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RADIAL SOLUTION OF ASYMPTOTICALLY LINEAR ELLIPTIC EQUATION WITH MIXED BOUNDARY VALUE IN ANNULAR DOMAIN*

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Abstract In this paper, we study nonlinear elliptic equation with mixed boundary value condition in annular domain. It is assumed that the nonlinearity is asymptotically linear and depends on the derivative term. Some results on the existence of solution are established by nonlinear analysis methods.

Keywords Mixed boundary value, annular domain, radial solution, gradient term, iterative method.

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1. Introduction

In this paper, we study the following nonlinear elliptic equation with gradient term in annular domain

$$\begin{cases} -\Delta u = f(|x|, u, \dfrac{x}{|x|} \cdot \nabla u) & \text{in } B_2 \backslash \overline{B_1}, \\ u\Big|_{\partial B_1} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial B_2} = 0, \end{cases} \tag{M}$$

where $B_i := \{x \in \mathbb{R}^n : |x| < i\}$, $i = 1, 2$, $n > 2$, $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. $f = f(r, s, \xi)$ is continuous and C^1 with respect to (s, ξ). $\partial/\partial\nu$ denotes the outward normal derivative.

Because of the wide interest in mathematics and applied mathematics, the existence of solution of elliptic equation in annular domains has been investigated by many authors, see [1, 2, 6, 8–10, 13–19, 24–26, 30] and the references cited therein. When $f(r, s, \xi) = s^p$, the equation is known as Lane-Emden equation. In [18], Ni and Nussbaum established numerous results concerning the uniqueness and nonuniqueness for positive radial solution, when the domain is a ball or an annulus. When the domain Ω is star-shaped and $f(r, s, \xi) = s^{\frac{n+2}{n-2}}$, the well-known Pohozaev identity implies that the problem has no solution (see [20]). Brezis and Nirenberg [4] proved that the perturbation of lower-term can reverse this situation. If Ω is an annulus, Pohozaev theorem does not work any more since the annulus is not a star-shaped domain. Therefore, it is possible that the constraints for the growth of f can

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be removed. Provided $f(r,s,\xi) = -s + s^{2N+1}$, Coffman [5] pointed out that there are many rotationally nonequivalent positive solutions and the number of these solutions is unbounded as $r \to +\infty$. The case $f(r,s,\xi) = g(r)h(s)$ was considered by Lin [15] and the case $f(r,s,\xi) = \lambda k(r)g(s)$ was studied by Wang [24]. Uniqueness of solutions was also studied when $f(r,s,\xi) = f(s)$ (see [17]) or $f(r,s,\xi) = s^p + s^q$ (see [30]). Recently, Dong and Wei [9] studied the existence of radial solution for elliptic equation with Dirichlet boundary value condition.

However, all of the above papers are devoted to the superlinear problems. In this paper, we focus on the asymptotically linear equation with mixed boundary value condition. There are some known papers related to asymptotically linear problems, such as [12,21] for second order elliptic equation, [29] for non-local elliptic equation, [27] for fourth-order elliptic equation and so on. For Sturm-Liouville equation involving p-Laplacian with mixed boundary condition, we refer to [22].

To state our main results, we introduce the following assumptions:

(F0) For $(r,\xi) \in [1,2] \times \mathbb{R}$, $f(r,0,\xi)$ is uniformly bounded and $f(r,0,\xi) \neq 0$;

(F1) There exists $k \in \mathbb{Z}^+$, and two continuous functions $\underline{\alpha}(r)$, $\overline{\alpha}(r)$, such that either of the following holds uniformly for $(r,s,\xi) \in [1,2] \times \mathbb{R} \times \mathbb{R}$:

i. $$(k-\frac{1}{2})^2\pi^2(n-2)^2C_n^2 < \underline{\alpha}(r) \leqslant r^{2n-2}f_s(r,s,\xi) \leqslant \overline{\alpha}(r) < k^2\pi^2(n-2)^2C_n^2;$$

ii. $$k^2\pi^2(n-2)^2C_n^2 < \underline{\alpha}(r) \leqslant r^{2n-2}f_s(r,s,\xi) \leqslant \overline{\alpha}(r) < (k+\frac{1}{2})^2\pi^2(n-2)^2C_n^2,$$

where
$$C_n = \frac{2^{n-2}}{2^{n-2}-1}$$
is a constant.

The first main result of this paper is given as follows.

Theorem 1.1. *Assume that (F0)-(F1) hold. Then equation (M) has at least one nontrivial radial solution.*

Remark 1.1. We give a concrete example to illustrate the above result. Let $n = 3$. Consider the following boundary value problem:

$$\begin{cases} -\Delta u = \dfrac{4}{|x|^4}(k+\varepsilon)^2\pi^2 u + h(\dfrac{x}{|x|}\cdot\nabla u) \quad \text{in } B_2\backslash\overline{B_1}, \\ u\Big|_{\partial B_1} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial B_2} = 0, \end{cases}$$

where $0 < |\varepsilon| < \frac{1}{2}$, $k \in \mathbb{Z}^+$, h is C^1 continuous and there exist constants $m_1, m_2 > 0$ such that $0 < m_1 < h(\zeta) < m_2$ for any $\zeta \in \mathbb{R}$. It is easy to know (F0) is satisfied. Besides, $\varepsilon < 0$ and $\varepsilon > 0$ correspond to the case i and ii of (F1), respectively. Theorem 1.1 implies that the above problem has at least one radial solution.

Some other asymptotically linear cases can also be considered with the following assumptions:

(F2) $f(r,0,\xi)=0$, for all $r\in[1,2],\xi\in\mathbb{R}$;

(F3)
$$\begin{aligned}0&\leqslant \liminf_{s\to 0} r^{2n-2}f_s(r,s,\xi)\leqslant \limsup_{s\to 0} r^{2n-2}f_s(r,s,\xi)<\frac{\pi^2(n-2)^2C_n^2}{4}\\&<\liminf_{|s|\to+\infty} r^{2n-2}f_s(r,s,\xi)\leqslant \limsup_{|s|\to+\infty} r^{2n-2}f_s(r,s,\xi)<+\infty\end{aligned}$$

uniformly for $(r,\xi)\in[1,2]\times\mathbb{R}$;

(F4) there exists $M_0>0$, such that for any $r\in[1,2],s\in\mathbb{R},\xi\in\mathbb{R}$,

$$\left|\frac{f(r,s,\xi)}{s}\right|\leqslant M_0;$$

(F5) f satisfies the local Lipschitz condition: there exist constants L and K, such that

$$r^{2n-2}|f_s(r,s,\xi)|\leqslant L(n-2)^2C_n^2$$

and

$$r^{n-1}|f_\xi(r,s,\xi)|\leqslant K(n-2)C_n$$

for any $r\in[1,2],|s|\leqslant\bar\rho_1,|\xi|\leqslant\bar\rho_2$, where $\bar\rho_1$, $\bar\rho_2$ are positive constants, which will be determined later. Moreover,

$$L<\frac{\pi^2}{4},\quad K<\frac{\pi}{2}-\frac{2L}{\pi}.$$

Theorem 1.2. *Assume that (F2)-(F5) hold. Then equation (M) has at least two nontrivial radial solutions. One of them is positive, and the other one is negative.*

