Journal of Applied Analysis and Computation
Volume 10, Number 6, December 2020, 2722–2733
Website:http://www.jaac-online.com
DOI:10.11948/20200092

BIFURCATIONS OF TRAVELING WAVE SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH FOURTH-ORDER DISPERSION AND CUBIC-QUINTIC NONLINEARITY*

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Abstract For the nonlinear schrödinger equation with fourth-order dispersion and cubic-quintic nonlinearity, by using the method of dynamical systems, the dynamics and bifurcations of the corresponding traveling wave system are studied. Under different parametric conditions, twenty exact parametric representations of the traveling wave solutions are obtained.

Keywords Peakon, periodic peakon, sawtooth cusp wave, kink wave, bifurcation.

MSC(2010) 35Q51, 35Q53, 34C23, 34C37.

1. Introduction

In this paper, we consider the nonlinear Schrödinger equation with fourth-order dispersion and cubic-quintic nonlinearity as follows:

$$iE_z - \frac{\beta_2}{2}E_{tt} + \gamma_1|E|^2E = i\frac{\beta_3}{6}E_{ttt} + \frac{\beta_4}{24}E_{tttt} - \gamma_2|E|^4E + i\alpha_1(|E|^2E)_t + i\alpha_2 E(|E|^2)_t. \tag{1.1}$$

This equation govern wave dynamics of optical fiber system (see [8]). The authors of [8] derived analytic soliton solutions (bright and dark soliton and other soliton solutions) of equation (1.1), by using an algebraic method with an auxiliary equation.

As far as we are concerned, the dynamical behavior of traveling wave solutions of equation (1.1) has not been studied in the literature. In this paper, we investigate the bifurcations of phase portraits of traveling system and the dynamical behavior of traveling wave solutions for equation (1.1) by applying the method of dynamical systems (see [2–7, 9–13]).

Following [8], we let

$$E(z,t) = \phi(\xi)\exp(i\theta), \quad \xi = pz - t, \ \theta = kz - ct. \tag{1.2}$$

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*The authors were supported by National Natural Science Foundation of China (No. 11871231).

Substituting (1.2) into equation (1.1), separating the real and imaginary parts, and reducing the fourth order derivative term $\phi^{(4)}$, we obtain

$$[\beta_4(6p-6\beta_2c-3\beta_3c^2+\beta_4c^3)-(\beta_3-\beta_4c)(12\beta_2+12\beta_3c-6\beta_4c^2)]\phi'' \\ +\beta_4(18\alpha_1+12\alpha_2)[2\phi(\phi')^2+\phi^2\phi'']-(\beta_3-\beta_4c)[(24k-12\beta_2c^2-4\beta_3c^3+\beta_4c^4)\phi \\ -24(\gamma_1-\alpha_1c)\phi^3-24\gamma_2\phi^5]=0. \quad (1.3)$$

Assume that $A=\beta_4(18\alpha_1+12\alpha_2)\neq 0$. We write

$$a=-\frac{1}{A}\left[\beta_4(6p-6\beta_2c-3\beta_3c^2+\beta_4c^3)-(\beta_3-\beta_4c)(12\beta_2+12\beta_3c-6\beta_4c^2)\right],$$
$$r=\frac{1}{A}(\beta_3-\beta_4c)(24k-12\beta_2c^2-4\beta_3c^3+\beta_4c^4),$$
$$q=-\frac{24}{A}(\beta_3-\beta_4c)(\gamma_1-\alpha_1c),\quad p=-\frac{24\gamma_2}{A}(\beta_3-\beta_4c).$$

Equation (1.3) becomes

$$(a-\phi^2)\phi''=2\phi(\phi')^2-r\phi-q\phi^3-p\phi^5. \quad (1.4)$$

Equation (1.4) is equivalent to the following integrable system

$$\frac{d\phi}{d\xi}=y,\quad \frac{dy}{d\xi}=\frac{2\phi y^2-r\phi-q\phi^3-p\phi^5}{a-\phi^2} \quad (1.5)$$

with the first integral

$$H(\phi,y)=(a-\phi^2)^2y^2+ar\phi^2+\frac{1}{2}(aq-r)\phi^4+\frac{1}{3}(ap-q)\phi^6-\frac{1}{4}p\phi^8=h. \quad (1.6)$$

When $a>0$, system (1.5) is a singular traveling wave system with the singular straight lines $\phi=\mp\sqrt{a}$ (see [2]).

