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STABILIZED TWO-GRID DISCRETIZATIONS OF LOCKING FREE FOR THE ELASTICITY EIGENVALUE PROBLEM*

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Abstract In this paper, we propose two stabilized two-grid finite element discretizations for nearly incompressible elasticity eigenvalue problem and give the error estimates of eigenvalues and eigenfunctions for the schemes. Numerical experiments are provided to validate our theoretical analysis and exhibit that our schemes are locking free and highly efficient.

Keywords Elasticity eigenvalue problem, Two-grid discretizations, Stabilization, Nonconforming Crouzeix-Raviart finite element, Locking free

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1. Introduction

Consider the elasticity eigenvalue problem (see [27]): Find $\gamma \in \mathbb{R}$, $(\underline{\boldsymbol{\sigma}}, \mathbf{u}) \in \underline{\boldsymbol{\Sigma}} \times \mathbf{V}$ satisfying

$$\begin{cases} \mathcal{A}\underline{\boldsymbol{\sigma}} - \underline{\varepsilon}(\mathbf{u}) = \underline{\mathbf{0}}, & \text{in } \Omega, \\ -\mathrm{div}\underline{\boldsymbol{\sigma}} = \gamma\rho\mathbf{u}, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in which $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain, $\mathbf{V} = \mathbf{H}_0^1(\Omega)$, $\underline{\boldsymbol{\Sigma}} = \{\underline{\boldsymbol{\tau}} \in L^2(\Omega; \mathcal{S});\ \int_\Omega \mathrm{tr}\underline{\boldsymbol{\tau}} \mathrm{d}\mathbf{x} = 0\}$ with the notations $L^2(\Omega)$, $\mathbf{H}_0^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v}|_{\partial\Omega} = \mathbf{0}\}$ and $\mathcal{S}$ being the scalar-valued Lebesgue function space, the vector-valued Sobolev space and the real symmetric matrices of order 2×2, respectively. $\mathbf{u}$ and $\underline{\boldsymbol{\sigma}}$ denote the displacement and stress, respectively, and ρ is the mass density which we assume that $\rho \equiv 1$ throughout this paper. The strain tensor $\underline{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\underline{\nabla}\mathbf{u} + (\underline{\nabla}\mathbf{u})^T)$ where $\underline{\nabla}\mathbf{u}$ is the displacement gradient tensor. The tensor $\mathcal{A}$ is the compliance tensor of fourth order defined by

$$\mathcal{A}\underline{\boldsymbol{\sigma}} = \frac{1}{2\mu}\left(\underline{\boldsymbol{\sigma}} - \frac{\lambda}{2\mu + 2\lambda}\mathrm{tr}\underline{\boldsymbol{\sigma}}\ \underline{\boldsymbol{I}}\right), \tag{1.2}$$

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where $\underline{\boldsymbol{I}} = (I_{ij})_{2\times 2}$ is the identity tensor, μ and λ are two positive Lamé parameters satisfying

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{1.3}$$

with the Poisson ratio $\nu \in (0, \frac{1}{2})$ and Young′s modulus E. Notice that the coefficients $(\mu, \lambda) \in [\mu_1, \mu_2] \times (0, +\infty)$ and $0 < \mu_1 < \mu_2 < +\infty$.

When the Lamé parameter λ tends to $+\infty$, namely, the material becomes incompressible, the performance of the linear conforming finite element deteriorates ([10, 28]), leading to the so-called locking phenomenon [4, 5]. So, it is important to design robust numerical methods to solve the elasticity problem. In order to deal with the locking phenomenon, there have been quite a few numerical methods, including the nonconforming finite element method [10, 11], p-version finite element method [28], discontinuous Galerkin method [18, 20, 23, 29], the mixed method [22, 24, 25, 35], and so on.

Up to now, the literatures [6, 7, 16, 21, 25, 27, 36] have done a lot of excellent works for the linear elasticity eigenvalue problem. Bertrand et al. in [6] analyzed the approximation of eigenvalues coming from the Hellinger-Reissner variational principle for elasticity problem and discussed an adaptive scheme. Meddahi et al. in [25] analyzed a mixed variational formulation with reduced symmetry, and proved quasi-optimal error estimates of the eigenvalues and eigenfunctions for the elasticity eigenvalue problem. Russo in [27] gave the theory of nonconforming methods for the approximation of mixed eigenvalue problems and applied it to discuss the Hellinger-Reissner elasticity mixed eigenvalue problem, and obtained the optimal error estimates of eigenvalues and eigenvectors. Inzunza et al. in [21] applied Raviart-Thomas elements to analyze a mixed elasticity eigenvalue problem. In this paper, we discuss two stabilized two-grid discretizations for the nearly incompressible elasticity eigenvalue problem, and our work are as follows:

(1) Inspired by the work of Zhang and Zhao in [35], for the elasticity eigenvalue problem (1.1), based on the discrete form of the original mixed variational formulation, we adopt the piecewise constant space and the nonconforming Crouzeix-Raviart (CR) element for the stress and the displacement, respectively, and introduce the jump penalty term to the displacement to estiblish a locking free and stablilized nonconforming mixed finite element method. Besides, we present an a priori error analysis in the $\mathbf{L}^2$-norm for stress and in the broken $\mathbf{H}^1$-norm for displacement. It is worth noting that, our mixed formulation can be reduced to the stabilized nonconforming finite element method in [18], in which the error estimates was analyzed in a mesh dependent energy-like norm for the elasticity problem.

(2) One expects to obtain approximate solutions as accurate as possible with less computation costs, and two-grid discretization is one of the efficient methods to achieve this goal. It has been successfully used to solve eigenvalue problems (see, e.g., [1, 7, 12, 15, 17, 19, 30–33, 37, 38], etc). In this paper, we propose two two-grid discretization schemes for solving the elasticity eigenvalue problem which belongs to the second type mixed eigenvalue problems (see [8]). With our schemes, the solution of the elasticity eigenvalue problem on a fine mesh π_h is reduced to solving an eigenvalue problem on a much coarser mesh π_H as well as solving a linear algebraic system on the fine mesh π_h, and the resulting solution remains an asymptotically optimal accuracy. Numerical experiments suggest that it takes significantly less time to obtain nearly the same accurate approximations by two-grid discretization

schemes than direct computation on a fine mesh.

(3) As we know, it is difficult to calculate elasticity eigenvalues in large parameter cases. In existing literatures there are few numerical experiments reports for the elasticity eigenvalue problem in the case of large parameter λ (the Poisson ratio $\nu \to \frac{1}{2}$). Here we implement numerical experiments especially with respect to large parameter λ. Numerical results obtained by using the stabilized nonconforming mixed finite element method and two-grid discretization schemes tend to be stable as λ increases, which indicates that our methods are locking free. The advantage of two-grid schemes becomes more and more obvious as λ increases.

Throughout the paper, C denotes a positive constant independent of the mesh size h and Lamé parameter λ, which may represent different values in different occurrences.

2. The linear elasticity eigenvalue problem and its mixed nonconforming finite element approximation

In this section, we introduce the linear elasticity problem and the corresponding eigenvalue problem and its nonconforming mixed finite element approximation.

2.1. Notations

Let $\mathbf{x} = (x_1, x_2)^T$, $\mathbf{v}(\mathbf{x}) = (v_1(\mathbf{x}), v_2(\mathbf{x}))^T$, and $\partial_i = \frac{\partial}{\partial x_i}$. The standard notation $H^s(\Omega)$ is used to denote the Sobolev space with norms $\|\cdot\|_{s,\Omega}$ and seminorms $|\cdot|_{s,\Omega}$, and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. For simplicity, we use $\|\cdot\|_s$ and $|\cdot|_s$ instead of $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$, respectively. We use $\mathbf{L}^2(\Omega)$, $\underline{\mathbf{H}}^s(\Omega)$ and $\underline{\mathbf{H}}^s(\Omega; \mathcal{S})$ to denote the vector-, tensor- and symmetric tensor-valued spaces, respectively. For vectors $\mathbf{w}$, $\mathbf{v} \in \mathbb{R}^2$ and matrices $\underline{\boldsymbol{\sigma}}$, $\underline{\boldsymbol{\tau}} \in \mathcal{S}$, denote $(\underline{\nabla}\mathbf{u})_{ij} = \partial_j u_i$, $(\mathrm{div}\underline{\boldsymbol{\sigma}})_i = \sum_{j=1}^{2} \partial_j \sigma_{ij}$ and $\underline{\boldsymbol{\sigma}} : \underline{\boldsymbol{\tau}} = \sum_{i,j=1}^{2} \sigma_{ij}\tau_{ij}$.

$\pi_h := \{\kappa\}$ is used to denote a regular mesh of Ω. For an edge (face) E, h_E is the diameter of E, h_κ is the diameter of element κ, and the mesh diameter $h = \max_{\kappa\in\pi_h} h_\kappa$. Let $\mathcal{E}_h^i$ be the set of all interior edges (faces) of π_h, $\mathcal{E}_h^b$ be the set of all edges (faces) on the boundary, and $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$. For any edge (face) $E \in \mathcal{E}_h^i$, let $\mathbf{n}^+$ be the unit normal of E pointing from κ^+ to κ^- and $\mathbf{n}^+ = -\mathbf{n}^-$, and for any function $\mathbf{w}$, the jump of $\mathbf{w}$ through E is defined by

$$[\![\mathbf{w}]\!]|_E = (\mathbf{w}|_{\kappa^+})|_E - (\mathbf{w}|_{\kappa^-})|_E.$$

For any edge (face) $E \in \mathcal{E}_h^b$ on the boundary, we define

$$[\![\mathbf{w}]\!]|_E = \mathbf{w}|_E.$$

Given a mesh π_h, define the following spaces:

$$\underline{\boldsymbol{\Sigma}}_h = \{\underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{\Sigma}} : \ \underline{\boldsymbol{\tau}}_h|_\kappa \in \mathbb{P}_0(\kappa; \mathcal{S}), \ \forall \kappa \in \pi_h\},$$

$$\mathbf{V}_h = \left\{ \mathbf{v}_h \in L^2(\Omega) : \ \mathbf{v}_h|_\kappa \in \mathbb{P}_1(\kappa), \ \forall \kappa \in \pi_h \text{ and } \int_{\mathrm{E}} [\![\mathbf{v}]\!] \mathrm{ds} = 0, \ \forall \mathrm{E} \in \mathcal{E}_{\mathrm{h}} \right\},$$

where $\mathbf{V}_h$ is the nonconforming CR finite element space [14], $\mathbb{P}_m(\kappa)$ is the space of polynomials of degree at most m $(m = 0, 1)$ in κ and $\mathbb{P}_m(\kappa; \mathcal{S})$ is the space of symmetric tensor in $\mathbb{P}_m(\kappa)$.

Define the broken H^1-norm $\|\cdot\|_{1,h}$ on $\mathbf{V}_h$:

$$\|\mathbf{v}\|_{1,h} = \sqrt{\sum_{\kappa \in \pi_h} |\mathbf{v}|^2_{1,\kappa}}.$$

2.2. Nonconforming mixed finite element approximation for the linear elasticity problem

Consider the following linear elasticity problem:

$$\begin{cases} \mathcal{A}\underline{\boldsymbol{\sigma}} - \underline{\varepsilon}(\mathbf{u}) = \underline{\mathbf{0}}, & \text{in } \Omega, \\ -\mathrm{div}\underline{\boldsymbol{\sigma}} = \mathbf{g}, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

here $\mathbf{g}$ is the load function.

