left[\arccos\left(\frac{2\omega_1^2(d_2) - 2d_1d_2\lambda_1^2 + (1-\beta^2)d_2\lambda_1}{r\beta(1-\beta^2)}\right)\right],$$

and taking derivative of this with respect of d_2, we obtain

$$\begin{aligned}\frac{d[\tau_{10}(d_2)]}{d(d_2)} &= -\frac{\omega_1^{'}(d_2)}{\omega_1^2(d_2)}\arccos\left(\frac{2\omega_1^2(d_2)-2d_1d_2\lambda_1^2+(1-\beta^2)d_2\lambda_1}{r\beta(1-\beta^2)}\right)\\ &\quad -\frac{4\omega_1(d_2)\omega_1^{'}(d_2)-2d_1\lambda_1^2+(1-\beta^2)\lambda_1}{r\beta(1-\beta^2)\omega_1(d_2)\sqrt{1-\left(\frac{2\omega_1^2(d_2)-2d_1d_2\lambda_1^2+(1-\beta^2)d_2\lambda_1}{r\beta(1-\beta^2)}\right)^2}}.\end{aligned} \tag{2.18}$$

In the same way as (i),

$$\omega_1^{'}(d_2) = -\frac{2\lambda_1^2 d_2\omega_1^2(d_2)+2d_2[-d_1\lambda_1^2+\frac{1}{2}(1-\beta^2)\lambda_1]^2}{4\omega_1^3(d_2)+2\omega_1(d_2)B_1}.$$

It follows from the expression of B_k that $B_1 \geq 0$, so $\omega_1^{'}(d_2) < 0$.

From $l^2 < \frac{2d_1}{1-\beta^2}$ and $\lambda_1 = \frac{1}{l^2}$, we have $-2d_1\lambda_1^2 + (1-\beta^2)\lambda_1 < 0$. Combining with (2.18), we obtain

$$\frac{d[\tau_{10}(d_2)]}{d(d_2)} > 0,$$

which shows that τ_{10} about d_2 is monotonically increasing. The proof is completed. □

Let $\tau_* = min\{\tau_{00}, \tau_{10}\}$. The following Lemma gives a detailed description for the minimum critical value of delay.

Lemma 2.5. *Suppose that* $d_1 > 0,\ d_2 \geq 0,\ r > 0,\ \frac{\sqrt{3}}{3} < \beta < 1$ *and* $l^2 < \frac{2d_1}{1-\beta^2}$ *hold. Let* d_2^* *be the unique positive root of the following equation*

$$\omega_1\left[\arccos\left(\frac{2\omega_0^2}{r\beta(1-\beta^2)}\right)\right] = \omega_0\left[\arccos\left(\frac{2\omega_1^2-2d_1d_2\lambda_1^2+(1-\beta^2)d_2\lambda_1}{r\beta(1-\beta^2)}\right)\right]. \tag{2.19}$$

we have the following results:

(i) if $0 < l^2 < \frac{2d_1}{3\beta-\beta^2}$*, then* $\tau_* = \tau_{00}$*;*

(ii) if $l^2 = \frac{2d_1}{3\beta-\beta^2}$*, then* $\tau_* = \tau_{10} = \tau_{00}$*;*

(iii) if $\frac{2d_1}{3\beta-\beta^2} < l^2 < \frac{2d_1}{1-\beta^2}$*, then*

$$\tau_* = \begin{cases} \tau_{10}, & for\ 0 \leq d_2 < d_2^*,\\ \tau_{00}, & for\ d_2 > d_2^*,\\ \tau_{10} = \tau_{00}, & for\ d_2 = d_2^*. \end{cases}$$

Proof. According to (2.6), we have

$$\tau_{00} = \frac{1}{\omega_0}\left[\arccos\left(\frac{2\omega_0^2}{r\beta(1-\beta^2)}\right)\right],$$

and

$$\tau_{10}=\frac{1}{\omega_1}\left[\arccos\left(\frac{2\omega_1^2-2d_1d_2\lambda_1^2+(1-\beta^2)d_2\lambda_1}{r\beta(1-\beta^2)}\right)\right].$$

We first consider the case of $d_2=0$, $d_1>0$.

$$\tau_{10}(0)=\tau_{10}|_{d_2=0}=\frac{1}{\omega_1(0)}\left[\arccos\left(\frac{2\omega_1^2(0)}{r\beta(1-\beta^2)}\right)\right],$$

where $\omega_1(0)=\omega_1|_{d_2=0}$. From (2.10) and (2.11), it is easy to know that ω_0 and $\omega_1(0)$ satisfy the following equations:

$$\begin{cases}\omega_0^4+(\frac{3}{2}\beta-\frac{1}{2})^2\omega_0^2-\frac{1}{4}r^2\beta^2(1-\beta^2)^2=0,\\ \omega_1^4(0)+(\lambda_1d_1-\frac{1-\beta^2}{2})^2\omega_1^2(0)-\frac{1}{4}r^2\beta^2(1-\beta^2)^2=0,\end{cases}$$

which implies that $\omega_0<\omega_1(0)$ when $(\frac{3}{2}\beta-\frac{1}{2})^2>(\lambda_1d_1-\frac{1-\beta^2}{2})^2$.

Thanks to $\frac{\sqrt{3}}{3}<\beta<1$, $l^2<\frac{2d_1}{1-\beta^2}$ and $\lambda_1=\frac{1}{l^2}$, by the analysis and calculation, we immediately prove that $\omega_0<\omega_1(0)$ when $\frac{2d_1}{3\beta-\beta^2}<l^2<\frac{2d_1}{1-\beta^2}$; $\omega_0=\omega_1(0)$ when $l^2=\frac{2d_1}{3\beta-\beta^2}$ and $\omega_0>\omega_1(0)$ when $0<l^2<\frac{2d_1}{3\beta-\beta^2}$.

Let

$$h(\omega)=\frac{1}{\omega}\left[\arccos\left(\frac{2\omega^2}{r\beta(1-\beta^2)}\right)\right],$$

then we have

$$h'(\omega)=-\frac{1}{\omega^2}\arccos\left(\frac{2\omega^2}{r\beta(1-\beta^2)}\right)-\frac{4}{r\beta(1-\beta^2)\sqrt{1-\left(\frac{2\omega^2}{r\beta(1-\beta^2)}\right)^2}}<0,$$

which implies that $h(\omega)$ is a monotonically decreasing function of ω. Therefore, we have $\tau_{00}>\tau_{10}(0)$ when $\omega_0<\omega_1(0)$; $\tau_{00}<\tau_{10}(0)$ when $\omega_0>\omega_1(0)$ and $\tau_{00}=\tau_{10}(0)$ when $\omega_0=\omega_1(0)$.

For the case of $d_2>0$ and $0<d_1<\frac{(1-\beta^2)d_2}{8r\beta}$, by the conclusion (ii) of Lemmma 2.4, $\tau_{10}(d_2)$ is increasing with respect to d_2. So $\tau_{10}(d_2)>\tau_{10}(0)>\tau_{00}$ when $0<l^2<\frac{2d_1}{3\beta-\beta^2}$. When $\frac{2d_1}{3\beta-\beta^2}<l^2<\frac{2d_1}{1-\beta^2}$, let

$$d_2^c=\frac{r\beta(1-\beta^2)}{2d_1\lambda_1^2-(1-\beta^2)\lambda_1}. \tag{2.20}$$

Thanks to $l^2<\frac{2d_1}{1-\beta^2}$ and $\lambda_1=\frac{1}{l^2}$, we have $2d_1\lambda_1^2-(1-\beta^2)\lambda_1>0$, which means that $d_2^c>0$.