Remark 1.2. Consider the case $n=3$, $f(r,s,\xi)=\dfrac{1}{r^4}h(s)(1+\tau\gamma(\xi))$, where $|\tau|<\dfrac12$, $\gamma\in C^1(\mathbb{R}^n)$, $|\gamma(\xi)|<1$,

$$h(s)=\begin{cases}\dfrac{\pi^2}{4}(8s+6\Lambda+3), & s\leqslant-\Lambda-1;\\ -\dfrac{\pi^2}{4}(3(s+\Lambda)^2-2s), & -\Lambda-1<s<-\Lambda;\\ \dfrac{\pi^2}{2}s, & |s|\leqslant\Lambda;\\ \dfrac{\pi^2}{4}(3(s-\Lambda)^2+2s), & \Lambda<s<\Lambda+1;\\ \dfrac{\pi^2}{4}(8s-6\Lambda-3), & s\geqslant\Lambda+1.\end{cases}$$

Obviously, h is a C^1 function. It is easy to know that for τ small enough and Λ big enough, all assumptions of Theorem 1.2 are satisfied.

The approaches of the present paper are based on some methods of nonlinear analysis. We derive an equivalent ordinary differential equation for (M), and then, deal with the corresponding ordinary differential equation. Some fixed point theorems are used and some iterative methods are also introduced to overcome the difficulty caused by the gradient term. Schauder' s fixed point theorem is essential to the proof of Theorem 1.1. Meanwhile, Mountain pass theorem and iterative technique are applied to prove Theorem 1.2.

This paper is organized as follows. In Section 2, we derive an equivalent ordinary differential equation and introduce some function spaces. Section 3 is devoted to proving Theorem 1.1. We first study the special problem provided that the nonlinearity does not contain the gradient term. Then, the general case involving gradient term is considered. To prove the second main theorem, we apply Mountain pass theorem to establish existence of solutions for the non-gradient problem in Section 4. Finally, the proof of Theorem 1.2 is given in Section 5.

2. Preliminaries and Equivalent ODE

For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, denote $r = |x|$. Then

$$\begin{aligned}
&r = \sqrt{x_1^2 + \cdots + x_n^2},\\
&\nabla u = (\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}) = (\frac{\mathrm{d}u}{\mathrm{d}r}\frac{x_1}{r}, \cdots, \frac{\mathrm{d}u}{\mathrm{d}r}\frac{x_n}{r}) = \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}x,\\
&\frac{x}{|x|} \cdot \nabla u = \frac{x}{r} \cdot \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}x = \frac{1}{r^2}\frac{\mathrm{d}u}{\mathrm{d}r}|x|^2 = \frac{\mathrm{d}u}{\mathrm{d}r},\\
&\Delta u = \mathrm{div}(\nabla u) = \mathrm{div}(\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}x) = \frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{n-1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}.
\end{aligned}$$

Hence, the elliptic problem (M) is equivalent to the following second order differential equation

$$-u''(r) - \frac{n-1}{r}u'(r) = f(r, u(r), u'(r)) \tag{2.1}$$

with mixed boundary condition

$$u(1) = u'(2) = 0.$$

Let $t = t(r)$, which will be determined later. Then (2.1) implies

$$-(t'(r))^2 u''(t) - t''(r)u'(t) - \frac{n-1}{r}t'(r)u'(t) = f(r(t), u(t), u'(t)t'(r)). \tag{2.2}$$

To make the gradient term in the left of (2.2) vanish, we choose $t(r)$ such that

$$t''(r) + \frac{n-1}{r}t'(r) = 0 \tag{2.3}$$

and

$$t(1) = 0, \quad t(2) = 1. \tag{2.4}$$

Let

$$t(r) = C_n\Big(1 - \frac{1}{r^{n-2}}\Big), \tag{2.5}$$

where

$$C_n = \frac{2^{n-2}}{2^{n-2} - 1}.$$

Therefore,

$$t'(r) = \frac{(n-2)C_n}{r^{n-1}}. \tag{2.6}$$

Meanwhile, (2.5) implies

$$r = \Big(1 - \frac{t}{C_n}\Big)^{-\frac{1}{n-2}},$$

and

$$t'(r) = (n-2)C_n\Big(1 - \frac{t}{C_n}\Big)^{\frac{n-1}{n-2}}. \tag{2.7}$$

From (2.2) we have

$$-u''(t) = \frac{f(r, u(t), u'(t)t'(r))}{(t'(r))^2}. \tag{2.8}$$

Let $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$g(t, s, \eta) := \frac{(r(t))^{2n-2}}{(n-2)^2(C_n)^2} f\Big((1 - \frac{t}{C_n})^{-\frac{1}{n-2}}, s, (n-2)C_n(1 - \frac{t}{C_n})^{\frac{n-1}{n-2}}\eta\Big).$$

It follows from (2.6) that

$$g(t, s, \eta) = \frac{(r(t))^{2n-2}}{(n-2)^2(C_n)^2} f(r(t), s, \frac{\eta}{r'(t)}),$$
$$g_s(t, s, \eta) = \frac{(r(t))^{2n-2}}{(n-2)^2(C_n)^2} f_s(r(t), s, \frac{\eta}{r'(t)}),$$

and

$$g_\eta(t, s, \eta) = \frac{(r(t))^{n-1}}{(n-2)C_n} f_\xi(r(t), s, \frac{\eta}{r'(t)}).$$

Since

$$g(t, u, u'(t)) = \frac{r^{2n-2}}{(n-2)^2C_n^2} f(r, u, u'(r)),$$

(M) becomes the following problem

$$\begin{cases} -u''(t) = g(t, u(t), u'(t)), \\ u(0) = u'(1) = 0. \end{cases} \tag{M$_{ODE}$}$$

From (F0)-(F5), g satisfies the following conditions:

(G0) $g(t, 0, \eta)$ is uniformly bounded for $(t, \eta) \in [0, 1] \times \mathbb{R}$ and $g(t, 0, \eta) \neq 0$;

(G1) There exists $k \in \mathbb{Z}^+$ and two continuous functions $\underline{\beta}(t)$ and $\overline{\beta}(t)$, such that either of the following holds uniformly for $(t, s, \eta) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$:

i. $(k - \frac{1}{2})^2\pi^2 < \underline{\beta}(t) \leqslant g_s(t, s, \eta) \leqslant \overline{\beta}(t) < k^2\pi^2$;

ii. $k^2\pi^2 < \underline{\beta}(t) \leqslant g_s(t, s, \eta) \leqslant \overline{\beta}(t) < (k + \frac{1}{2})^2\pi^2$;

(G2) $g(t,0,\eta)=0$, for all $t\in[0,1],\eta\in\mathbb{R}$;