The primal mixed variational formulation of (2.1) is stated to seek $(\underline{\boldsymbol{\sigma}}^{\mathbf{g}}, \mathbf{u}^{\mathbf{g}}) \in \underline{\boldsymbol{\Sigma}} \times \mathbf{V}$ staisfying

$$a(\underline{\boldsymbol{\sigma}}^{\mathbf{g}}, \underline{\boldsymbol{\tau}}) + b(\underline{\boldsymbol{\tau}}, \mathbf{u}^{\mathbf{g}}) = 0, \quad \forall \underline{\boldsymbol{\tau}} \in \underline{\boldsymbol{\Sigma}}, \tag{2.2}$$

$$b(\underline{\boldsymbol{\sigma}}^{\mathbf{g}}, \mathbf{v}) = -(\mathbf{g}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.3}$$

where

$$a(\underline{\boldsymbol{\sigma}}^{\mathbf{g}}, \underline{\boldsymbol{\tau}}) = \int_\Omega \mathcal{A}\underline{\boldsymbol{\sigma}}^{\mathbf{g}} : \underline{\boldsymbol{\tau}} d\mathbf{x} = \int_\Omega \frac{1}{2\mu} \left(\underline{\boldsymbol{\sigma}}^{\mathbf{g}} - \frac{\lambda}{2\mu + d\lambda} \mathrm{tr}\underline{\boldsymbol{\sigma}}^{\mathbf{g}} \underline{\boldsymbol{I}} \right) : \underline{\boldsymbol{\tau}} d\mathbf{x},$$

$$b(\underline{\boldsymbol{\tau}}, \mathbf{u}^{\mathbf{g}}) = -\int_\Omega \underline{\boldsymbol{\tau}} : \underline{\varepsilon}(\mathbf{u}^{\mathbf{g}}) d\mathbf{x}, \ \ (\mathbf{g}, \mathbf{v}) = \int_\Omega \mathbf{g} \cdot \mathbf{v} d\mathbf{x}.$$

We use $\|A\|_{0,\Omega}$ to denote the $\mathbf{L}^2$-norm of A (if A is a matrix, $\|A\|_{0,\Omega} = \sqrt{\int_\Omega A : A d\mathbf{x}}$; if A is a vector, $\|A\|_{0,\Omega} = \sqrt{\int_\Omega A \cdot A d\mathbf{x}}$). We also use $\|\cdot\|_0$ instead of $\|\cdot\|_{0,\Omega}$ for simplicity. From *K-ellipticity* and *Inf-sup condition* (see (3.2) and (3.3) in [35]), by the Brezzi-Babuska theory for mixed methods, we know that the problem (2.2)-(2.3) has a unique solution $(\underline{\boldsymbol{\sigma}}^{\mathbf{g}}, \mathbf{u}^{\mathbf{g}}) \in \underline{\boldsymbol{\Sigma}} \times \mathbf{V}$ such that

$$\|\underline{\boldsymbol{\sigma}}^{\mathbf{g}}\|_0 + \|\mathbf{u}^{\mathbf{g}}\|_1 \le C\|\mathbf{g}\|_0. \tag{2.4}$$

A stabilized nonconforming mixed finite element formulation of (2.2)-(2.3) is: Find $(\underline{\boldsymbol{\sigma}}_h^{\mathbf{g}}, \mathbf{u}_h^{\mathbf{g}}) \in \underline{\boldsymbol{\Sigma}}_h \times \mathbf{V}_h$ such that

$$a(\underline{\boldsymbol{\sigma}}_h^{\mathbf{g}}, \underline{\boldsymbol{\tau}}) + b_h(\underline{\boldsymbol{\tau}}, \mathbf{u}_h^{\mathbf{g}}) = 0, \quad \forall \underline{\boldsymbol{\tau}} \in \underline{\boldsymbol{\Sigma}}_h, \tag{2.5}$$

$$b_h(\underline{\boldsymbol{\sigma}}_h^{\mathbf{g}}, \mathbf{v}) + c_h(\mathbf{u}_h^{\mathbf{g}}, \mathbf{v}) = -(\mathbf{g}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h, \tag{2.6}$$

where

$$b_h(\underline{\boldsymbol{\tau}}, \mathbf{u}_h^{\mathbf{g}}) = -\sum_{\kappa\in\pi_h} \int_\kappa \underline{\boldsymbol{\tau}} : \underline{\varepsilon}(\mathbf{u}_h^{\mathbf{g}}) d\mathbf{x},$$

$$c_h(\mathbf{u}_h^{\mathbf{g}}, \mathbf{v}) = -\theta \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E [\![\mathbf{u}_h^{\mathbf{g}}]\!] \cdot [\![\mathbf{v}]\!] ds,$$

where θ is a positive constant.

Remark 1. Noticing that for finite λ, $\mathcal{A}$ is invertible, and $\underline{\boldsymbol{\sigma}}^{\mathbf{g}} = \mathcal{A}^{-1}\underline{\varepsilon}(\mathbf{u}_h^{\mathbf{g}})$ with $\underline{\varepsilon}_h$ being the elementwise version of $\underline{\varepsilon}$. Substituting it into (2.6), we can obtain

$$\sum_{\kappa\in\pi_h} \int_\kappa \mathcal{A}^{-1}\underline{\varepsilon}(\mathbf{u}_h^{\mathbf{g}}) : \underline{\varepsilon}(\mathbf{v}) d\mathbf{x} + \theta \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E [\![\mathbf{u}_h^{\mathbf{g}}]\!] \cdot [\![\mathbf{v}]\!] ds = (\mathbf{g}, \mathbf{v}), \tag{2.7}$$

which indicates that the stabilized nonconforming mixed finite element method (2.5)-(2.6) is exactly the stabilized nonconforming CR element method in [18] in which the authors analyzed the error estimates for the elasticity problem in an energy-like norm that is related to mesh and Lamé parameters, while in this paper, we analyze the errors of stress in $\mathbf{L}^2$-norm and displacement in the broken $\mathbf{H}^1$-norm, which are completely independent of the mesh size and Lamé parameters.

We can easily prove that $a(\cdot,\cdot)$ and $c_h(\cdot,\cdot)$ are symmetric, and from [35] we know the following conditions hold:

$$\begin{aligned} &a(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}) \le C\|\underline{\boldsymbol{\sigma}}_h\|_0 \|\underline{\boldsymbol{\tau}}\|_0, \quad \forall \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}} \in \underline{\boldsymbol{\Sigma}}_h, \\ &b_h(\underline{\boldsymbol{\tau}}, \mathbf{u}_h) \le C\|\underline{\boldsymbol{\tau}}\|_0 \|\mathbf{u}_h\|_{1,h}, \quad \forall \underline{\boldsymbol{\tau}} \in \underline{\boldsymbol{\Sigma}}_h, \mathbf{u}_h \in \mathbf{V}_h. \end{aligned} \tag{2.8}$$

$\mathrm{Z_h}$-*ellipticity*:

$$a(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\tau}}_h) \ge M\|\underline{\boldsymbol{\tau}}_h\|_0^2, \quad \forall \underline{\boldsymbol{\tau}}_h \in \underline{\mathbf{Z}}_h, \tag{2.9}$$

where $M > 0$ is a constant and $\underline{\mathbf{Z}}_h = \{\underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{\Sigma}}_h | b_h(\underline{\boldsymbol{\tau}}_h, \mathbf{v}_h) = 0, \mathbf{v}_h \in \mathbf{V}_h\}$.

Inf-sup condition:

$$\sup_{\underline{\mathbf{0}}\neq\underline{\boldsymbol{\tau}}_h\in\underline{\boldsymbol{\Sigma}}_h} \frac{b_h(\underline{\boldsymbol{\tau}}_h, \mathbf{v}_h)}{\|\underline{\boldsymbol{\tau}}_h\|_0} \ge \beta_1\|\mathbf{v}_h\|_{1,h} - \beta_2 \left(\sum_{E\in\mathcal{E}_h} h_E^{-1} \|[\![\mathbf{v}_h]\!]\|_{0,E}^2 \right)^{\frac{1}{2}}, \tag{2.10}$$

where $\beta_1,\ \beta_2$ are two positive constants.

From Brezzi-Babuska theorem, we know that the stabilized formulation (2.5)-(2.6) admits a unique solution $(\underline{\boldsymbol{\sigma}}_h^{\mathbf{g}}, \mathbf{u}_h^{\mathbf{g}}) \in \underline{\boldsymbol{\Sigma}}_h \times \mathbf{V}_h$.

We refer to [2,22,26] to give the following regularity assumption. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain. For any $\mathbf{g} \in \mathbf{L}^2(\Omega)$, (2.2)-(2.3) has a unique solution $\underline{\boldsymbol{\sigma}}^{\mathbf{g}} \in \underline{\mathbf{H}}(\Omega; \mathcal{S})$ and $\mathbf{u}^{\mathbf{g}} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ such that

$$\|\underline{\boldsymbol{\sigma}}^{\mathbf{g}}\|_1 + \|\mathbf{u}^{\mathbf{g}}\|_2 \le C\|\mathbf{g}\|_0. \tag{2.11}$$

Let $(\underline{\boldsymbol{\sigma}}^{\mathbf{g}}, \mathbf{u}^{\mathbf{g}})$ and $(\underline{\boldsymbol{\sigma}}_h^{\mathbf{g}}, \mathbf{u}_h^{\mathbf{g}})$ be the solution of (2.2)-(2.3) and (2.5)-(2.6), respectively. The consistency term of nonconforming mixed finite element is defined by:

$$E_h(\underline{\boldsymbol{\sigma}}^{\mathbf{g}}, \mathbf{u}^{\mathbf{g}}, \mathbf{v}) = b_h(\underline{\boldsymbol{\sigma}}^{\mathbf{g}}, \mathbf{v}) + (\mathbf{g}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V} + \mathbf{V}_h. \tag{2.12}$$

Referring to [35] we have the following estimate:

$$|E_h(\underline{\boldsymbol{\sigma}}^{\mathbf{g}}, \mathbf{u}^{\mathbf{g}}, \mathbf{v})| \le Ch|\underline{\boldsymbol{\sigma}}^{\mathbf{g}}|_1\|\mathbf{v}\|_{1,h}, \quad \forall \mathbf{v} \in \mathbf{V} + \mathbf{V}_h. \tag{2.13}$$

Proposition 1. Assume that $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain, $(\underline{\boldsymbol{\sigma}}^{\mathbf{g}}, \mathbf{u}^{\mathbf{g}})$ and $(\underline{\boldsymbol{\sigma}}_h^{\mathbf{g}}, \mathbf{u}_h^{\mathbf{g}})$ are the solution of (2.2)-(2.3) and (2.5)-(2.6), respectively, then it is valid that

$$\|\underline{\boldsymbol{\sigma}}_h^{\mathbf{g}} - \underline{\boldsymbol{\sigma}}^{\mathbf{g}}\|_0 + \|\mathbf{u}_h^{\mathbf{g}} - \mathbf{u}^{\mathbf{g}}\|_{1,h} \le Ch\|\mathbf{g}\|_0, \tag{2.14}$$

$$\|\mathbf{u}_h^{\mathbf{g}} - \mathbf{u}^{\mathbf{g}}\|_0 \le Ch^2\|\mathbf{g}\|_0, \tag{2.15}$$

where C is a positive constant.

2.3. Nonconforming mixed finite element approximation for the linear elasticity eigenvalue problem

The weak formulation of (1.1) is: Seek $\gamma \in \mathbb{R}$, $(\underline{\boldsymbol{\sigma}}, \mathbf{u}) \in \underline{\boldsymbol{\Sigma}} \times \mathbf{V}$ satisfying

$$a(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) + b(\underline{\boldsymbol{\tau}}, \mathbf{u}) = 0, \quad \forall \underline{\boldsymbol{\tau}} \in \underline{\boldsymbol{\Sigma}}, \tag{2.16}$$

$$b(\underline{\boldsymbol{\sigma}}, \mathbf{v}) = -\gamma(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}. \tag{2.17}$$

A stabilized nonconforming mixed finite element formulation of (2.16)-(2.17) reads: Find $\gamma_h \in \mathbb{R}$, $(\underline{\boldsymbol{\sigma}}_h, \mathbf{u}_h) \in \underline{\boldsymbol{\Sigma}}_h \times \mathbf{V}_h$ such that

$$a(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}) + b_h(\underline{\boldsymbol{\tau}}, \mathbf{u}_h) = 0, \quad \forall \underline{\boldsymbol{\tau}} \in \underline{\boldsymbol{\Sigma}}_h, \tag{2.18}$$

$$b_h(\underline{\boldsymbol{\sigma}}_h, \mathbf{v}) + c_h(\mathbf{u}_h, \mathbf{v}) = -\gamma_h(\mathbf{u}_h, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h. \tag{2.19}$$

Define the operators $\underline{\mathbf{S}} : \underline{\mathbf{L}}^2(\Omega) \to \underline{\boldsymbol{\Sigma}}$ and $\mathbf{T} : \mathbf{L}^2(\Omega) \to \mathbf{V} \overset{c}{\hookrightarrow} \mathbf{L}^2(\Omega)$ by

$$a(\underline{\mathbf{S}}\mathbf{g}, \underline{\boldsymbol{\tau}}) + b(\underline{\boldsymbol{\tau}}, \mathbf{T}\mathbf{g}) = 0, \quad \forall \underline{\boldsymbol{\tau}} \in \underline{\boldsymbol{\Sigma}}, \tag{2.20}$$

$$b(\underline{\mathbf{S}}\mathbf{g}, \mathbf{v}) = -(\mathbf{g}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.21}$$

then $\underline{\mathbf{S}}\mathbf{g} = \underline{\boldsymbol{\sigma}}^{\mathbf{g}}$, $\mathbf{T}\mathbf{g} = \mathbf{u}^{\mathbf{g}}$.

Define $\underline{\mathbf{S}}_h :\ \underline{\mathbf{L}}^2(\Omega; \mathcal{S}) \to \underline{\boldsymbol{\Sigma}}_h$ and $\mathbf{T}_h :\ \mathbf{L}^2(\Omega) \to \mathbf{V}_h$ by

$$a(\underline{\mathbf{S}}_h\mathbf{g}, \underline{\boldsymbol{\tau}}) + b_h(\underline{\boldsymbol{\tau}}, \mathbf{T}_h\mathbf{g}) = 0, \quad \forall \underline{\boldsymbol{\tau}} \in \underline{\boldsymbol{\Sigma}}_h, \tag{2.22}$$

$$b_h(\underline{\mathbf{S}}_h\mathbf{g}, \mathbf{v}) + c_h(\mathbf{T}_h\mathbf{g}, \mathbf{v}) = -(\mathbf{g}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h, \tag{2.23}$$

then $\underline{\mathbf{S}}_h\mathbf{g} = \underline{\boldsymbol{\sigma}}_h^{\mathbf{g}}$, $\mathbf{T}_h\mathbf{g} = \mathbf{u}_h^{\mathbf{g}}$. Thus, (2.16)-(2.17) and (2.18)-(2.19) have the following equivalent operator forms, respectively:

$$\underline{\boldsymbol{\sigma}} = \underline{\mathbf{S}}(\gamma\mathbf{u}), \quad \gamma\mathbf{T}\mathbf{u} = \mathbf{u}, \tag{2.24}$$

$$\underline{\boldsymbol{\sigma}}_h = \underline{\mathbf{S}}_h(\gamma_h\mathbf{u}_h), \quad \gamma_h\mathbf{T}_h\mathbf{u}_h = \mathbf{u}_h, \tag{2.25}$$

and $\frac{1}{\gamma}$ and $\frac{1}{\gamma_h}$ are the eigenvalues of $\mathbf{T}$ and $\mathbf{T}_h$, respectively.

