Monotonicity of the ratios of two Abelian integrals for Hamiltonian systems with parameters[1]

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Abstract

We study the monotonicity of the ratios of two Abelian integrals $\oint_{\gamma_i(h)} ydx \setminus \oint_{\gamma_i(h)} xydx$ over three period annuli $\{\gamma_i(h)\}$, for $i = 1, 2, 3$, defined by a seventh-degree hyperelliptic Hamiltonian $H(x, y) = y^2 + \Psi(x)$ with a parameter. The parameter makes the problem more challenging to analyze. To overcome the difficulty, we apply some criterion with the help of transformations, tools in computer algebra such as boundary polynomial theory to determine the monotonicity of the ratios. Our results establish the existence and uniqueness of limit cycle bifurcated from each period annulus.

Keywords: Abelian integral; Monotonicity; Hamiltonian system.

1. Introduction and main description

A weak version of Hilbert's 16th problem [1] asks for the maximal number of zeros of the Abelian integral

$$I(h) = \oint_{\Gamma_h} f(x, y)dx + g(x, y)dy, \ h \in J,$$

along the closed curves $\Gamma_h = \{(x, y) : H(x, y) = h, \ h \in J\}$, where $H(x, y)$, $f(x, y)$ and $g(x, y)$ are polynomials, satisfying $\deg(H) = n + 1$, $\max\{\deg f, \deg g\} = m$. The weak version of the

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problem is still difficult. It has only been completely solved for $n = 2$. There are at most two zeros of the corresponding Abelian integral for the generic quadratic Hamiltonian system, and at most one zero for the degenerate quadratic Hamiltonian systems. When considering $f(x,y) = (a_0 + a_1 x)y$, $g(x,y) = 0$ and $H(x,y) = y^2 + P_{n+1}(x)$, where $P_{n+1}(x)$ is a $n+1$th degree polynomial, $I(h)$ has a simper form

$$I(h) = a_0 \oint_{\Gamma_h} ydx + a_1 \oint_{\Gamma_h} xydx \doteq a_0 I_0(h) + a_1 I(h), \quad h \in J. \tag{1}$$

In this setting, $I_0(h) = \oint_{\Gamma_h} ydx = \oint_{\Gamma_h} dxdy > 0$ by Green formula, and then the ratio is well defined

$$r(h) = \frac{I_1(h)}{I_0(h)}. \tag{2}$$

When $r(h)$ is monotone, then we can claim that $I(h)$ has at most one zero, and any zero can be reached at $h = h^*$ by taking $a_0 = -r(h^*)a_1$. Therefore, studying the monotonicity of the ratios of two Abelian integral is a much simper version of weak Hilbert's problem, it has been considered in a series papers [2, 3, 4, 5, 6]. We note that the monotonicity of the ratios of two Abelian integrals has intensive application. It is an important topic for studying the existence of periodic traveling waves [7, 8] and some reaction diffusion models with memory effect [10]. In addition, the monotonicity determines the number of negative eigenvalues when studying the spectral stability of the periodic waves [11]. Applying the monotonicity criterion for studying the spectral stability would be much more interesting for the periodic traveling waves in nonlinear dispersion drinfel'd-sokolov $D(m,n)$ system [12] and periodic ultra waves for in nanoscale optics [13].

The study on the monotonicity of $r(h)$ appeared in BT bifurcation study when determining the limit cycle that emerges from a singularity and die in a homoclinic loop bifurcation, and for studying weak Hilbert's problem on degenerate quadratic Hamiltonian system perturbed by quadratic polynomials [14]. Later, Li and Zhang [2] proposed a monotonicity criterion for the ratio of two general Abelian integrals. Li-Zhang criterion has been successfully applied for ratios of two Abelian integrals when studying three-dimensional linear space of three integrals by a geometric method [15, 16, 17, 18]. This criterion has been generalized for studying Abelian integrals with more generating elements [20]. Two improved criteria were proposed by Liu and Xiao in [3] and [4], which has simper from and is convenient to use. Using those criteria combined some skill analysis, the monotonicity of $r(h)$ has been studied for quartic hyperelliptic Hamiltonian system [19], quintic hyperelliptic Hamiltonian system [5, 6] and higher order Hamiltonian systems [21, 22, 23, 24], as well as for exploring the existence of solitary wave solution and coexistence periodic waves and solitary waves [7, 8]. In [21, 22], the monotonicity concerned on the sixth order hyperelliptic Hamiltonian systems have been partially determined. However, there exist some problems left.

In this paper, we solve partially the problems left in [21, 22], studying the monotonicity of $r(h)$ for sixth order hyperelliptic Hamiltonian systems by applying some new techniques including polynomial boundary theory and bounding the possible roots on a cubic set. In detail, we study $r(h)$ on the Hamiltonian

$$H(x,y) = \frac{y^2}{2} + \int x(x-\alpha)(x-\beta)(x-\gamma)(x-\lambda)(x-1)dx,$$

and the Hamiltonian system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x(x-\alpha)(x-\beta)(x-\gamma)(x-\lambda)(x-1). \end{cases} \tag{3}$$

where $0 \le \alpha \le \beta \le \gamma \le \lambda \le 1$. We take the parameters within our interest in the three items.

1. When $\alpha = \beta = \gamma = 0$ and $0 < \lambda < 1$, $(1,0)$ is an elementary center, $(\lambda, 0)$ is a hyperelliptic saddle and $(0,0)$ is a cusp of order 2; The period annulus $\mathbf{P}_1$, see Figure 2(a), is defined by

$$\{\gamma_h\} = \{(x,y) : H(x,y) = h, \ h \in (h_c, h_s)\}$$

with

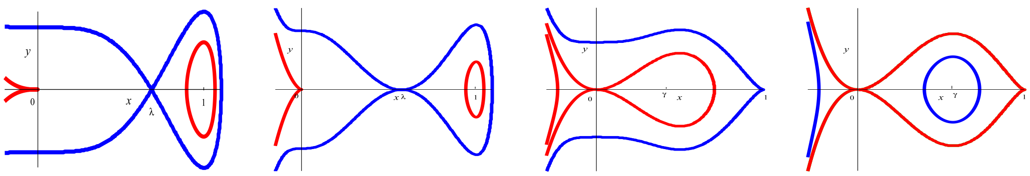
$$h_c = -\frac{1}{42} + \frac{1}{30}\lambda, \quad h_s = -\frac{1}{210}\lambda^6(5\lambda - 7).$$

2. When $\alpha = 0$ and $0 < \beta = \gamma = \lambda < 1$, $(1,0)$ is an elementary center, $(\lambda, 0)$ is a nilpotent saddle and $(0,0)$ is a cusp; The period annulus $\mathbf{P}_2$, see Figure 2(b), is determined by the Hamiltonian, at the saddle and center, respectively,

$$h_c = -\frac{1}{42} + \frac{1}{10}\lambda - \frac{3}{20}\lambda^2 + \frac{1}{12}\lambda^3, \quad h_s = -\frac{1}{420}\lambda^6(3\lambda - 7).$$

3. When $\alpha = \beta = 0$, $0 < \gamma \le \frac{4}{7}$ and $\lambda = 1$, $(\gamma, 0)$ is an elementary center, $(0,0)$ is a nilpotent saddle and $(1,0)$ is a cusp of order one; The period annulus $\mathbf{P}_3$, see Figure 2(c) and (d), is determined by the Hamiltonian, at the saddle and center, respectively,

$$h_c = -\frac{1}{420}\gamma^5(10\gamma^2 - 28\gamma + 21), \quad h_s = 0.$$



(a) (b) (c) (d)

Figure 1: The level set of $H(x, y) = h$: (a) for $\alpha = \beta = \gamma = 0$ and $0 < \lambda < 1$; (b) $\alpha = 0$ and $0 < \beta = \gamma = \lambda < 1$; (c) and (d) for $\alpha = \beta = 0$, $0 < \gamma \leq \frac{4}{7}$ and $\lambda = 1$.

Our main results on the ratio of the two Abelian integrals

$$\mathscr{R}(h) = \frac{\oint_{\gamma_h} xydx}{\oint_{\gamma_h} ydx}$$

are summarized as follows:

Theorem 1.1. *If $\alpha = \beta = \gamma = 0$ and $0<\lambda<1$, then $\mathscr{R}(h)$ is monotonic in the open interval (h_c, h_s), where $h_c = -\frac{1}{42} + \frac{1}{30}\lambda$, $h_s = -\frac{1}{210}\lambda^6(5\lambda - 7)$.*

Theorem 1.2. *If $\alpha = 0$ and $0 <\beta = \gamma = \lambda <1$, then $\mathscr{R}(h)$ is monotonic in the open interval (h_c, h_s), where $h_c = -\frac{1}{42} + \frac{1}{10}\lambda - \frac{3}{20}\lambda^2 + \frac{1}{12}\lambda^3$, $h_s = -\frac{1}{420}\lambda^6(3\lambda - 7)$.*

Theorem 1.3. *When $\alpha = \beta = 0$, $0 <\gamma \leq \frac{4}{7}$ and $\lambda = 1$, then $\mathscr{R}(h)$ is monotonic in the open interval (h_c, h_s), where $h_c = -\frac{1}{420}\gamma^5(10\gamma^2 - 28\gamma + 21)$, $h_s = 0$.*

Based on the above results, we can conclude that the perturbed Hamiltonian system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x(x - \alpha)(x - \beta)(x - \gamma)(x - \lambda)(x - 1) + \epsilon(a_0 + a_1 x)y. \end{cases}$$

with ϵ sufficiently small and a_0 and a_1 are bounded parameters, has at most one limit cycle by Poincaré bifurcation when the parameters are located in the region stated above.