When $d_2=d_2^c$, from (2.11), we have

$$\omega_1^4(d_2^c)+[\lambda_1^2(d_1^2+(d_2^c)^2)-\lambda_1d_1(1-\beta^2)+\frac{1}{4}(1-\beta^2)^2]\omega_1^2(d_2^c)=\omega_1^2(d_2^c)(\omega_1^2(d_2^c)+B_1(d_2^c))=0.$$

According to the expression of B_k , we know that $B_1 \geq 0$, which means that $\omega_1^2(d_2^c)+B_1(d_2^c) > 0$. Thus we have

$$\lim_{d_2 \to d_2^c} \omega_1(d_2) = 0,$$

which implies that $\lim_{d_2 \to d_2^c} \tau_{10}(d_2) = +\infty$.

In addition, notice that $\tau_{10}(0) < \tau_{00}$ when $\frac{2d_1}{3\beta - \beta^2} < l^2 < \frac{2d_1}{1 - \beta^2}$, we conclude that there exists unique positive real number d_2^* such that $\tau_{10} = \tau_{00}$. So, when $\frac{2d_1}{3\beta - \beta^2} < l^2 < \frac{2d_1}{1 - \beta^2}$, we have $\tau_* = \tau_{10}$ for $0 < d_2 < d_2^*$, $\tau_* = \tau_{00}$ for $d_2 > d_2^*$, and $\tau_* = \tau_{00} = \tau_{10}$ for $d_2 = d_2^*$.

For the case of $d_2 > 0$ and $d_1 \geq \frac{(1-\beta^2)d_2}{8r\beta}$, we have the same discussion and results as in the case $d_2 \geq 0$ and $0 < d_1 < \frac{(1-\beta^2)d_2}{8r\beta}$ above. This completes the proof. □

In the following section, we will discuss the properties of curves $d_2 = d_2^c$ and $d_2 = d_2^*$.

Lemma 2.6. *Suppose that* $d_1 > 0,\ d_2 \geq 0,\ r > 0,\ \frac{\sqrt{3}}{3} < \beta < 1$ *and* $l^2 < \frac{2d_1}{1-\beta^2}$ *hold.* d_2^c *and* d_2^* *are defined by* (2.20) *and* (2.19) *respectively. Taking* d_2^c *and* d_2^* *as functions of* d_1*, we have the following conclusions.*

(i) d_2^c *is strictly monotonically decreasing with respect to* d_1*;*

(ii) d_2^c *is always greater than* d_2^**;*

(iii) d_2^* *is strictly monotonically decreasing with respect to* d_1*;*

(iv) $d_2^*(d_1)$ *is defined on intevel* $[\frac{(1-\beta^2)l^2}{2}, \frac{(3\beta-\beta^2)l^2}{2}]$*, and* $d_2^*(\frac{(1-\beta^2)l^2}{2}) = \frac{(3\beta-1)l^2}{2}$.

Proof. From the expression of d_2^c, it is easy to prove that the conclusion (i) is true.

To prove the conclusion (ii), we define the function

$$\begin{aligned} s(d_2) &= \tau_{10}(d_2) - \tau_{00}(d_2) \\ &= \frac{1}{\omega_1}\left[\arccos\left(\frac{2\omega_1^2 - 2d_1 d_2 \lambda_1^2 + (1-\beta^2)d_2\lambda_1}{r\beta(1-\beta^2)}\right)\right] - \frac{1}{\omega_0}\left[\arccos\left(\frac{2\omega_0^2}{r\beta(1-\beta^2)}\right)\right]. \end{aligned} \tag{2.21}$$

It follows from the conclusion (ii) of Lemma 2.4 that $\tau_{10}(d_2)$ is monotonically increasing with respect to d_2, which means that $s(d_2)$ is monotonically increasing with respect to d_2. Substituting $d_2 = d_2^c$ and $d_2 = d_2^*$ into (2.21) respectively, we obtain $s(d_2^c) = +\infty$ and $s(d_2^*) = 0$. That is to say, $s(d_2^c) > s(d_2^*)$, which means that $d_2^c > d_2^*$. This completes the proof of (ii).

For the conclusion (iii), when $s(d_2) = 0$, taking d_2 regard as a function of d_1 and taking the derivative with respect to d_1 on both sides of equation (2.21), we obtain

$$
\begin{aligned}
&-\frac{\omega_1^{'}(d_1)}{\omega_1^2(d_1)}\arccos\big(\frac{2\omega_1(d_1)^2-2d_1d_2\lambda_1^2+(1-\beta^2)d_2\lambda_1}{r\beta(1-\beta^2)}\big)\\
&-\frac{4\omega_1(d_1)\omega_1^{'}(d_1)-2d_2\lambda_1^2-2d_1d_2^{'}\lambda_1^2+(1-\beta^2)d_2^{'}\lambda_1}{\omega_1(d_1)r\beta(1-\beta^2)\sqrt{1-\big(\frac{2\omega_1^2(d_1)-2d_1d_2\lambda_1^2+(1-\beta^2)d_2\lambda_1}{r\beta(1-\beta^2)}\big)^2}}=0,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
d_2^{'}(d_1)=&\frac{r\beta(1-\beta^2)\sqrt{1-\left(\frac{2\omega_1^2(d_1)-2d_1d_2\lambda_1^2+(1-\beta^2)d_2\lambda_1}{r\beta(1-\beta^2)}\right)^2}}{2d_1\lambda_1^2-(1-\beta^2)\lambda_1}\\
&\times\frac{\omega_1^{'}(d_1)}{\omega_1(d_1)}\arccos\left(\frac{2\omega_1^2(d_1)-2d_1d_2\lambda_1^2+(1-\beta^2)d_2\lambda_1}{r\beta(1-\beta^2)}\right)\\
&+\frac{4\omega_1(d_1)\omega_1^{'}(d_1)-2d_2\lambda_1^2}{2d_1\lambda_1^2-(1-\beta^2)\lambda_1}.
\end{aligned}
\tag{2.22}
$$

The proof process is similar to (i) of Lemma 2.4. We can obtain

$$
\omega_1^{'}(d_1)=\frac{(1-\beta^2-2\lambda_1 d_1)(\omega_1(d_1)^2\lambda_1+d_2^2\lambda_1^3)}{4\omega_1^3(d_1)+2\omega_1(d_1)B_1}.
$$

Thanks to $B_1\geq 0$, $l^2<\frac{2d_1}{1-\beta^2}$ and $\lambda_1=\frac{1}{l^2}$, we have $2d_1\lambda_1-(1-\beta^2)>0$, which means that $\omega_1^{'}(d_1)<0$. Combining with (2.22), we can prove that $d_2^{'}(d_1)<0$. This completes the proof of (iii).

From Lemma 2.5 (iii), we know that there exists d_2^* when $\frac{2d_1}{3\beta-\beta^2}<l^2<\frac{2d_1}{1-\beta^2}$. So if d_2^* is regarded as a function of d_1, then d_2^* is defined on interval $[\frac{(1-\beta^2)l^2}{2},\frac{(3\beta-\beta^2)l^2}{2}]$. Furthermore, substituting $d_1=\frac{(1-\beta^2)l^2}{2}$ into (2.11) and combining with (2.10), we have

$$
\begin{cases}
\omega_0^4+(\frac{3}{2}\beta-\frac{1}{2})^2\omega_0^2-\frac{1}{4}r^2\beta^2(1-\beta^2)^2=0,\\
\omega_1^4+\lambda_1^2d_2^2\omega_1^2-\frac{1}{4}r^2\beta^2(1-\beta^2)^2=0,
\end{cases}
$$

which implies that $\omega_1=\omega_0$ when $d_2^*=d_2(\frac{(1-\beta^2)l^2}{2})=\frac{(3\beta-1)l^2}{2}$. Therefore, we complete the proof of the conclusion (iv). □

In order to easily describe the main results of this section, we define the following areas in the d_1-d_2 plane.

$$
R_{10}=\{(d_1,d_2)|\frac{1-\beta^2}{2}l^2<d_1<\frac{3\beta-\beta^2}{2}l^2,0\leq d_2<d_2^*\},
$$

$$
\begin{aligned}
R_{01}=&\{(d_1,d_2)|\frac{1-\beta^2}{2}l^2<d_1<\frac{3\beta-\beta^2}{2}l^2,d_2^*<d_2<d_2^c\}\\
&\cup\{(d_1,d_2)|d_1>\frac{3\beta-\beta^2}{2}l^2,0\leq d_2<d_2^c\},
\end{aligned}
$$

$$R_{00} = \{(d_1, d_2) | d_1 > \frac{1-\beta^2}{2} l^2, d_2 \geq d_2^c\}, \quad R_0 = R_{01} \cup R_{00}.$$

From Lemmas 2.5 and 2.6, we have

$$\tau_* = \begin{cases} \tau_{00}, & (d_1, d_2) \in R_0, \\ \tau_{10}, & (d_1, d_2) \in R_{10}. \end{cases}$$

By Lemmas 2.3, 2.5 and 2.6 and Hopf bifurcation theory for partial funcational differential equations, we obtain the following results on the stability and Hopf bifurcation of the system (1.5).