(G3)
$$0\leqslant\liminf_{s\to0} g_s(t,s,\eta)\leqslant\limsup_{s\to0} g_s(t,s,\eta)<\frac{\pi^2}{4};$$
$$<\liminf_{|s|\to+\infty} g_s(t,s,\eta)\leqslant\limsup_{|s|\to+\infty} g_s(t,s,\eta)<\ +\infty;$$

uniformly for $(t,\eta)\in[0,1]\times\mathbb{R}$;

(G4) there exists $M>0$, such that for any $t\in[0,1],s\in\mathbb{R},\eta\in\mathbb{R}$,
$$|\frac{g(t,s,\eta)}{s}|\leqslant M;$$

(G5) g satisfies local Lipschitz condition: there exist constants L and K, such that
$$|g_s(t,s,\eta)|\leqslant L$$
and
$$|g_\eta(t,s,\eta)|\leqslant K$$
for any $t\in[0,1],|s|\leqslant\rho_1,|\eta|\leqslant\rho_2$, where ρ_1, ρ_2 are positive constants, related to $\bar{\rho}_1$, $\bar{\rho}_2$ in (F5), which will be determined later in Lemma 4.7. Moreover,
$$L<\frac{\pi^2}{4},\quad K<\frac{\pi}{2}-\frac{2L}{\pi}.$$

Now we introduce the working spaces. Define $I:=(0,1)$. Let $C^1(I)$ be the space of continuously differentiable functions in I, and
$$C^1_M(I):=\{u\in C^1(I),u(0)=u'(1)=0\},$$
equipped with the norm
$$\|u\|=\max_{t\in[0,1]}|u(t)|+\max_{t\in[0,1]}|u'(t)|. \tag{2.9}$$
It is easy to know that $C^1(I)$ and $C^1_M(I)$ are Banach spaces. Denote by $H^1_M(I)$ the closure of $C^1_M(I)$ in Hilbert space $H^1(I)$ equipped with the scalar product of $H^1(I)$. Since $C^1_M(I)$ is densely imbedded into $H^1_M(I)$, $H^1_M(I)$ is the completion of $C^1_M(I)$ by H^1 norm. Hence, $H^1_M(I)$ is also a Hilbert space equipped with the scalar product of $H^1(I)$.

By a standard argument, we know that the eigenvalue problem
$$\begin{cases}-u''=\lambda u,\\ u(0)=u'(1)=0\end{cases} \tag{2.10}$$
possesses a class of eigenvalues $\{\lambda_k\}$, where
$$\lambda_k=(k-\frac{1}{2})^2\pi^2,\quad k=1,2,\cdots.$$

It is well known that
$$||u||_{H^1}:=(\int_I|u'(t)|^2\mathrm{d}t)^{\frac{1}{2}}$$

is an equivalent norm in $H^1(I)$. Notice that $\lambda_1 = \dfrac{\pi^2}{4}$, and the corresponding eigenfunction of λ_1 is denoted by φ_1, which is positive in I. Moreover, from

$$\lambda_1 = \inf_{u\in H^1_M(I), u\neq 0} \frac{||u||^2_{H^1}}{||u||^2_{L^2}},$$

we know

$$||u||_{L^2} \leqslant \frac{2}{\pi}||u||_{H^1}, \quad \forall\ u \in H^1_M(I). \tag{2.11}$$

3. Proof of Theorem 1.1

In this section, we first consider some special cases, where the nonlinearities do not contain the gradient terms. The similar argument can also be found in [28]. Consider the following problem:

$$\begin{cases} -u'' = h(t,u), \\ u(0) = u'(1) = 0, \end{cases} \tag{3.1}$$

where h: $[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and C^1 with respect to u.

The following comparison theorem is known as Theorem of Strum-Picone (see [23]), which describes the location of the zero points of nontrivial solution.

Lemma 3.1. *Let $x = x(t)$ and $y = y(t)$ be the solutions of equations*

$$x''(t) + P(t)x = 0 \tag{3.2}$$

and

$$y''(t) + Q(t)y = 0,$$

respectively. Assume that there exist t_1 and t_2, $t_1 < t_2$, such that $y(t_1) = y(t_2) = 0$, and $P(t) \geqslant Q(t), P(t) \not\equiv Q(t), t \in [t_1, t_2]$. Then $x(t)$ has a zero point in (t_1, t_2).

Lemma 3.2. *Assume that $m^2\pi^2 \leqslant P(t) \leqslant M^2\pi^2$, $P(t) \not\equiv m^2\pi^2, M^2\pi^2$, $t \in [0,1]$, where m, M are positive constants. Then for any successive zero points of $x(t)$, denote by t_1 and t_2, $0 \leqslant t_1 < t_2 \leqslant 1$, the following holds:*

$$\frac{1}{M} < t_2 - t_1 < \frac{1}{m}.$$

Proof. See [9]. □

The following lemma is essential to prove the existence of sulution concerning (3.1).

Lemma 3.3. *Assume that either of the following holds:*

i. $(k-\frac{1}{2})^2\pi^2 \leqslant P(t) \leqslant k^2\pi^2, P(t) \not\equiv (k-\frac{1}{2})^2\pi^2, k^2\pi^2;$

ii. $k^2\pi^2 \leqslant P(t) \leqslant (k+\frac{1}{2})^2\pi^2, P(t) \not\equiv k^2\pi^2, (k+\frac{1}{2})^2\pi^2.$

Then problem

$$\begin{cases} -x'' = P(t)x, \\ x(0) = x'(1) = 0, \end{cases}$$

only has the trivial solution $x(t) \equiv 0$.

Proof. Assume that there exists a solution $x(t) \neq 0$ satisfying $x(0) = x'(1) = 0$. To prove the lemma we make some extensions by

$$\bar{x}(t) = \begin{cases} x(t), & t \in [0,1]; \\ x(2-t), & t \in (1,2], \end{cases}$$

and

$$\bar{P}(t) = \begin{cases} P(t), & t \in [0,1]; \\ P(2-t), & t \in (1,2]. \end{cases}$$

Hence, it is easy to check that

$$-\bar{x}'' = \bar{P}(t)\bar{x}, \quad \bar{x}(0) = \bar{x}(2) = 0.$$

For Case 1, we notice that $y(t) = \sin(k-\frac{1}{2})\pi t$ is a solution of $y''+(k-\frac{1}{2})^2\pi^2 y = 0$. Now we compare $\bar{x}(t)$ with $y(t)$ by Lemma 3.2. Obviously, the zeros of $y(t)$ in $[0,2]$ are $t_i = \frac{2i}{2k-1}, 0 \leqslant i \leqslant 2k-1, i \in \mathbb{Z}$. Hence, $y(t)$ has $2k$ zero points in $[0,2]$, which implies $\bar{x}(t)$ has at least $2k-1$ zero points in $(0,2)$. Since $\bar{x}(0) = \bar{x}(2) = 0$, we know that $\bar{x}(t)$ has at least $2k+1$ zero points in $[0,2]$. Denote by $\bar{t}_1, \cdots, \bar{t}_{2k+1}$ the zero points of $\bar{x}(t)$ in $[0,2]$ such that $0 = \bar{t}_1 \leqslant \bar{t}_2 \leqslant \cdots \leqslant \bar{t}_{2k+1} = 2$. For any successive zeros $\bar{t}_i, \bar{t}_{i+1} \in [0,2]$, Lemma 3.2 implies that

$$\bar{t}_{i+1} - \bar{t}_i > \frac{1}{k}, \quad i = 1, \cdots, 2k.$$

Then, we get

$$2 - 0 = \bar{t}_{2k+1} - \bar{t}_1 = \sum_{i=1}^{2k} (\bar{t}_{i+1} - \bar{t}_i) > 2k \cdot \frac{1}{k} = 2,$$

which leads to a contradiction.