We organize the paper as follows. In Section 2, we give an introduction on polynomial boundary theory, which will be applied to determine whether or not the related parameter-algebraic systems have roots when the parameter is located in a certain interval. In next three sections, we prove the above theorems by verifying that $\mathscr{R}'(h) \neq 0$ in the related intervals. A conclusion is drawn in section 6.

2. Preliminaries

Considering a Hamiltonian

$$H(x,y) = y^2 + \Psi(x), \tag{4}$$

where $\Psi(x)$ is analytic on some open interval (α, A). Suppose there exists a value $a \in (\alpha, A)$ such that

$$\Psi^{'}(x)(x-\alpha) > 0, \quad x \in (\alpha, A)\setminus\{\alpha\}, \tag{5}$$

implying the Hamiltonian has a local minimum at $(a, 0)$, and there exist a family of continuous closed curves defined by $\{\Gamma_h\} = \{(x,y) : H(x,y) = h, h \in (h_1, h_2)\}$, where $h_1 = \Psi(a)$ and $h_2 = \Psi(A) = \Psi(\alpha)$. For $h \in (h_1, h_2)$, there exist two functions $\mu(h)$ and $v(h)$ satisfying that $\Psi(\mu(h)) = \Psi(v(h)) = h$ with $\alpha{<}\mu(h){<}a{<}v(h){<}A$. Introducing the function

$$U(h) \triangleq \mu(h) + v(h).$$

Then there exists the following criterion for determining the monotonicity of $r(h)$.

Theorem 2.1. ([3]) *Assume that $H(x,y)$ has the form given in (4). Then $r(h)$ is increasing (or decreasing) in (h_1, h_2) when $U^{'}(h) > 0$ (or $U^{'}(h) < 0$) in (h_1, h_2).*

We will apply the criterion to study our problem, however, we do not apply the method by a straightforward analysis. We transform the criterion in an algebraic version to determine whether the related parameter-algebraic system have roots or not. Therefore, we first give an introduction on a method to count roots of parameter-algebraic system. Consider κ be a field, n ordered variables $x_1{<}x_2{<}\cdots{<}x_n$ and polynomial ring $R = k[x_1, \cdots, x_n]$ on κ. The main variable is represented by $mvar(f)$, which refers to the greatest variable x_i in $f(x_1, \cdots, x_i)$. The leading coefficient is represented by $lc(f)$, which refers to the coefficient of the main variable of f.

Definition 2.1. *A semi-algebraic system (SAS for short) is a conjunctive polynomial formula of the following form*

$$\begin{cases} p_1(x_1, x_2, \cdots, x_n) = 0, \cdots, p_s(x_1, x_2, \cdots, x_n) = 0, \\ g_1(x_1, x_2, \cdots, x_n) \geq 0, \cdots, g_r(x_1, x_2, \cdots, x_n) \geq 0, \\ g_{r+1}(x_1, x_2, \cdots, x_n) > 0, \cdots, g_t(x_1, x_2, \cdots, x_n) > 0, \\ h_1(x_1, x_2, \cdots, x_n) \neq 0, \cdots, h_m(x_1, x_2, \cdots, x_n) \neq 0, \end{cases}$$

where $n, s \geq 1, t \geq r \geq 0, m \geq 0$, all $p_i, g_i, h_i \in R(u, x)$ are polynomials with integer coefficients.

An SAS is expressed as $[F, N, P, H]$, and $F = [p_1, \cdots, p_s]$, $N = [g_1, \cdots, g_r]$, $P = [g_{r+1}, \cdots, g_t]$, $H = [h_1, \cdots, h_m]$. It is known as a parametric SAS if $s<n$ (where the first s x_is are variables and the last $n-s$ ones are parameters). One method for counting the roots of parameter-semi-algebraic system is to triangulate SAS into one or more TSAs: $T_1, \cdots, T_L$,

$$TSA, T_j : \begin{cases} f_1^j(u, x_1) = 0, f_2^j(u, x_1, x_2) = 0, \cdots, f_s^j(x_1, \cdots, x_s) = 0, \\ g_1(x_1, x_2, \cdots, x_n) \geq 0, \cdots, g_r(x_1, x_2, \cdots, x_n) \geq 0, \\ g_{r+1}(x_1, x_2, \cdots, x_n)>0, \cdots, g_t(x_1, x_2, \cdots, x_n)>0, \\ h_1(x_1, x_2, \cdots, x_n) \neq 0, \cdots, h_m(x_1, x_2, \cdots, x_n) \neq 0, \end{cases}$$

where $\left\{f_1^j, f_2^j, \cdots, f_s^j\right\}$ represents a triangular set, or a normal ascending chain. Let $\mathrm{dis}(f_i, x_i)$ represents the discriminant of the polynomial f_i with respect to x_i, $\mathrm{res}(\cdot,*,x_j)$ represents the Sylvester resultant between $\cdot$ and $*$ with respect to x_j, and $\mathrm{gcd}(f_1, \cdots, f_s)$ denotes the greatest common factor of $f_1, \cdots, f_s$.

Definition 2.2. *For a parametric TSA T, we define*

$$\begin{aligned} B_{T_j} =& lc(f_1, x_1) \cdot dis(f_1, x_1) \cdot \\ & \Pi_{2 \leq i \leq s} res(lc(f_i, x_i) \cdot dis(f_i, x_i); f_{i-1}, \cdots, f_1) \cdot \\ & \Pi_{1 \leq j \leq t} res(g_j; f_s, \cdots, f_1) \cdot \\ & \Pi_{1 \leq k \leq m} res(h_k; f_s, \cdots, f_1), \end{aligned}$$

the above equation is the boundary polynomial of TSA.

For parametric semi-lgebraic systems TSA: T_j and $T_{\tilde{j}}$, we introduce the notations:

$$r_i^{j\tilde{j}} = gcd(res(f_i^j; f_i^{\tilde{j}}, f_{i-1}^{\tilde{j}}, \cdots, f_1^{\tilde{j}}), res(f_i^{\tilde{j}}; f_i^j, f_{i-1}^j, \cdots, f_1^j)),\ 1 \leq i \leq s,$$

$$C_{j\tilde{j}} = gcd(r_i^{j\tilde{j}}, \cdots, r_s^{j\tilde{j}}).$$

Definition 2.3. *If a parametric semi-algebraic system S is equivalentally transformed to a regular TSAs $\{T_1, T_2, \cdots, T_l\}$, then*

$$B_s = \Pi_{1 \leq j \leq \tilde{j} \leq l} C_{j\tilde{j}} \cdot \Pi_{1 \leq j \leq l} B_{T_j}$$

is called the boundary polynomial of SAS.

Lemma 2.1. ([25, 26]) *The number of distinct real solutions of the semi-algebraic system S is invariant in each connected component of the complement of $B_S = 0$ in R^{n-s}.*

Remark 2.1. When the parameters are located in the boundary set of $B_s = 0$, i.e. the zero set of B_s, we need have a further analysis.

In the next three sections, we mainly apply these methodologies combined with other techniques to prove our results.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Taking $\alpha = \beta = \gamma = 0$ and $0<\lambda<1$, system (3) becomes

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x^4(x-\lambda)(x-1), \end{cases} \tag{6}$$

which has a Hamiltonian function

$$H_1(x,y) = \frac{1}{2}y^2 + \frac{1}{5}\lambda x^5 - \frac{1}{6}(1+\lambda)x^6 + \frac{1}{7}x^7.$$

The ovals $\gamma_1(h)$ surrounds the center $(1,0)$. We take a transform $x = \tilde{x}+1$, $y = y$ to move the center to the origin without any other change of the period annulus $\mathbf{P}_1$ and ovals $\gamma_1(h)$, and the outer boundary is homoclinic to a saddle $(-a,0)$. Then system (6) becomes

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x(x+1)^4(x+a), \end{cases} \tag{7}$$

where $a = 1-\lambda$, and the related Hamiltonian has the form

$$\mathcal{H}_1(x,y) = h_c + \frac{1}{2}y^2 + \frac{1}{2}ax^2 + \frac{1}{3}(1+4a)x^3 + \frac{1}{4}(4+6a)x^4 + \frac{1}{5}(6+4a)x^5 + \frac{1}{6}(4+a)x^6 + \frac{1}{7}x^7.$$