Theorem 2.7. *Assume that* $d_1 > 0, d_2 \geq 0, r > 0, \frac{\sqrt{3}}{3} < \beta < 1$ *and* $l^2 < \frac{2d_1}{1-\beta^2}$ *hold.* τ_{ki} *and* d_2^* *are defined by* (2.6) *and* (2.19) *respectively, we have the following results on the stability and Hopf bifurcation of the system* (1.5)*:*

1. *The positive equilibrium* (u_*, v_*) *is locally asymptotically stable for* $\tau \in [0, \tau_*)$ *and unstable for* $\tau \in (\tau_*, +\infty)$*.*

2. *The system* (1.5) *undergoes Hopf bifurcations at* $\tau = \tau_{ki}$*. For* $d_1 > 0$*,* $d_2 = 0$*, there exist the critical value* τ_{ki} *of spatially non-homogeneous Hopf bifurcations for any* $k \in \mathbb{N}$*. But for* $d_1 > 0$*,* $d_2 > 0$*, there exist the critical value* τ_{ki} *of spatially non-homogeneous Hopf bifurcations for finite wave numbers* $k \in \mathbb{N}$ *and* $N_1 < k \leq N_2 (N_1 < k < N_2)$*, where* N_1 *and* N_2 *are defined by Lemma 2.1.*

3. *About the homogeneous/non-homgeneous Hopf bifurcation, we conclude as follows:*

   (i) when $(d_1, d_2) \in R_{00}$*, the spatially non-homogeneous Hopf bifurcations will not occur, and only spatially homogeneous Hopf bifurcations occurs at* τ_{0i} *and* $\tau_* = \tau_{00}$*, thus the bifurcating periodic orbits from the first critical value is spatially homogeneous;*

   (ii) when $(d_1, d_2) \in R_{01}$*, both spatially non-homogeneous and spatially homogeneous Hopf bifurcations occur and the bifurcating periodic orbits from the first critical value* τ_{00} *is spatially homogeneous;*

   (iii) when $(d_1, d_2) \in R_{10}$*, both spatially non-homogeneous and spatially homogeneous Hopf bifurcations occur and the bifurcating periodic orbits from the first critical value* τ_{10} *is spatially non-homogeneous;*

   (iv) when $\frac{(1-\beta^2)l^2}{2} < d_1 < \frac{(3\beta-\beta^2)l^2}{2}$*, and* $d_2 = d_2^*$*, the spatially homogeneous Hopf bifurcations at* $\tau_* = \tau_{00}$ *and spatially non-homogeneous Hopf bifurcations at* τ_{10} *appear at the same time, and there exists a double Hopf bifurcation.*

From Lemma 2.5 and Lemma 2.6, as shown in Figure 1, we draw a sketch of curves $d_2 = d_2^*$ and $d_2 = d_2^c$ in the $d_1 - d_2$ plane. It follows from Theorem 2.7 that when (d_1, d_2) falls in region R_{10}, the first Hopf bifurcation point is τ_{10}, which is a spatially non-homogeneous Hopf bifurcation point. When (d_1, d_2) falls in other regions of the plane, the first Hopf bifurcation point is spatially homogeneous.

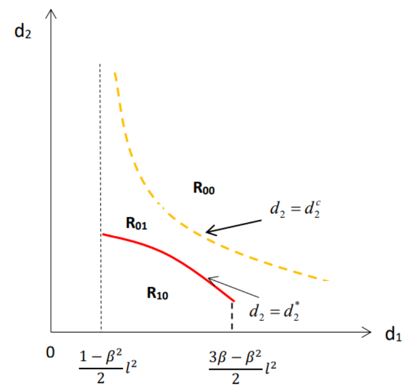

Figure 1: The distribution of the first Hopf bifurcation value τ_* in the $d_1 - d_2$ plane. The solid (red) line represents $d_2 = d_2^*$ and the dotted (yellow) line represents $d_2 = d_2^c$. We have $\tau_* = \tau_{10}$ in the region R_{10} and $\tau_* = \tau_{00}$ in the region R_0.

3 Normal form of Hopf bifurcation

In this section, we investigate the stability and direction of Hopf bifurcation. Because the system (1.5) has both delay and spatial average, we can't use the normal form theory for partial functional differential equations developed by Faria [27]. Recently, Song and Shi [28] derived an explicit algorithm to determine the direction of Hopf bifurcation depending on the coefficients of the original system for a general reaction-diffusion system with delay and spatial average. So we compute the normal form for the system (1.5) by using the theory developed by Song and Shi. In the following, we only give the main results for the system (1.5). For a more detailed process of calculation, readers can refer to the Section 3.1 in the literature [28].

For convenience, we rewrite the system (1.5) as follows:

$$\begin{cases} u_t = d_1 u_{xx} + f^{(1)}(u, \tilde{u}, v), & x \in (0, l\pi), t > 0, \\ v_t = d_2 v_{xx} + f^{(2)}(u, u_\tau, v), & x \in (0, l\pi), t > 0, \end{cases} \tag{3.1}$$

where

$$f^{(1)}(u, \tilde{u}, v) = (u + u_*)(1 - \tilde{u} - u_*) - \sqrt{u + u_*}(v + v_*),\ f^{(2)}(u, u_\tau, v) = r(v + v_*)(-\beta + \sqrt{u_\tau + u_*}).$$

Obviously, $(0,0)$ is always a equilibrium for system (3.1).

For Hopf bifurcation, we have the following assumption condition:

(AC) when $\tau = \tau_*$, there exists a $n_* \in \mathbb{N}_0$ such that Eq.(2.5) has a pair of simple purely imaginary roots $\pm i\omega_*$, and the corresponding transversality condition holds.

