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EXISTENCE OF SOLUTIONS FOR DUAL SINGULAR INTEGRAL EQUATIONS WITH CONVOLUTION KERNELS IN CASE OF NON-NORMAL TYPE

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Abstract This paper is devoted to the study of dual singular integral equations with convolution kernels in the case of non-normal type. Via using the Fourier transforms, we transform such equations into Riemann boundary value problems. To solve the equation, we establish the regularity theory of solvability. The general solutions and the solvable conditions of the equation are obtained. Especially, we investigate the asymptotic property of solutions at nodes. This paper will have a significant meaning for the study of improving and developing complex analysis, integral equations and Riemann boundary value problems.

Keywords Singular integral equations, Riemann boundary value problems, convolution kernel, regularity theory, dual equations.

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1. Introduction

It is well known that there are rather complete investigations on the method of solution for equations of Cauchy type as well as integral equations of convolution type. Singular integral equations and Riemann boundary value problems have a lot of applications, e.g. in elasticity theory, fluid dynamics, quantum mechanics. In recent years, many mathematicians have studied singular integral equations and formed a relatively systematic theoretical system (see [1, 4, 6, 7, 10, 29, 30] and references therein). [5] first began to study singular integral equation of Wiener-Hopf type with continuous coefficients. [11] discussed the Noether theory of singular integral equations of convolution type. [14, 16, 18, 19, 25–27] dealt with the invertibility of singular integral operators with discontinuous coefficients, and then considered the solvability theory and the general solutions for some classes of singular integral equations with convolution kernels on the whole real axis (or, on the unit circle) in the case of normal-type. For operators containing both Cauchy principal value integral and convolution, the conditions of their Noethericity were discussed in [8, 23, 28, 33] in more general cases. For applications, the problems to find their solutions is very important. Therefore, singular integral equations of convolution type, mathematically, belong to an interesting subject in the theory of integral equations.

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Motivated by the above works, we investigate the existence of solutions for one class of dual singular integral equations with convolution kernels in the cases of non-normal type. In the process of studying equations, we find that the methods used in [5,10,29] are no longer suitable for the case of non-normal type, that is, it is difficult to use only the Fourier transform technique to study the case of non-normal type. Hence, we shall introduce a new method to complete our research. In this paper we apply Fourier analysis theory and boundary value method in the theory of analytic functions to deal with the solvability of the equations. Our approach is novel and effective, different from the ones in classical cases. Therefore, this paper generalizes and improves the theories of integral equations and the classical Riemann boundary value problems.

2. Some classes of functions and Fourier transforms

In this section, we present some definitions and lemmas, and we mainly introduce the concepts of classes $\{\{0\}\}$ ($((0))$, $\ll 0 \gg$) and $\{0\}$ ((0), $< 0 >$).

Definition 2.1. The Fourier transform $\mathcal{F}$ and the inverse transform $\mathcal{F}^{-1}$ are defined as follows

$$(\mathcal{F}f)(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)e^{ist}dt; \quad (\mathcal{F}^{-1}F)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(s)e^{-ist}ds. \tag{2.1}$$

For simplification, in (2.1), we denote them as $F(s) = (\mathcal{F}f)(s)$, $f(t) = (\mathcal{F}^{-1}F)(t)$, respectively.

Definition 2.2. We say that $F(s) \in \{\{0\}\}$, if (1)$F(s) \in \widehat{H}$, i.e., it satisfies the Hölder condition on the whole real domain $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$; (2) $F(s) \in L^2(\mathbb{R})$.

Definition 2.3. A function $f(t) \in \{0\}$, if its Fourier transform $F(s)$ belongs to $\{\{0\}\}$.

Definition 2.4. Let $F(s)$ be continuous on $\mathbb{R}$, if the following conditions are fulfilled: (1) $F(s) \in \hat{H}$; (2) $F(s) = O(|s|^{-\sigma}), \sigma > \frac{1}{2}$, where $|s|$ is sufficiently large, then we call $F(s) \in ((0))^\sigma$ or $((0))$.

If $F(s) \in ((0))^\sigma$ or $((0))$, we call that $f(t) \in (0)^\sigma$ or (0).