Lemma 2.1. *For any* $\mathbf{g} \in \mathbf{L}^2(\Omega)$*, the operators*

$$\mathbf{T} :\ \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega) \text{ and } \mathbf{T}_h :\ \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega) \tag{2.26}$$

are both self-adjoint.

Proof. Since $a(\cdot,\cdot)$ is symmetric, by a similar argument with that on page 753 in [3], we can verify that $\mathbf{T}$ is self-adjoint. Next we will prove $\mathbf{T}_h$ is self-adjoint. For any $\mathbf{g},\ \mathbf{f}\in\mathbf{L}^2(\Omega)$, let $\mathbf{v}=\mathbf{T}_h\mathbf{f}$ in (2.23) and we can obtain

$$b_h(\underline{\mathbf{S}}_h\mathbf{g},\mathbf{T}_h\mathbf{f})+c_h(\mathbf{T}_h\mathbf{g},\mathbf{T}_h\mathbf{f})=-(\mathbf{g},\mathbf{T}_h\mathbf{f}). \tag{2.27}$$

Let $\mathbf{g}=\mathbf{f}$ and $\underline{\boldsymbol{\tau}}=\underline{\mathbf{S}}_h\mathbf{g}$ in (2.22), then we have

$$a(\underline{\mathbf{S}}_h\mathbf{f},\underline{\mathbf{S}}_h\mathbf{g})+b_h(\underline{\mathbf{S}}_h\mathbf{g},\mathbf{T}_h\mathbf{f})=0, \tag{2.28}$$

which together with (2.27) yields

$$(\mathbf{g},\mathbf{T}_h\mathbf{f})=a(\underline{\mathbf{S}}_h\mathbf{f},\underline{\mathbf{S}}_h\mathbf{g})-c_h(\mathbf{T}_h\mathbf{g},\mathbf{T}_h\mathbf{f}). \tag{2.29}$$

Denote $\mathbf{g}=\mathbf{f}$ and $\mathbf{v}=\mathbf{T}_h\mathbf{g}$ in (2.23), we get

$$b_h(\underline{\mathbf{S}}_h\mathbf{f},\mathbf{T}_h\mathbf{g})+c_h(\mathbf{T}_h\mathbf{f},\mathbf{T}_h\mathbf{g})=-(\mathbf{f},\mathbf{T}_h\mathbf{g}), \tag{2.30}$$

and let $\underline{\boldsymbol{\tau}}=\underline{\mathbf{S}}_h\mathbf{f}$ in (2.22) we have

$$a(\underline{\mathbf{S}}_h\mathbf{g},\underline{\mathbf{S}}_h\mathbf{f})+b_h(\underline{\mathbf{S}}_h\mathbf{f},\mathbf{T}_h\mathbf{g})=0. \tag{2.31}$$

From (2.30) and (2.31) we obtain

$$(\mathbf{f},\mathbf{T}_h\mathbf{g})=a(\underline{\mathbf{S}}_h\mathbf{g},\underline{\mathbf{S}}_h\mathbf{f})-c_h(\mathbf{T}_h\mathbf{f},\mathbf{T}_h\mathbf{g}), \tag{2.32}$$

which together with the symmetry of $a(\cdot,\cdot)$, $c_h(\cdot,\cdot)$ and (2.32) we derive

$$\begin{aligned}(\mathbf{T}_h\mathbf{g},\mathbf{f})&=(\mathbf{f},\mathbf{T}_h\mathbf{g})=a(\underline{\mathbf{S}}_h\mathbf{g},\underline{\mathbf{S}}_h\mathbf{f})-c_h(\mathbf{T}_h\mathbf{f},\mathbf{T}_h\mathbf{g})\\&=a(\underline{\mathbf{S}}_h\mathbf{f},\underline{\mathbf{S}}_h\mathbf{g})-c_h(\mathbf{T}_h\mathbf{g},\mathbf{T}_h\mathbf{f})=(\mathbf{g},\mathbf{T}_h\mathbf{f}),\end{aligned} \tag{2.33}$$

i.e., $\mathbf{T}_h$ is self-adjoint. □

Lemma 2.2. *For any* $\mathbf{g}\in\mathbf{L}^2(\Omega)$, *there hold*

$$\|\mathbf{T}_h\mathbf{g}\|_{1,h}\le C\|\mathbf{g}\|_0, \tag{2.34}$$
$$\|\underline{\mathbf{S}}_h\mathbf{g}\|_0\le C\|\mathbf{g}\|_0. \tag{2.35}$$

Proof. By the triangle inequality, (2.14) and (2.4) we derive

$$\begin{aligned}\|\mathbf{T}_h\mathbf{g}\|_{1,h}&=\|\mathbf{T}_h\mathbf{g}-\mathbf{T}\mathbf{g}+\mathbf{T}\mathbf{g}\|_{1,h}\le\|\mathbf{T}_h\mathbf{g}-\mathbf{T}\mathbf{g}\|_{1,h}+\|\mathbf{T}\mathbf{g}\|_1\\&\le Ch\|\mathbf{g}\|_0+C\|\mathbf{g}\|_0\le C\|\mathbf{g}\|_0.\end{aligned} \tag{2.36}$$

Using the similar arguments to (2.36) we have

$$\|\underline{\mathbf{S}}_h\mathbf{g}\|_0\le\|\underline{\mathbf{S}}_h\mathbf{g}-\underline{\mathbf{S}}\mathbf{g}\|_0+\|\underline{\mathbf{S}}\mathbf{g}\|_0\le C\|\mathbf{g}\|_0, \tag{2.37}$$

The proof is completed. □

From (2.14) and (2.15) we get

$$\begin{aligned}\|\mathbf{T}_h-\mathbf{T}\|_0&=\sup_{\mathbf{g}\in\mathbf{L}^2(\Omega)\backslash\{\mathbf{0}\}}\frac{\|\mathbf{T}_h\mathbf{g}-\mathbf{T}\mathbf{g}\|_0}{\|\mathbf{g}\|_0}\le Ch^2\to0(h\to0),\\ \|\underline{\mathbf{S}}_h-\underline{\mathbf{S}}\|_0&=\sup_{\mathbf{g}\in\mathbf{L}^2(\Omega)\backslash\{\mathbf{0}\}}\frac{\|\underline{\mathbf{S}}_h\mathbf{g}-\underline{\mathbf{S}}\mathbf{g}\|_0}{\|\mathbf{g}\|_0}\le Ch.\end{aligned} \tag{2.38}$$

By (2.38) and $\mathbf{T}_h$ is a finite rank operator, we know that $\mathbf{T}$ is completely continuous operator. According to the spectral approximation theory, the eigenvalues of (2.16)-(2.17) can be arranged as

$$0<\gamma_1\le\gamma_2\le\cdots\le\gamma_k\le\cdots\nearrow+\infty,$$

and the corresponding eigenfunctions are

$$(\underline{\boldsymbol{\sigma}}_1,\mathbf{u}_1),\ (\underline{\boldsymbol{\sigma}}_2,\mathbf{u}_2),\cdots,(\underline{\boldsymbol{\sigma}}_k,\mathbf{u}_k),\cdots$$

where $(\mathbf{u}_i,\mathbf{u}_j)=\delta_{ij}$. The eigenvalues of (2.18)-(2.19) can be arranged as

$$0<\gamma_{1,h}\le\gamma_{2,h}\le\cdots\le\gamma_{k,h}\le\cdots\le\gamma_{N,h},$$

and the corresponding eigenfunctions are

$$(\underline{\boldsymbol{\sigma}}_{1,h},\mathbf{u}_{1,h}),\ (\underline{\boldsymbol{\sigma}}_{2,h},\mathbf{u}_{2,h}),\cdots,(\underline{\boldsymbol{\sigma}}_{k,h},\mathbf{u}_{k,h}),\cdots,(\underline{\boldsymbol{\sigma}}_{N,h},\mathbf{u}_{N,h}),$$

where $(\mathbf{u}_{i,h},\mathbf{u}_{j,h})=\delta_{ij}$, $N=\dim\mathbf{V}_\mathrm{h}$.

Let $\widetilde{\gamma}=\frac{1}{\gamma}$, $\widetilde{\gamma}_{k,h}=\frac{1}{\gamma_{k,h}}$. In the following discussion, γ and γ_h all denote the kth eigenvalue.

Suppose that $\{\gamma_k\}$ and $\{\gamma_{k,h}\}$ are eigenvalues of (2.16)-(2.17) and (2.18)-(2.19), respectively. We use $\gamma=\gamma_k$ to denote the kth eigenvalue with the algebraic multiplicity q, that is $\gamma=\gamma_k=\gamma_{k+1}=\cdots=\gamma_{k+q-1}$. The space spanned by all eigenfunctions corresponding to γ is written as $\mathbf{M}(\gamma)$, and direct sum of eigenspaces corresponding to all eigenvalues of (2.18)-(2.19) converge to γ is written as $\mathbf{M}_h(\gamma)$.

Lemma 2.3. *Assume that $(\gamma_h,\underline{\boldsymbol{\sigma}}_{\boldsymbol{h}},\mathbf{u}_h)$ is the kth eigenpair of (2.18)-(2.19) and $\|\mathbf{u}_h\|_0=1$, and γ is the kth eigenvalue of (2.16)-(2.17). Then $\gamma_h\to\gamma$ when $h\to0$, and there exists an eigenpair $(\underline{\boldsymbol{\sigma}},\mathbf{u})$ corresponding to γ satisfying*

$$|\gamma_h-\gamma|\le Ch^2, \tag{2.39}$$
$$\|\mathbf{u}_h-\mathbf{u}\|_0\le Ch^2, \tag{2.40}$$
$$\|\underline{\boldsymbol{\sigma}}_h-\underline{\boldsymbol{\sigma}}\|_0\le Ch, \tag{2.41}$$
$$\|\mathbf{u}_h-\mathbf{u}\|_{1,h}\le Ch. \tag{2.42}$$

Moreover, let $\mathbf{u}\in\mathbf{M}(\gamma)$ and $\|\mathbf{u}\|_0=1$, then we have $\mathbf{u}_h\in\mathbf{M}_h(\gamma)$ such that

$$\|\mathbf{u}_h-\mathbf{u}\|_{1,h}\le Ch. \tag{2.43}$$

Proof. We refer to [3, 34] to complete the proof. From Lemma 1 in [34] we have $\gamma_h\to\gamma$ $(h\to0)$. By Theorem 7.2 in [3] and (2.38) we get

$$\begin{aligned}|\gamma_h-\gamma|&\le C((\mathbf{T}_h-\mathbf{T})\mathbf{u},\mathbf{u})+\|(\mathbf{T}_h-\mathbf{T})|_{\mathbf{M}(\gamma)}\|_0^2\\&\le C\|(\mathbf{T}_h-\mathbf{T})\mathbf{u}\|_0+\|(\mathbf{T}_h-\mathbf{T})|_{\mathbf{M}(\gamma)}\|_0^2\le Ch^2,\end{aligned} \tag{2.44}$$

i.e., (2.39) holds. From Lemma 2 in [34] we get

$$\|\mathbf{u}_h-\mathbf{u}\|_0\le C\|(\mathbf{T}_h-\mathbf{T})|_{\mathbf{M}(\gamma)}\|_0, \tag{2.45}$$

which together with (2.38) yields (2.40). Using the triangle inequality, (2.24), (2.25), (2.39), (2.40), (2.35) and (2.14), we derive (2.41) as follows

$$\begin{aligned}\|\underline{\boldsymbol{\sigma}}_h-\underline{\boldsymbol{\sigma}}\|_0&\le\|\underline{\mathbf{S}}_h\gamma_h\mathbf{u}_h-\underline{\mathbf{S}}_h\gamma\mathbf{u}_h+\underline{\mathbf{S}}_h\gamma\mathbf{u}_h-\underline{\mathbf{S}}_h\gamma\mathbf{u}+\underline{\mathbf{S}}_h\gamma\mathbf{u}-\underline{\mathbf{S}}\gamma\mathbf{u}\|_0\\&\le C(|\gamma_h-\gamma|\|\underline{\mathbf{S}}_h\mathbf{u}_h\|_0+\gamma\|\mathbf{u}_h-\mathbf{u}\|_0)+\|\underline{\mathbf{S}}_h\gamma\mathbf{u}-\underline{\mathbf{S}}\gamma\mathbf{u}\|_0\\&\le Ch.\end{aligned} \tag{2.46}$$