Following the procedure in [28], define the real-valued Sobolev space

$$X = \{(u,v)^\top \in (W^{2,2}(0,\pi))^2, \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \ at \ x = 0, l\pi\}.$$

Then $\mathcal{C} := C([-1,0];X)$ is the Banach space of continuous mappings from $[-1,0]$ to X with the sup norm. Letting $\alpha = \tau - \tau_*$ and normalizing time scale by the transformation $t \to \frac{t}{\tau}$, we can rewrite the system (3.1) as follows on $\mathcal{C}$:

$$\frac{\partial U(t)}{\partial t} = \tau_* dU_{xx}(t) + L_0(U_t(\theta), \tilde{U}(0)) + F(U_t(\theta), \tilde{U}(0), \alpha), \tag{3.2}$$

where $U(t) = (u(x,t), v(x,t))^\top$, with the inner product definded by

$$[U,V] = \int_0^{l\pi} U^\top V dx, \ U,V \in X,$$

$$U_t(\theta) = U(x, t+\theta), \ -1 \le \theta \le 0, \ \tilde{U}(0) = \frac{1}{l\pi}\int_0^{l\pi} U(x,t)dx,$$

$$L_0(\varphi, \tilde{\varphi}(0)) = \tau_* \begin{pmatrix} f_u^{(1)}\varphi_1(0) + f_{\tilde{u}}^{(1)}\tilde{\varphi}_1(0) + f_v^{(1)}\varphi_2(0) \\ f_u^{(2)}\varphi_1(0) + f_{u_\tau}^{(2)}\varphi_1(-1) + f_v^{(2)}\varphi_2(0) \end{pmatrix},$$

$$F(\varphi, \tilde{\varphi}(0), \alpha) = \alpha d\Delta\varphi(0) + L(\alpha)(\varphi, \tilde{\varphi}(0)) + f(\varphi, \tilde{\varphi}(0), \alpha),$$

with

$$L(\alpha)(\varphi, \tilde{\varphi}(0)) = \frac{\alpha}{\tau_*} L_0(\varphi, \tilde{\varphi}(0)),$$

and

$$f(\varphi, \tilde{\varphi}(0), \alpha) = (\tau_* + \alpha) \begin{pmatrix} \sum\limits_{i+j+k\ge 2} \frac{1}{i!j!k!} f_{ijk}^{(1)} \varphi_1^i(0)\tilde{\varphi}_1^j(0)\varphi_2^k(0) \\ \sum\limits_{i+j+k\ge 2} \frac{1}{i!j!k!} f_{ijk}^{(2)} \varphi_1^i(0)\varphi_1^j(-1)\varphi_2^k(0) \end{pmatrix}.$$

Here

$$f_{ijk}^{(1)} = \frac{\partial^{i+j+k} f^{(1)}}{\partial u^i \partial \tilde{u}^j \partial v^k}(0,0), \quad f_{ijk}^{(2)} = \frac{\partial^{i+j+k} f^{(2)}}{\partial u^i \partial u_\tau^j \partial v^k}(0,0).$$

For the system (3.2), by computation, we have

$$\begin{aligned} & f_u^{(1)} = \frac{1-\beta^2}{2}, \ f_{\tilde{u}}^{(1)} = -\beta^2, \ f_v^{(1)} = -\beta, \\ & f_u^{(2)} = 0, \ f_{u_\tau}^{(2)} = \frac{r(1-\beta^2)}{2}, \ f_v^{(2)} = 0, \\ & f_{110}^{(1)} = -1, \ f_{101}^{(1)} = -\frac{1}{2\beta}, \ f_{011}^{(1)} = 0, \ f_{200}^{(1)} = \frac{1-\beta^2}{4\beta^2}, \ f_{020}^{(1)} = 0, \ f_{002}^{(1)} = 0, \\ & f_{110}^{(2)} = 0, \ f_{101}^{(2)} = 0, \ f_{011}^{(2)} = \frac{r}{2\beta}, \ f_{200}^{(2)} = 0, \ f_{020}^{(2)} = -\frac{r(1-\beta^2)}{4\beta^2}, \ f_{002}^{(2)} = 0. \end{aligned} \tag{3.3}$$

The characteristic equation of the linearized system of (3.2) is

$$\prod_{k\in\mathbb{N}_0}\Gamma_k(\mu)=0, \tag{3.4}$$

where $\Gamma_k(\mu)=det(\mathcal{N}_k(\mu))$ with

$$\mathcal{N}_k(\mu)=\mu I_2-\tau_* M_k-\tau_* A_0-\tau_* A_1 e^{-\mu\tau}-\tau_*\chi_k A_2. \tag{3.5}$$

Here M_k, A_0, A_1, A_2 and χ_k are given in Section 2. Therefore, from the assumption condition (AC), we know that there exists a n_* such that (3.4) has a pair of simple purely imaginary roots $\pm i\omega_c$ with $\omega_c=\tau_*\omega_*$.

Let $C:=C([-1,0],R^2)$, $C^*:=C([-1,0],R^{2*})$, where R^{2*} is the two-dimensional space of row vectors. We define $\eta_k\in BV([-1,0];R^2)$ such that

$$M_k\varphi(0)+L_0(\varphi(\theta),\widetilde{\varphi}(0))=\int_{-1}^{0}d\eta_k\varphi(\theta),\varphi\in C,$$

and the following adjoint bilinear form on $C^*\times C$

$$\langle\psi(s),\varphi(\theta)\rangle=\psi(0)\phi(0)-\int_{-1}^{0}\int_{0}^{\theta}\psi(\xi-\theta)d\eta_k\varphi(\xi)d\xi, for\ \psi\in C^*,\varphi\in C.$$

Choose $\Phi(\theta)=(\xi e^{i\omega_c\theta},\bar{\xi}e^{-i\omega_c\theta}),\Psi(s)=col(\eta^\top e^{-i\omega_c s},\bar{\eta}^\top e^{i\omega_c s})$. Here $\xi\in C^2$ is the eigenvector corresponding to the eigenvalue $i\omega_c$ of (2.2), and $\eta\in C^2$ is the corresponding adjoint eigenvector, satisfying $<\Psi(s),\Phi(\theta)>=I_2$, where

$$\xi=\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\begin{pmatrix}1\\ -\dfrac{2i\omega_*+2d_1\lambda_{k*}-(1-\beta^2)+2\beta^2\chi_{k*}}{2\beta}\end{pmatrix},$$

$$\eta=\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\eta_1\begin{pmatrix}1\\ \dfrac{2i\omega_*+2d_1\lambda_{k*}-(1-\beta^2)+2\beta^2\chi_{k*}}{r(1-\beta^2)e^{-i\omega_c}}\end{pmatrix},$$

with

$$\eta_1=\left(1-\frac{(2i\omega_*+2d_1\lambda_{k*}-(1-\beta^2)+2\beta^2\chi_{k*})^2}{2r\beta(1-\beta^2)e^{-i\omega_c}}+\frac{\tau_* e^{i\omega_*\tau_*}(2i\omega_*+2d_1\lambda_{k*}-(1-\beta^2)+2\beta^2\chi_{k*})}{2e^{-i\omega_c}}\right)^{-1}.$$

Similar to the Section 3.2 of the literature [28], through calculation, we obtain

$$A_{20}^{(1)}=\tau_*\begin{pmatrix}-\dfrac{1}{\beta}\xi_1\xi_2+\dfrac{1-\beta^2}{4\beta^2}\xi_1^2\\ \dfrac{r}{\beta}\xi_1\xi_2e^{-i\omega_c}-\dfrac{r(1-\beta^2)}{4\beta^2}\xi_1^2e^{-2i\omega_c}\end{pmatrix},\ A_{20}^{(2)}=\tau_*\begin{pmatrix}-2\xi_1^2\\ 0\end{pmatrix},\ A_{20}^{(3)}=\tau_*\begin{pmatrix}0\\ 0\end{pmatrix},$$

$$A_{11}^{(1)}=\tau_*\begin{pmatrix}-\dfrac{1}{\beta}(\xi_1\bar{\xi}_2+\bar{\xi}_1\xi_2)+\dfrac{1-\beta^2}{2\beta^2}|\xi_1|^2\\ \dfrac{r}{\beta}(\xi_1\bar{\xi}_2e^{-i\omega_c}+\bar{\xi}_1\xi_2e^{i\omega_c})-\dfrac{r(1-\beta^2)}{2\beta^2}|\xi_1|^2\end{pmatrix},\ A_{11}^{(2)}=\tau_*\begin{pmatrix}-4|\xi_1|^2\\ 0\end{pmatrix},\ A_{11}^{(3)}=\tau_*\begin{pmatrix}0\\ 0\end{pmatrix},$$

and $A_{02}^{(i)} = \bar{A}_{20}^{(i)}, i = 1, 2, 3$.