And the Abelian integrals $I_0(h)$ and $I_1(h)$ have new expressions

$$I_0(h) = \oint_{\gamma_1(h)} y dx = \oint_{\gamma_1(h)} y d\tilde{x} \triangleq I_{01}(h),$$

$$I_1(h) = \oint_{\gamma_1(h)} xy dx = \oint_{\gamma_1(h)} (\tilde{x}+1)y d\tilde{x} \doteq I_{01}(h) + I_{11}(h).$$

Then we need only prove $\frac{I_{11}(h)}{I_{01}(h)}$ is monotonic in the interval (h_c, h_s) to prove Theorem 1.1. Introducing

$$\Psi_1(x) = \frac{1}{2}ax^2 + \frac{1}{3}(1+4a)x^3 + \frac{1}{4}(4+6a)x^4 + \frac{1}{5}(6+4a)x^5 + \frac{1}{6}(4+a)x^6 + \frac{1}{7}x^7,$$

and the notations for the involutions $x(h)$ and $z_1(x)$, satisfying

$$\Psi_1(x(h)) = \Psi_1(z_1(h)) = h, \ -a<x(h) < 0<z_1(h)<\kappa_a,$$

where $\Psi_1(-a) = \Psi_1(\kappa_a)$. Introducing

$$U_1(h) = x(h) + z_1(h),$$

we need only prove $U_1'(h) \neq 0$ when the parameter $a \in (0,1)$ by Theorem 2.1. Since $\Phi_1(z_1(h)) = h$ and $\Phi_1^{'}(z_1(h)) > 0$, then $z_1^{'}(h) > 0$ in (h_c, h_s). Thus, $z_1(h)$ has an inverse function in the form of $h = h^{-1}(z_1)$, which is substituted into $x(h)$ to get $x(h) = x(z_1(h))$, where $x(z_1)$ is defined by $\phi(x) - \phi(z_1) = 0$, and implicitly defined by

$$\begin{aligned} q(z_1, x, a) = & -30x^6 + (-35a - 140 - 30z_1)x^5 + (-30z_1^2 + (-35a - 140)z_1 - 168a - 252)x^4 \\ & + (-30z_1^3 + (-35a - 140)z_1^2 + (-168a - 252)z_1 - 315a - 210)x^3 + (-30z_1^4 + (\\ & - 35a - 140)z_1^3 + (-168a - 252)z_1^2 + (-315a - 210)z_1 - 280a - 70)x^2(-30z_1^5 \\ & + (-35a - 140)z_1^4 + (-168a - 252)z_1^3 + (-315a - 210)z_1^2 + (-280a - 70)z_1 \\ & - 105a)x - 30z_1^6 + (-35a - 140)z_1^5 + (-168a - 252)z_1^4 + (-315a - 210)z_1^3 \\ & + (-280a - 70)z_1^2 - 105az_1. \end{aligned}$$

We know that $z_1(x)$ is implicitly defined by $q(z_1, x, a)$. Therefore

$$U_1'(h) = \Big(\frac{dx}{dz_1} + 1\Big)x'(h) = \Big(-\frac{q_x}{q_{z_1}} + 1\Big)x'(h) = -2(x - z_1)\frac{U_{11}(z_1, x, a)}{U_{12}(z_1, x, a)}x'(h),$$

where

$$\begin{aligned} U_{11}(z_1, x, a) = & 70ax^3 + 105ax^2z_1 + 105axz_1^2 + 70az_1^3 + 75x^4 + 120x^3z_1 + 135x^2z_1^2 + 120xz_1^3 \\ & + 75z_1^4 + 252ax^2 + 336axz_1 + 252az_1^2 + 280x^3 + 420x^2z_1 + 420xz_1^2 + 280z_1^3 \\ & + 315ax + 315az_1 + 378x^2 + 378z_1^2 + 504xz_1 + 140a + 210x + 210z_1 + 35, \end{aligned}$$

$$\begin{aligned} U_{12}(z_1, x, a) = & 35ax^4 + 70ax^3z_1 + 105ax^2z_1^2 + 140axz_1^3 + 175az_1^4 + 30x^5 + 60x^4z_1 + 90x^3z_1^2 \\ & + 120x^2z_1^3 + 150xz_1^4 + 180z_1^5 + 168ax^3 + 336ax^2z_1 + 504axz_1^2 + 672az_1^3 \\ & + 140x^4 + 280x^3z_1 + 420x^2z_1^2 + 560xz_1^3 + 700z_1^4 + 315ax^2 + 630axz_1 \\ & + 945az_1^2 + 252x^3 + 504x^2z_1 + 756xz_1^2 + 1008z_1^3 + 280ax + 560az_1 \\ & + 210x^2 + 420xz_1 + 630z_1^2 + 105a + 70x + 140z_1. \end{aligned}$$

It is sufficient to prove that $U_{1i}(z_1, x, \lambda) \neq 0$ on D when $i = 1, 2$, where

$$D = \{(z_1, x, a)| -a<x<0<z_1<\kappa_a, 0<a<1\}.$$

First, we get the resultant between $U_{12}(z_1, x, a)$ and $q(z_1, x, a)$ with respect to x,

$$r_0(z_1) = -2977309629000000000000 {z_1}^5 (z_1 + 1)^{20} (z_1 + a)^5.$$

Obviously, $r_0(z_1)$ has no zero in the interval $(0, \kappa_a)$. So $U_{12}(z_1, x, a)$ and $q(z_1, x, a)$ have no common roots on D, which means $U_{12}(z_1, x, a) \neq 0$ on D and the ratio $\frac{U_{11}}{U_{12}}$ is well defined.

Similarly, we can calculate the resultant of $U_{11}(z_1, x, a)$ and $q(z_1, x, a)$ with respect to x and z_1, respectively, we obtain

$$R_1(z_1, a) = g(z_1, a), \quad R_2(x, a) = g(x, a), \tag{8}$$

where $g(w, a)$ is a polynomial with a little long expression

$$g(w, a) = 121550625(w+1)^8(112546875\, a^8 w^8 + 321562500\, a^7 w^9 + 493062500\, a^6 w^{10} + \cdots + 19600).$$

Taking $x = -\frac{a}{1+t}$ and $a = \frac{1}{5(1+s)}$ with $t, s > 0$, then $a \in (0, \frac{1}{5})$, and $g(x, a)$ has the form

$$g(x, a) = \frac{194481(5st + 5s + 5t + 4)^8}{6103515625(1 + t)^{24}(1 + s)^{24}} g^*(t, s),$$

where all coefficients of $g^*(t, s)$ are greater than zero, and $\tilde{g}^*(t, 0) > 0$, we here include $a = \frac{1}{5}$, similar problems in other systems, $g^*(0, 0) = 0$. Hence, $g^*(t, s){>}0$ for $t > 0, s > 0$, implying that the resultant $g(x, a)$ does not vanish for $x \in (-a, 0)$. Therefore, $U_{11}(z_1, x, a)$ and $q(z_1, x, a)$ have no common roots on D_1, where

$$D_1 = \left\{(z_1, x, a)| -a<x<0<z_1<\kappa_a, 0<a \leq \frac{1}{5}\right\}.$$

Hence, $U_{11}(z_1, x, a) \neq 0$ on D_1. We can claim the result given below.

Proposition 3.1. *$\frac{I_{11}(h)}{I_{01}(h)}$ is monotonic in (h_c, h_s) when $a \in (0, \frac{1}{5}]$.*

Next, we study the algebraic problem on the region

$$D \setminus D_1 = \left\{(z_1, x, a)| -a<x<0<z_1<\kappa_a, \frac{1}{5}<a<1\right\},$$

on which the methodology above deficits. We construct a semi-algebraic system,

$$S_A:\begin{cases} q_1(z_1,x,a)=0, \quad q_1(a,\kappa,a)=0, \quad U_{11}(z_1,x,a)=0, \\ z_1>0, \ x>0, \quad x+a>0, \quad \kappa-z_1>0, \quad \kappa>0. \end{cases} \tag{9}$$

We compute the boundary polynomials with the help of Maple software (2021),

$$\begin{aligned} B_{S_A}=&a\times\Big(a-1\Big)\Big(a-\frac{2}{7}\Big)\Big(a-\frac{1}{5}\Big)\Big(a+\frac{1}{4}\Big)\Big(a+\frac{2}{5}\Big)\Big(a^4-\frac{28}{5}a^3+\frac{63}{5}a^2-14a+7\Big)\times \\ &\Big(a^6-\frac{732}{625}a^5+\frac{117228}{153125}a^4+\frac{597248}{765625}a^3-\frac{238944}{765625}a^2-\frac{71616}{765625}a+\frac{23872}{765625}\Big)\times G_1(a)\times G_2(a), \end{aligned}$$

where $G_1(a)$ and $G_2(a)$ are polynomials in a with degree 18 and 44, respectively. $B_{S_A}=0$ has two zeros $a_1^*=\frac{2}{7}$ and $a_2^*=0.36172\cdots$ in $(\frac{1}{5},1)$, where a_2^* is the root of $G_1(a)$. The degenerate point in B_{S_A} a_1^* and a_2^* divide $(\frac{1}{5},1)$ into five sets

$$\left(\frac{1}{5},a_1^*\right)\cup\{a_1^*\}\cup(a_1^*,a_2^*)\cup\{a_2^*\}\cup(a_2^*,1).$$