Using a similar argument as (2.46), we can get

$$\begin{aligned}\|\mathbf{u}_h-\mathbf{u}\|_{1,h}\le C(|\gamma_h-\gamma|\|\mathbf{T}_h\mathbf{u}_h\|_{1,h}+\gamma\|\mathbf{u}_h-\mathbf{u}\|_0)+\|\mathbf{T}_h\gamma\mathbf{u}-\mathbf{T}\gamma\mathbf{u}\|_{1,h}\\ \le Ch,\end{aligned} \tag{2.47}$$

that is (2.42). Next we will prove (2.43). The eigenfunctions $\{\mathbf{u}_{j,h}\}$ is used to denote an orthonormal system of $\mathbf{M}_h(\gamma)$ with respect to the inner product $(\cdot,\cdot)$. Combining with (2.40) we have a basis $\{\mathbf{u}_j^0\}_k^{k+q-1}$ for $\mathbf{M}(\gamma)$ satisfying $\|\mathbf{u}_j^0\|_0=1$ and making (2.40) hold. Hence, for any $\mathbf{u}\in\mathbf{M}(\gamma)$ with $\|\mathbf{u}\|_0=1$, let $\mathbf{u}=\sum\limits_{j=k}^{k+q-1}\alpha_j\mathbf{u}_j^0$, then we have

$$1=\|\mathbf{u}\|_0^2=\sum_{j=k}^{k+q-1}\alpha_j^2+\sum_{i\neq j,i,j=k}^{k+q-1}\alpha_i\alpha_j(\mathbf{u}_i^0,\mathbf{u}_j^0). \tag{2.48}$$

By (2.40) we have $(\mathbf{u}_i^0,\mathbf{u}_j^0)=(\mathbf{u}_i^0,\mathbf{u}_j^0)-(\mathbf{u}_{i,h},\mathbf{u}_{j,h})\to 0$ $(h\to 0)$ when $i\neq j$, thus, $\sum\limits_{j=k}^{k+q-1}\alpha_j^2\to 1$ $(h\to 0)$. Let $\mathbf{u}_h=\sum\limits_{j=k}^{k+q-1}\alpha_j\mathbf{u}_{j,h}$, we have $\mathbf{u}_h\in\mathbf{M}_h(\gamma)$. By (2.40) we derive

$$\|\mathbf{u}_h-\mathbf{u}\|_0\le C\sum_{j=k}^{k+q-1}\alpha_j^2\|\mathbf{u}_{j,h}-\mathbf{u}_j^0\|_0\le Ch^2, \tag{2.49}$$

combining the triangle inequality, the inverse estimate and the interpolation error estimation, we derive

$$\begin{aligned}\|\mathbf{u}_h-\mathbf{u}\|_{1,h}&\le\|\mathbf{u}_h-\mathbf{I}_h\mathbf{u}\|_{1,h}+\|\mathbf{I}_h\mathbf{u}-\mathbf{u}\|_{1,h}\\ &\le Ch^{-1}\|\mathbf{u}_h-\mathbf{I}_h\mathbf{u}\|_0+Ch\le Ch,\end{aligned} \tag{2.50}$$

i.e., (2.43) holds. Here $\mathbf{I}_h\mathbf{u}$ is a linear interpolation function. □

Suppose that $(\underline{\boldsymbol{\sigma}},\mathbf{u})$ is an eigenfunction of (2.16)-(2.17), then for any $\mathbf{v}\in\mathbf{V}+\mathbf{V}_h$,

$$E_h(\underline{\boldsymbol{\sigma}},\mathbf{u},\mathbf{v})=b_h(\underline{\boldsymbol{\sigma}},\mathbf{v})+(\gamma\mathbf{u},\mathbf{v}). \tag{2.51}$$

For any $(\underline{\boldsymbol{\sigma}}^{\mathbf{0}},\ \mathbf{u}^0)\in\underline{\boldsymbol{\Sigma}}_h\times\mathbf{V}_h$, $\mathbf{u}^0\neq 0$, define the Rayleigh quotient

$$\gamma^r=\frac{a(\underline{\boldsymbol{\sigma}}^0,\underline{\boldsymbol{\sigma}}^0)+2b_h(\underline{\boldsymbol{\sigma}}^0,\mathbf{u}^0)+c_h(\mathbf{u}^0,\mathbf{u}^0)}{-(\mathbf{u}^0,\mathbf{u}^0)}. \tag{2.52}$$

Lemma 2.4. *For any* $(\underline{\boldsymbol{\sigma}}^{\mathbf{0}},\ \mathbf{u}^0)\in\underline{\boldsymbol{\Sigma}}_h\times\mathbf{V}_h$, $\mathbf{u}^0\neq 0$, *assume that* $(\gamma,\ \underline{\boldsymbol{\sigma}},\ \mathbf{u})$ *is an eigenpair of* (2.16)-(2.17), *then the Rayleigh quotient* γ^r *satisfies*

$$\begin{aligned}\gamma^r-\gamma=&\frac{a(\underline{\boldsymbol{\sigma}}^0-\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\sigma}}^0-\underline{\boldsymbol{\sigma}})+2b_h(\underline{\boldsymbol{\sigma}}^0-\underline{\boldsymbol{\sigma}},\mathbf{u}^0-\mathbf{u})+c_h(\mathbf{u}^0-\mathbf{u},\mathbf{u}^0-\mathbf{u})}{-(\mathbf{u}^0,\mathbf{u}^0)}\\ &+\gamma\frac{(\mathbf{u}^0-\mathbf{u},\mathbf{u}^0-\mathbf{u})}{-(\mathbf{u}^0,\mathbf{u}^0)}+\frac{2E_h(\underline{\boldsymbol{\sigma}},\mathbf{u},\mathbf{u}^0)}{-(\mathbf{u}^0,\mathbf{u}^0)}.\end{aligned} \tag{2.53}$$

Proof. From (2.16)-(2.17), (2.18)-(2.19), the symmetry of $a(\cdot,\cdot)$ and (2.12), we can derive

$$\begin{aligned}&a(\underline{\boldsymbol{\sigma}}^0-\underline{\boldsymbol{\sigma}},\underline{\boldsymbol{\sigma}}^0-\underline{\boldsymbol{\sigma}})+2b_h(\underline{\boldsymbol{\sigma}}^0-\underline{\boldsymbol{\sigma}},\mathbf{u}^0-\mathbf{u})+c_h(\mathbf{u}^0-\mathbf{u},\mathbf{u}^0-\mathbf{u})+\gamma(\mathbf{u}^0-\mathbf{u},\mathbf{u}^0-\mathbf{u})\\ &=a(\underline{\boldsymbol{\sigma}}^0,\underline{\boldsymbol{\sigma}}^0)+2b_h(\underline{\boldsymbol{\sigma}}^0,\mathbf{u}^0)+c_h(\mathbf{u}^0,\mathbf{u}^0)+\gamma(\mathbf{u}^0,\mathbf{u}^0)\end{aligned}$$

$$
\begin{aligned}
&\quad - [a(\underline{\boldsymbol{\sigma}}^0, \underline{\boldsymbol{\sigma}}) + b_h(\underline{\boldsymbol{\sigma}}^0, \mathbf{u}) + b_h(\underline{\boldsymbol{\sigma}}, \mathbf{u}^0) + c_h(\mathbf{u}^0, \mathbf{u}) + \gamma(\mathbf{u}^0, \mathbf{u})] \\
&\quad - [a(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\sigma}}^0) + b_h(\underline{\boldsymbol{\sigma}}^0, \mathbf{u}) + b_h(\underline{\boldsymbol{\sigma}}, \mathbf{u}^0) + c_h(\mathbf{u}, \mathbf{u}^0) + \gamma(\mathbf{u}, \mathbf{u}^0)] \\
&\quad + [a(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\sigma}}) + b_h(\underline{\boldsymbol{\sigma}}, \mathbf{u}) + b_h(\underline{\boldsymbol{\sigma}}, \mathbf{u}) + c_h(\mathbf{u}, \mathbf{u}) + \gamma(\mathbf{u}, \mathbf{u})] \\
&= a(\underline{\boldsymbol{\sigma}}^0, \underline{\boldsymbol{\sigma}}^0) + 2b_h(\underline{\boldsymbol{\sigma}}^0, \mathbf{u}^0) + c_h(\mathbf{u}^0, \mathbf{u}^0) + \gamma(\mathbf{u}^0, \mathbf{u}^0) \\
&\quad - [a(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\sigma}}^0) + b_h(\underline{\boldsymbol{\sigma}}^0, \mathbf{u})] - 2E_h(\underline{\boldsymbol{\sigma}}, \mathbf{u}, \mathbf{u}^0) \\
&= a(\underline{\boldsymbol{\sigma}}^0, \underline{\boldsymbol{\sigma}}^0) + 2b_h(\underline{\boldsymbol{\sigma}}^0, \mathbf{u}^0) + c_h(\mathbf{u}^0, \mathbf{u}^0) + \gamma(\mathbf{u}^0, \mathbf{u}^0) - 2E_h(\underline{\boldsymbol{\sigma}}, \mathbf{u}, \mathbf{u}^0), \qquad (2.54)
\end{aligned}
$$

dividing by $-(\mathbf{u}^0, \mathbf{u}^0)$ on both sides of (2.54) yields (2.53). □

3. Two-grid discretizations for the linear elasticity eigenvalue problem

In this section, two two-grid discretizations are presented to solve the linear elasticity eigenvalue problem. Suppose that $\pi_H(H \in (0,1))$ is a sequence of shape-regular coarse triangulation, and a fine grid refined from π_H is written as π_h $(h \ll H)$.

3.1. Two-grid discretization based on inverse iteration

We refer to [30] to establish the following two-grid scheme.

Scheme 3.1. Two-grid discretization scheme based on inverse iteration.

Step 1. Solve the mixed eigenvalue problem (2.18)-(2.19) on $\underline{\boldsymbol{\Sigma}}_H \times \mathbf{V}_H$: Seek $(\gamma_H, \underline{\boldsymbol{\sigma}}_H, \mathbf{u}_H) \in \mathbb{R} \times \underline{\boldsymbol{\Sigma}}_H \times \mathbf{V}_H,\ \|\mathbf{u}_H\|_0 = 1$, satisfying

$$
\begin{aligned}
&a(\underline{\boldsymbol{\sigma}}_H, \underline{\boldsymbol{\tau}}) + b_H(\underline{\boldsymbol{\tau}}, \mathbf{u}_H) = 0, \quad \forall \underline{\boldsymbol{\tau}} \in \underline{\boldsymbol{\Sigma}}_H, \\
&b_H(\underline{\boldsymbol{\sigma}}_H, \mathbf{v}) + c_H(\mathbf{u}_H, \mathbf{v}) = -\gamma_H(\mathbf{u}_H, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_H.
\end{aligned} \tag{3.1}
$$

Step 2. Compute a linear system on $\underline{\boldsymbol{\Sigma}}_h \times \mathbf{V}_h$: Seek $(\underline{\boldsymbol{\sigma}}^h, \mathbf{u}^h) \in \underline{\boldsymbol{\Sigma}}_h \times \mathbf{V}_h$ such that

$$
\begin{aligned}
&a(\underline{\boldsymbol{\sigma}}^h, \underline{\boldsymbol{\tau}}_h) + b_h(\underline{\boldsymbol{\tau}}_h, \mathbf{u}^h) = 0, \quad \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{\Sigma}}_h, \\
&b_h(\underline{\boldsymbol{\sigma}}^h, \mathbf{v}_h) + c_h(\mathbf{u}^h, \mathbf{v}_h) = -\gamma_H(\mathbf{u}_H, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h,
\end{aligned} \tag{3.2}
$$

Step 3. Calculate the Rayleigh quotient

$$
\gamma^h = \frac{a(\underline{\boldsymbol{\sigma}}^h, \underline{\boldsymbol{\sigma}}^h) + 2b_h(\underline{\boldsymbol{\sigma}}^h, \mathbf{u}^h) + c_h(\mathbf{u}^h, \mathbf{u}^h)}{-(\mathbf{u}^h, \mathbf{u}^h)}.
$$

Let $(\gamma_H, \underline{\boldsymbol{\sigma}}_H, \mathbf{u}_H)$ be the kth eigenpair of (2.18)-(2.19), then $(\gamma^h, \underline{\boldsymbol{\sigma}}^h, \mathbf{u}^h)$ obtained by Scheme 3.1 is the kth eigenpair approximation of (2.16)-(2.17).

Theorem 3.1. *Assume that $(\gamma^h,\ \underline{\boldsymbol{\sigma}}^h,\ \mathbf{u}^h)$ is an approximate eigenpair obtained by Scheme 3.1. Then we have $\mathbf{u} \in \mathbf{M}(\gamma)$ satisfying*

$$
\|\mathbf{u}^h - \mathbf{u}\|_{1,h} \le C(H^2 + h), \tag{3.3}
$$

$$
\|\underline{\boldsymbol{\sigma}}^h - \underline{\boldsymbol{\sigma}}\|_0 \le C(H^2 + h), \tag{3.4}
$$

$$
|\gamma^h - \gamma| \le C(H^4 + h^2), \tag{3.5}
$$

where C is a positive constant independent of mesh size and λ.