Let $k_{n,20}(\theta) = (k_{n,20}^{(1)}(\theta), k_{n,20}^{(2)}(\theta))^\top, k_{n,11}(\theta) = (k_{n,11}^{(1)}(\theta), k_{n,11}^{(2)}(\theta))^\top, k_{n,02}(\theta) = (k_{n,02}^{(1)}(\theta), k_{n,02}^{(2)}(\theta))^\top$. From the Appendix of the literature [28], we know that

when $n_* = 0$,

$$\begin{aligned} k_{0,11}(\theta) =& \frac{1}{\sqrt{l\pi} i\omega_c}(\xi\eta^\top e^{i\omega_c\theta} - \bar{\xi}\bar{\eta}^\top e^{-i\omega_c\theta})(A_{11}^{(1)} + A_{11}^{(2)} + A_{11}^{(3)}) \\ &+ \frac{\mathcal{N}_0^{-1}(0)}{\sqrt{l\pi}}\left(I - \frac{\mathcal{N}_0(i\omega_c)}{i\omega_c}\xi\eta^\top + \frac{\mathcal{N}_0(-i\omega_c)}{i\omega_c}\bar{\xi}\bar{\eta}^\top\right)(A_{11}^{(1)} + A_{11}^{(2)} + A_{11}^{(3)}), \end{aligned}$$

$$\begin{aligned} k_{0,20}(\theta) =& -\frac{1}{\sqrt{l\pi} i\omega_c}(\xi\eta^\top e^{i\omega_c\theta} + \frac{1}{3}\bar{\xi}\bar{\eta}^\top e^{-i\omega_c\theta})(A_{20}^{(1)} + A_{20}^{(2)} + A_{20}^{(3)}) \\ &+ \frac{e^{2i\omega_c\theta}\mathcal{N}_0^{-1}(2i\omega_c)}{\sqrt{l\pi}}\left(I + \frac{\mathcal{N}_0(i\omega_c)}{i\omega_c}\xi\eta^\top + \frac{\mathcal{N}_0(-i\omega_c)}{3i\omega_c}\bar{\xi}\bar{\eta}^\top\right)(A_{20}^{(1)} + A_{20}^{(2)} + A_{20}^{(3)}), \end{aligned}$$

and when $n_* \neq 0$,

$$\begin{aligned} k_{0,11}(\theta) &= \frac{1}{\sqrt{l\pi}}\mathcal{N}_0^{-1}(0)A_{11}^{(1)}, \\ k_{0,20}(\theta) &= \frac{1}{\sqrt{l\pi}}\mathcal{N}_0^{-1}(2i\omega_c)A_{20}^{(1)}e^{2i\omega_c\theta}, \\ k_{2n_*,11}(\theta) &= \frac{1}{\sqrt{2l\pi}}\mathcal{N}_{2n_*}^{-1}(0)A_{11}^{(1)}, \\ k_{2n_*,20}(\theta) &= \frac{1}{\sqrt{2l\pi}}\mathcal{N}_{2n_*}^{-1}(2i\omega_c)A_{20}^{(1)}e^{2i\omega_c\theta}. \end{aligned}$$

Then we continue to compute the S_2 terms:

$$S_2(\xi e^{i\omega_c\theta}, k_{n,11}(\theta)) = 2\tau_* \begin{pmatrix} -\frac{1}{2\beta}(\xi_1 k_{n,11}^{(2)}(0) + \xi_2 k_{n,11}^{(1)}(0)) + \frac{1-\beta^2}{4\beta^2}\xi_1 k_{n,11}^{(1)}(0) \\ \frac{r}{2\beta}(\xi_2 k_{n,11}^{(1)}(-1) + \xi_1 k_{n,11}^{(2)}(0)e^{-i\omega_c}) - \frac{r(1-\beta^2)}{4\beta^2}\xi_1 k_{n,11}^{(1)}(-1)e^{-i\omega_c} \end{pmatrix},$$

$$\tilde{S}_2^{(1)}(\xi, k_{n,11}(\theta)) = 2\tau_* \begin{pmatrix} -\xi_1 k_{n,11}^{(1)}(0) \\ 0 \end{pmatrix},$$

$$\tilde{S}_2^{(2)}(\xi e^{i\omega_c\theta}, k_{0,11}(0)) = 2\tau_* \begin{pmatrix} -\xi_1 k_{0,11}^{(1)}(0) \\ 0 \end{pmatrix},$$

$$\tilde{S}_2^{(3)}(\xi, k_{0,11}(0)) = 2\tau_* \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$S_2(\bar{\xi} e^{-i\omega_c\theta}, k_{n,20}(\theta)) = 2\tau_* \begin{pmatrix} -\frac{1}{2\beta}(\bar{\xi}_1 k_{n,20}^{(2)}(0) + \bar{\xi}_2 k_{n,20}^{(1)}(0)) + \frac{1-\beta^2}{4\beta^2}\bar{\xi}_1 k_{n,20}^{(1)}(0) \\ \frac{r}{2\beta}(\bar{\xi}_2 k_{n,20}^{(1)}(-1) + \bar{\xi}_1 k_{n,20}^{(2)}(0)e^{i\omega_c}) - \frac{r(1-\beta^2)}{4\beta^2}\bar{\xi}_1 k_{n,20}^{(1)}(-1)e^{i\omega_c} \end{pmatrix},$$

$$\tilde{S}_2^{(1)}(\bar{\xi}, k_{n,20}(\theta)) = 2\tau_* \begin{pmatrix} -\bar{\xi}_1 k_{n,20}^{(1)}(0) \\ 0 \end{pmatrix},$$

$$\tilde{S}_2^{(2)}(\bar{\xi} e^{i\omega_c\theta}, k_{0,20}(0)) = 2\tau_* \begin{pmatrix} -\bar{\xi}_1 k_{0,20}^{(1)}(0) \\ 0 \end{pmatrix},$$

$$\tilde{S}_2^{(3)}(\bar{\xi}, k_{0,20}(0)) = 2\tau_* \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

According to the expression of C_{21} in [28],

$$C_{21} = \begin{cases} \dfrac{1}{6l\pi}\eta^T(B_{21}^{(1)} + B_{21}^{(2)} + B_{21}^{(3)} + B_{21}^{(4)}), & n_* = 0, \\ \dfrac{1}{4l\pi}\eta^T B_{21}^{(1)}, & n_* \neq 0. \end{cases}$$

By computation, we have

$$B_{21}^{(1)} = \tau_* \begin{pmatrix} -\dfrac{9(1-\beta^2)}{8\beta^4}|\xi_1|^2\xi_1 + \dfrac{3}{4\beta^3}(\xi_1^2\bar{\xi}_2 + 2|\xi_1|^2\xi_2) \\ -\dfrac{9(1-\beta^2)r}{8\beta^4}|\xi_1|^2\xi_1 e^{-i\omega_c} - \dfrac{3r}{4\beta^3}(\xi_1^2\bar{\xi}_2 e^{-2i\omega_c} + 2|\xi_1|^2\xi_2) \end{pmatrix},$$

and $B_{21}^{(2)} = B_{21}^{(3)} = B_{21}^{(4)} = (0,0)^\top$.