The main result of this paper is the following conclusion.

Theorem 1.1. *Suppose that in equation* (1.1)*, the parameters satisfy the condition* $A=\beta_4(18\alpha_1+12\alpha_2)\neq 0$ *and in system* (1.5)*, the the parameters satisfy the conditions* $a>0,\Delta_1=q^2-6pr>0$. *Then, we have the following results.*

(i) When $p>0,q<0,r>0$, *as the parameter* a *is varied, system* (1.5) *has the bifurcations of phase portraits shown in Fig. 1(a)–(g). When* $p<0,q>0,r<0$, *as the parameter* a *is varied, system* (1.5) *has the bifurcations of phase portraits shown in Fig. 2(a)–(g).*

(ii) Under different parameter conditions, corresponding to the level curves defined by $H(\phi,y)=h_s$ *with different types, system* (1.5) *has periodic wave solutions, kink wave solutions, periodic peakon solutions, and peakon solutions with exact parametric representations given by* (3.4)–(3.23).

(iii) Equation (1.1) *has 20 exact traveling wave solutions* $E(z,t)=\phi(pz-t)\exp(i(kz-ct))=\phi(\xi)\exp(i(kz-ct))$, *where* $\phi(\xi)$ *is given by* (3.4)–(3.23).

The proof of the theorem 1.1 will be seen in next two sections.

This paper is organized as follows. In section 2, we consider the bifurcations of phase portraits of system (1.5). In section 3 we give all possible exact solutions of $\phi(\xi)$, under different parameter conditions.

2. Bifurcations of phase portraits of system (1.5) when there exist five equilibrium points on the ϕ−axis

We consider the following associated regular system of (1.5):

$$\frac{d\phi}{d\zeta} = (a-\phi^2)y, \quad \frac{dy}{d\zeta} = 2\phi y^2 - r\phi - q\phi^3 - p\phi^5 \equiv 2\phi y^2 - \phi f(\phi), \tag{2.1}$$

where $d\xi = (a-\phi^2)d\zeta$, $f(\phi) = r + q\phi^2 + p\phi^4$. System (2.1) has the same level curves as system (1.5), but systems (1.5) and (2.1) define different vector fields on the two sides of the singular straight lines $\phi = \phi_s$ and $\phi = -\phi_s$, where $\phi_s = \sqrt{a}$.

Obviously, system (2.1) has the equilibrium point $O(0,0)$. In addition, let $E_j(\phi_j, 0)$ be another equilibrium point of system (1.5). Then, we have $f(\phi_j) = 0$, i.e., ϕ_j is a zero point of $f(\phi)$.

When $pq < 0, pr > 0$ and $\Delta = q^2 - 4pr > 0$, on the ϕ−axis, system (1.5) has five equilibrium points $O(0,0)$, $E_{1\mp}(\mp\phi_1, 0)$ and $E_{2\mp}(\mp\phi_2, 0)$, where $\phi_{1,2} = \left(\frac{-q\mp\sqrt{q^2-4pr}}{2p}\right)^{\frac{1}{2}}$. When $pr < 0$ (or $r = 0$, $pq < 0$), system (1.5) has three equilibrium points.

When $Y_s = \frac{1}{2}f(\sqrt{a}) > 0$, system (2.1) has two equilibrium points $S_{1\mp}(-\phi_s, \mp\sqrt{Y_s})$ and $S_{2\mp}(\phi_s, \mp\sqrt{Y_s})$ on the two singular lines $\phi = \mp\phi_s$.