For Case 2, we notice that $y(t) = \sin k\pi t$ is a solution of $y'' + k^2\pi^2 y = 0$. Then the zeros of $y(t)$ in $[0,2]$ are $t_i = \frac{i}{k}, 0 \leqslant i \leqslant 2k, i \in \mathbb{Z}$. Hence, $y(t)$ has $2k+1$ zero points in $[0,2]$, which implies $\bar{x}(t)$ has at least $2k$ zero points in $(0,2)$. Since $\bar{x}(0) = \bar{x}(2) = 0$, we know that $\bar{x}(t)$ has at least $2k+2$ zero points in $[0,2]$. Denote $\bar{t}_1, \cdots, \bar{t}_{2k+2}$ as the zero points of $\bar{x}(t)$ in $[0,2]$ such that $0 = \bar{t}_1 \leqslant \bar{t}_2 \leqslant \cdots \leqslant \bar{t}_{2k+2} = 2$. For any successive zeros $\bar{t}_i, \bar{t}_{i+1} \in [0,2]$, Lemma 3.2 implies that

$$\bar{t}_{i+1} - \bar{t}_i > \frac{1}{k+\frac{1}{2}}, \quad i = 1, \cdots, 2k+1.$$

Then, we get

$$2 - 0 = \bar{t}_{2k+2} - \bar{t}_1 = \sum_{i=1}^{2k+1} (\bar{t}_{i+1} - \bar{t}_i) > (2k+1) \cdot \frac{1}{k+\frac{1}{2}} = 2,$$

which is also a contradiction. $\square$

Remark 3.1. It should be pointed out that this result is quite different from the result for Dirichlet problem. For Dirichlet problem, it is well-known that when $P(t)$ locates between two successive eigenvalues, the linear equation only has trivial solution. This difference is mainly owing to the fact that Dirichlet condition ensures the right endpoint is also a zero point of the solution, so more accurate estimate about the distance between two zero points can be obtained. However, the same argument can not be applied to mixed boundary value problem, because the right endpoint 1 is not a zero point of the solution any more. We use the extension to treat the problem as a Dirichlet problem in $[0, 2]$. Actually, if the assumptions in the lemma are replaced by $(k - \frac{1}{2})^2\pi^2 \leqslant P(t) \leqslant (k + \frac{1}{2})^2\pi^2$, $P(t)$ may cross an eigenvalue of Dirichlet problem in $[0, 2]$, which can not ensure the result. For example, if $k = 1$, $\frac{\pi^2}{4} \leqslant P(t) \leqslant \frac{9\pi^2}{4}$, then $P(t)$ may cross π^2, which is an eigenvalue of Dirichlet problem in $[0, 2]$.

The following lemma ensures the existence and uniqueness for problem (3.1).

Lemma 3.4. *There exist two continuous functions $\underline{\beta}(t)$ and $\overline{\beta}(t)$, such that either of the following holds uniformly:*

$$i. \quad (k - \frac{1}{2})^2\pi^2 < \underline{\beta}(t) \leqslant h_u(t, u) \leqslant \overline{\beta}(t) < k^2\pi^2;$$

$$ii. \quad k^2\pi^2 < \underline{\beta}(t) \leqslant h_u(t, u) \leqslant \overline{\beta}(t) < (k + \frac{1}{2})^2\pi^2.$$

Then the equation (3.1) *has a unique solution.*

To prove the above lemma, we first show the following results.

Lemma 3.5. *Assume that all of the assumptions of Lemma 3.4 hold. The equation* (3.1) *has at most one solution.*

Proof. Assume that $u_1(t), u_2(t)$ are the solutions of (3.1), namely,

$$-u_1'' = h(t, u_1), \quad -u_2'' = h(t, u_2).$$

Let $u = u_1 - u_2$. Hence,

$$-u'' = -u_1'' + u_2'' = h(t, u_1) - h(t, u_2) = h_u(t, u_2 + \theta(u_1 - u_2))u, \quad 0 \leqslant \theta \leqslant 1,$$

and $u(0) = u'(1) = 0$. According to Lemma 3.3, we know $u \equiv 0$. □

Now, we consider the existence of solutions for equation (3.1). Rewrite equation (3.1) in the following form:

$$-u'' = h(t, u) - h(t, 0) + h(t, 0) = \Big(\int_0^1 h_u(t, \theta u)\mathrm{d}\theta\Big)u + h(t, 0).$$

For any $u \in C_M^1(I)$, from Lemma 3.3 we know that the linear boundary value problem

$$-v'' = \Big(\int_0^1 h_u(t, \theta u)\mathrm{d}\theta\Big)v + h(t, 0), \quad v(0) = v'(1) = 0 \tag{3.3}$$

has a unique solution $v \in C_M^1(I)$.

Define operator $P : C_M^1(I) \to C_M^1(I)$. For $u \in C_M^1(I)$,

$$P[u](t) = v(t)$$

is the unique solution of equation (3.3). Then the existence of solution is equivalent to the existence of fixed point of P in $C^1_M(I)$.

Lemma 3.6. *The operator P is continuous.*

Proof. For any sequence $\{u_n\} \subset C^1_M(I)$ satisfying $u_n \to u_0$ as $n \to \infty$, let $v_n = Pu_n$, then we have

$$-v_n'' = \Big(\int_0^1 h_u(t, \theta u_n)\mathrm{d}\theta\Big) v_n + h(t, 0). \tag{3.4}$$

Claim that $\{v_n\}$ is bounded in $C^1_M(I)$. If not, $\|v_n\| \to \infty$. Let $\omega_n = v_n/\|v_n\|$. Then $\{\omega_n\} \subset C^1_M(I)$, $||\omega_n|| = 1$, and

$$-\omega_n'' = \Big(\int_0^1 h_u(t, \theta u_n)\mathrm{d}\theta\Big)\omega_n + \frac{h(t, 0)}{\|v_n\|}. \tag{3.5}$$

Hence,

$$\|\omega_n''\| \leqslant \max_{t\in[0,1]} \overline{\beta}(t) + 1 < \infty.$$

From

$$\omega_n'(t) = \omega_n'(0) + \int_0^t \omega_n''(s)\mathrm{d}s, \tag{3.6}$$

$$\omega_n(t) = \omega_n(0) + \int_0^t \omega_n'(s)\mathrm{d}s, \tag{3.7}$$

$\{\omega_n'\}$ and $\{\omega_n\}$ are uniformly bounded and equicontinuous sequence of functions. By Ascoli-Arzelà Theorem, $\{\omega_n'\}$ and $\{\omega_n\}$ contain a uniformly convergent subsequence respectively (for convenience we also use the same notation), such that

$$\omega_n \longrightarrow \omega_0, \quad \omega_n' \longrightarrow \varphi.$$

It is easy to know $\omega_0 \in C^1_M(I)$.