Correspondingly, we have five regions

$$\begin{cases} D_2 &= \left\{(z_1,x,a)|-a<x<0<z_1<\kappa_a, \frac{1}{5}<a<a_1^*\right\}, \\ D_3 &= \left\{(z_1,x,a)|-a<x<0<z_1<\kappa_a, a=a_1^*=\frac{2}{7}\right\}, \\ D_4 &= \left\{(z_1,x,a)|-a<x<0<z_1<\kappa_a, a_1^*<a<a_2^*\right\}, \\ D_5 &= \left\{(z_1,x,a)|-a<x<0<z_1<\kappa_a, a=a_2^*\right\}, \\ D_6 &= \left\{(z_1,x,a)|-a<x<0<z_1<\kappa_a, a_2^*<a<1\right\}. \end{cases}$$

By applying Theorem 2.1, the number of roots of SAS (9) is invariant in each D_2, D_4 and D_6. We can choose any sample point in each interval $(\frac{1}{5},\frac{2}{7})$, $(\frac{2}{7},a_2^*)$ and $(a_2^*,1)$ to determine the number of roots of (9), and obtain that, there exist no root of (9) on D_2, D_3, D_4 and D_6 (where we take $a=\frac{2}{7}$ for D_3). Hence, we claim that

Proposition 3.2. *$\frac{I_{11}(h)}{I_{01}(h)}$ is monotonic when $a\in\left(\frac{1}{5},a_1^*\right)\cup\{a_1^*\}\cup(a_1^*,a_2^*)\cup(a_2^*,1)$.*

Proof. We only prove the case when $a\in(\frac{1}{5},\frac{2}{7})$, and other cases can be proved similarly. We fix a value of a in $(\frac{1}{5},\frac{2}{7})$ to investigate whether $U_{11}(z_1,x,a)$ and $q(z_1,x,a)$ have common roots on

D_2. Take $a = \frac{3}{14} \in (\frac{1}{5}, \frac{2}{7})$ and substitute it into (8) yields

$$\begin{aligned} R_1\Big(z_1, \frac{3}{14}\Big) =& 121550625\Big(z_1+1\Big)^8\Big(182250000 z_1^{16} + \frac{11299500000}{7} z_1^{15} + \frac{313743375000}{49} z_1^{14} \\ &+ \frac{5115120046875}{343} z_1^{13} + \frac{215670583546875}{9604} z_1^{12} + \frac{435693574265625}{19208} z_1^{11} \\ &+ \frac{84787385421875}{5488} z_1^{10} + \frac{1279390596046875}{76832} z_1^{9} + \frac{66027538509375}{153664} z_1^{7} \\ &+ \frac{41092975003875}{614656} z_1^{6} + \frac{104159365125}{38416} z_1^{5} - \frac{240012697607}{307328} z_1^{4} + \frac{18025490111}{153664} z_1^{3} \\ &+ \frac{4735667889}{614656} z_1^{2} - \frac{56502745}{76832} z_1 + \frac{56502745}{614656}\Big). \end{aligned}$$

By applying Sturm's Theorem, we know that $R_1\Big(z_1, \frac{3}{14}\Big) \neq 0$ in the interval $z_1 \in (0,1)$, while

$$\kappa_a = \kappa_{\frac{3}{14}} \in \Big[\frac{83845}{1048576}, \frac{41923}{524288}\Big] \approx \Big[0.07996082306, 0.07996177673\Big] \subset (0,1),$$

so $R_1\Big(z_1, \frac{3}{14}\Big) \neq 0$ in the interval $\Big(0, \kappa_{\frac{3}{14}}\Big)$. This implies that $U_{11}(z_1, x, \frac{3}{14})$ and $q(z_1, x, \frac{3}{14})$ have no common zero. Therefore, $U_{11}(z_1, x, a)$ and $q(z_1, x, a)$ have no root on D_2 by Theorem 2.1, so $U_{11}(z_1, x, a) \neq 0$ when a is located in $(\frac{1}{5}, \frac{2}{7})$. This completes the proof. □

The remainder of this part is to investigate whether $U_{11}(z_1, x, a) \neq 0$ on D_5, where by taking $a = a_2^*$. As a_2^* is the root of $G_1(a)$ given by below in boundary polynomial sets B_{S_A},

$$\begin{aligned} G_1(a) =& a^{18} - \frac{78}{5} a^{17} + \frac{145740264603}{1290427321} a^{16} - \frac{16250513677416}{32260683025} a^{15} + \frac{49984100828991}{32260683025} a^{14} \\ &- \frac{112334770109478}{32260683025} a^{13} + \frac{951760584025461}{161303415125} a^{12} - \frac{154457666571279156}{20162926890625} a^{11} \\ &+ \frac{773677792766606148}{100814634453125} a^{10} - \frac{3001540898662649152}{504073172265625} a^{9} + \frac{45360914633493055872}{12601829306640625} a^{8} \\ &- \frac{10763234426146247001 6}{63009146533203125} a^{7} + \frac{7819125046431334272}{12601829306640625} a^{6} - \frac{1782203864297278464}{12601829306640625} a^{5} \\ &+ \frac{208811914453201 92}{12601829306640625} a^{4} + \frac{135839936479629312}{12601829306640625} a^{3} - \frac{2180845238820003 84}{63009146533203125} a^{2} \\ &+ \frac{3108111445008384}{12601829306640625} a - \frac{339339524722688}{12601829306640625}, \end{aligned}$$

and we have $a_2^* \in \Big[\frac{11853}{32768}, \frac{94825}{262144}\Big]$ by real root isolation. We calculate the resultant with respect to a between R_1 and $G_1(a)$, then calculate the resultant with respect to a between R_2 and $G(a)$.

By applying Sturm's Theorem to $res(R_1, G_1, a)$ and $res(R_2, G_1, a)$, $res(R_1, G_1, a)$ has a unique root $z_1^* \in \left[\frac{61923}{1048576}, \frac{123847}{2097152}\right]$, and $res(R_2, G_1, a) = 0$ has two roots $x_* \in \left[-\frac{15621}{131072}, -\frac{124967}{1048576}\right]$, $x_\dagger \in \left[-\frac{124047}{2097152}, -\frac{62023}{1048576}\right]$. Thus, if $U_{11}(z_1, x, a)$ and $q_1(z_1, x, a)$ have common roots for $-a<x<0<z_1<\kappa_{a_2^*}$, then the common roots must be located on the following two regions:

$$\mathcal{C}_1 : \Big[\frac{61923}{1048576}, \frac{123847}{2097152}\Big] \times \Big[-\frac{15621}{131072}, -\frac{124967}{1048576}\Big] \times \Big[\frac{11853}{32768}, \frac{94825}{262144}\Big],$$

$$\mathcal{C}_2 : \Big[\frac{61923}{1048576}, \frac{123847}{2097152}\Big] \times \Big[-\frac{124047}{2097152}, -\frac{62023}{1048576}\Big] \times \Big[\frac{11853}{32768}, \frac{94825}{262144}\Big].$$

Now we prove that $q(z_1, x, a)$ has no zeros on the above two cubes. We apply tools in polynomial optimization to compute the maximum and the minimum values of $q(z_1, x, a)$ on each region $\mathcal{C}_1$ and $\mathcal{C}_2$, and obtain that the maximum $q(z_1, x, a) = \frac{50670586312339979362583027506047327 9}{66461399789245793645190353014017228 8}$ and minimum $q(z_1, x, a) = \frac{3242709232550702438644881396122017319 5}{4253529586511730793292182592897102643 2}$ of $\mathcal{C}_1$ are all positive. Therefore, we have no root of $q(z_1, x, a)$, and $U_{11}(z_1, x, a)$ and $q(z_1, x, a)$ have no common roots on $\mathcal{C}_1$ and $\mathcal{C}_2$. Hence, $U_{11}(z_1, x, a) \neq 0$ on D_5. We note that the tools *Maximize* and *Minimize* in polynomial optimization package get exact values at the vertices, therefore, the algorithm is accurate even though it is based on numerical analysis, as there are no critical points in the two cubes. We can also use a standard analysis to prove $q(z_1, x, a) \neq 0$, which will be given in Appendix A. We can now claim that

Proposition 3.3. $\frac{I_{11}(h)}{I_{01}(h)}$ *is monotonic when* $a = a_2^*$.