Definition 2.5. If (1) $F(s) \in \hat{H}$; (2) $F(s) \in H^\sigma(N_\infty)$, $\sigma > \frac{1}{2}$, i.e., it belongs to H in the neighborhood N_∞ of ∞, and $F(\infty) = 0$, then we call $F(s) \in \ll 0 \gg^\sigma$ or $\ll 0 \gg$, and $f(t) \in < 0 >^\sigma$ or $< 0 >$.

Definition 2.6. For two functions $k(t)$ and $f(t)$, their convolution is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} k(t-\tau)f(\tau)d\tau, \quad -\infty < t < +\infty, \tag{2.2}$$

we denote it as $k * f$. It is well known that [5,29]

$$\mathcal{F}(k * f(t)) = \mathcal{F}k(t) \cdot \mathcal{F}f(t) = K(s)F(s). \tag{2.3}$$

Definition 2.7. We also introduce the operator ϵ of Cauchy principal value integral

$$\epsilon f(t) = \text{P.V.}\frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{f(\tau)}{\tau - t}d\tau = \lim_{\varrho\to 0} \frac{1}{\pi i} \int_{|\tau-t|>\varrho} \frac{f(\tau)}{\tau - t}d\tau, \quad -\infty < t < +\infty. \tag{2.4}$$

It follows from [8,16,33] that ϵ maps $\{0\}$ and $<0>$ into themselves respectively and $\epsilon^2 = I$ (identity).

Definition 2.8. We define operators N and S as follows

$$Nf(t) = f(-t), \quad Sf(t) = f(t)\text{sgn}t, \quad -\infty < t < +\infty. \tag{2.5}$$

Lemmas 2.1 and 2.2 are obvious facts and we omit their proof here.

Lemma 2.1. *(1) If $k, f \in \{0\}$ $(<0>)$, then $k * f \in \{0\}$ $(<0>)$; (2) If $f \in \{0\}$ and $k \in (0)$ $(<0>)$, then $k * f \in (0)$ $(<0>)$.*

Lemma 2.2 (see [8,33])**.** *The operators $\mathcal{F}, \mathcal{F}^{-1}, \epsilon, N, S$ are as the before, then we have*

$$(1) N^2 = S^2 = I; \ (2)\mathcal{F}^2 = N; \ (3)\mathcal{F}S = \epsilon\mathcal{F}; \ (4)SN = -NS; \ (5)\mathcal{F}^{-1} = N\mathcal{F} = \mathcal{F}N. \tag{2.6}$$

The following lemma 2.3 plays an important role and it is used to get our some results in this paper.

Lemma 2.3. *Let $f(t) \in \{0\}, F(s) = \mathcal{F}f(t)$, then we have*

$$\mathcal{F}[\epsilon f(t)] = -SF(s), \ i.e., \ \mathcal{F}\epsilon = -S\mathcal{F}. \tag{2.7}$$

Proof. By Lemma 2.3, we have $\epsilon = \mathcal{F}S\mathcal{F}^{-1}$, but $\mathcal{F}^{-1} = N\mathcal{F} = \mathcal{F}N$, $\mathcal{F}^2 = N$, thus we obtain $\mathcal{F}\epsilon = -S\mathcal{F}$. □

Lemma 2.4. *If $f \in \{0\}$, (0) or $<0>$ and $\mathcal{F}f(0) = 0$, then ϵf belongs to the same class.*

Proof. By Lemma 2.3 and assumptions, and note that

$$\mathcal{F}f(\infty) = \mathcal{F}f(0) = 0, \tag{2.8}$$

thus Lemma 2.4 can be proved. □

In Lemma 2.4, note that $F(0) = 0$ is a necessary condition, otherwise the lemma is invalid.

In the following section, we shall focus on the theory of Noether solvability and the methods of solution for dual singular integral equations with convolution kernels in the non-normal type case.

3. Singular integral equations of dual type

Consider the equation

$$\begin{cases} a_1\omega(t) + \frac{b_1}{\pi i}\int_{-\infty}^{+\infty}\frac{\omega(\tau)}{\tau - t}d\tau + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_1(t-\tau)\omega(\tau)d\tau = g(t), & 0 < t < +\infty; \\ a_2\omega(t) + \frac{b_2}{\pi i}\int_{-\infty}^{+\infty}\frac{\omega(\tau)}{\tau - t}d\tau + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_2(t-\tau)\omega(\tau)d\tau = g(t), & -\infty < t < 0. \end{cases} \tag{3.1}$$