From (3.5) and (3.6), we obtain

$$\omega_n'(t) = \omega_n'(0) - \int_0^t \Big(\int_0^1 h_u(s, \theta u_n)\mathrm{d}\theta\omega_n + \frac{h(s, 0)}{\|v_n\|}\Big)\mathrm{d}s. \tag{3.8}$$

Let $n \to \infty$, from (3.7) and (3.8), we have

$$\omega_0(t) = \omega_0(0) + \int_0^t \varphi(s)\mathrm{d}s,$$
$$\varphi(t) = \varphi(0) - \int_0^t \Big(\int_0^1 h_u(s, \theta u_0)\mathrm{d}\theta\omega_0\Big)\mathrm{d}s.$$

Therefore,

$$-\omega_0'' = \int_0^1 h_u(t, \theta u_0)\mathrm{d}\theta\omega_0.$$

By Lemma 3.4, we have $\omega_0 \equiv 0$, which is a contradiction with $\|\omega_0\| = 1$, so $\{v_n\}$ is a bounded sequence. Then, by (3.5) we know $\{v_n''\}$ is bounded and $\{v_n'\}$, $\{v_n\}$ are bounded and equicontinuous sequences of functions. By Ascoli-Arzelà Theorem,

$$v_n \longrightarrow v_0, \quad v_n' \longrightarrow \varphi_0.$$

Then we know

$$
\begin{aligned}
v_n'(t) &= v_n'(0) + \int_0^t v_n''(s)\mathrm{d}s \\
&= v_n'(0) - \int_0^t \Big(\big(\int_0^1 h_u(s,\theta u_n)\mathrm{d}\theta \big) v_n + h(s,0) \Big)\mathrm{d}s, \\
v_n(t) &= v_n(0) + \int_0^t v_n'(s)\mathrm{d}s.
\end{aligned}
$$

Let $n \to \infty$, from the above we obtain

$$
-v_0'' = \Big(\int_0^1 h_u(t,\theta u_0)\mathrm{d}\theta \Big) v_0 + h(t,0).
$$

By the uniqueness we know $v_0 = Pu_0$, which completes the proof. □

Lemma 3.7. *P is a compact operator.*

Proof. For any bounded set $S \subset C_M^1(I)$, we claim that $P(S)$ is bounded in $C_M^1(I)$. Otherwise, by an analogous manner as the proof of Lemma 3.6 we will get a contradiction. For every $u \in S$, $v = Pu$ is defined by (3.3). Since $\|u\|$, $\|h_u\|$ are all bounded, then $\|v''\| < \infty$. Then we conclude that $\{v'\}$, $\{v\}$ are bounded and equicontinuous. By Ascoli-Arzelà Theorem, P is a compact operator. □

Lemma 3.8. *$P(C_M^1(I))$ is bounded in $C_M^1(I)$.*

Proof. If not, there exists a sequance $\{u_n\}$, $\|Pu_n\| \to \infty$ $(n \to \infty)$. Let $v_n = Pu_n$. Then (3.4) holds. Take $\omega_n = v_n/\|v_n\|$. Then $\{\omega_n\} \subset C_M^1(I)$, $\|\omega_n\| = 1$, (3.5), (3.6), (3.7) and (3.8) hold. Then we get

$$
\omega_n \longrightarrow \omega_0, \quad \omega_n' \longrightarrow \phi
$$

and $\|\omega_0\| = 1$. Since $\{\int_0^1 h_u(t,\theta u_n)\mathrm{d}\theta\}$ is bounded in $L^2(I)$,

$$
\int_0^1 h_u(t,\theta u_n)\mathrm{d}\theta \rightharpoonup h_1(t)
$$

in $L^2(I)$. Obviously,

$$
\underline{\beta}(t) \leqslant h_1(t) \leqslant \overline{\beta}(t),
$$

where $\underline{\beta}(t), \overline{\beta}(t)$ satisfies either i or ii in Lemma 3.4. Let $k \to \infty$, from (3.7) and (3.8), for a.e. $t \in I$,

$$
-\omega_0''(t) = h_1(t)\omega_0(t), \quad \omega_0(0) = \omega_0'(1) = 0.
$$

Hence, $\omega_0 \equiv 0$, which is a contradiction to $\|\omega_0\| = 1$. □

Proof of Lemma 3.4. The uniqueness is given in Lemma 3.5. To obtain the existence, assume $D = \{u \in C_M^1(I), \|u\| \leqslant K+1\}$, where K is given in Lemma 3.8. The continuity and compactness are established in Lemma 3.6 and Lemma 3.7, respectively. By Schauder's fixed point theorem, the operator $P : D \to D$ has at least one fixed point. □

Next, we study the existence of boundary value problem involving the gradient term. For any $v \in C^1_M(I)$, consider the following problem:

$$\begin{cases} -u'' = g(t, u, v'), \\ u(0) = u'(1) = 0. \end{cases} \tag{M$_v$}$$

Lemma 3.9. *Assume that $g(t, 0, p) \neq 0$ and either of the following holds uniformly:*

$$\text{i.}\quad (k - \frac{1}{2})^2\pi^2 < \underline{\beta}(t) \leqslant g_u(t, u, p) \leqslant \overline{\beta}(t) < k^2\pi^2;$$
$$\text{ii.}\quad k^2\pi^2 < \underline{\beta}(t) \leqslant g_u(t, u, p) \leqslant \overline{\beta}(t) < (k + \frac{1}{2})^2\pi^2.$$

Then, for any $v \in C^1_M(I)$, problem (M)$_v$ has a unique nontrivial solution u_v.

Proof. The proof can be obtained by Lemma 3.4. □

Lemma 3.10. *Let all of the assumptions of Lemma 3.9 hold. Then, for any $v \in C^1_M(I)$, there exists a positive constant ρ, independent of v, such that*

$$||u_v|| \leqslant \rho$$

for all solutions u_v obtained in Lemma 3.9.