Combining Proposition 3.1, 3.2 and 3.3 proves Theorem 1.1.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Taking $\alpha = 0$ and $0<\beta = \lambda = \gamma<1$, system (3) becomes

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x^2(x-\lambda)^3(x-1), \end{cases} \tag{10}$$

which has a Hamiltonian function

$$H_2(x, y) = \frac{1}{2}y^2 + \frac{1}{3}\lambda^3 x^3 - \frac{1}{4}\left(\lambda^3 + 3\lambda^2\right)x^4 + \frac{1}{5}\left(3\lambda^2 + 3\lambda\right)x^5 - \frac{1}{6}\left(1 + 3\lambda\right)x^6 + \frac{1}{7}x^7.$$

The ovals $\gamma_2(h)$ surrounds the center $(1, 0)$. We move the center $(1, 0)$ to the origin $(0, 0)$ and take a flip with respect to $y-$axis, $x = -\tilde{x} + 1$. Then the system (10) becomes by dropping the tilde,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x(-x+1)^2(x-b)^3, \end{cases} \tag{11}$$

where $b = 1 - \lambda \in (0,1)$, and the associated Hamiltonian has the form

$$\mathcal{H}_2(x,y) = \frac{1}{2}y^2 + \Psi_2(x)$$

with

$$\Psi_2(x) = h_c + \frac{1}{2}b^3x^2 - \frac{1}{3}\left(2b^3+3b^2\right)x^3 + \frac{1}{4}\left(b^3+6a^2+3b\right)x^4 - \frac{1}{5}\left(3b^2+6b+1\right)x^5 + \frac{1}{6}\left(2+3b\right)x^6 - \frac{1}{7}x^7.$$

The ovals $\gamma_2(h)$ surrounds the elementary center $(0,0)$. And the Abelian integrals $I_0(h)$ and $I_1(h)$ have new expressions

$$I_0(h) = \oint_{\gamma_2(h)} ydx = \oint_{\gamma_2(h)} yd\tilde{x} \triangleq I_{02}(h),$$

$$I_1(h) = \oint_{\gamma_2(h)} xydx = \oint_{\gamma_2(h)} (-\tilde{x}+1)yd\tilde{x} \doteq -I_{02}(h) + I_{12}(h).$$

Then we need only prove $\frac{I_{12}(h)}{I_{02}(h)}$ is monotonic in the interval (h_c, h_s) to prove Theorem 1.2.

Similar to above analysis, there exist a pair of analytic involutions $x(h)$ and $z_2(h)$ defined by

$$\Psi_2(x(h)) = \Psi_2(z_2(h)) = h, \ \kappa_b < x(h) < 0 < z_2(h) < b,$$

where $\Psi_2(\kappa_b) = \Psi_2(b)$, they are implicitly determined by

$$\begin{aligned} \tilde{q}(z_2,x,b) =& 60x^6 + (-210b+60z_2-140)x^5 + \cdots - 105z_2(-\frac{4z_2^5}{7} + \frac{1}{3}(6b+4)z_2^4 \\ &+ \frac{1}{5}(-12b^2-24b-4)z_2^3 + (b^3+6b^2+3b)z_2^2 + \frac{1}{3}(-8b^3-12b^2)z_2 + 2b^3). \end{aligned}$$

Introducing

$$U_2(h) = x(h) + z_2(h),$$

we have

$$U_2'(h) = \Big(\frac{dz_2}{dx}+1\Big)x'(h) = \Big(-\frac{\tilde{q}_x}{\tilde{q}_{z_2}}+1\Big)x'(h) = 2(z_2-x)\frac{U_{21}(z_2,x,b)}{U_{22}(z_2,x,b)}x'(h),$$

where

$$U_{21}(z_2,x,b) = 105b^3x + 105b^3z_2 - 378b^2x^2 + -504b^2xz_2 - 378b^2z_2^2 + 420bx^3 + 630bx^2z_2 + 630bxz_2^2 + 420bz_2^3 - 150x^4 - 240x^3z_2 - 270x^2z_2^2 - 240xz_2^3 - 150z_2^4 - 140b^3 + 630b^2x + 630b^2z_2 - 756bx^2 - 1008bxz_2 - 756bz_2^2 + 280x^3 + 420x^2z_2 + 420xz_2^2 + 280z_2^3 - 210b^2 + 315bx + 315bz_2 - 126x^2 - 168xz_2 - 126z_2^2,$$

$$U_{22}(z_2,x,b) = 105b^3x^2 + 210b^3xz_2 + 315b^3z_2^2 + -252b^2x^3 - 504b^2x^2z_2 - 756b^2xz_2^2 - 1008b^2z_2^3 + 210bx^4 + 420bx^3z_2 + 630bx^2z_2^2 + 840bxz_2^3 + 1050bz_2^4 - 60x^5 - 120x^4z_2 - 180x^3z_2^2 - 240x^2z_2^3 - 300xz_2^4 - 360z_2^5 - 280b^3x - 560b^3z_2 + 630b^2x^2 + 1260b^2xz_2 + 1890b^2z_2^2 - 504bx^3 + 210b^3 - 1008bx^2z_2 - 1512bxz_2^2 - 2016bz_2^3 + 140x^4 + 280x^3z_2 + 420x^2z_2^2 + 560xz_2^3 + 700z_2^4 - 420b^2x - 840b^2z_2 + 315bx^2 + 630bxz_2 + 945bz_2^2 - 84x^3 - 168x^2z_2 - 252xz_2^2 - 336z_2^3.$$

It is sufficient to prove $U_{2i}(z_2,x,b) \neq 0$ on $\tilde{D}$ when $i = 1, 2$, where

$$\tilde{D} = \{(z_2,x,b) | \kappa_b < x < 0 < z_2 < b, 0 < b < 1\}.$$

First, we get the resultant between $U_{22}(z_2,x,b)$ and $\tilde{q}(z_2,x,b)$ with respect to x,

$$r_0(z_2) = -609753012019200000000000{z_2}^5 (z_2 - 1)^{10} (b - z_2)^{15}.$$

Obviously, $r_0(z_2)$ has no zero in the interval $(0, b)$. So $U_{22}(z_2,x,b)$ and $\tilde{q}(z_2,x,b)$ have no common roots on $\tilde{D}$, which means $U_{22}(z_2,x,b) \neq 0$ on $\tilde{D}$ and the ratio $\frac{U_{21}}{U_{22}}$ is well defined.

Similarly, we can calculate the resultant of $U_{21}(z_2,x,b)$ and $\tilde{q}(z_2,x,b)$ with respect to x and z_2, respectively, we obtain

$$\tilde{R}_1(z_2,b) = \tilde{g}(z_2,b), \quad \tilde{R}_2(x,b) = \tilde{g}(x,b), \tag{12}$$

where $\tilde{g}(w,b)$ is a polynomial with a little long expression

$$\tilde{g}(w,b) = 3889620000(b-w)^4(18753525\, b^{16}w^4 + 30005640\, b^{15}w^5 + 119951118\, b^{14}w^6 + \cdots + 99574272w^4).$$

Taking $z_2 = \frac{b}{1+t}$ and $b = \frac{1}{3(1+s)}$ with $t, s > 0$, ensuring $b \in (0, \frac{1}{3})$ and $z_2 \in (0, b)$, and we have the form,

$$\tilde{R}_1(z_2,b) = \frac{1}{4782969(1+s)^{24}(1+t)^{24}} \tilde{g}^*(t,s),$$

where all coefficients of $\tilde{g}^*(t,s)$ are greater than zero, and $\tilde{g}^*(t,0) > 0$, we here include $b = \frac{1}{3}$, similar problems in other systems, $\tilde{g}^*(0,0) = 0$. Hence, $\tilde{g}^*(t,s) > 0$ for $t > 0, s > 0$, implying that the resultant $\tilde{R}_1(z_2, b)$ does not vanish for $z_2 \in (0, b)$. Therefore, $U_{21}(z_2, x, b)$ and $\tilde{q}(z_2, x, b)$ have no common roots on $\tilde{D}_1$, where

$$\tilde{D}_1 = \left\{(z_2, x, b) | \kappa_b < x < 0 < z_2 < b, 0 < b \leq \frac{1}{3}\right\}.$$

Hence, $U_{21}(z_2, x, b) \neq 0$ on $\tilde{D}_1$, we can claim the result given below.

Proposition 4.1. $\frac{I_{12}(h)}{I_{02}(h)}$ *is monotonic in* (h_c, h_s) *when* $b \in (0, \frac{1}{3}]$.