In addition, we can compute D_{21} by the expression of D_{21} in [28].

$$D_{21} = \frac{1}{6i\omega_c}(-a_{20}a_{11} + |a_{11}|^2 + \frac{2}{3}|a_{02}|^2),$$

where

$$a_{20} = \begin{cases} \dfrac{1}{\sqrt{l\pi}}\eta^T(A_{20}^{(1)} + A_{20}^{(2)} + A_{20}^{(3)}), & n_* = 0, \\ 0 & n_* \neq 0, \end{cases}$$

$$a_{11} = \begin{cases} \dfrac{1}{\sqrt{l\pi}}\eta^T(A_{11}^{(1)} + A_{11}^{(2)} + A_{11}^{(3)}), & n_* = 0, \\ 0 & n_* \neq 0, \end{cases}$$

and

$$a_{02} = \begin{cases} \dfrac{1}{\sqrt{l\pi}}\eta^T(A_{02}^{(1)} + A_{02}^{(2)} + A_{02}^{(3)}), & n_* = 0, \\ 0 & n_* \neq 0. \end{cases}$$

Finally, E_{21} and H_{21} can be calculated by the following expression:

$$E_{21} = \begin{cases} \dfrac{1}{6\sqrt{l\pi}}\eta^T\big(S_2(\xi e^{i\omega_c\theta}, k_{0,11}(\theta)) + S_2(\bar{\xi}e^{-i\omega_c\theta}, k_{0,20}(\theta)) \\ +\tilde{S}_2^{(1)}(\xi, k_{0,11}(\theta)) + \tilde{S}_2^{(1)}(\bar{\xi}, k_{0,20}(\theta))\big), & n_* = 0, \\ \dfrac{1}{6\sqrt{l\pi}}\eta^T\left(S_2(\xi e^{i\omega_c\theta}, k_{0,11}(\theta)) + S_2(\bar{\xi}e^{-i\omega_c\theta}, k_{0,20}(\theta))\right) \\ +\dfrac{1}{6\sqrt{2l\pi}}\eta^T\left(S_2(\xi e^{i\omega_c\theta}, k_{2n_*,11}(\theta)) + S_2(\bar{\xi}e^{-i\omega_c\theta}, k_{2n_*,20}(\theta))\right), & n_* \neq 0. \end{cases}$$

$$H_{21}=\begin{cases}\frac{1}{6\sqrt{l\pi}}\eta^T\big(\tilde{S}_2^{(2)}(\xi e^{i\omega_c\theta},k_{0,11}(0))+\tilde{S}_2^{(2)}(\bar{\xi}e^{-i\omega_c\theta},k_{0,20}(0))\\ +\tilde{S}_2^{(3)}(\xi,k_{0,11}(0))+\tilde{S}_2^{(3)}(\bar{\xi}^T,k_{0,20}(0))\big), & n_*=0,\\ \frac{1}{6\sqrt{l\pi}}\eta^T\left(\tilde{S}_2^{(2)}(\xi e^{i\omega_c\theta},k_{0,11}(0))+\tilde{S}_2^{(2)}(\bar{\xi}e^{-i\omega_c\theta},k_{0,20}(0))\right), & n_*\neq 0.\end{cases}$$

Let

$$R_1=i\omega_*\eta^\top\xi,\ \ R_{21}=C_{21}+\frac{3}{2}(D_{21}+E_{21}+H_{21})$$

and

$$\delta_1=Re(R_1),\ \ \delta_2=Re(R_{21}),$$

then we can calculate the value of δ_2 and $\delta_1\delta_2$ according to the above expression.

On the one hand, the sign of $\delta_1\delta_2$ determines the direction of Hopf bifurcation. The bifurcation is forward when $\delta_1\delta_2<0$ and the bifurcation is backward when $\delta_1\delta_2>0$. On the other hand, the sign of δ_2 determines the stability of the nontrivial periodic orbit. The nontrivial periodic orbit is stable when $\delta_2<0$ and the nontrivial periodic orbit is unstable when $\delta_2>0$. Therefore, we can determine the direction and stability of Hopf bifurcation at $\tau=\tau_*$ according to the given parameters in the system (1.5).

4 Numerical simulations

From Theorem 2.7, we know that when $\frac{\sqrt{3}}{3}<\beta<1,\ d_1>0,\ d_2\geq 0$ and $0<l^2<\frac{2d_1}{3\beta-\beta^2}$, $\tau_*=\tau_{00}$; when $d_1>0,\ \frac{2d_1}{3\beta-\beta^2}<l^2<\frac{2d_1}{1-\beta^2}$, $\tau_*=\tau_{00}$ for $d_2>d_2^*$ and $\tau_*=\tau_{10}$ for $0\leq d_2<d_2^*$. This shows that system (1.5) will generate spatially homogeneous and non-homogeneous periodic orbits when $\frac{2d_1}{3\beta-\beta^2}<l^2<\frac{2d_1}{1-\beta^2}$. In this section, we present the results of some numerical simulations for the cases of $\frac{2d_1}{3\beta-\beta^2}<l^2<\frac{2d_1}{1-\beta^2}$ and $0<l^2<\frac{2d_1}{3\beta-\beta^2}$, respectively.

4.1 Simultaneous occurrence of spatially homogeneous and non-homogeneous Hopf bifurcation

Choosing parameters $\beta=0.8,\ r=3$ and $l=1.5$, we have $\lambda_1=(\frac{1}{l})^2=\frac{4}{9}$ and $u_*=0.64$, $v_*=0.2304$. If we take $d_1=1$, then the condition $\frac{2d_1}{3\beta-\beta^2}<l^2<\frac{2d_1}{1-\beta^2}$ is satisfied. In Fig.2, $\tau=\tau_{00}$ is the homogeneous Hopf bifurcation curve and $\tau=\tau_{10}$ is non-homogeneous Hopf bifurcation curve. The two bifurcation curves intersect at point $P(0.43,1.1485)$, which is the double Hopf bifurcation point and shows that $d_2^*=0.43$. Taking three points $P_1(0.38,1.1)$,

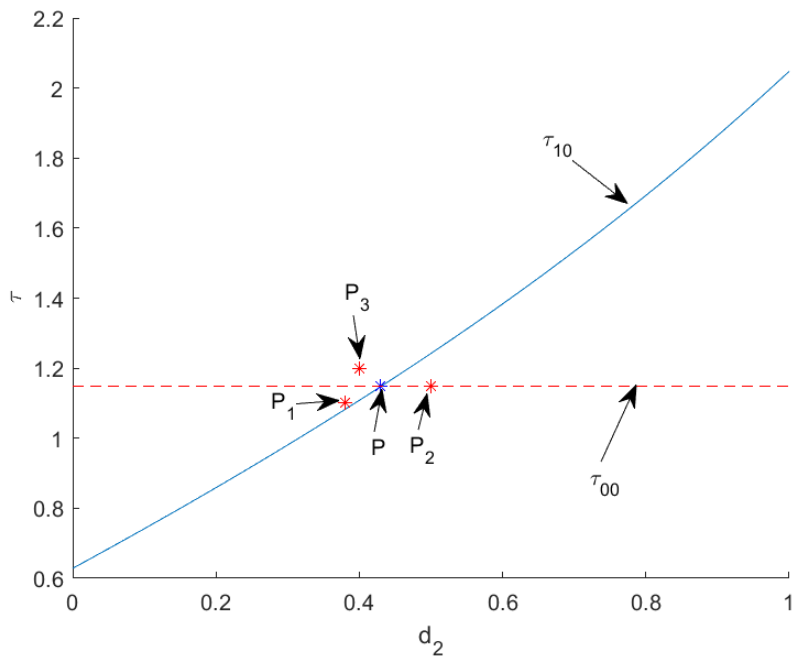

Figure 2: Bifurcation curves diagram for the system (1.5). Parameter values are $d_1 = 1,\ \beta = 0.8,\ r = 3,\ l = 1.5$.

$P_2(0.5, 1.15)$ and $P_3(0.4, 1.2)$ near point $P(0.43, 1.1485)$(indicated by '*' in Fig.2), we perform numerical simulations.

Choosing $d_2 = 0.38 < d_2^*$, we have $\tau_* = \tau_{10} = 1.0823$, which implies that the first Hopf bifurcation point is spatially non-homogeneous. From the calculation of the normal form in Section 3, we obtain $\delta_1 = 0.2249$, $\delta_2 = -0.2091$, which shows that the non-homogeneous Hopf bifurcation is forward and the bifurcating spatially non-homogeneous periodic solutions are stable. The top row in Fig.3 presents the stable spatially non-homogeneous periodic solutions when $d_2 = 0.38$ and $\tau = 1.1$ $(i.e., P_1)$.