Proof. Assume that there exists $\{v_n\}$ such that $\|u_{v_n}\| \to \infty$. Then

$$-u''_{v_n} = \Big(\int_0^1 g_u(t, \theta u_{v_n}, v'_n)\mathrm{d}\theta\Big)u_{v_n} + g(t, 0, v'_n).$$

Denote $\omega_n = u_{v_n}/\|u_{v_n}\|$ and then $||\omega_n|| = 1$. Since (G0) holds, the second term in the above equation is bounded. Then a similar argument can be obtained, as the proof of Lemma 3.8. □

Proof of Theorem 1.1. Define $B_\rho := \{x \in C^1_M(I), ||x|| \leqslant \rho\}$, where $\rho > 0$ is the uniform bound in Lemma 3.10. We consider the operator $Q : B_\rho \to B_\rho$. For every v, Qv denotes the solution u_v of (M)$_v$ determined by Lemma 3.9. By Schauder's fixed point theorem, Q has at least one fixed point. □

4. Variational method

In this section, we consider (M)$_{ODE}$ by means of variational methods. In fact, the problem (M)$_{ODE}$ is non-variational because of the influence of the gradient term. We first study auxiliary problem (M)$_v$. For any fixed $v \in C^1_M(I)$, we call $u \in H^1_M(I)$ a weak solution, if

$$\int_I u'(t)\varphi'(t)\mathrm{d}t = \int_I g(t, u(t), v'(t))\varphi(t)\mathrm{d}t, \quad \forall\ \varphi \in C^\infty_M(I).$$

Then the weak solutions are equivalent to the critical points of the Euler-Lagrange functional $J_v : H^1_M(I) \to \mathbb{R}$,

$$J_v(u) = \frac{1}{2}\int_I |u'(t)|^2\mathrm{d}t - \int_I G(t, u, v')\mathrm{d}t,$$

where

$$G(t,u,\eta):=\int_0^u g(t,s,\eta)\mathrm{d}s.$$

Let $u^+=\max\{u,0\}$, $u^-=\min\{u,0\}$. Consider the following problem

$$\begin{cases} -u''=g^{\pm}(t,u,v'), \\ u(0)=u'(1)=0. \end{cases} \tag{4.1}$$

where

$$g^+(t,s,\eta)=\begin{cases} g(t,s,\eta), & s\geqslant 0, \\ 0, & s<0; \end{cases} \qquad g^-(t,s,\eta)=\begin{cases} 0, & s>0, \\ g(t,s,\eta), & s\leqslant 0. \end{cases}$$

Define the corresponding functional $J_v^{\pm}:H_M^1(I)\to\mathbb{R}$ as follows:

$$J_v^{\pm}(u)=\frac{1}{2}\|u\|_{H^1}^2-\int_I G^{\pm}(t,u,v')\mathrm{d}t, \quad u\in H_M^1(I),$$

where $G^{\pm}(t,u,\eta)=\int_0^u g^{\pm}(t,s,\eta)ds$. Obviously, $J_v^{\pm}\in C^1(H_M^1(I),\mathbb{R})$. Let u be a critical point of $J_v^{\pm}$, which implies that u is a weak solution of (4.1). Furthermore, by the weak maximum principle it follows that $u\geqslant 0(\leqslant 0)$ in I. Thus u is also a solution of problem $(\mathrm{M})_v$. Hence, a nontrivial critical point of $J_v^+(J_v^-)$ is actually a positive (negative) solution of $(\mathrm{M})_v$.

Lemma 4.1. *Under the assumptions (G3) and (G4), $J_v^{\pm}$ is unbounded from below.*

Proof. (G3) and (G4) imply that there exist $\varepsilon>0$ and $C_\varepsilon>0$ such that

$$G^{\pm}(t,s^{\pm},\eta)\geqslant\frac{1}{2}(\frac{\pi^2}{4}+\varepsilon)|s^{\pm}|^2-C_\varepsilon, \quad \forall\, t\in I,\ s\in\mathbb{R},\ \eta\in\mathbb{R}. \tag{4.2}$$

From (4.2) we obtain

$$\begin{aligned} J_v^{\pm}(\pm k\varphi_1) &\leqslant \frac{1}{2}\|k\varphi_1\|_{H^1}^2-\frac{1}{2}(\frac{\pi^2}{4}+\varepsilon)\int_I k^2\varphi_1^2\mathrm{d}t+\int_I C_\varepsilon\mathrm{d}t \\ &\leqslant \frac{k^2}{2}\|\varphi_1\|_{H^1}^2-\frac{k^2}{2}(\frac{\pi^2}{4}+\varepsilon)\|\varphi_1\|_{L^2}^2+C_\varepsilon \\ &\leqslant \frac{k^2}{2}(1-\frac{\frac{\pi^2}{4}+\varepsilon}{\frac{\pi^2}{4}})\|\varphi_1\|_{H^1}^2+C_\varepsilon. \end{aligned} \tag{4.3}$$

Then $\lim\limits_{k\to+\infty}J_v^{\pm}(k\varphi_1)=-\infty$. □

Remark 4.1. Obviously, there exists $\gamma>0$ independent of v, such that

$$J_v^{\pm}(\pm k\varphi_1)\leqslant 0, \quad \text{for all} \quad k\geqslant\gamma.$$

Lemma 4.2. *Assume that (G2)-(G4) hold. Then there exist $\rho,R>0$ such that $J_v^{\pm}(u)\geqslant R$, if $\|u\|_{H^1}=\rho$.*

Proof. From (G2)-(G4), we can take $\varepsilon_0 > 0$, $C_0 > 0$, $\tau > 2$ such that

$$G^{\pm}(t, s, \eta) \leqslant \frac{1}{2}(\frac{\pi^2}{4} - \varepsilon_0)|s|^2 + C_0|s|^{\tau}. \tag{4.4}$$

Then Poincaré inequality and Sobolev inequality imply

$$\begin{aligned} J_v^{\pm}(u) &\geqslant \frac{1}{2}\|u\|_{H^1}^2 - \frac{\frac{\pi^2}{4} - \varepsilon_0}{2}\int_I |u|^2 \mathrm{d}t - C_0\int_I |u|^{\tau}\mathrm{d}t \\ &\geqslant \frac{1}{2}(1 - \frac{\frac{\pi^2}{4} - \varepsilon_0}{\frac{\pi^2}{4}})\|u\|_{H^1}^2 - C_s C_0\|u\|_{H^1}^{\tau}, \end{aligned} \tag{4.5}$$

where C_s is the Sobolev constant. Choosing $\|u\|_{H^1} = \rho$ small enough, we obtain $J_v^{\pm}(u) \geqslant R > 0$. □

Lemma 4.3. *Suppose that (G3) and (G4) hold. Then every Palais-Smale sequence of $J_v^{\pm}$ has a convergent subsequence in $H_M^1(I)$.*

Proof. It suffices to show that every (PS) sequence $\{u_n\}$ is bounded in $H_M^1(I)$. We prove the case of J_v^+, and the case of J_v^- can be proved analogously. Assume that $\{u_n\} \subset H_M^1(I)$ is a (PS) sequence, i.e.,

$$J_v^+(u_n) \to c, \quad (J_v^+)'(u_n) \to 0 \quad \text{as } n \to +\infty. \tag{4.6}$$

From (G3) and (G4) we know that

$$|g^+(t, s, v')s| \leqslant C(1 + |s|^2).$$

(4.6) implies that for all $\varphi \in H_M^1(I)$,

$$\int_I \Big(u_n'\varphi' - g^+(t, u_n, v')\varphi\Big)\mathrm{d}t \to 0. \tag{4.7}$$