Next, we study the algebraic problem on the region

$$\tilde{D} \setminus \tilde{D}_1 = \left\{(z_2, x, b) | \kappa_b < x < 0 < z_2 < b, \frac{1}{3} < b < 1\right\},$$

on which the methodology above deficits. We construct a semi-algebraic system,

$$\tilde{S}_A : \begin{cases} \tilde{q}(z_2, x, b) = 0, \quad \tilde{q}(b, \kappa_b, b) = 0, \quad U_{21}(z_2, x, b) = 0, \\ z_2 > 0, \quad -x > 0, \quad b - z_2 > 0, \quad x - \kappa_b > 0, \quad -\kappa_b > 0. \end{cases} \tag{13}$$

We compute the boundary polynomials with the help of Maple software (2021),

$$\begin{aligned} \tilde{B}_{\tilde{S}_A} =& b \times \Big(b - 1\Big) \times \Big(b - \frac{1}{3}\Big) \times \Big(b + \frac{3}{2}\Big) \times \Big(b + \frac{4}{3}\Big) \times \Big(b^2 - \frac{14}{3}b + 7\Big) \times \Big(b^3 - \frac{6}{5}b^2 + \frac{3}{5}b - \frac{4}{35}\Big) \times \\ & \Big(b^9 - 10b^8 + \frac{904}{21}b^7 - \frac{130916}{1323}b^6 + \frac{166256}{1323}b^5 - \frac{325288}{3969}b^4 + \frac{14720}{567}b^3 - \frac{32624}{5103}b^2 + \frac{5968}{2187}b \\ & - \frac{23872}{19683}\Big) \times \Big(b^9 - 5b^8 + \frac{3366028}{382107}b^7 + \frac{1007656}{382107}b^6 - \frac{22502048}{1146321}b^5 + \frac{267457472}{10316889}b^4 \\ & - \frac{1657174400}{567}b^3 + \frac{697125632}{92852001}b^2 - \frac{507179008}{278556003}b + \frac{18113536}{92852001}\Big) \times \tilde{G}(b), \end{aligned}$$

where $\tilde{G}(b)$ is a polynomial in b of degree 74. $\tilde{B}_{\tilde{S}_A} = 0$ has a zero $b^* = 0.4189\cdots$ in $(\frac{1}{3}, 1)$, where b^* is the root of $\Big(b^3 - \frac{6}{5}b^2 + \frac{3}{5}b - \frac{4}{35}\Big)$. The degenerate point b^* divides $(\frac{1}{3}, 1)$ into three sets

$$(\frac{1}{3}, b^*) \cup \{b^*\} \cup (b^*, 1).$$

Correspondingly, we have three regions

$$\begin{cases} \tilde{D}_2 &= \left\{(z_2, x, b) | \kappa_b < x < 0 < z_2 < b, \frac{1}{3} < b < b^*\right\}, \\ \tilde{D}_3 &= \{(z_2, x, b) | \kappa_b < x < 0 < z_2 < b, b = b^*\}, \\ \tilde{D}_4 &= \{(z_2, x, b) | \kappa_b < x < 0 < z_2 < b, b^* < b < 1\}. \end{cases}$$

By applying Theorem 2.1, the number of roots of SAS (13) is invariant in each $\tilde{D}_2$ and $\tilde{D}_4$. We can choose any sample point in each interval $(\frac{1}{3}, b^*)$ and $(b^*, 1)$ to determine the number of roots of (13), and obtain that, there exist no root of (13) on $\tilde{D}_2$ and $\tilde{D}_4$. Hence, we claim that

Proposition 4.2. $\frac{I_{12}(h)}{I_{02}(h)}$ *is monotonic when* $b \in (\frac{1}{3}, b^*) \cup (b^*, 1)$.

Proof. We only prove the case when $b \in (\frac{1}{3}, b^*)$, and the other case can be proved similarly. We fix a value of b in $(\frac{1}{3}, b^*)$ to investigate whether $U_{21}(z_1, x, b)$ and $\tilde{q}(z_2, x, b)$ have common roots on $\tilde{D}_2$. Take $b = \frac{4}{10} \in (\frac{1}{3}, b^*)$ and substitute it into (12) yields

$$\begin{aligned}
\tilde{R}_2\Big(x, \frac{4}{10}\Big) =& 3889620000(\frac{2}{5} - x)^4(5382000000\, x^{20} - 65318400000\, x^{19} + 338722560000\, x^{18} \\
& - 1079362368000x^{17} + 2365469568000x^{16} - 3780692858880x^{15} + 4558780930816x^{14} \\
& - 4229357511680x^{13} + 3047887376128x^{12} - \frac{1067078495254912}{625}x^{11} \\
& + \frac{57546797138631808}{78125}x^{10} - \frac{93442558322608768}{390625}x^9 + \frac{1362419487200499 1008}{244140625}x^8 \\
& - \frac{10319709812193624192}{1220703125}x^7 + \frac{3769156189960936512}{6103515625}x^6 + \frac{727652923638018 56}{6103515625}x^5 \\
& - \frac{8354072519376832}{6103515625}x^4 - \frac{3926231230668544}{6103515625}x^3 - \frac{19732202973056}{6103515625}x^2 + \frac{1065204839552}{1220703125}x \\
& + \frac{532602419776}{6103515625}).
\end{aligned}$$

By applying Sturm's Theorem, we know that $\tilde{R}_2\Big(x, \frac{4}{10}\Big) \neq 0$ in the interval $x \in (-1, 0)$, while

$$\kappa_b = \kappa_{\frac{4}{10}} \in \Big[-\frac{44511}{524288}, -\frac{89021}{1048576}\Big] \approx \Big[-0.08489799500, -0.08489704132\Big] \subset (-1, 0),$$

so $\tilde{R}_2\Big(x, \frac{4}{10}\Big) \neq 0$ in the interval $\Big(\kappa_{\frac{4}{10}}, 0\Big)$. This implies that $U_{21}(z_2, x, \frac{4}{10})$ and $\tilde{q}(z_2, x, \frac{4}{10})$ have no common zero. Therefore, $U_{21}(z_2, x, b)$ and $\tilde{q}(z_2, x, b)$ have no root on $\tilde{D}_2$ by Theorem 2.1, so $U'(h) \neq 0$ when b is located in $(\frac{1}{3}, b^*)$. This completes the proof. □

The remainder of this part is to investigate whether $U_{21}(z_2, x, b) \neq 0$ on $\tilde{D}_3$, where taking $b = b^*$. As b^* is the root of the factor in $\tilde{B}_{\tilde{S}_A}$,

$$p(b) = b^3 - \frac{6}{5}b^2 + \frac{3}{5}b - \frac{4}{35},$$

and we have $b^* = 0.4189\cdots$. By calculation we know that $res(\tilde{R}_2, p(b), b)$ has no zero in $(\kappa_{b^*}, 0) \in (\kappa_{b=1}, 0)$, where $\kappa_{b^*} = -0.0882\cdots$, $\kappa_{b=1} = -0.1666\cdots$. Therefore, $U_{21}(z_2, x, b^*) \neq 0$ on $\tilde{D}_3$. Thus, we claim that

Proposition 4.3. $\frac{I_{12}(h)}{I_{02}(h)}$ *is monotonic when* $b = b^*$.

Combining Proposition 4.1, 4.2 and 4.3 proves Theorem 1.2.

5. Proof of Theorem 1.3

When $\alpha = \beta = 0$, $0<\gamma \leq \frac{4}{7}$ and $\lambda = 1$, system (3) becomes

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x^3(x-\gamma)(x-1)^2, \end{cases} \tag{14}$$

with a Hamiltonian

$$H_3(x,y) = \frac{1}{2}y^2 - \frac{1}{4}\gamma x^4 + \frac{1}{5}(1+2\gamma)x^5 - \frac{1}{6}(2+\gamma)x^6 + \frac{1}{7}x^7,$$

The continuous family of ovals $\gamma_3(h)$ surrounds the center $(\gamma, 0)$. We make the transformation $x = \tilde{x} + \gamma$ to move the center $(\gamma, 0)$ to the origin $(0,0)$. Then system (14) becomes by dropping the tilde,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x(x+\gamma)^3(x-1+\gamma)^2, \end{cases}$$

with the Hamiltonian

$$\mathcal{H}_3(x,y) = \frac{y^2}{2} + \Psi_3(x),$$

where

$$\begin{aligned} \Psi_3(x) = h_c &+ \frac{1}{2}\gamma^3(\gamma-1)^2 x^2 + \frac{1}{3}\left(5\gamma^4 - 8\gamma^3 + 3\gamma^2\right)x^3 + \frac{1}{4}\left(10\gamma^3 + 12\gamma^2 + 3\gamma\right)x^4 \\ &+ \frac{1}{5}\left(10\gamma^2 - 8\gamma + 1\right)x^5 + \frac{1}{6}(5\gamma-2)x^6 + \frac{1}{7}x^7, \end{aligned}$$

The ovals $\gamma_3(h)$ surround $(0,0)$ and the Abelian integrals have the forms,

$$I_0(h) = \oint_{\gamma_3(h)} y dx = \oint_{\gamma_{03}(h)} y d\tilde{x} \triangleq I_{03}(h),$$

$$I_1(h)=\oint_{\gamma_3(h)} xydx=\oint_{\gamma_{03}(h)}(\tilde{x}+\gamma)yd\tilde{x}\triangleq I_{03}(h)+I_{13}(h).$$

Now we're going to study the monotonicity of $\frac{I_{13}(h)}{I_{03}(h)}$ for the proof of Theorem 1.3. Similar to above analysis, there are two analytical involutions $x(h)$ and $z_3(h)$ satisfying

$$\Psi_3(x(h))=\Psi_3(z_3(h))=h,\quad -\gamma<x(h)<0<z_3(h)<\kappa_\gamma, 0<\gamma\leq\frac{4}{7}$$

where $\Psi_3(-\gamma)=\Psi_3(\kappa_\gamma)$, $x(h)$ and $z_3(h)$ are implicitly determined by

$$\begin{aligned}\hat{q}(z_3,x,\gamma)=&-60x^6+(-350\gamma+140-60z_3)x^5+\cdots-60z_3(z_3^5+\frac{1}{6}(35\gamma-14)z_3^4\\&+\frac{1}{5}(70\gamma^2-56\gamma+7)z_3^3+\frac{1}{4}(70\gamma^3-84\gamma^2+21\gamma)z_3^2+\frac{1}{3}(35\gamma^4-56\gamma^3\\&+21\gamma^2)z_3+\frac{7}{2}\gamma^3(\gamma-1)^2).\end{aligned}$$