Choosing $d_2 = 0.5 > d_2^*$, we have $\tau_* = \tau_{00} = 1.1485$, which implies that the first Hopf bifurcation point is spatially homogeneous. From the calculation of the normal form in the Section 3, we obtain $\delta_1 = 0.2459$, $\delta_2 = -0.4598$, which shows that the homogeneous Hopf bifurcation is also forward and the the bifurcating spatially homogeneous periodic solutions are stable. The middle row in Fig.3 presents the stable spatially homogeneous periodic solutions when $d_2 = 0.5$ and $\tau = 1.15$ $(i.e., P_2)$.

When we take $d_2 = 0.4, \tau = 1.2$ (i.e., point P_3 in Fig2), The bottom row in Fig.3 presents the stable spatially non-homogeneous periodic solutions.

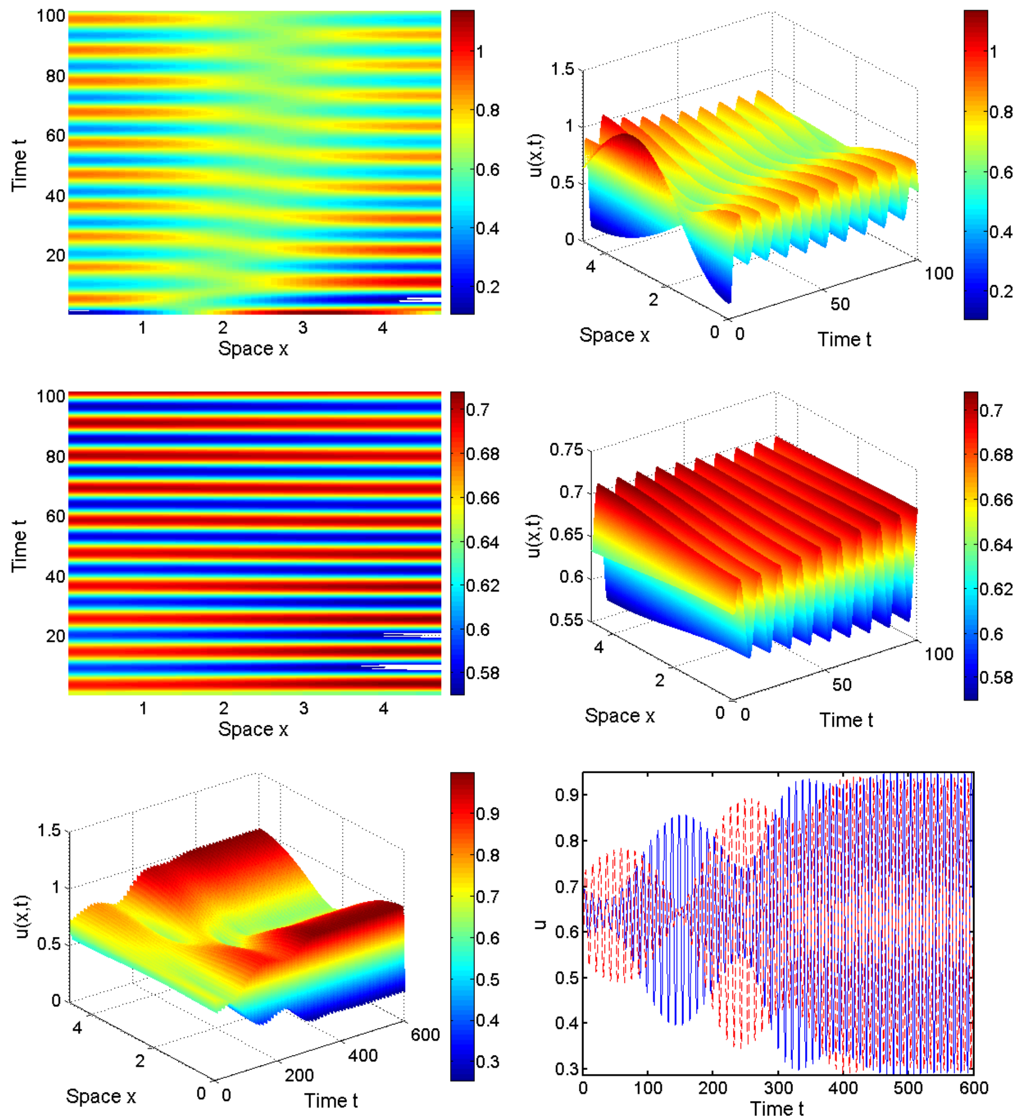

Figure 3: The simulations for species u of the system (1.5). Parameter values are $d_1 = 1,\ \beta = 0.8,\ r = 3,\ l = 1.5$. (The top row): $d_2 = 0.38,\ \tau = 1.1$ which corresponds to P_1 and the corresponding initial conditions are $u(x,t) = 0.64 - 0.5cos(0.5x)$, $v(x,t) = 0.2304 - 0.1cos(0.5x)$ for $t \in [-\tau, 0]$; (The middle row): $d_2 = 0.5,\ \tau = 1.15$ which corresponds to P_2 and the corresponding initial conditions are $u(x,t) = 0.64 + 0.01cos(0.5x)$, $v(x,t) = 0.2304 + 0.01cos(0.5x)$ for $t \in [-\tau, 0]$; (The bottom row): $d_2 = 0.4,\ \tau = 1.2$ which corresponds to P_3 and the corresponding initial conditions are $u(x,t) = 0.64 + 0.05cos(0.5x)$, $v(x,t) = 0.2304 + 0, 01cos(0.5x)$ for $t \in [-\tau, 0]$. When $x = 0.785$, the solution is plotted (blue solid curve) and $x = 3.925$, the solution is also plotted (red dotted curve).

4.2 Occurrence of only spatially homogeneous Hopf bifurcation

The values of parameters β, r and l are the same as those in Section 4.1. If we take $d_1 = 3$, then $0 < l^2 < \dfrac{2d_1}{3\beta - \beta^2}$ is satisfied. Therefore $\tau_* = \tau_{00}$ for any $d_2 \geq 0$. The homogeneous Hopf bifurcation curve $\tau = \tau_{00}$ and non-homogeneous Hopf bifurcation curve $\tau = \tau_{10}$ in the plane $d_2 - \tau$ are shown in Fig.4. It can be seen that these two Hopf bifurcation curves do not intersect. Taking $d_2 = 0.2$, direct calculation means that $\tau_{00} = 1.1485$ and $\tau_{10} = 1.16214$. We choose three points $P_4(0.2, 0.6)$, $P_5(0.2, 1.16)$ and $P_6(0.2, 1.63)$(represented by '*') in Fig.4 for numerical simulations.

Choosing $d_2 = 0.2$, we have $\tau_* = \tau_{00} = 1.1485$. If we take $\tau = 0.6 < \tau_*$, then the positive equilibrium is stable. In the top row of Fig.5 $(i.e., P_4)$, we show numerical simulation, which is consistent with the theoretical results. If we take $\tau = 1.16 > \tau_*$, the first Hopf bifurcation point is spatially homogeneous. From the calculation steps of the normal form in the Section 3, we obtain $\delta_1 = 0.2456$, $\delta_2 = -10.8069$, which implies that the homogeneous Hopf bifurcation is forward and the bifurcating spatially homogeneous periodic solutions are stable, as shown in the middle row of Fig.5 $(i.e., P_5)$. If we take $\tau = 1.63 > \tau_{10} > \tau_{00} = \tau_*$, the first Hopf bifurcation point is spatially homogeneous, the bifurcating spatially homogeneous periodic solutions are still stable, as shown in the bottom row of Fig.5 $(i.e., P_6)$.