Setting $\varphi = u_n$ and using Hölder inequality we have

$$\begin{aligned} \|u_n\|_{H^1}^2 &= \int_I g^+(t, u_n, v')u_n \mathrm{d}t + \langle (J_v^+)'(u_n), u_n\rangle \\ &\leqslant \int_I g^+(t, u_n, v')u_n \mathrm{d}t + o(1)\|u_n\|_{H^1} \\ &\leqslant C + C\|u_n\|_{L^2}^2 + o(1)\|u_n\|_{H^1}. \end{aligned} \tag{4.8}$$

We claim that $\|u_n\|_{L^2}$ is bounded. Otherwise, passing to a subsequence,

$$\|u_n\|_{L^2}^2 \to +\infty, \quad \text{as } n \to +\infty.$$

We put $\omega_n := \dfrac{u_n}{\|u_n\|_{L^2}}$. Then $\|\omega_n\|_{L^2} = 1$. Moreover, from (4.8) we know

$$\|\omega_n\|_{H^1}^2 \leqslant o(1) + C + \frac{o(1)}{\|u_n\|_{L^2}} \cdot \frac{\|u_n\|_{H^1}}{\|u_n\|_{L^2}} \leqslant o(1) + C + o(1)\|\omega_n\|_{H^1}.$$

Hence, $\|\omega_n\|_{H^1}$ is bounded. Passing to a subsequence, we may assume that there exists $\omega \in H_M^1(I)$, $\|\omega\|_{L^2} = 1$ such that

$$\omega_n \rightharpoonup \omega, \quad \text{weakly in } H_M^1(I), \quad n \to +\infty,$$

$$\omega_n \to \omega, \quad \text{strongly in } L^2(I), \quad n \to +\infty.$$

From (4.7) it follows

$$\int_I \omega_n' \varphi' \mathrm{d}t - \int_I \frac{g^+(t, u_n, v')}{\|u_n\|_{L^2}} \varphi \mathrm{d}t = o(1), \quad \forall \ \varphi \in H_M^1(I). \tag{4.9}$$

Taking $\varphi = \omega_n^-$, we know $\|\omega_n^-\|_{H^1} = o(1)$, which implies $\omega^-(t) = 0$, a.e. in I and thus $\omega(t) \geqslant 0$.

If $\omega(t) = 0$, from (G4) we get

$$\frac{|g^+(t, u_n, v')|}{\|u_n\|_{L^2}} = |\frac{g^+(t, u_n, v')}{u_n}|\omega_n \leqslant M\omega_n \to 0.$$

We have

$$\lim_{n\to+\infty} \frac{g^+(t, u_n, v')}{\|u_n\|_{L^2}} = 0.$$

If $\omega(t) > 0$, $u_n = \omega_n \|u_n\|_{L^2} \to +\infty$. (G3) implies that there exists $\delta > 0$ such that

$$\liminf_{n\to+\infty} \frac{g^+(t, u_n, v')}{\|u_n\|_{L^2}} = \liminf_{n\to+\infty} \frac{g^+(t, u_n, v')}{u_n}\omega_n \geqslant (\frac{\pi^2}{4} + \delta)\omega.$$

From the above two cases, for all $t \in I$,

$$\liminf_{n\to+\infty} \frac{g^+(t, u_n, v')}{\|u_n\|_{L^2}} \geqslant (\frac{\pi^2}{4} + \delta)\omega. \tag{4.10}$$

Taking $\varphi = \varphi_1$ in (4.7), since $\varphi_1 > 0, \omega \geqslant 0$, from Fatou's Lemma we derive

$$\begin{aligned}
\frac{\pi^2}{4}\int_I \omega\varphi_1 \mathrm{d}t &= \int_I \omega'\varphi_1' \mathrm{d}t \\
&= \lim_{n\to+\infty}\int_I \omega_n'\varphi_1' \mathrm{d}t \\
&= \lim_{n\to+\infty}\int_I \frac{g^+(t, u_n, v')}{\|u_n\|_{L^2}}\varphi_1 \mathrm{d}t \\
&\geqslant \int_I \liminf_{n\to+\infty} \frac{g^+(t, u_n, v')}{\|u_n\|_{L^2}}\varphi_1 \mathrm{d}t \\
&\geqslant (\frac{\pi^2}{4} + \delta)\int_I \omega\varphi_1 \mathrm{d}t,
\end{aligned}$$

which follows that $\omega \equiv 0$. This conclusion contradicts with $\|\omega_n\|_{L^2} = 1$, so $\|u_n\|_{L^2}$ is bounded. Then, from (4.8) we know that $\{u_n\}$ is bounded in $H_M^1(I)$. □

Lemma 4.4. *Let (G2)-(G4) hold. Then, for any $v \in C_M^1(I)$, problem $(M)_v$ has at least one positive weak solution and one negative weak solution $u_v^\pm \in H_M^1(I)$.*

Proof. Let

$$c_v^\pm = \inf_{\psi\in\Psi^\pm} \max_{s\in[0,1]} J_v^\pm(\psi(s)), \tag{4.11}$$

where

$$\Psi^\pm = \{\psi \in C([0,1], H_M^1(I)) : \ \psi(0) = 0, \ \psi(1) = \pm\gamma\varphi_1\},$$

γ is given by Remark 4.1. Since Lemma 4.3 holds, Mountain pass theorem implies that there exists a weak solution $u_v^\pm$ such that

$$(J_v^\pm)'(u_v^\pm) = 0, \quad J_v^\pm(u_v^\pm) = \inf_{\psi\in\Psi^\pm} \max_{s\in[0,1]} J_v^\pm(\psi(s)).$$

The proof is completed. $\square$

Lemma 4.5. *Let $v \in C_M^1(I)$. Then there exists a positive constant c_0 independent of v such that*

$$||u_v^\pm||_{H^1} \geqslant c_0$$

for all solutions $u_v^\pm$ of (M)$_v$ obtained in Lemma 4.4.

Proof. Since $u_v^\pm$ is a solution of (M)$_v$, we have

$$\int_I |(u_v^\pm)'|^2 \mathrm{d}t = \int_I g^\pm(t, u_v^\pm, v') u_v^\pm \mathrm{d}t.$$

From (G3) and (G4) we know there exist $\epsilon > 0$, $c_\epsilon > 0$ such that

$$|g^\pm(t, s^\pm, \eta)| \leqslant (\frac{\pi^2}{4} - \epsilon)|s^\pm| + c_\epsilon |s^\pm|^{2^*-1}, \quad \text{for any } t \in I,\ s \in \mathbb{R},\ \eta \in \mathbb{R}^n.$$

Hence,

$$\int_I |(u_v^\pm)'|^2 \mathrm{d}t \leqslant (\frac{\pi^2}{4} - \epsilon) \int_I |u_v^\pm|^2 \mathrm{d}t + c_\epsilon \int_I |u_v^\pm|^{2^*} \mathrm{d}t.$$

By Poincaré inequality and Sobolev embedding, we obtain

$$(1 - \frac{\frac{\pi^2}{4} - \epsilon}{\frac{\pi^2}{4}})||u_v^\pm||_{H^1}^2 \leqslant c_\epsilon \|u_v^\pm\|_{L^{2^*}}^{2^*} \leqslant c_\epsilon c_0^{2^*} ||u_v^\pm||_{H^1}^{2^*},$$

which implies the conclusion. $\square$

Lemma 4.6. *Let (H1)-(H3) hold. Then there exists a positive constant $\overline{\rho}$, which is independent of v, such that*

$$||u_v^\pm||_{H^1} \leqslant \overline{\rho}$$

for all solutions $u_v^\pm$ obtained in Lemma 4.4.