where $a_j, b_j (j = 1, 2)$ are constants and b_1, b_2 are not equal to zero simultaneously. $k_1, k_2, g \in <0>^{\beta}$ (or $(0)^{\beta}, 0 < \beta < 1$) and the unknown function $\omega(t)$ is requred to be in $\{0\}$. After simplification, (3.1) may be written as

$$\begin{cases} a_1\omega(t) + b_1\epsilon\omega(t) + k_1 * \omega(t) = g(t), & 0 < t < +\infty; \\ a_2\omega(t) + b_2\epsilon\omega(t) + k_2 * \omega(t) = g(t), & -\infty < t < 0. \end{cases} \tag{3.2}$$

Extending t in the first equation of (3.2) to $-\infty < t < 0$, and in the second one of (3.2) to $0 < t < +\infty$, i.e., we add $-\phi_-(t)$ and $+\phi_+(t)$ to (3.2), then (3.2) can be rewritten as

$$\begin{cases} a_1\omega(t) + b_1\epsilon\omega(t) + k_1 * \omega(t) = g(t) - \phi_-(t); \\ a_2\omega(t) + b_2\epsilon\omega(t) + k_2 * \omega(t) = g(t) + \phi_+(t), \end{cases} \quad -\infty < t < +\infty, \tag{3.3}$$

where

$$\phi_+(t) = \begin{cases} \phi(t), & t \geq 0, \\ 0, & t < 0, \end{cases} \quad \phi_-(t) = \begin{cases} 0, & t \geq 0, \\ -\phi(t), & t < 0, \end{cases}$$

and $\phi \in \{0\}$ is an undetermined function, obviously $\phi(t) = \phi_+(t) - \phi_-(t)$.

We firstly use the Fourier transform to convert Eq.(3.3) into a Riemann boundary value problem. By Lemmas 2.2 and 2.3, we get

$$\begin{cases} \Psi^+(s) + G(s) = E_2(s)\Omega(s); \\ \Psi^-(s) + G(s) = E_1(s)\Omega(s), \end{cases} \quad -\infty < s < +\infty, \tag{3.4}$$

where

$$\Omega = \mathcal{F}\omega,\ G = \mathcal{F}g,\ K_j = \mathcal{F}k_j,\ \Psi^{\pm} = \mathcal{F}\phi_{\pm},\ E_j(s) = a_j - b_j sgns + K_j(s),\ j = 1, 2.$$

Note that, from equation (3.3) to equation (3.4), by taking the Fourier transform to $-\phi_-(t)$ we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(-\phi_-(t))e^{ist}dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}(-\phi_-(t))e^{ist}dt + \frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty}(-\phi_-(t))e^{ist}dt$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\phi(t)e^{ist}dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\phi_-(t)e^{ist}dt = \Psi^-(s).$$

From (3.4) we have

$$\frac{\Psi^+(s) + G(s)}{E_2(s)} = \frac{\Psi^-(s) + G(s)}{E_1(s)} = \Omega(s). \tag{3.5}$$

Thus, we should only solve the following Riemann boundary value problem (3.6) in place of (3.1).

$$\Psi^+(s) = E(s)\Psi^-(s) + W(s), \quad -\infty < s < +\infty, \tag{3.6}$$

in which

$$E(s) = \frac{E_2(s)}{E_1(s)}, \quad W(s) = (E(s) - 1)G(s). \tag{3.7}$$

Now we assume that $E_1(s)$ has some zero-points $e_1, e_2, \cdots, e_n$ with the orders $\xi_1, \xi_2, \cdots, \xi_n$ respectively; $E_2(s)$ has some zero-points $c_1, c_2, \cdots, c_q$ with the orders $\eta_1, \eta_2, \cdots, \eta_q$ respectively, where ξ_j, η_j are the non-negative integers. In this case, we say that (3.6) is a Riemann boundary value problem of non-normal type.

Put

$$\sum_{j=1}^{n}\xi_j = n_1, \quad \sum_{j=1}^{q}\eta_j = n_2, \quad V_1(s) = \Pi_{j=1}^{n}(s - e_j)^{\xi_j}, \quad V_2(s) = \Pi_{j=1}^{q}(s - c_j)^{\eta_j},$$

then we can rewrite (3.6) in the form

$$\Psi^+(s) = \frac{V_2(s)}{V_1(s)} D(s)\Psi^-(s) + W(s), \quad -\infty < s < +\infty, \tag{3.8}$$

where $E(s) = \frac{V_2(s)}{V_1(s)} D(s)$ and $D(s) \neq 0$. In view of the values of $a_j \pm b_j$, we have the following several cases.