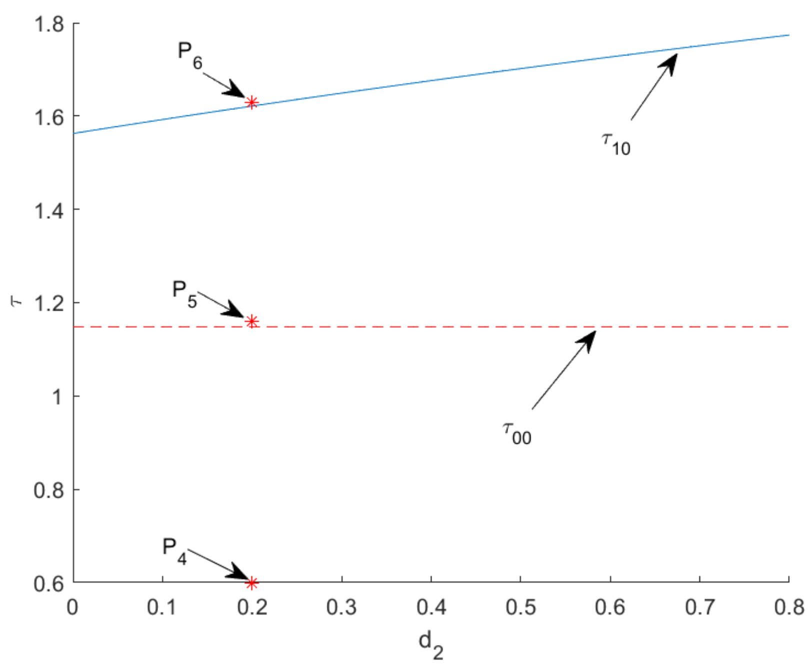

Figure 4: Bifurcation curves diagram for the system (1.5). Parameter values are $d_1 = 3$, $\beta = 0.8$, $r = 3$, $l = 1.5$.

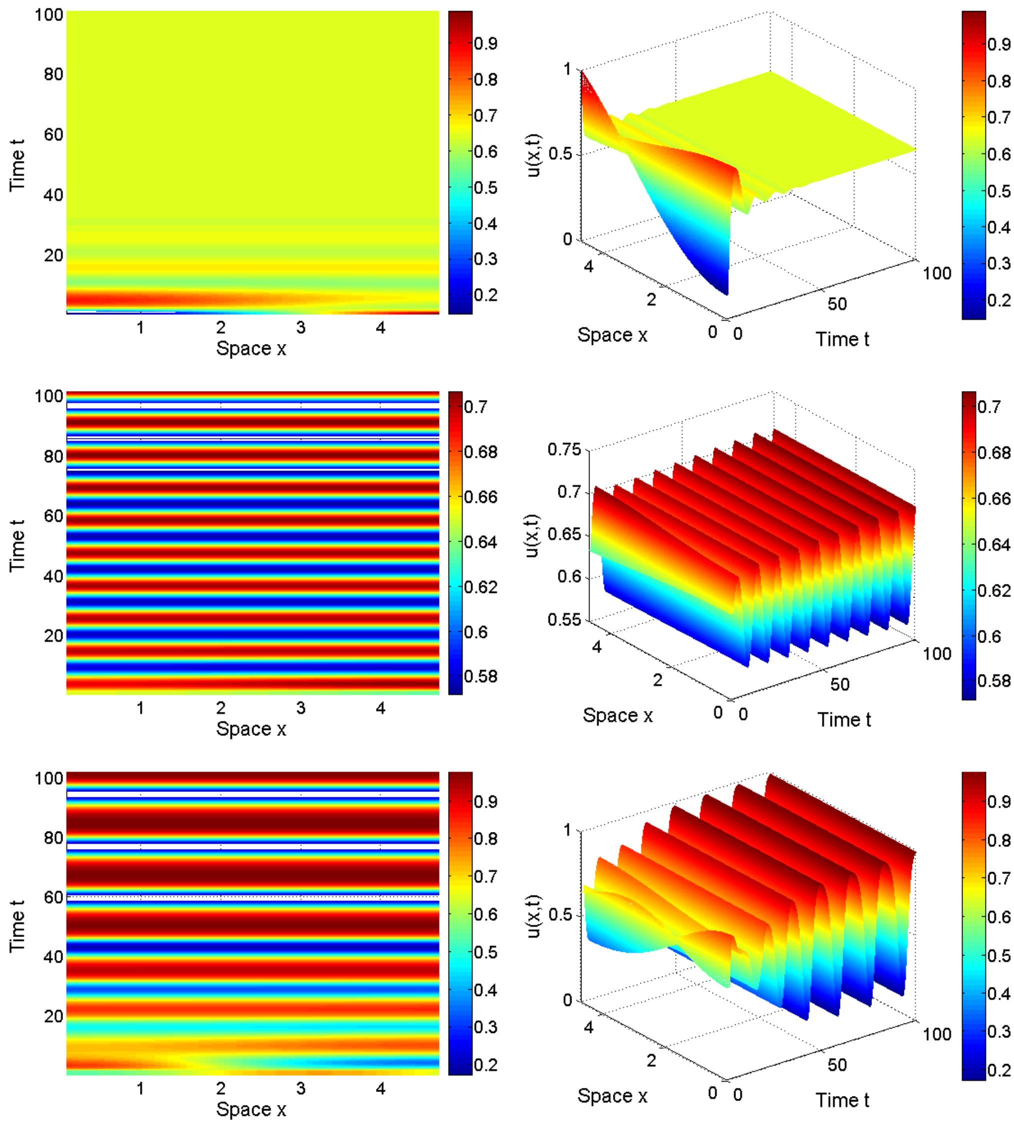

Figure 5: The simulations for species u of the system (1.5). Parameter values are $d_1 = 3,\ \beta = 0.8,\ r = 3,\ l = 1.5$. (The top row): $d_2 = 0.2,\ \tau = 0.6$ which corresponds to P_4 and the corresponding initial conditions are $u(x,t) = 0.64 - 0.5cos(0.5x),\ v(x,t) = 0.2304 - 0.1cos(0.5x)$ for $t \in [-\tau, 0]$; (The middle row): $d_2 = 0.2,\ \tau = 1.16$ which corresponds to P_5 and the corresponding initial conditions are $u(x,t) = 0.64 + 0.01cos(0.5x),\ v(x,t) = 0.2304 + 0.01cos(0.5x)$ for $t \in [-\tau, 0]$; (The bottom row): $d_2 = 0.2,\ \tau = 1.63$ which corresponds to P_6 and the corresponding initial conditions are $u(x,t) = 0.64 - 0.1cos(x),\ v(x,t) = 0.2304 - 0.5cos(x)$ for $t \in [-\tau, 0]$.

5 Conclusion

In this paper, we introduce both time delay and nonlocal prey intraspecific competition into a diffusive predator-prey systems with herd behaviour. We first prove the stability of the positive equilibrium (u_*, v_*) of the system (1.5) when $\tau \in [0, \tau_*)$ and $l^2 < \dfrac{2d_1}{1-\beta^2}$, which implies the influence of delay and nonlocal competition on stability. We also find that, for the different ranges of diffusive coefficients d_1 and d_2, under the together action of time delay and nonlocal competition, the first critical value of Hopf bifurcation may be homogeneous or non-homogeneous. As is known to all, the properties of Hopf bifurcation can be determined by the normal form. Thus we use the algorithm of calcating the normal form of delay-induced homogeneous/non-homogeneous Hopf bifurcation for the reaction-diffusion system with delay and spatial average established by Song and Shi [28] to the system (1.5). It can be seen from Fig2 that the double Hopf bifurcation exists for the system (1.5) with delay and spatial average when the diffusive coefficient d_1 is small. Finally, the spatially stable homogeneous or non-homogeneous periodic solutions are shown by numerical simulations.

In addition, the nonlocal term appears in the reaction term in this paper. More recently, Song et al. [29] established a diffusive consumer-resource model with nonlocal perception of resource availability, where the nonlocal term appears in the diffusion term. The biological meanings of the two modeling methods are completely different. We hope that our next work will be to apply the new methods developed in reference [29] to our specific model.

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