Let $M(\phi_j, 0)$ be the coefficient matrix of the linearized system of (2.1) at the equilibrium point $(\phi_j, 0)$. We have

$$J(0,0) = \det M(0,0) = ar, \quad J(\phi_{1,2}, 0) = \det M(\phi_{1,2}, 0) = (a - \phi_{1,2}^2)\phi_{1,2} f'(\phi_{1,2}),$$
$$J(\phi_s, \sqrt{Y_s}) = -8Y_s\phi_s^2 < 0.$$

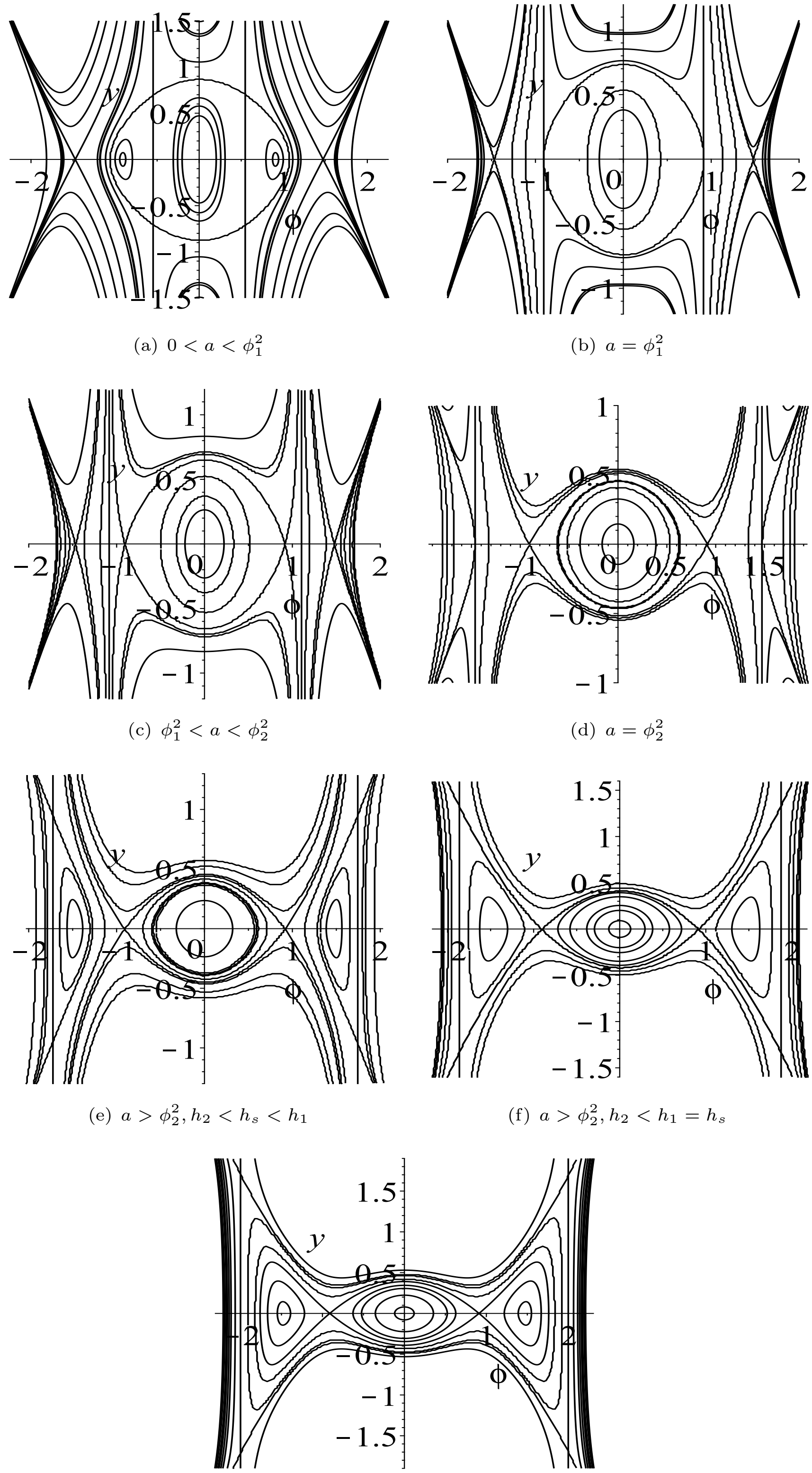
By the theory in the planar dynamical systems and using the above information, we can determine an equilibrium point is a center or saddle point.