Define the function

$$U_3(h)=x(h)+z_3(h),$$

and we have

$$U_3'(h)=\Big(-\frac{\bar{q}_x}{\bar{q}_{z_3}}+1\Big)x'(h)=2(z_3-x)\frac{U_{31}(z_3,x,\gamma)}{U_{32}(z_3,x,\gamma)}x'(h),$$

where

$$\begin{aligned}U_{31}(z_3,x,\gamma)=&350\gamma^4+1050\gamma^3x+1050\gamma^3z_3+1260\gamma^2x^2+1680\gamma^2xz_3+1260\gamma^2z_3^2+700\gamma x^3\\&+1050\gamma x^2z_3+1050\gamma xz_3^2+700\gamma z_3^3+150x^4+240x^3z_3+270x^2z_3^2+240xz_3^3\\&+150z_3^4-560\gamma^3-1260\gamma^2x-1260z_3\gamma^2-1008\gamma z_3^2-1344\gamma xz_3-1008\gamma z_3^2\\&-280x^3-420x^2z_3-420xz_3^2-280z_3^3+210\gamma^2+315\gamma z_3+315\gamma x+126x^2\\&+168xz_3+126z_3^2,\end{aligned}$$

$$\begin{aligned}U_{32}(z_3,x,\gamma)=&210\gamma^5+700\gamma^4x+1400\gamma^4z_3+1050\gamma^3x^2+2100\gamma^3xz_3+3150\gamma^3z_3^2+840\gamma^2x^3\\&+1680\gamma^2x^2z_3+2520\gamma^2xz_3^2+3360\gamma^2z_3^3+350\gamma x^4+700\gamma x^3z_3+1050\gamma x^2z_3^2\\&+1400\gamma xz_3^3+1750\gamma z_3^4+60x^5+120x^4z_3+180x^3z_3^2+240x^2z_3^3+300xz_3^4\\&+360z_3^5-420\gamma^4-1120\gamma^3z_3-2240\gamma^3z_3-1260\gamma^2x^2-2520\gamma^2xz_3-3780\gamma^2z_3^2\\&-672\gamma x^3-1344\gamma x^2z_3-2016\gamma xz_3^2-2688\gamma z_3^3-140x^4-280x^3z_3-420x^2z_3^2\\&-560xz_3^3-700z_3^4+210\gamma^3+420\gamma^2x+840\gamma^2z_3+315\gamma x^2+630\gamma xz_3+945\gamma z_3^2\\&+84x^3+168x^2z_3+252xz_3^2+336z_3^3.\end{aligned}$$

It is suffice to prove that $U_{3i}(z_3,x,\gamma)\neq 0$ for $i=1,2$ on $\hat{D}_1\cup\hat{D}_2$, where

$$\begin{aligned}\hat{D}_1 &= \left\{(z_3,x,\gamma)|-\gamma<x<0<z_3<\kappa_\gamma, 0<\gamma<\tfrac{4}{7}\right\},\\ \hat{D}_2 &= \left\{(z_3,x,\gamma)|-\tfrac{4}{7}<x<0<z_3<\tfrac{3}{7}, \gamma=\tfrac{4}{7}\right\}.\end{aligned}$$

Caculating the resultant between $U_{32}(z_3,x,\gamma)$ and $\hat{q}(z_3,x,\gamma)$ with respect to x, we have

$$\hat{r}_0(z_3) = -609753012019200000000000 z_3{}^5\left(\gamma+z_3-1\right)^{10}\left(\gamma+z_3\right)^{15}.$$

Obviously, $\hat{r}_0(z_3)\neq 0$ on $(0,\kappa_\gamma)$, which means that $U_{32}(z_3,x,\gamma)$ and $q_3(z_3,x,\gamma)$ have no common root on $\hat{D}_1\cup\hat{D}_2$. So $U_{32}(z_3,x,\gamma)\neq 0$ on $\hat{D}_1\cup\hat{D}_2$.

Similarly, we can calculate the resultant of $U_{31}(z_3,x,\gamma)$ and $q_3(z_3,x,\gamma)$ with respect to x and z_3, respectively, and obtain as follows

$$\hat{R}_1(z_3,\gamma)=\hat{g}(z_3,\gamma),\quad \hat{R}_2(x,\gamma)=\hat{g}(x,\gamma), \tag{15}$$

where $\hat{g}(w,\gamma)$ is a polynomial and has the following expression

$$\hat{g}(w,\gamma) = 3889620000(\gamma+w)^4(17500000\,\gamma^{20}-70000000\,\gamma^{19}w-413000000\,\gamma^{18}w^2+\cdots+99574272\,w^4).$$

Because the expression for $\hat{g}(w,b)$ is too long, it's not given in detail here. Taking $x=-\frac{\gamma}{1+t}$ and $\gamma=\frac{4}{7(1+s)}$ with $s>0, t>0$ (satisfying $-\gamma<x<0$ and $0<\gamma<\frac{4}{7}$) yields

$$\hat{g}(x,\gamma)=\frac{1}{79792266297612001(1+s)^{24}(1+t)^{24}}\hat{g}^*(t,s),$$

where all coefficients of $\hat{g}^*(t,s)$ are greater than zero, and $\hat{g}^*(t,0)>0$, we here include $\gamma=\frac{4}{7}$, $\hat{g}^*(0,0)=0$. Hence, $\hat{g}^*(t,s)>0$ on $\{(t,s): t\in(0,+\infty), s\in[0,+\infty)\}$, which means that $U_{31}(z_3,x,\gamma)$ and $\hat{q}(z_3,x,\gamma)$ have no common roots on $\hat{D}_1\cup\hat{D}_2$. Hence, $\hat{U}_1(z_3,x,\gamma)\neq 0$ on $\hat{D}_1$ as well as $U'(h)\neq 0$. Therefore we can claim that

Proposition 5.1. $\frac{I_{13}(h)}{I_{03}(h)}$ *is monotonic in the interval* $(h_c,0)$ *when* $\gamma\in(0,\frac{4}{7}]$.

This completes the proof of Theorem 1.3.

6. Conclusion

In this work, we determine the monotonicity of the ratios of two Abelian integrals along different topological period annuli. We use a criterion in an algebraic way, and fully apply some techniques in polynomial boundary theory and bounding roots of algebraic system to overcome the difficulty arising from the parametric Hamiltonian. However, there still exist some problems left, for example, how can we determine the monotonicity when the Hamiltonian has two or more parameters and how can we get a monotonicity criterion for the ratio of two general integral integrals $I_i(h)$ and $I_j(h)$, $i, j \in \mathbb{N}^+$. We have also found that the monotonicity of $r(h)$ is determined by the sign of $q_x - q_z$. The monotonicity of other ratios $I_i(h) \setminus I_j(h)$ may also depend the signs of a certain combination of q_x and q_z. We have utilized symbolic computations to verify the non-vanishing property of related algebraic system, some analytic methods [27, 28] may be helpful to reduce the related computational analysis. The related studies need further analysis and more techniques.

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Appendix A. In this section, we show $q(z_1, x, a) > 0$ on $\mathcal{C}_1$ and $q(z_1, x, a) < 0$ on $\mathcal{C}_2$. However, we only prove the former claim and the later one can be proved similarly. We note that the proof is based on computational analysis. Firstly, we have

Theorem A. *$q(z_1, x, a)$ has no critical points inside the cube $\mathcal{C}_1$.*

We note that, when we say "inside" a set, implying taking points in the open set of the cube $\mathcal{C}_1$ excluding the six surfaces or a set excluding the boundaries. When we use "on" a set, implying taking points in the closed set of the cube $\mathcal{C}_1$ including the six surfaces or a set including the boundaries. For convenience, we denote the surface $\left[\frac{61923}{1048576}, \frac{123847}{2097152}\right] \times \left[-\frac{15621}{131072}, -\frac{124967}{1048576}\right]$ with $a = \frac{11853}{32768}$ by $AA'B'B$, see Figure 2.