Proof. From Lemma 2.3 we have $\mathbf{u} \in \mathbf{M}(\gamma)$ making $\mathbf{u}_H - \mathbf{u}$ satisfy (2.42). Combining (2.23) and (3.2) we have $\mathbf{u}^h = \gamma_H \mathbf{T}_h \mathbf{u}_H$. Using (2.24) and the triangle inequality yields

$$\begin{aligned} \|\mathbf{u}^h - \mathbf{u}\|_{1,h} &= \|\gamma_H \mathbf{T}_h \mathbf{u}_H - \gamma \mathbf{T}\mathbf{u}\|_{1,h} \\ &= \|\gamma_H \mathbf{T}_h \mathbf{u}_H - \gamma \mathbf{T}_h \mathbf{u}_H + \gamma \mathbf{T}_h \mathbf{u}_H - \gamma \mathbf{T}_h \mathbf{u} + \gamma \mathbf{T}_h \mathbf{u} - \gamma \mathbf{T}\mathbf{u}\|_{1,h} \\ &\le |\gamma_H - \gamma| \|\mathbf{T}_h \mathbf{u}_H\|_{1,h} + \gamma \|\mathbf{T}_h(\mathbf{u}_H - \mathbf{u})\|_{1,h} + \|\mathbf{T}_h \gamma \mathbf{u} - \mathbf{T}\gamma \mathbf{u}\|_{1,h}, \end{aligned} \tag{3.6}$$

which together with (2.34), (2.14) and Lemma 2.3 gives

$$\|\mathbf{u}^h - \mathbf{u}\|_{1,h} \le C(H^2 + \|\mathbf{u}_H - \mathbf{u}\|_0 + h) \le C(H^2 + h), \tag{3.7}$$

i.e., (3.3) holds. Using (2.24), (2.25), the triangle inequality, (2.35), (2.39), (2.40) and (2.38), we deduce

$$\begin{aligned} \|\underline{\boldsymbol{\sigma}}^h - \underline{\boldsymbol{\sigma}}\|_0 &= \|\gamma_H \underline{\mathbf{S}}_h \mathbf{u}_H - \gamma \underline{\mathbf{S}}_h \mathbf{u}_H + \gamma \underline{\mathbf{S}}_h \mathbf{u}_H - \gamma \underline{\mathbf{S}}_h \mathbf{u} + \gamma \underline{\mathbf{S}}_h \mathbf{u} - \gamma \underline{\mathbf{S}}\mathbf{u}\|_0 \\ &\le |\gamma_H - \gamma| \|\underline{\mathbf{S}}_h \mathbf{u}_H\|_0 + |\gamma| \|\underline{\mathbf{S}}_h(\mathbf{u}_H - \mathbf{u})\|_0 + \|\underline{\mathbf{S}}_h \gamma \mathbf{u} - \underline{\mathbf{S}}\gamma \mathbf{u}\|_0 \\ &\le C(H^2 + \|\mathbf{u}_H - \mathbf{u}\|_0 + h) \\ &\le C(H^2 + h), \end{aligned} \tag{3.8}$$

i.e., (3.4) holds. Finally, we prove (3.5). From (5.4) in [35] we have

$$|c_h(\mathbf{u}_h, \mathbf{u}_h)| \le C\|\mathbf{u}_h\|_{1,h}^2. \tag{3.9}$$

Since $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and $\mathbf{u}^h \in \mathbf{V}_h$ is a piecewise $\mathbf{H}^1$-function, by (1.5) in [9] we obtain

$$\|\mathbf{u}^h - \mathbf{u}\|_0 \le C\|\mathbf{u}^h - \mathbf{u}\|_{1,h}. \tag{3.10}$$

By (2.51), $E_h(\underline{\boldsymbol{\sigma}}, \mathbf{u}, \mathbf{u}) = 0$, (2.13) and (3.3) we obtain

$$\begin{aligned} |E_h(\underline{\boldsymbol{\sigma}}, \mathbf{u}, \mathbf{u}^h)| &= |E_h(\underline{\boldsymbol{\sigma}}, \mathbf{u}, \mathbf{u}^h - \mathbf{u})| \\ &\le Ch|\underline{\boldsymbol{\sigma}}|_1 \|\mathbf{u}^h - \mathbf{u}\|_{1,h} \le C(H^2 h + h^2). \end{aligned} \tag{3.11}$$

In Lemma 2.4, let $\gamma^r = \gamma^h$, $\underline{\boldsymbol{\sigma}}^0 = \underline{\boldsymbol{\sigma}}^h$ and $\mathbf{u}^0 = \mathbf{u}^h$, then, by (2.8), (3.9), (3.10), (3.11), (3.3) and (3.4) we have

$$\begin{aligned} |\gamma^h - \gamma| &= \left| \frac{a(\underline{\boldsymbol{\sigma}}^h - \underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\sigma}}^h - \underline{\boldsymbol{\sigma}}) + 2b_h(\underline{\boldsymbol{\sigma}}^h - \underline{\boldsymbol{\sigma}}, \mathbf{u}^h - \mathbf{u}) + c_h(\mathbf{u}^h - \mathbf{u}, \mathbf{u}^h - \mathbf{u})}{-(\mathbf{u}^h, \mathbf{u}^h)} \right. \\ &\quad \left. + \gamma \frac{(\mathbf{u}^h - \mathbf{u}, \mathbf{u}^h - \mathbf{u})}{-(\mathbf{u}^h, \mathbf{u}^h)} + \frac{2E_h(\underline{\boldsymbol{\sigma}}, \mathbf{u}, \mathbf{u}^h)}{-(\mathbf{u}^h, \mathbf{u}^h)} \right| \\ &\le C \left(\frac{\|\underline{\boldsymbol{\sigma}}^h - \underline{\boldsymbol{\sigma}}\|_0^2 + 2\|\underline{\boldsymbol{\sigma}}^h - \underline{\boldsymbol{\sigma}}\|_0 \|\mathbf{u}^h - \mathbf{u}\|_{1,h} + \|\mathbf{u}^h - \mathbf{u}\|_{1,h}^2}{\|\mathbf{u}^h\|_0^2} \right. \\ &\quad \left. + \frac{\|\mathbf{u}^h - \mathbf{u}\|_0^2}{\|\mathbf{u}^h\|_0^2} + \frac{E_h(\underline{\boldsymbol{\sigma}}, \mathbf{u}, \mathbf{u}^h)}{\|\mathbf{u}^h\|_0^2} \right) \\ &\le C \left[\frac{(\|\underline{\boldsymbol{\sigma}}^h - \underline{\boldsymbol{\sigma}}\|_0 + \|\mathbf{u}^h - \mathbf{u}\|_{1,h})^2}{\|\mathbf{u}^h\|_0^2} + \frac{E_h(\underline{\boldsymbol{\sigma}}, \mathbf{u}, \mathbf{u}^h)}{\|\mathbf{u}^h\|_0^2} \right] \\ &\le C[(H^2 + h)^2 + H^2 h + h^2] \\ &\le C(H^4 + h^2), \end{aligned} \tag{3.12}$$

that is (3.5) holds. □

3.2. Two-grid discretization based on the shifted-inverse iteration

Referring to the literature [32], we have the following scheme.

Scheme 3.2. Two-grid discretization scheme based on the shifted-inverse iteration.

Step 1. Solve the mixed eigenvalue problem (2.18)-(2.19) on $\underline{\mathbf{\Sigma}}_H \times \mathbf{V}_H$: Seek $(\gamma_H, \underline{\boldsymbol{\sigma}}_H, \mathbf{u}_H) \in \mathbb{R} \times \underline{\mathbf{\Sigma}}_H \times \mathbf{V}_H,\ \|\mathbf{u}_H\|_0 = 1$, satisfying

$$\begin{aligned} &a(\underline{\boldsymbol{\sigma}}_H, \underline{\boldsymbol{\tau}}) + b_H(\underline{\boldsymbol{\tau}}, \mathbf{u}_H) = 0, \quad \forall \underline{\boldsymbol{\tau}} \in \underline{\mathbf{\Sigma}}_H, \\ &b_H(\underline{\boldsymbol{\sigma}}_H, \mathbf{v}) + c_H(\mathbf{u}_H, \mathbf{v}) = -\gamma_H(\mathbf{u}_H, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_H. \end{aligned} \tag{3.13}$$

Step 2. Compute a linear system on $\underline{\mathbf{\Sigma}}_h \times \mathbf{V}_h$: Seek $(\underline{\boldsymbol{\sigma}}', \mathbf{u}') \in \underline{\mathbf{\Sigma}}_h \times \mathbf{V}_h$ such that

$$\begin{aligned} &a(\underline{\boldsymbol{\sigma}}', \underline{\boldsymbol{\tau}}_h) + b_h(\underline{\boldsymbol{\tau}}_h, \mathbf{u}') = 0, \quad \forall \underline{\boldsymbol{\tau}}_h \in \underline{\mathbf{\Sigma}}_h, \\ &b_h(\underline{\boldsymbol{\sigma}}', \mathbf{v}_h) + c_h(\mathbf{u}', \mathbf{v}_h) + \gamma_H(\mathbf{u}', \mathbf{v}_h) = -(\mathbf{u}_H, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \end{aligned} \tag{3.14}$$

and set $\mathbf{u}^h = \frac{\mathbf{u}'}{\|\mathbf{u}'\|_0}$, $\underline{\boldsymbol{\sigma}}^h = \frac{\underline{\boldsymbol{\sigma}}'}{\|\mathbf{u}'\|_0}$.

Step 3. Calculate the Rayleigh quotient

$$\gamma^h = \frac{a(\underline{\boldsymbol{\sigma}}^h, \underline{\boldsymbol{\sigma}}^h) + 2b_h(\underline{\boldsymbol{\sigma}}^h, \mathbf{u}^h) + c_h(\mathbf{u}^h, \mathbf{u}^h)}{-(\mathbf{u}^h, \mathbf{u}^h)}.$$

Denote

$$dist(\mathbf{w}, \mathbf{W}) = \inf_{\mathbf{u} \in \mathbf{W}} \|\mathbf{w} - \mathbf{u}\|_{1,h}.$$

For the sake of simplicity, we also denote $(\gamma_H, \mathbf{u}_H) = (\gamma_{k,H}, \mathbf{u}_{k,H})$ and $(\gamma^h, \mathbf{u}^h) = (\gamma_k^h, \mathbf{u}_k^h)$ for simplicity. The following lemma comes from Lemma 3.1 in [37] and can be proved similarly.

Lemma 3.1. *Assume that $(\widetilde{\gamma}_*, \mathbf{w}_*)$ is an approximation for the kth eigenpair $(\widetilde{\gamma}, \mathbf{u})$ where $\widetilde{\gamma}_*$ is not an eigenvalue of $\mathbf{T}_h$, $\mathbf{w}_* \in \mathbf{V}_h$ with $\|\mathbf{w}_*\|_0 = 1$, and $\mathbf{u}_* = \frac{\mathbf{T}_h \mathbf{w}_*}{\|\mathbf{T}_h \mathbf{w}_*\|_0}$. Suppose that*

(A1) $\inf_{\mathbf{v} \in \mathbf{M}_h(\gamma)} \|\mathbf{w}_* - \mathbf{v}\|_0 \le \frac{1}{2}$;

(A2) $|\widetilde{\gamma}_* - \widetilde{\gamma}| \le \frac{\rho}{4}$, $|\widetilde{\gamma}_{j,h} - \widetilde{\gamma}_j| \le \frac{\rho}{4}$ *for* $j = k-1, k, k+q (j \neq 0)$, *where* $\rho = \min_{j \neq k} |\widetilde{\gamma}_j - \widetilde{\gamma}|$ *is the separation constant of the kth eigenvalue* $\widetilde{\gamma}$;

(A3) Let $\mathbf{u}' \in \mathbf{V}_h$ and $\mathbf{u}^h \in \mathbf{V}_h$ satisfying

$$(\widetilde{\gamma}_* - \mathbf{T}_h)\mathbf{u}' = \mathbf{u}_*, \quad \mathbf{u}^h = \frac{\mathbf{u}'}{\|\mathbf{u}'\|_0}. \tag{3.15}$$

Then

$$dist(\mathbf{u}^h, \mathbf{M}_h(\gamma)) \le \frac{4}{\rho} \max_{k \le j \le k+q-1} |\widetilde{\gamma}_* - \widetilde{\gamma}_{j,h}| dist(\mathbf{w}_*, \mathbf{M}_h(\gamma)). \tag{3.16}$$

Theorem 3.2. *Assume that $(\gamma^h,\ \underline{\boldsymbol{\sigma}}^h,\ \mathbf{u}^h)$ is an approximate eigenpair obtained by Scheme 3.2. Then we have $\mathbf{u} \in \mathbf{M}(\gamma)$ satisfying*

$$\|\mathbf{u}^h - \mathbf{u}\|_{1,h} \le C(H^3 + h), \tag{3.17}$$

$$\|\underline{\boldsymbol{\sigma}}^h - \underline{\boldsymbol{\sigma}}\|_0 \le C(H^2 + h), \tag{3.18}$$

$$|\gamma^h - \gamma| \le C(H^4 + h^2), \tag{3.19}$$

where C is a positive constant independent of mesh size and λ.