For the first integral $H(\phi, y) = h$ defined by (1.6), we write that

$$h_0 = H(0,0) = 0, \quad h_s = H(\mp\phi_s, \mp\sqrt{Y_s}) = \frac{a^2}{12}(pa^2 + 2aq + 6r).$$
$$h_1 = H(\phi_1, 0) = \frac{\phi_1^2}{24p^2}\left[-(16arp^2 - 2apq^2 + 5pqr - q^3) + (2apq - 3pr + q^2)\sqrt{\Delta}\right],$$
$$h_2 = H(\phi_2, 0) = \frac{\phi_2^2}{24p^2}\left[(16arp^2 - 2apq^2 + 5pqr - q^3) + (2apq - 3pr + q^2)\sqrt{\Delta}\right].$$

It is easy to see from the right hand of h_s that if $\Delta_1 = q^2 - 6pr > 0$, then, when $a = \frac{1}{p}(-q \mp \sqrt{\Delta_1})$, we have $h_s = 0$.

We next consider more interesting cases. Assume that $a > 0, pr > 0, \Delta_1 > 0$ and for a fixed parameter group (p, q, r) such that system (2.1) has five equilibrium points on the ϕ−axis, then, by varying the parameter $a > 0$, i.e., by changing the relative positions between the singular straight lines $\phi = \mp\sqrt{a}$ and the equilibrium points $E_{j\mp}(\mp\phi_j, 0), (j = 1, 2), O(0,0)$, on the basis of qualitative analysis, under different parameter conditions, we have the bifurcations of phase portraits of system (2.1) shown in Fig. 1(a)–(g) and Fig. 2(a)–(g).