(1) If $a_1 \pm b_1 \neq 0$, and $a_2 \pm b_2$ are not equal to zero simultaneously, then (3.6) is a Riemann boundary value problem with nodes $s = 0, \infty$.

(2) If $a_j - b_j = 0, a_j + b_j \neq 0$ $(j = 1, 2)$, then (3.6) is a Riemann boundary value problem with node $s = 0$.

(3) If $a_1 \pm b_1 \neq 0$, $a_2 \pm b_2 = 0$, then (3.6) is a Riemann boundary value problem with node $s = \infty$.

Without loss of generality, we only consider the case (1). Other cases can be discussed similarly. Since $\omega(t) \in \{0\}$, thus $\Omega(s) = \mathcal{F}\omega(t) \in \{\{0\}\}$, and by [10] we must have $\mathcal{F}\omega(0) = 0$.

Thus the solution $\Psi(s)$ of (3.6) should be at least continuous along the whole real axis and

$$\Psi^\pm(0) = -G(0). \tag{3.9}$$

We denote

$$\gamma_0 = \alpha_0 + i\beta_0 = \frac{1}{2\pi i} \ln D(s)|_{-0}^{+0}. \tag{3.10}$$

Define by $\kappa = [\alpha_0]$ the index of the problem (3.6), then we have $0 \leq \alpha = \alpha_0 - \kappa < 1$. Set

$$\gamma = \gamma_0 - \kappa = \alpha + i\beta_0. \tag{3.11}$$

Next we discuss the solvability of (3.8). We first define a sectionally holomorphic function $X(z)$:

$$X(z) = \begin{cases} e^{\Gamma(z)}, & z \in \mathbb{C}^+; \\ \frac{(z+i)^{n_1}}{(z-i)^{n_2}} e^{\Gamma(z)}, & z \in \mathbb{C}^-. \end{cases} \tag{3.12}$$

in which we have put

$$\Gamma(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\ln D_0(t) dt}{t - z}, \quad z \in \mathbb{C}^+ \cup \mathbb{C}^- \tag{3.13}$$

and

$$D_0(t) = \left(\frac{t+i}{t-i}\right)^\kappa D(t),$$

here we have taken the definite branch of

$$\ln D_0(t) = \kappa \ln \frac{t+i}{t-i} + \ln D(t),$$

provided we have chosen $\ln \frac{t+i}{t-i} \; |_{t=\pm 0} = \pm i\pi$. It is easy to verify that $X(z)$ is a canonical function and its boundary values satisfy

$$\frac{X^+(s)}{X^-(s)} = \frac{(s-i)^{n_2}}{(s+i)^{n_1}} D_0(s). \tag{3.14}$$

Thus, (3.8) could also be rewrite as

$$\Psi^+(s) = \frac{V_2(s)(s+i)^{n_1}X^+(s)}{V_1(s)(s-i)^{n_2}X^-(s)}\Psi^-(s) + W(s), \quad -\infty < s < +\infty. \tag{3.15}$$

We again put

$$\gamma_\infty = \alpha_\infty + i\beta_\infty = \frac{1}{2\pi i}\ln D(s)|_{-\infty}^{+\infty}, \tag{3.16}$$

where $\ln D(s)$ is taken to be continuous branch for $s > 0$ and $s < 0$ respectively such that it is continuous at $s = \infty$, and $0 \leq \alpha_\infty < 1$. Without loss of generality, we assume $a_1b_2 \neq a_2b_1$, then $\gamma_\infty \neq 0$. If $a_1b_2 = a_2b_1$, the only difference lies in that γ_∞ and γ may be zero, then in which cases the analysis will be even simpler, here we do not discuss it. We first consider the homogeneous problem of (3.15) given by

$$\Psi^+(s) = \frac{V_2(s)(s+i)^{n_1}X^+(s)}{V_1(s)(s-i)^{n_2}X^-(s)}\Psi^-(s). \tag{3.17}$$