Proof. The proof is similar to that of Theorem 3.1 in [37]. We briefly describe as follows for the convenience of reading. Denote $\widetilde{\gamma}_* = \frac{1}{\gamma_H}$ and $\mathbf{w}_* = \mathbf{u}_H$, then $\mathbf{u}_* = \mathbf{T}_h\mathbf{u}_H/\|\ \mathbf{T}_h\mathbf{u}_H\|_0$. From Lemma 2.3 we have $\widetilde{\mathbf{u}} \in \mathbf{M}(\gamma)$ making $\mathbf{u}_H - \widetilde{\mathbf{u}}$ satisfy (2.42). Since $\mathbf{v} \in \mathbf{M}_h(\gamma)$ and $\mathbf{u}_H \in \mathbf{V}_H$ are piecewise $\mathbf{H}^1$-functions, using (1.5) in [9], the triangle inequality, (2.42) and (2.43), we derive

$$\begin{aligned}\inf_{\mathbf{v}\in\mathbf{M}_h(\gamma)}\|\mathbf{w}_*-\mathbf{v}\|_0 &\le dist(\mathbf{u}_H,\mathbf{M}_h(\gamma))\\ &\le \|\mathbf{u}_H-\widetilde{\mathbf{u}}\|_{1,h}+dist(\widetilde{\mathbf{u}},\mathbf{M}_h(\gamma))\le CH,\end{aligned}\tag{3.20}$$

which shows that Condition (A1) in Lemma 3.2 is true when H is properly small. Because $\widetilde{\gamma}_*$ and $\widetilde{\gamma}_{j,h}$ are approximations of γ, by Lemma 2.3 we have

$$|\widetilde{\gamma}_*-\widetilde{\gamma}|=\frac{|\gamma_H-\gamma|}{|\gamma_H\gamma|}\le\frac{\rho}{4},$$

and for $\widetilde{\gamma}_{j,h}=\frac{1}{\gamma_{j,h}}$, $j=k,\,k+1,\,...,\,k+q-1$

$$|\widetilde{\gamma}_{j,h}-\widetilde{\gamma}_j|=\frac{|\gamma_{j,h}-\gamma_j|}{|\gamma_{j,h}\gamma_j|}\le\frac{\rho}{4},$$

which means that Condition (A2) is satisfied when H is properly small. From (3.14) we have

$$\begin{aligned}&a(\underline{\boldsymbol{\sigma}}',\underline{\boldsymbol{\tau}}_h)+b_h(\underline{\boldsymbol{\tau}}_h,\mathbf{u}')=0,\quad\forall\underline{\boldsymbol{\tau}}_h\in\underline{\boldsymbol{\Sigma}}_h,\\ &b_h(\underline{\boldsymbol{\sigma}}',\mathbf{v}_h)+c_h(\mathbf{u}',\mathbf{v}_h)=-(\mathbf{u}_H+\gamma_H\mathbf{u}',\mathbf{v}_h),\quad\forall\mathbf{v}_h\in\mathbf{V}_h,\end{aligned}\tag{3.21}$$

which together with (2.24) and (2.25) yields

$$\underline{\boldsymbol{\sigma}}'=\mathbf{S}_h(\mathbf{u}_H+\gamma_H\mathbf{u}'),\tag{3.22}$$

$$\mathbf{u}'=\mathbf{T}_h(\mathbf{u}_H+\gamma_H\mathbf{u}'),\tag{3.23}$$

then from (3.23) we have

$$(\frac{1}{\gamma_H}-\mathbf{T}_h)\mathbf{u}'=\frac{1}{\gamma_H}\mathbf{T}_h\mathbf{u}_H,\quad\mathbf{u}^h=\frac{\mathbf{u}'}{\|\mathbf{u}'\|_0}.\tag{3.24}$$

Note that $\mathbf{u}_*$ differs from $\mathbf{T}_h\mathbf{u}_H=\|\mathbf{T}_h\mathbf{u}_H\|_0\mathbf{u}_*$ by a constant, then

$$(\frac{1}{\gamma_H}-\mathbf{T}_h)\mathbf{u}'=\mathbf{u}_*,\quad\mathbf{u}^h=\frac{\mathbf{u}'}{\|\mathbf{u}'\|_0},$$

i.e., Condition (A3) true. Thus, we have

$$dist(\mathbf{u}^h,\mathbf{M}_h(\gamma))\le\frac{4}{\rho}\max_{k\le j\le k+q-1}|\widetilde{\gamma}_*-\widetilde{\gamma}_{j,h}|dist(\mathbf{u}_H,\mathbf{M}_h(\gamma)).\tag{3.25}$$

Using the triangle inequality and (2.39), for $j=k,\,k+1,\,...,\,k+q-1$ we get

$$|\widetilde{\gamma}_*-\widetilde{\gamma}_{j,h}|=|\frac{1}{\gamma_H}-\frac{1}{\gamma_{j,h}}|\le C(|\gamma_H-\gamma|+|\gamma-\gamma_{j,h}|)\le C(H^2+h^2).\tag{3.26}$$

Since the dimension of space $\mathbf{M}_h(\gamma)$ is q, we have $\mathbf{u}^0\in\mathbf{M}_h(\gamma)$ such that

$$\|\mathbf{u}^h-\mathbf{u}^0\|_{1,h}=dist(\mathbf{u}^h,\mathbf{M}_h(\gamma)),\tag{3.27}$$

which together with (3.25), (3.26), (3.20) and (2.39) yields

$$\|\mathbf{u}^h - \mathbf{u}^0\|_{1,h} \le CH(H^2 + h^2) \le CH^3. \tag{3.28}$$

From (2.42) we have $\mathbf{u} \in \mathbf{M}(\gamma)$ satisfying

$$\|\mathbf{u}^0 - \mathbf{u}\|_{1,h} \le Ch. \tag{3.29}$$

Combining the triangle inequality, (3.28) and (3.29) we derive

$$\|\mathbf{u}^h - \mathbf{u}\|_{1,h} \le \|\mathbf{u}^h - \mathbf{u}^0\|_{1,h} + \|\mathbf{u}^0 - \mathbf{u}\|_{1,h} \le C(H^3 + h), \tag{3.30}$$

that is (3.17) holds. Using (2.43) we have $\mathbf{u}_h \in \mathbf{M}_h(\gamma)$ satisfying

$$\|(\widetilde{\gamma}_* - \mathbf{T}_h)^{-1}\mathbf{T}_h\mathbf{u}_h\|_0 = \|(\widetilde{\gamma}_* - \widetilde{\gamma}_{j,h})^{-1}\mathbf{T}_h\mathbf{u}_h\|_0. \tag{3.31}$$

Since $\mathbf{u}_h \in \mathbf{M}_h(\gamma)$ and $\mathbf{u}_H \in \mathbf{V}_H$ are piecewise $\mathbf{H}^1$-functions, using (1.5) in [9] and (2.34) we derive

$$\begin{aligned}
&\|(\widetilde{\gamma}_* - \mathbf{T}_h)^{-1}\mathbf{T}_h(\mathbf{u}_H - \mathbf{u}_h)\|_0 = \|\mathbf{T}_h(\widetilde{\gamma}_* - \mathbf{T}_h)^{-1}(\mathbf{u}_H - \mathbf{u}_h)\|_0 \\
&\quad \le C\|\mathbf{T}_h(\widetilde{\gamma}_* - \mathbf{T}_h)^{-1}(\mathbf{u}_H - \mathbf{u}_h)\|_{1,h} \\
&\quad \le C\|(\widetilde{\gamma}_* - \mathbf{T}_h)^{-1}\|\|\mathbf{u}_H - \mathbf{u}_h\|_0 \\
&\quad \le C|(\widetilde{\gamma}_* - \gamma_h^{-1})^{-1}|\|\mathbf{u}_H - \mathbf{u}_h\|_0 \\
&\quad \le C|\gamma_H - \gamma|^{-1}\|\mathbf{u}_H - \mathbf{u}_h\|_0.
\end{aligned} \tag{3.32}$$

By (3.24) and $\widetilde{\gamma}_* = \frac{1}{\gamma_H}$, the triangle inequality, (3.31) and (3.32), we obtain

$$\|\mathbf{u}'\|_0 = \|(\widetilde{\gamma}_* - \mathbf{T}_h)^{-1}(\widetilde{\gamma}_*\mathbf{T}_h\mathbf{u}_H)\|_0 \ge C\|\frac{1}{\gamma_h - \gamma_H}\mathbf{u}_h\|_0. \tag{3.33}$$

Because $\mathbf{u}^0 \in \mathbf{H}^1(\Omega)$ and $\mathbf{u}^h \in \mathbf{V}_h$ is a piecewise $\mathbf{H}^1$-function, using (1.5) in [9] and (3.28) we have

$$\|\mathbf{u}^h - \mathbf{u}^0\|_0 \le C\|\mathbf{u}^h - \mathbf{u}^0\|_{1,h} \le CH^3. \tag{3.34}$$

Denote $\underline{\widetilde{\boldsymbol{\sigma}}}_h = \mathbf{S}_h(\gamma_h\mathbf{u}^0)$. Combining $\underline{\boldsymbol{\sigma}}^h = \frac{\underline{\boldsymbol{\sigma}}'}{\|\mathbf{u}'\|_0}$, (3.22), $\mathbf{u}^h = \frac{\mathbf{u}'}{\|\mathbf{u}'\|_0}$, (2.35), the triangle inequality, (3.33) and (3.34), we derive

$$\begin{aligned}
\|\underline{\boldsymbol{\sigma}}^h - \underline{\widetilde{\boldsymbol{\sigma}}}_h\|_0 &= \|\frac{\underline{\boldsymbol{\sigma}}'}{\|\mathbf{u}'\|_0} - \mathbf{S}_h(\gamma_h\mathbf{u}^0)\|_0 \\
&\le C\left(\|\frac{\mathbf{u}_H}{\|\mathbf{u}'\|_0}\|_0 + \|\gamma_H\mathbf{u}^h - \gamma_H\mathbf{u}^0\|_0 + \|\gamma_H\mathbf{u}^0 - \gamma_h\mathbf{u}^0\|_0\right) \\
&\le C(|\gamma_h - \gamma_H| + \|\mathbf{u}^h - \mathbf{u}^0\|_0) \\
&\le C(H^2 + h^2).
\end{aligned} \tag{3.35}$$

Combining the triangle inequality, (3.35) and (2.41), we have

$$\|\underline{\boldsymbol{\sigma}}^h - \underline{\boldsymbol{\sigma}}\|_0 \le C(\|\underline{\boldsymbol{\sigma}}^h - \underline{\widetilde{\boldsymbol{\sigma}}}_h\|_0 + \|\underline{\widetilde{\boldsymbol{\sigma}}}_h - \underline{\boldsymbol{\sigma}}\|_0) \le C(H^2 + h), \tag{3.36}$$

i.e., (3.18) holds. Using the similar proof of (3.5), from (3.17), (3.18) and (3.11), we obtain (3.19). The proof is completed.

In the above proof, C is independent of mesh diameters and λ. □

Remark 2. Bi *et al*. in [7] analyzed the regularity estimate for the elastic equations in concave domains. For $\Omega \subset \mathbb{R}^3$, Inzunza *et al*. in [21] presented the regularity estimate. Hence, the conclusions and analysis in this paper can be extended to the case of concave and three-dimensional domains.

4. Numerical experiments

In this section, some numerical experiments will be reported to show our theoretical analysis and the high efficiency of two-grid schemes. In practical computation, we adopt the command ('\') to solve linear equations. Thanks to the package of iFEM [13]. We use MATLAB 2018b on a DELL inspiron15 7510 with 32G memory to solve the discrete mixed eigenvalue problems. In the following we specify the notations appeared in our tables.

H, h: H and h is the mesh size of π_H and π_h, respectively.

$\gamma_{k,h}$: the kth eigenvalue of (2.18)-(2.19) on π_h obtained by using the nonconforming mixed finite element directly.

$\gamma_{Si}^{k,h}(i = 1, 2)$: the kth eigenvalue obtained by Schemes 3.1 and 3.2 on π_h, respectively.

$-$: the calculation cannot continue because the computer runs out of memory.

t(s): The CPU time(s) from the program starting until the calculation results appear.

Example 4.1. Consider the elasticity eigenvalue problem (2.16)-(2.17) with the density $\rho \equiv 1$ on three test domains. When the domain Ω is the unit square $\Omega_S = (0, 1)^2$ we take $\theta = 1$, the regular hexagon Ω_H with side length of 1 take $\theta = 0.08$ and the L-shaped domain $\Omega_L = (0, 1)^2/[\frac{1}{2}, 1]^2$ take $\theta = 1$.

Referring to [29] we set Young′s modulus $E = 3$, the Poisson ratio $\nu = 0.49999$, $0.499999, 0.4999999$ in formula (1.3), respectively, and calculate in three cases of Lamé parameters: Case 1: $\mu = 1.0000$, $\lambda = 49999.3333$; Case 2: $\mu = 1.0000$, $\lambda = 499999.3333$; Case 3: $\mu = 1.0000$, $\lambda = 4999999.3333$, respectively. We compute the first two numerical eigenvalues of (2.16)-(2.17), and list the solutions in Tables 1 - 6. Because the exact eigenvalues of (2.16)-(2.17) are unknown, the formula $ratio(\gamma_{k,h}) = lg|\frac{\gamma_{k,h}-\gamma_{k,\frac{h}{2}}}{\gamma_{k,\frac{h}{2}}-\gamma_{k,\frac{h}{4}}}|/lg2$ is used to calculate the approximate convergence order.

Table 1. The numerical results on Ω_S obtained by direct calculation.