(a) $0 < a < \phi_1^2$

(b) $a = \phi_1^2$

(c) $\phi_1^2 < a < \phi_2^2$

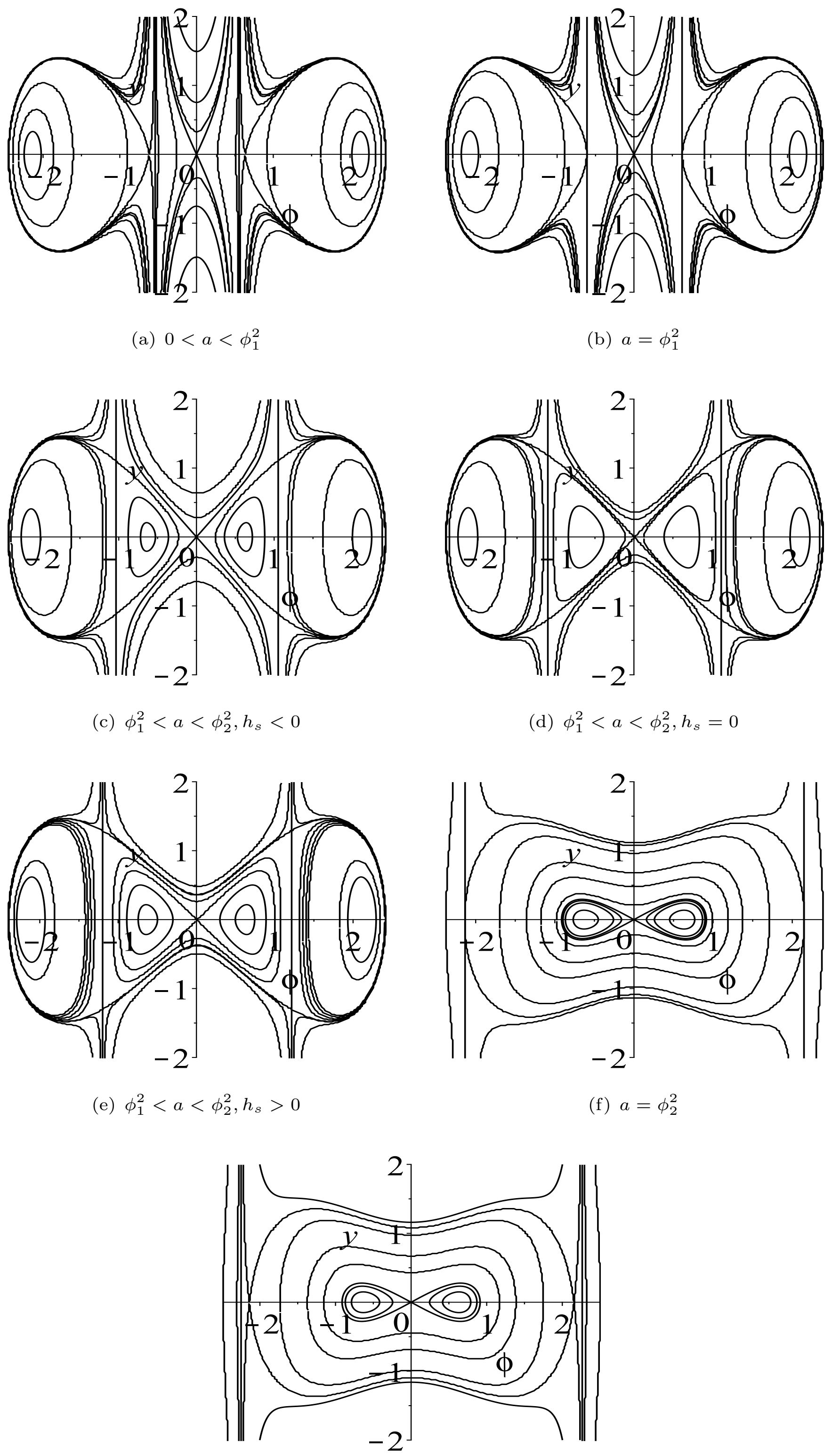
(d) $a = \phi_2^2$

(e) $a > \phi_2^2, h_2 < h_s < h_1$

(f) $a > \phi_2^2, h_2 < h_1 = h_s$

(g) $a > \phi_2^2, h_2 < h_1 < h_s$

Figure 1. The bifurcations of phase portraits of system (1.5) when $p > 0, q < 0, r > 0$ and $\Delta_1 > 0$.



(a) $0 < a < \phi_1^2$

(b) $a = \phi_1^2$

(c) $\phi_1^2 < a < \phi_2^2, h_s < 0$

(d) $\phi_1^2 < a < \phi_2^2, h_s = 0$

(e) $\phi_1^2 < a < \phi_2^2, h_s > 0$

(f) $a = \phi_2^2$

(g) $a > \phi_2^2$

Figure 2. The bifurcations of phase portraits of system (1.5) when $p < 0, q > 0, r < 0$ and $\Delta_1 > 0$.

3. Some exact traveling wave solutions of equation (1.1)

In this section, we calculate some possible exact parametric representations of the orbits defined by $H(\phi, y) = h$ of system (1.5) and give some traveling wave solutions for equation (1.1).

We see from (1.6) that

$$y^2 = \frac{p\phi^8 - \frac{4}{3}(pa-q)\phi^6 - 2(aq-r)\phi^4 - 4ar\phi^2 + 4h}{4(a-\phi^2)^2} \equiv \frac{pG(\phi)}{4(a-\phi^2)^2}. \tag{3.1}$$

By using the first equation of (1.5), we have

$$\xi = \int_{\phi_0}^{\phi} \frac{2|a-\phi^2|d\phi}{\sqrt{p\phi^8 - \frac{4}{3}(pa-q)\phi^6 - 2(aq-r)\phi^4 - 4ar\phi^2 + 4h}} \equiv \int_{\phi_0}^{\phi} \frac{2|a-\phi^2|d\phi}{\sqrt{pG(\phi)}}. \tag{3.2}$$

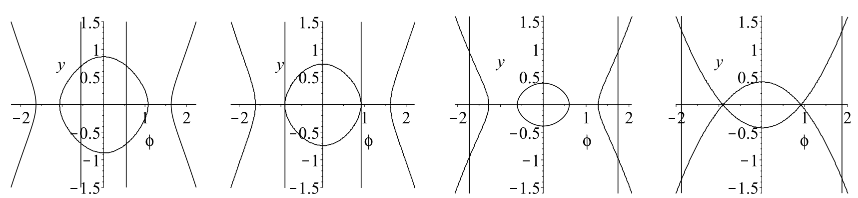
Making the transformation $\psi = \phi^2$, (3.2) becomes

$$\xi = \int_{\psi_0}^{\psi} \frac{|a-\psi|d\psi}{\sqrt{\psi\left[p\psi^4 - \frac{4}{3}(pa-q)\psi^3 - 2(aq-r)\psi^2 - 4ar\psi + 4h\right]}} \equiv \int_{\psi_0}^{\psi} \frac{2|a-\psi|d\psi}{\sqrt{p\psi\tilde{G}(\psi)}}. \tag{3.3}$$