Via using the principle of analytic continuation [16, 27], we obtain an analytic solution of (3.17):

$$Y_1(z) = \begin{cases} \frac{X(z)V_2(z)(z+i)^{n_1}P_\vartheta(z)}{(z+i)^\kappa}, & z \in \mathbb{C}^+, \\ \frac{X(z)V_1(z)(z-i)^{n_2}P_\vartheta(z)}{(z+i)^\kappa}, & z \in \mathbb{C}^-, \end{cases} \tag{3.18}$$

in (3.18), when $\vartheta \geq 0$, $P_\vartheta(z)$ is a polynomial of degree ϑ with arbitrary complex coefficients; when $\vartheta < 0$, $P_\vartheta(z) \equiv 0$, where $\vartheta = \kappa - n_1 - n_2$.

Now we solve the non-homogeneous problem (3.15) in class $\{0\}$. To do this, we consider the following function

$$\eta(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{V_1(s)W(s)}{(s+i)^{n_1}X^+(s)(s-z)}ds, \quad \forall z \in \mathbb{C}^+ \cup \mathbb{C}^-. \tag{3.19}$$

We will apply SokhotskiPlemelj formula and generalized Liouville theorem [16,33] to the boundary value problem (3.15), which has a singularity at e_j and c_k. Therefore, we need to construct a Hermite interpolation polynomial $H_\rho(z)$ with the degree ρ, and we can assume that

$$H_\rho(z) = \sum_{j=0}^{\rho} A_j z^{\rho-j}, \quad \rho = n_1 + n_2 - 1$$

which has some zero-points of the orders ξ_j, η_k $(1 \leq j \leq n, 1 \leq k \leq q)$ at e_j, c_k, respectively, where A_l $(0 \leq l \leq \rho)$ are constants.

Making use of (3.19) and $H_\rho(z)$, we can define the following function:

$$Y_2(z) = \begin{cases} \frac{X(z)(z+i)^{n_1}[\eta(z)(z+i)^\kappa - H_\rho(z)]}{(z+i)^\kappa V_1(z)}, & z \in \mathbb{C}^+; \\ \frac{X(z)(z-i)^{n_2}[\eta(z)(z+i)^\kappa - H_\rho(z)]}{(z+i)^\kappa V_2(z)}, & z \in \mathbb{C}^-. \end{cases} \tag{3.20}$$

By means of the classical Riemann boundary value problem, we can verify that (3.20) is the particular solution of (3.15). In view of the solvability of linear equations, we obtain a general solution of (3.15):

$$\Psi(z) = \sum_{j=1}^{2} Y_j(z). \tag{3.21}$$

From (3.18) and (3.20), $\Psi(z)$ can also be written as the explicit solution:

$$\Psi(z) = \begin{cases} \frac{X(z)(z+i)^{n_1}}{(z+i)^{\kappa}V_1(z)}[\eta(z)(z+i)^{\kappa} - H_\rho(z) + V_1(z)V_2(z)P_\vartheta(z)], & z \in \mathbb{C}^+; \\ \frac{X(z)(z-i)^{n_2}}{(z+i)^{\kappa}V_2(z)}[\eta(z)(z+i)^{\kappa} - H_\rho(z) + V_1(z)V_2(z)P_\vartheta(z)], & z \in \mathbb{C}^-. \end{cases} \tag{3.22}$$

In the following, we discuss the conditions of solvability and the properties of solution for Eq. (3.15).

First, we consider the behaviors of solution near $s = 0$. Similar to the discussion in [2, 14], by using SokhotskiPlemelj formula to $X(z)$ in (3.12), we can obtain

$$X^+(s) = \sqrt{D_0(s)}e^{\Gamma(s)}, \quad X^-(s) = \frac{e^{\Gamma(s)}}{\sqrt{D_0(s)}}, \tag{3.23}$$

where $\sqrt{D_0(s)} = \exp\{\frac{1}{2}\log D_0(s)\}$ has a definite value.