		Case 1			
h	$\gamma_{1,h}$	$ratio(\gamma_{1,h})$	$\gamma_{2,h}$	$ratio(\gamma_{2,h})$	t(s)
$\frac{\sqrt{2}}{8}$	51.6878	2.20	87.4997	2.21	0.06
$\frac{\sqrt{2}}{16}$	52.1999	2.13	91.1013	2.09	0.11
$\frac{\sqrt{2}}{32}$	52.3114	2.11	91.8805	2.05	0.71
$\frac{\sqrt{2}}{64}$	52.3369		92.0642		8.27
$\frac{\sqrt{2}}{128}$	52.3428		92.1086		109.49
$\frac{\sqrt{2}}{256}$	-		-		-
		Case 2			
h	$\gamma_{1,h}$	$ratio(\gamma_{1,h})$	$\gamma_{2,h}$	$ratio(\gamma_{2,h})$	t(s)
$\frac{\sqrt{2}}{8}$	51.6878	2.20	87.5009	2.21	0.06
$\frac{\sqrt{2}}{16}$	52.2000	2.13	91.1024	2.09	0.14
$\frac{\sqrt{2}}{32}$	52.3114	2.11	91.8816	2.05	0.79
$\frac{\sqrt{2}}{64}$	52.3369		92.0653		8.29
$\frac{\sqrt{2}}{128}$	52.3428		92.1097		107.96
$\frac{\sqrt{2}}{256}$	-		-		-
		Case 3			
h	$\gamma_{1,h}$	$ratio(\gamma_{1,h})$	$\gamma_{2,h}$	$ratio(\gamma_{2,h})$	t(s)
$\frac{\sqrt{2}}{8}$	51.6878	2.20	87.5010	2.21	0.06
$\frac{\sqrt{2}}{16}$	52.2000	2.13	91.1025	2.09	0.14
$\frac{\sqrt{2}}{32}$	52.3114	2.11	91.8817	2.05	0.80
$\frac{\sqrt{2}}{64}$	52.3369		92.0654		8.28
$\frac{\sqrt{2}}{128}$	52.3428		92.1098		108.18
$\frac{\sqrt{2}}{256}$	-		-		-

Table 2. The numerical results on Ω_H obtained by direct calculation.

h	$\gamma_{1,h}$	$ratio(\gamma_{1,h})$	$\gamma_{2,h}$	$ratio(\gamma_{2,h})$	t(s)
Case 1					
$\frac{1}{4}$	16.7400	1.79	19.4273	1.22	0.07
$\frac{1}{8}$	17.9836	2.06	28.0543	1.94	0.21
$\frac{1}{16}$	18.3440	2.28	31.7512	2.03	0.85
$\frac{1}{32}$	18.4306		32.7142		6.88
$\frac{1}{64}$	18.4482		32.9498		95.25
$\frac{1}{128}$	-		-		-
Case 2					
h	$\gamma_{1,h}$	$ratio(\gamma_{1,h})$	$\gamma_{2,h}$	$ratio(\gamma_{2,h})$	t(s)
$\frac{1}{4}$	16.7400	1.79	19.4275	1.22	0.07
$\frac{1}{8}$	17.9836	2.06	28.0546	1.94	0.21
$\frac{1}{16}$	18.3440	2.28	31.7515	2.03	0.90
$\frac{1}{32}$	18.4306		32.7146		6.68
$\frac{1}{64}$	18.4483		32.9501		97.70
$\frac{1}{128}$	-		-		-
Case 3					
h	$\gamma_{1,h}$	$ratio(\gamma_{1,h})$	$\gamma_{2,h}$	$ratio(\gamma_{2,h})$	t(s)
$\frac{1}{4}$	16.7400	1.79	19.4275	1.22	0.09
$\frac{1}{8}$	17.9836	2.06	28.0546	1.94	0.20
$\frac{1}{16}$	18.3440	2.28	31.7516	2.03	0.93
$\frac{1}{32}$	18.4306		32.7146		6.91
$\frac{1}{64}$	18.4483		32.9501		100.12
$\frac{1}{128}$	-		-		-

Table 3. The numerical results on Ω_L obtained by direct calculation.

		Case 1			
h	$\gamma_{1,h}$	$ratio(\gamma_{1,h})$	$\gamma_{2,h}$	$ratio(\gamma_{2,h})$	t(s)
$\frac{\sqrt{2}}{8}$	109.566	1.43	134.088	2.17	0.06
$\frac{\sqrt{2}}{16}$	120.998	1.23	144.819	1.94	0.09
$\frac{\sqrt{2}}{32}$	125.229	1.16	147.211	1.86	0.43
$\frac{\sqrt{2}}{64}$	127.036		147.834		4.60
$\frac{\sqrt{2}}{128}$	127.843		148.005		105.93
$\frac{\sqrt{2}}{256}$	-		-		-
		Case 2			
h	$\gamma_{1,h}$	$ratio(\gamma_{1,h})$	$\gamma_{2,h}$	$ratio(\gamma_{2,h})$	t(s)
$\frac{\sqrt{2}}{8}$	109.567	1.43	134.089	2.17	0.06
$\frac{\sqrt{2}}{16}$	120.999	1.23	144.820	1.94	0.09
$\frac{\sqrt{2}}{32}$	125.230	1.16	147.212	1.86	0.46
$\frac{\sqrt{2}}{64}$	127.036		147.835		4.63
$\frac{\sqrt{2}}{128}$	127.844		148.006		107.91
$\frac{\sqrt{2}}{256}$	-		-		-
		Case 3			
h	$\gamma_{1,h}$	$ratio(\gamma_{1,h})$	$\gamma_{2,h}$	$ratio(\gamma_{2,h})$	t(s)
$\frac{\sqrt{2}}{8}$	109.567	1.43	134.089	2.17	0.06
$\frac{\sqrt{2}}{16}$	120.999	1.23	144.820	1.94	0.11
$\frac{\sqrt{2}}{32}$	125.230	1.16	147.212	1.86	0.46
$\frac{\sqrt{2}}{64}$	127.037		147.835		4.59
$\frac{\sqrt{2}}{128}$	127.844		148.007		104.81
$\frac{\sqrt{2}}{256}$	-		-		-

Table 4. The numerical results on Ω_S obtained by Schemes 3.1 and 3.2.

H	h	$\gamma_{S1}^{1,h}$	$\gamma_{S1}^{2,h}$	t(s)	$\gamma_{S2}^{1,h}$	$\gamma_{S2}^{2,h}$	t(s)	$\gamma_{1,h}$	$\gamma_{2,h}$	t(s)
					Case 1					
$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{64}$	52.3397	92.1107	0.82	52.3369	92.0644	0.87	52.3369	92.0641	8.27
$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{128}$	52.3429	92.1104	4.40	52.3428	92.1086	4.39	52.3428	92.1086	109.49
$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{256}$	52.3443	92.1214	46.54	52.3442	92.1196	48.08	-	-	-
$\frac{\sqrt{2}}{32}$	$\frac{\sqrt{2}}{512}$	52.3445	92.1224	374.10	52.3445	92.1223	379.59	-	-	-
					Case 2					
H	h	$\gamma_{S1}^{1,h}$	$\gamma_{S1}^{2,h}$	t(s)	$\gamma_{S2}^{1,h}$	$\gamma_{S2}^{2,h}$	t(s)	$\gamma_{1,h}$	$\gamma_{2,h}$	t(s)
$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{64}$	52.3397	92.1118	0.84	52.3369	92.0655	0.87	52.3369	92.0653	8.29
$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{128}$	52.3430	92.1115	4.50	52.3428	92.1097	4.66	52.3428	92.1097	107.96
$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{256}$	52.344367	92.1225	46.16	52.3442	92.1207	47.49	-	-	-
$\frac{\sqrt{2}}{32}$	$\frac{\sqrt{2}}{512}$	52.3446	92.1235	373.75	52.3446	92.1234	379.20	-	-	-
					Case 3					
H	h	$\gamma_{S1}^{1,h}$	$\gamma_{S1}^{2,h}$	t(s)	$\gamma_{S2}^{1,h}$	$\gamma_{S2}^{2,h}$	t(s)	$\gamma_{1,h}$	$\gamma_{2,h}$	t(s)
$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{64}$	52.3397	92.1119	0.90	52.3369	92.0656	0.89	52.3369	92.0654	8.28
$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{128}$	52.3430	92.1116	4.49	52.3428	92.1098	4.75	52.3428	92.1098	108.18
$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{256}$	52.3444	92.1226	45.41	52.3442	92.1208	47.94	-	-	-
$\frac{\sqrt{2}}{32}$	$\frac{\sqrt{2}}{512}$	52.3446	92.1237	369.71	52.3446	92.1235	380.81	-	-	-

Table 5. The numerical results on Ω_H obtained by Schemes 3.1 and 3.2.

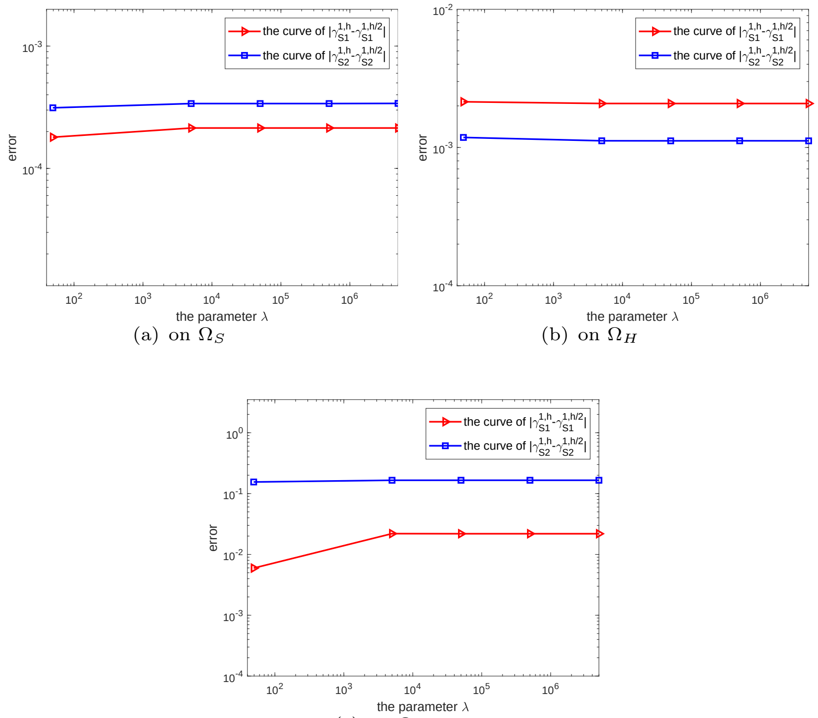
H	h	γ_{S1}^{1}	γ_{S1}^{2}	t(s)	γ_{S2}^{1}	γ_{S2}^{2}	t(s)	$\gamma_{1,h}$	$\gamma_{2,h}$	t(s)
					Case 1					
$\frac{1}{8}$	$\frac{1}{32}$	18.4489	32.8040	0.72	18.4306	32.7172	0.74	18.4306	32.7142	6.88
$\frac{1}{16}$	$\frac{1}{64}$	18.4493	32.9516	4.13	18.4483	32.9498	4.23	18.4483	32.9498	95.25
$\frac{1}{16}$	$\frac{1}{128}$	18.4510	33.0058	25.31	18.4500	33.0046	24.82	-	-	-
$\frac{1}{32}$	$\frac{1}{256}$	18.4489	33.0157	199.80	18.4489	33.0158	208.73	-	-	-
					Case 2					
H	h	γ_{S1}^{1}	γ_{S1}^{2}	t(s)	γ_{S2}^{1}	γ_{S2}^{2}	t(s)	$\gamma_{1,h}$	$\gamma_{2,h}$	t(s)
$\frac{1}{8}$	$\frac{1}{32}$	18.4489	32.8043	0.74	18.4306	32.7175	0.74	18.4306	32.7146	6.68
$\frac{1}{16}$	$\frac{1}{64}$	18.4493	32.9519	4.13	18.4483	32.9501	4.25	18.4483	32.9501	97.70
$\frac{1}{16}$	$\frac{1}{128}$	18.4510	33.0062	23.72	18.4500	33.0047	24.03	-	-	-
$\frac{1}{32}$	$\frac{1}{256}$	18.4489	33.0160	196.92	18.4489	33.0160	204.90	-	-	-
					Case 3					
H	h	$\gamma_{S1}^{1,h}$	$\gamma_{S1}^{2,h}$	t(s)	$\gamma_{S2}^{1,h}$	$\gamma_{S2}^{2,h}$	t(s)	$\gamma_{1,h}$	$\gamma_{2,h}$	t(s)
$\frac{1}{8}$	$\frac{1}{32}$	18.4490	32.8043	0.77	18.4306	32.7175	0.80	18.4306	32.7146	6.91
$\frac{1}{16}$	$\frac{1}{64}$	18.4493	32.9512	4.09	18.4483	32.9501	4.38	18.4483	32.9501	100.12
$\frac{1}{16}$	$\frac{1}{128}$	18.4510	33.0062	23.39	18.4500	33.0047	24.49	-	-	-
$\frac{1}{32}$	$\frac{1}{256}$	18.4489	33.0161	196.23	18.4489	33.0160	202.33	-	-	-

Table 6. The numerical results on Ω_L obtained by Schemes 3.1 and 3.2.