Proof. We compute the partial derivatives of $q(z_1, x, a)$ with respect to the three variables as follows,

$$\begin{aligned} q_{z_1} =& -30x^5 + (-60z_1 - 35a - 140)x^4 + (-90z_1^2 + 2(-35a - 140)z_1 - 168a - 252)x^3 \\ &+ 3(-30z_1^3 + (-35a - 140)z_1^2 + (-168a - 252)z_1 - 315a - 210)x^2 + 2(-30z_1^4 + (\\ &- 35a - 140)z_1^3 + (-168a - 252)z_1^2 + (-315a - 210)z_1 - 280a - 70)x - 30z_1^5 + (\\ &- 35a - 140)z_1^4 + (-168a - 252)z_1^3 + (-315a - 210)z_1^2 + (-280a - 70)z_1 - 105a, \end{aligned}$$

$$\begin{aligned} q_x =& -180x^5 + 5(-35a - 140 - 30z_1)x^4 + 4(-30z_1^2 + (-35a - 140)z_1 - 168a - 252)x^3 \\ &+ 3(-30z_1^3 + (-35a - 140)z_1^2 + (-168a - 252)z_1 - 315a - 210)x^2 + 2(-30z_1^4 + (\\ &- 35a - 140)z_1^3 + (-168a - 252)z_1^2 + (-315a - 210)z_1 - 280a - 70)x - 30z_1^5 + (\\ &- 35a - 140)z_1^4 + (-168a - 252)z_1^3 + (-315a - 210)z_1^2 + (-280a - 70)z_1 - 105a, \end{aligned}$$

$$
\begin{aligned}
q_a =& -35x^5 + (-35z_1 - 168)x^4 + (-35z_1^2 - 168z_1 - 315)x^3 + (-35z_1^3 - 168z_1^2 - 315z_1 \\
& - 280)x^2 + (-35z_1^4 - 168z_1^3 - 315z_1^2 - 280z - 105)x - 35z_1^5 - 168z_1^4 - 315z_1^3 \\
& - 280z_1^2 - 105z_1.
\end{aligned}
$$

We calculate the resultant of q_{z_1} and q_x with respect to a, and obtain that

$$
resultant(q_{z_1}, q_x, a) = 1050x^9 + 3150x^8z + 6300x^7z^2 + \cdots - 7350z \doteq Q^*(z_1, x).
$$

We will show that $Q^*(z_1, x) < 0$ on $\left[\frac{61923}{1048576}, \frac{123847}{2097152}\right] \times \left[-\frac{15621}{131072}, -\frac{124967}{1048576}\right]$. In fact, the resultant between $Q^*_{z_1}$ and Q^*_x with respect to x has no zero in $\left[\frac{61923}{1048576}, \frac{123847}{2097152}\right]$ by a straightforward computation. Therefore, the maximum and minimum values of $Q^*(z_1, x)$ are taken on the four boundaries of $\left[\frac{61923}{1048576}, \frac{123847}{2097152}\right] \times \left[-\frac{15621}{131072}, -\frac{124967}{1048576}\right]$. However, there exist no critical points of $Q^*(z_1, x)$ on the four boundaries by a computational analysis. Then the maximum and minimum values of $Q^*(z_1, x)$ are taken on the four vertexes. By comparing the values of $Q^*(z_1, x)$ on the four vertexs

$$
\max Q^*(z_1, x) = -\frac{67654044785339311457682659346029982699023830762871490 7215}{76624777043294442917917351357515459180936956109180 1088},
$$

and

$$
\min Q^*(z_1, x) = -\frac{34639095121367343257759555599566061504773570558808802 4554625}{39231885846166754773973683895047915100639721527900215 7056}.
$$

Therefore, $Q^*(z_1, x) < 0$ on $\left[\frac{61923}{1048576}, \frac{123847}{2097152}\right] \times \left[-\frac{15621}{131072}, -\frac{124967}{1048576}\right]$. Hence, $q_{z_1} \neq q_x$ on $\left[\frac{61923}{1048576}, \frac{123847}{2097152}\right] \times \left[-\frac{15621}{131072}, -\frac{124967}{1048576}\right]$, this implies that there exists no critical points inside the cube $\mathcal{C}_1$. $\square$

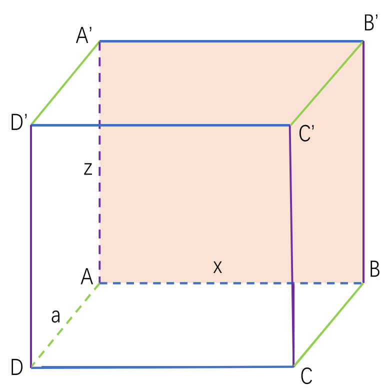

Figure 2: Cube $\mathcal{C}_1$.

Theorem B. *When one variable is fixed, the two-variable polynomial $q(z_1, x, a)$ has no critical points inside the six open surfaces of $\mathcal{C}_1$.*

Proof. We only prove the claim for the surface $AA'B'B$, where we fix $a = \frac{11853}{32768}$ on $\mathcal{C}_1$, and the other case can be proved similarly. We calculate the resultant between $q_{z_1}(z_1, x, \frac{11853}{32768})$ and $q_x(z_1, x, \frac{11853}{32768})$ with respect to z_1, and denote it as $Q^\dagger(x)$,

$$
\begin{aligned}
Q^\dagger(x) =& 12861977597280000000000x^{25} + \frac{935006789190289306640625}{4}x^{24} + \cdots \\
&+ \frac{26886400246847795715994281255068110524996266601 5625}{1361129467683753853853498429727072845824}.
\end{aligned}
$$

According to Sturm's Theory, $Q^\dagger(x)$ have no zero for $x \in \Big[-\frac{15621}{131072}, -\frac{124967}{1048576}\Big]$. This implies that $q(z_1, x, \frac{11853}{32768})$ has no critical points inside the surface $AA'B'B$. $\square$

Theorem C. *The maximum and minimum values of $q(z_1, x, a)$ on $\mathcal{C}_1$ are taken at the vertex $D\Big(\frac{61923}{1048576}, -\frac{15621}{131072}, \frac{94825}{262144}\Big)$ and $B'\Big(\frac{123847}{2097152}, -\frac{124967}{1048576}, \frac{11853}{32768}\Big)$.*

Proof. From Theorems A and B, the maximum and minimum values of $q(z_1, x, a)$ on $\mathcal{C}_1$ may be taken on the twelve edges. Inside the edge AB excluding the two endpoints, where we fix $z_1 = \frac{61923}{1048576}$ and $a = \frac{11853}{32768}$ and have

$$
q_x\Big(\frac{61923}{1048576}, x, \frac{11853}{32768}\Big) = -180x^5 - \frac{404834225}{524288}x^4 + \cdots - \frac{312426029938632338626878 12364365}{633825300114114700748351602688}.
$$

According Sturm's theorem reveals that

$$
q_x\Big(\frac{61923}{1048576}, x, \frac{11853}{32768}\Big) \neq 0, \quad x \in \Big(-\frac{15621}{131072}, -\frac{124967}{1048576}\Big).
$$

Similarly, we can prove that the one-variable polynomial $q(z_1, x, a)$ ($q(z_1, x, a)$ restricted on one edge) has no critical points inside other edges. Therefore, the maximum and minimum values of $q(z_1, x, a)$ can be taken at two of eight vertices of $\mathcal{C}_1$.

By direct computation, we get the values of the polynomial $q(z_1, x, a)$ at the eight vertices of

$\mathcal{C}_1$ as follows

$$
\begin{aligned}
q\Big(\frac{61923}{1048576},-\frac{15621}{131072},\frac{11853}{32768}\Big) &= \frac{5066966282953155529594939050563786 25}{664613997892457936451903530140172288},\\
q\Big(\frac{61923}{1048576},-\frac{124967}{1048576},\frac{11853}{32768}\Big) &= \frac{506686345373054602000436536620432765}{664613997892457936451903530140172288},\\
q\Big(\frac{61923}{1048576},-\frac{15621}{131072},\frac{94825}{262144}\Big) &= \frac{506705863123399793625830275060473279}{664613997892457936451903530140172288},\\
q\Big(\frac{61923}{1048576},-\frac{124967}{1048576},\frac{94825}{262144}\Big) &= \frac{506695580046184441260018365157011205}{664613997892457936451903530140172288},\\
q\Big(\frac{123847}{2097152},-\frac{15621}{131072},\frac{11853}{32768}\Big) &= \frac{3242775043523235559281320067952236 1265}{42535295865117307932921825928971026432},\\
q\Big(\frac{123847}{2097152},-\frac{124967}{1048576},\frac{11853}{32768}\Big) &= \frac{32427092325507024386448813961220173195}{42535295865117307932921825928971026432},\\
q\Big(\frac{123847}{2097152},-\frac{15621}{131072},\frac{94825}{262144}\Big) &= \frac{32428341455876232646559595333766084381}{42535295865117307932921825928971026432},\\
q\Big(\frac{123847}{2097152},-\frac{124967}{1048576},\frac{94825}{262144}\Big) &= \frac{32427683336233801584587332883715472751}{42535295865117307932921825928971026432}.
\end{aligned}
$$

Comparing the above values, we have

$$
\max q(z_1,x,a) = \frac{506705863123399793625830275060473279}{664613997892457936451903530140172288} > 0,
$$

and

$$
\min q(z_1,x,a) = \frac{32427092325507024386448813961220173195}{42535295865117307932921825928971026432} > 0,
$$

which have a same sign. Therefore $q(z_1,x,a) > 0$ on $\mathcal{C}_1$. This completes the proof.

□

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