If $s = 0$ is an ordinary node, then $0 < \alpha < 1$ and $\gamma \neq 0$. It is easy to verify that, in the neighborhood of $s = 0$,

$$\begin{aligned} \Psi^+(+0) &= \frac{e^{3\gamma\pi i}W(+0) - W(-0)}{2i\sin\gamma\pi}e^{-2\gamma\pi i}; \\ \Psi^+(-0) &= \frac{e^{3\gamma\pi i}W(+0) - W(-0)}{2i\sin\gamma\pi}e^{-\gamma\pi i}. \end{aligned} \tag{3.24}$$

By using (3.9) and $e^{\gamma\pi i} \neq 1$, from (3.24) we can obtain

$$\frac{W(+0)}{W(-0)} = e^{-3\gamma\pi i}. \tag{3.25}$$

If $s = 0$ is a special node, since $\Psi(s)$ is continuous at $s = 0$, we should have the following conditions of solvability

$$u_0 = \frac{1}{V_1(0)V_2(0)}[v_0 - \frac{i^{\kappa-1}}{2\pi}\int_{-\infty}^{+\infty}\frac{V_1(s)W(s)}{X^+(s)(s+i)^{n_1}s}ds] \tag{3.26}$$

as well as

$$\mathcal{F}g(0) = 0, \ i.e., \ G(0) = 0, \tag{3.27}$$

where u_0, v_0 are the constant terms of $P_\vartheta(z)$, $H_\rho(z)$, respectively.

Second, we consider the property of solution at $s = \infty$. Note that, it follows from [10, 26, 28] that, near $s = \infty$,

$$X(s) = \frac{\chi(s)}{|s|^{\alpha_\infty}} \tag{3.28}$$

and $\chi(s) \in H(N_\infty)$, i.e., $\chi(s)$ satisfies the Hölder condition in the neighbourhood N_∞ of ∞.

If $s = \infty$ is an ordinary node, i.e., $0 \le \alpha_\infty < 1$, $\gamma_\infty \neq 0$. Due to (3.27) and $W(s) \in \hat{H}$, we have $\eta(s) \in \hat{H}$, so, when $\frac{1}{2} < \alpha_\infty < \beta < 1$, we have $\lim_{s\to\infty} X(s)\eta(s)s^{\alpha_\infty} = 0$. This implies

$$X(s)\eta(s) = o(\frac{1}{|s|^{\alpha_\infty}}) \quad (s \to \infty). \tag{3.29}$$

When $\frac{1}{2} < \beta \leq \alpha_\infty < 1$, by [8, 19] we know that $X(s)\eta(s)s^{-\varepsilon+\alpha_\infty}$ is bounded at $s = \infty$, thus we have

$$X(s)\eta(s) = O(\frac{1}{|s|^{-\varepsilon+\alpha_\infty}}) \quad (s \to \infty), \tag{3.30}$$

where $\varepsilon > 0$ is arbitrarily small such that $-\varepsilon + \alpha_\infty > \frac{1}{2}$. Again set

$$B(s) = \frac{X^+(s)(s+i)^{n_1-\kappa}}{V_1(s)}[V_1(s)V_2(s)P_\vartheta(s) - H_\rho(s)]. \tag{3.31}$$

We now discuss the asymptotic property of $B(s)$ at $s = \infty$, and when $\vartheta \geq 0$, we know that $\kappa \geq n_1 + n_2$, and $\kappa > \rho = n_1 + n_2 - 1$, so we obtain that the following formula

$$\frac{(s+i)^{n_1-\kappa}}{V_1(s)}[V_1(s)V_2(s)P_\vartheta(s) - H_\rho(s)]$$

is bounded at $s = \infty$. From (3.12), (3.23), (3.28), and again by [11, 29], we have $|X^+(s)| \leq \frac{A}{|s|^{\alpha_\infty}}$, where $A \in \mathbb{R}^+$. Therefore, we get $|B(s)X^+(s)| \leq A$, that is,

$$B(s) = O(\frac{1}{|s|^{\alpha_\infty}}) \quad (s \to \infty). \tag{3.32}$$

Similar to the previous discussion, we have the following results:
when $\vartheta < 0$, since $\Psi(z)$ is bounded at $z = \infty$, one must have

$$A_j = 0, \quad \forall j \in \{0, 1, \ldots, -\vartheta - 1\}, \tag{3.33}$$

moreover, when $\kappa > 0$, we get

$$B(s) = o(\frac{1}{|s|^{\alpha_\infty}}) \quad (s \to \infty); \tag{3.34}$$

when $\kappa < 0$, we require that (3.26) holds, and to eliminate the singularity of $\Psi(z)$ at c_k, d_j, we also have