					Case 1					
H	h	$\gamma_{S1}^{1,h}$	$\gamma_{S1}^{2,h}$	t(s)	$\gamma_{S2}^{1,h}$	$\gamma_{S2}^{2,h}$	t(s)	$\gamma_{1,h}$	$\gamma_{2,h}$	t(s)
$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{64}$	128.284	148.087	0.52	127.165	147.838	0.52	127.036	147.834	4.60
$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{128}$	128.062	148.014	3.08	127.847	148.005	3.19	127.843	148.005	105.93
$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{256}$	128.455	148.062	25.99	128.216	148.053	25.92	-	-	-
$\frac{\sqrt{2}}{32}$	$\frac{\sqrt{2}}{512}$	128.433	148.067	146.52	128.381	148.067	144.53	-	-	-
					Case 2					
H	h	$\gamma_{S1}^{1,h}$	$\gamma_{S1}^{2,h}$	t(s)	$\gamma_{S2}^{1,h}$	$\gamma_{S2}^{2,h}$	t(s)	$\gamma_{1,h}$	$\gamma_{2,h}$	t(s)
$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{64}$	128.285	148.088	0.57	127.166	147.839	0.56	127.036	147.835	4.63
$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{128}$	128.063	148.015	3.22	127.848	148.006	3.17	127.844	148.006	107.91
$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{256}$	128.456	148.063	26.15	128.217	148.054	25.88	-	-	-
$\frac{\sqrt{2}}{32}$	$\frac{\sqrt{2}}{512}$	128.434	148.069	144.97	128.382	148.068	145.49	-	-	-
					Case 3					
H	h	$\gamma_{S1}^{1,h}$	$\gamma_{S1}^{2,h}$	t(s)	$\gamma_{S2}^{1,h}$	$\gamma_{S2}^{2,h}$	t(s)	$\gamma_{1,h}$	$\gamma_{2,h}$	t(s)
$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{64}$	128.285	148.088	0.52	127.166	147.839	0.52	127.037	147.833	4.59
$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{128}$	128.063	148.015	3.21	127.848	148.007	3.06	127.844	148.007	104.81
$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{256}$	128.456	148.063	25.95	128.217	148.054	25.74	-	-	-
$\frac{\sqrt{2}}{32}$	$\frac{\sqrt{2}}{512}$	128.434	148.069	146.25	128.382	148.068	145.18	-	-	-

From Tables 1 - 6 we can find that the numerical results obtained by the stabilized nonconforming finite element and Schemes 3.1 and 3.2 are convergent and tend to be stable as λ increases, which means that it is locking free to use the nonconforming finite element and two-grid schemes to solve the nearly incompressible elasticity eigenvalue problem. The results in Tables 1 - 3 show that the convergence order of the first two eigenvalues are approximately equal to 2.00 on Ω_S, 2.00 on Ω_H, and 1.00 on Ω_L, respectively. Comparing Schemes 3.1, 3.2 and direct calculation, we can find that the numerical results obtained by Schemes 3.1 and 3.2 are close to the results by direct calculation. Besides, in Tables 1 - 6, due to the computer memory limitation, direct calculation cannot be performed when the mesh becomes smaller and smaller, but two-grid schemes can still work. The results in Tables 4 - 6 also suggest that it takes significantly much less time to get the numerical results by Schemes 3.1 and 3.2 than directly calculation, and our two-grid discretization schemes have great advantage as the mesh size decreases with respect to lager Lam$\acute{e}$ parameter λ.

In order to illustrate the influence of the Lam$\acute{e}$ parameter λ on eigenvalues, we plot the error curves of approximations for the first two eigenvalues of (2.16)-(2.17) by taking $\mu = 1.0067,\ \lambda = 49.3289$; $\mu = 1.0001,\ \lambda = 4999.3333$; $\mu = 1.0000,\ \lambda = 49999.3333$; $\mu = 1.0000,\ \lambda = 499999.3333$; $\mu = 1.0000,\ \lambda = 4999999.3333$ while $h = \frac{\sqrt{2}}{256}$ on Ω_S and Ω_L, $h = \frac{1}{128}$ on Ω_H, respectively. Due to the exact eigenvalues are unknown, we adopt the formulas $|\gamma_{S1}^{k,h} - \gamma_{S1}^{k,\frac{h}{2}}|$ and $|\gamma_{S2}^{k,h} - \gamma_{S2}^{k,\frac{h}{2}}|$ to denote the "error" of the eigenvalue $\gamma_{S1}^{k,h}$ and $\gamma_{S2}^{k,h}$, respectively, and the "error" curves are depicted in Figures 1 - 2. From Figures 1 - 2 we can see that the "error" of the numerical results tend to be stable as λ increases, which means that the stabilized nonconforming finite element method and our schemes are locking free.



(a) on Ω_S (b) on Ω_H

(c) on Ω_L

Figure 1. Error curves of the first eigenvalue.

References

[1] A. Andreev, R. Lazarov and M. Racheva, *Postprocessing and higher order convergence of the mixed finite element approximations of biharmonic eigenvalue problems*, Am. J. Comput. Appl. Math., 2005, 182, 333–349.

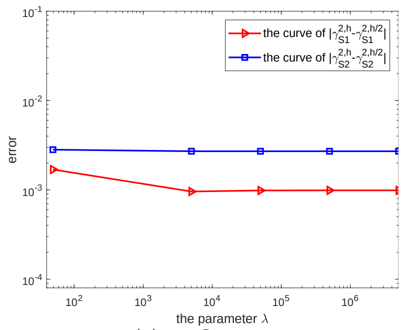
[2] D. Arnold, J. Douglas and C. Gupta, *A family of higher order mixed finite element methods for plane elasticity*, Numer. Math., 1984, 45, 1222.

[3] I. Babuška and J. Osborn, *Eigenvalue problems, in: P.G. Ciarlet, J.L. Lions (Eds.), Finite Element Methods (Part 1), Handbook of Numerical Analysis, vol. 2*, Elsevier Science Publishers, North-Holand, 1991.

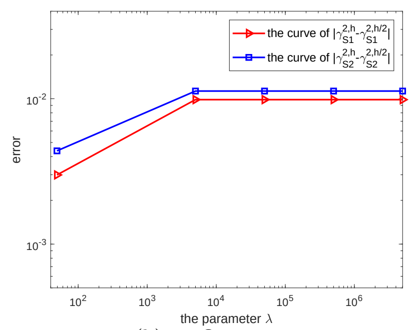
[4] I. Babuška and M. Suri, *Locking effects in the finite element approximation of elasticity problems*, Numer. Math., 1992, 62, 439–463.

[5] I. Babuška and M. Suri, *On locking and robustness in the finite element method*, SIAM J. Numer. Anal., 1992, 29, 1261–1293.

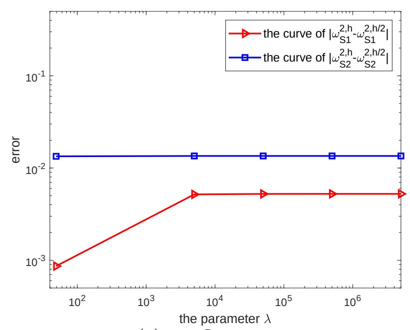
[6] F. Bertrand, D. Boffi and R. Ma, *An adaptive finite element scheme for the*



(a) on Ω_S



(b) on Ω_H



(c) on Ω_L

Figure 2. Error curves of the second eigenvalue.

hellinger-reissner elasticity mixed eigenvalue problem, Comput. Methods Appl. Math., 2021, 21(3), 501–512.

[7] H. Bi, X. Zhang and Y. Yang, *The nonconforming crouzeix-raviart element approximations and two-grid discretizations for the elastic eigenvalue problem*, J. Comput. Math., 2023, 41(6), 1041–1063.

[8] D. Boffi, *Finite element approximation of eigenvalue problems*, Acta Numer., 2010, 19, 1–120.

[9] S. Brenner, *Poincaré-friedrichs inequalities for piecewise h^1 functions*, SIAM J. Numer. Anal., 2003, 41, 306–324.

[10] S. Brenner and L. Scoot, *The mathematical theory of finite element methods, 3rd ed.*, Springer, New York, 2010.

[11] S. Brenner and L. Sung, *Linear finite element methods for planar linear elasticity*, Math. Comp., 1992, 59(200), 321–338,.

[12] J. Chen, Y. Xu and J. Zou, *An adaptive inverse iteration for maxwell eigenvalue problem based on edge elements*, J. Comput. Phys., 2010, 229, 2649–2658.

[13] L. Chen, *ifem: An innovative finite element methods package in matlab. technical report*, 2009. University of California at Irvine.

[14] M. Crouzeix and P. Raviart, *Conforming and nonconforming finite element methods for solving the stationary stokes equations*, RAIRO Anal. Numer., 1973, 3, 33–75.

[15] X. Dai and A. Zhou, *Three-scale finite element discretizations for quantum eigenvalue problems*, SIAM J. Numer. Anal., 2008, 46, 295–324.

[16] B. Gong, J. Han, J. Sun and Z. Zhang, *A shifted-inverse adaptive multigrid method for the elastic eigenvalue problem*, Commun. Comput. Phys., 2019, 27(1), 251–273.

[17] J. Han, Z. Zhang and Y. Yang, *A new adaptive mixed finite element method based on residual type a posterior error estimates for the stokes eigenvalue problem*, Numer. Methods Partial Differ. Equ., 2015, 31(1), 31–53.

[18] P. Hansbo and M. Larson, *Discontinuous galerkin and the crouzeix-raviart element: Application to elasticity*, M2AN Math. Model Numer. Anal., 2003, 37(1), 63–72.

[19] X. Hu and X. Cheng, *Corrigendum to: "acceleration of a two-grid method for eigenvalue problems"*, Math. Comput., 2015, 84, 2701–2704.

[20] X. Huang and J. Huang, *The compact discontinuous galerkin method for nearly incompressible linear elasticity*, J. Sci. Comput., 2013, 56, 291–318.

[21] D. Inzunza, F. Lepe and G. Rivera, *Displacement-pseudostress formulation for the linear elasticity spectral problem*, Numer. Methods Partial Differential Eq., 2023, 39, 1996–2017.

[22] K. Kim, *Analysis of some low-order nonconforming mixed finite elements for linear elasticity problem*, Numer. Methods Partial Differ. Equ., 2006, 22, 638–660.

[23] J. Lee and H. Kim, *Analysis of a staggered discontinuous galerkin method for linear elasticity*, J. Sci. Comput., 2016, 66, 625–649.

[24] M. Li, D. Shi and Z. Li, *The stabilized mixed finite element scheme of elasticity problem*, Comput. Appl. Math., 2017, 37(3), 2588–2604.

[25] S. Meddahi, D. Mora and R. Rodríguez, *Finite element spectral analysis for the mixed formulation of the elasticity equations*, SIAM J. Numer. Anal., 2013, 51(2), 1041–1063.

[26] M. Morley, *A family of mixed finite elements for linear elasticity*, Numer. Math., 1989, 55, 633–666.

[27] A. Russo, *Eigenvalue approximation by mixed non-conforming finite element methods*, Calcolo, 2014, 51(4), 563–597.

[28] M. Vogelius, *An analysis of the p-version of the finite element method for nearly incompressible materials. uniformly valid, optimal order estimates*, Numer. Math., 1983, 41, 39–53.

[29] F. Wang, S. Wu and J. Xu, *A mixed discontinuous galerkin method for linear elasticity with strongly imposed symmetry*, J. Sci. Comput., 2020, 83(2), 1–17.

[30] J. Xu and A. Zhou, *A two-grid discretization scheme for eigenvalue problems*, Math. Comput., 1999, 70(233), 17–26.

[31] J. Xu and A. Zhou, *Two-grid discretization scheme for eigenvalue problems*, Math. Comput., 2001, 70, 17–25.

[32] Y. Yang and H. Bi, *A two-grid discretization scheme based on the shifted-inverse power method*, SIAM J. Numer. Anal., 2011, 49(4), 1602–1624.

[33] Y. Yang, H. Bi, J. Han and Y. Yu, *The shifted-inverse iteration based on the multigrid discretizations for eigenvalue problems*, SIAM J. Sci. Comput., 2015, 37(6), A2583–A2606.

[34] Y. Yang and Z. Chen, *The order-preserving convergence for spectral approximation of self-adjoint completely continuous operators*, Sci. China Ser. A, 2008, 51, 1232–1242.

[35] B. Zhang and J. Zhao, *A mixed formulation of stabilized nonconforming finite element method for linear elasticity*, Adv. Appl. Math. Mech., 2020, 12(1), 278–300.

[36] X. Zhang, Y. Yang and Y. Zhang, *A locking-free shifted inverse iteration based on multigrid discretization for the elastic eigenvalue problem*, Math. Methods Appli. Sci., 2021, 44(7), 5821–5838.

[37] Y. Zhang, H. Bi and Y. Yang, *The two-grid discretization of ciarlet-raviart mixed method for biharmonic eigenvalue problems*, Appl. Numer. Math., 2019, 138, 94–113.

[38] J. Zhou, X. Hu, L. Zhong et al., *Two-grid methods for maxwell eigenvalue problems*, SIAM J. Numer. Anal., 2014, 52, 2027–2047.