Proof. We only give the proof of J_v^+, the case of J_v^- is similar. We suppose, by contradiction, there exist subsequences $\{v_j\}$ and $\{u_{v_j}\}$, such that $\{v_j\} \subset C_M^1(I)$, $\{u_{v_j}\} \subset H_M^1(I)$ and

$$(J_{v_j}^+)'(u_{v_j}) = 0, \quad \|u_{v_j}\|_{H^1} \to +\infty \quad \text{as } j \to +\infty.$$

Then for all $\varphi \in H_M^1(I)$,

$$\int_I \Big(u_{v_j}'\varphi' - g^+(t, u_{v_j}, v_j')\varphi\Big)\mathrm{d}t = 0. \tag{4.12}$$

From (4.12), a similar argument like in Lemma 4.3 will lead to a contradiction, which completes the proof. $\square$

Now, since f is continuous in all variables and $v \in C_0^1(I)$, using the regularity theory we know that $u_v^\pm$ is C^2, see [3]. As a consequence of Sobolev embedding theorem and Lemma 4.6, the following lemma is trivial.

Lemma 4.7. *Assume that $v \in C^1_M(I)$. Then there exists two positive constants ρ_1 and ρ_2, independent of v, such that*

$$\max_{t\in I} |u_v^{\pm}(t)| \leqslant \rho_1, \qquad \max_{t\in I} |(u_v^{\pm})'(t)| \leqslant \rho_2.$$

5. Iterative method and proof of Theorem 1.2

In this section, we prove Theorem 1.2 by some iterative arguments, which was established in [7]. Define map

$$T : H^1_M(I) \to H^1_M(I), \quad Tv \mapsto u_v,$$

with domain $D(T) = C^1_M(I) \subset H^1_M(I)$. Here u_v is the solution of $(\mathrm{M})_v$ given by Lemma 4.4. For any $v \in H^1_M(I)$, the map is well-defined, and actually, $D(C^1_M(I)) \subset C^1_M(I)$ because of the regularity theory. Moreover, denote $B_{\overline{\rho}} := \{x \in H^1_M(I), ||x|| \leqslant \overline{\rho}\}$, where $\overline{\rho} > 0$ is the uniform bound in Lemma 4.6. Then, $T(C^1_M(I)) \subset B_{\overline{\rho}}$. Hence, $T(C^1_M(I)) \subset B_{\overline{\rho}} \cap C^1_M(I)$. It should be pointed out that, in contrast to the proof of Lemma 3.4, T is a multivalued map because of the absence of uniqueness. Recall that x is a fixed point of map T, if and only if $x \in T(x)$.

Proof of Theorem 1.2. We prove the existence of positive solution and the negative one is similar. Construct a sequence $\{u_n\} \subset B_{\overline{\rho}} \cap C^1_M(I)$ as the solutions of

$$\begin{cases} -u_n'' = g^+(t, u_n, u_{n-1}'), \\ u_n(0) = u_n'(1) = 0, \end{cases} \tag{IE$_n$}$$

obtained by Lemma 4.4, and choose $u_0 \in B_{\overline{\rho}} \cap C^1_M(I)$. Hence, $u_n \in B_{\overline{\rho}} \cap C^1_M(I)$.

By $(\mathrm{IE})_n$ and $(\mathrm{IE})_{n+1}$, we know

$$\int_I u_n'(u_{n+1}' - u_n')\mathrm{d}t = \int_I g^+(t, u_n, u_{n-1}')(u_{n+1} - u_n)\mathrm{d}t,$$
$$\int_I u_{n+1}'(u_{n+1}' - u_n')\mathrm{d}t = \int_I g^+(t, u_{n+1}, u_n')(u_{n+1} - u_n)\mathrm{d}t.$$

Then

$$\begin{aligned} ||u_{n+1} - u_n||^2_{H^1} &= \int_I \big(g^+(t, u_{n+1}, u_n') - g^+(t, u_n, u_{n-1}')\big)(u_{n+1} - u_n)\mathrm{d}t \\ &= \int_I \big(g^+(t, u_{n+1}, u_n') - g^+(t, u_n, u_n')\big)(u_{n+1} - u_n)\mathrm{d}t \\ &\quad + \int_I g^+(t, u_n, u_n') - g^+(t, u_n, u_{n-1}')\big)(u_{n+1} - u_n)\mathrm{d}t \\ &= \int_I g_u^+(t, u_n + \theta(u_{n+1} - u_n), u_n')(u_{n+1} - u_n)^2\mathrm{d}t \\ &\quad + \int_I g_\eta^+(t, u_n, u_{n-1}' + \vartheta(u_n' - u_{n-1}'))(u_n' - u_{n-1}')(u_{n+1} - u_n)\mathrm{d}t, \end{aligned}$$

where $0 \leqslant \theta \leqslant 1$, $0 \leqslant \vartheta \leqslant 1$. Using hypothesis (G5), (2.11) as well as Cauchy-Schwarz inequality, we obtain

$$||u_{n+1} - u_n||^2_{H^1} \leqslant L \int_I (u_{n+1} - u_n)^2\mathrm{d}t + K \int_I (u_n' - u_{n-1}')(u_{n+1} - u_n)\mathrm{d}t$$

$$\leqslant \frac{4L}{\pi^2}||u_{n+1}-u_n||_{H^1}^2 + K||u_n-u_{n-1}||_{H^1}||u_{n+1}-u_n||_{L^2}$$
$$\leqslant \frac{4L}{\pi^2}||u_{n+1}-u_n||_{H^1}^2 + \frac{2K}{\pi}||u_n-u_{n-1}||_{H^1}||u_{n+1}-u_n||_{H^1}.$$

Hence,

$$||u_{n+1}-u_n||_{H^1} \leqslant \frac{2K\pi}{\pi^2-4L}||u_n-u_{n-1}||_{H^1}.$$

From (G5) we know that $L < \pi^2/4$ and $k := 2K\pi/(\pi^2-4L)$ satisfying $k \in (0,1)$. It can be easily seen that $\{u_n\} \subset H_M^1(I)$ is a Cauchy sequence, which implies that there exists $u^* \in H_M^1(I)$ such that $u^* \in T(u^*)$. According to the regularity theory, it follows that $u^* \in C^2(I)$, which is actually a classical solution. Finally, from Lemma 4.5 we know that $||u^*||_{H^1} \geqslant c_0$, which means that u^* is nontrivial. □

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