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{V_1(s)W(s)ds}{X^+(s)(s+i)^{n_1}(s-c_k)^r} = 0, \quad r = 1, 2, \ldots, \eta_k, \ k = 1, 2, \ldots q, \\ &\int_{-\infty}^{+\infty} \frac{V_1(s)W(s)ds}{X^+(s)(s+i)^{n_1}(s-e_j)^p} = 0, \quad p = 1, 2, \ldots, \xi_j, \ j = 1, 2, \ldots, n; \end{aligned} \tag{3.35}$$

when $\kappa = 0$, we require that (3.35) and the following (3.36) are fulfilled

$$\begin{aligned} \eta(c_k) &= \frac{u_0}{c_k + i}, \quad \forall\ k = 1, 2, \ldots q; \\ \eta(e_j) &= \frac{v_0}{d_j + i}, \quad \forall\ j = 1, 2, \ldots, n. \end{aligned} \tag{3.36}$$

Thus, when $\alpha_\infty > \frac{1}{2}$, we have

$$\Psi(s) = o(\frac{1}{|s|^v}) \quad (s \to \infty), \tag{3.37}$$

where $v > \min\{\beta, -\varepsilon + \alpha_\infty\}$; when $\alpha_\infty \leq \frac{1}{2}$, discussions may be made fully analogous to those in [2,9,19,33].

If $s = \infty$ is a special node, then $\alpha_\infty = 0$, $\gamma_\infty \neq 0$, one can translate it into the case that $\alpha_\infty < \frac{1}{2}$. Similar arguments can be used [14,31,33,34]. Note that when $\kappa < 0$, in order to eliminate a singularity of $\Psi(z)$ at $z = -i$, one must have

$$\int_{-\infty}^{+\infty} (\frac{E_2(s)}{E_1(s)} - 1)\frac{V_1(s)}{X^+(s)}\frac{G(s)}{(s+i)^{n_1+r}}ds = 0, \quad r = 1, 2, \ldots, -\kappa. \tag{3.38}$$

In conclusion, we can formulate the main results about solutions of Eq. (3.1) in the following form.

Theorem 3.1. *Under suppositions $a_1 \pm b_1 \neq 0$, in the case of non-normal type, the necessary condition of solvability to Eq. (3.1) is (3.27) in class $\{0\}$. Assume that this is fulfilled.*

(1) If $s = 0$ is an ordinary node, then (3.25) holds; if $s = 0$ is a special node, then (3.26) and (3.27) hold.

(2) Let $s = \infty$ be an ordinary node, if $\alpha_\infty > \frac{1}{2}$, one require that (3.29), (3.30), and (3.37) hold, then Eq. (3.1) has a solution; if $\alpha_\infty \leq \frac{1}{2}$, when $\vartheta > 0$, we rewrite $P_{\vartheta-1}(s)$ instead of $P_\vartheta(s)$ in (3.22), then Eq. (3.1) has $\vartheta - 1$ linearly independent solutions; when $\vartheta \leq 0$, (3.33) holds. Moreover, when $\kappa > 0$, (3.34) holds; when $\kappa < 0$, (3.26) and (3.35) hold; when $\kappa = 0$, (3.36) holds, then Eq. (3.1) has the unique solution.

Let $s = \infty$ be a special node, if $\kappa > 0$, then $A_{-\vartheta-1} = 0$; if $\kappa < 0$, then (3.35) and (3.38) hold; if $\kappa = 0$, the discuss is similar to the case that $\alpha_\infty > \frac{1}{2}$, then Eq. (3.1) has a unique solution.

(3) If $\vartheta > 0$, Eq. (3.1) has ϑ linearly independent solutions; if $\vartheta \leq 0$, Eq. (3.1) has a unique solution.

Thus (3.1) has the general solution

$$\omega(t) = \mathcal{F}^{-1}\Omega(s), \tag{3.39}$$

where $\Omega(s)$ is given by (3.5).

Finally, we give the following two remarks.

Remark 3.1. In Eq.(3.1), if $k_1, k_2, g \in< 0 >$, then $\omega \in< 0 >$. Similarly, if $k_1, k_2, g \in< 0 >^\sigma$, then $\omega \in< 0 >^\sigma$, where $0 < \sigma < 1$.

Remark 3.2. Indeed, we can also investigate the solvability of Eq. (3.1) in Clifford analysis, and the stability of solution for Eq. (3.1) (see [3,12,13,15,17,20–22,24,32]). Further discussion is omitted here.

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