Obviously, for a general h, we can not obtain the exact parametric representations for the level curves defined by (1.6) since $\psi\tilde{G}(\psi)$ is a fifth polynomial and the right hand of (3.3) is a hyperelliptic integral. Only in some special cases, we can get the exact parametric representations.

3.1. Assume that $p > 0, q < 0, r > 0$ and $\Delta_1 > 0$ (see Fig. 1(a)–(g)).

When $f(\sqrt{a}) > 0$, the level curves defined by $H(\phi, y) = h_s$ in Fig. 1 can be shown in Fig. 3 (a)-(d).



(a) $0 < a < \phi_1^2$ (b) $a = \phi_1^2$ (c) $a > \phi_2^2, h_s < h_1$ (d) $a > \phi_2^2, h_2 < h_1 = h_s$

Figure 3. The level curves defined by $H(\phi, y) = h_s$ of system (1.5).

(i) Corresponding to the level curves defined by $H(\phi, y) = h_s$ in Fig. 3 (a), there exists an oval orbit passing through two singular straight lines $\phi = \mp\phi_s$ and intersecting the $\phi-$axis at points $(\mp\phi_M, 0)$. In addition, there are two open orbits passing through the $\phi-$axis at points $(\mp\phi_L, 0)$. In this case, we have $G(\phi) = (\phi_L - \phi^2)(\phi_M - \phi^2)(a-\phi^2)^2$. Thus, for the oval orbit, (3.3) reduces to $\sqrt{p}\,\xi =$

$\int_{\psi}^{\psi_M} \frac{d\psi}{\sqrt{(\psi_L-\psi)(\psi_M-\psi)\psi}}$, where $\psi_L = \phi_L^2$ and $\psi_M = \phi_M^2$. It follows the parametric representation of a periodic solution of equation (1.1):

$$\begin{aligned} \phi(\xi) &= \left(\phi_L^2 - \frac{\phi_L^2-\phi_M^2}{\mathrm{dn}^2(\Omega_1\xi,k)}\right)^{\frac{1}{2}}, \quad \xi \in [-T_1, T_1], \\ \phi(\xi) &= -\left(\phi_L^2 - \frac{\phi_L^2-\phi_M^2}{\mathrm{dn}^2(\Omega_1\xi,k)}\right)^{\frac{1}{2}}, \quad \xi \in [T_1, 3T_1], \end{aligned} \tag{3.4}$$

where $\Omega_1 = \frac{1}{2}\phi_L\sqrt{p}, k^2 = \frac{\phi_M^2}{\phi_L^2}, T_1 = \frac{K(k)}{\Omega_1}$, and $K(k)$ is the complete elliptic integral of the first kind, $\mathrm{dn}(\cdot,k), \mathrm{sn}(\cdot,k), \mathrm{cn}(\cdot,k)$ are Jacobian elliptic functions (see [1]).

Notice that the area of the oval in Fig. 3 (a) is partitioned into three parts by the two singular straight lines $\phi = \mp\sqrt{a}$. The right arch is the limit curve of the family of periodic orbits of system (1.5) enclosing the equilibrium point $(\phi_1, 0)$, which gives rise to a lower periodic peakon solution (see Fig. 4 (b)) of equation (1.1) with the parametric representation

$$\phi(\xi) = \left(\phi_L^2 - \frac{\phi_L^2-\phi_M^2}{\mathrm{dn}^2(\Omega_1\xi,k)}\right)^{\frac{1}{2}}, \quad \xi \in [-\xi_{01}, \xi_{01}), \tag{3.5}$$

where $\xi_{01} = \mathrm{dn}^{-1}\left(\sqrt{\frac{\phi_L^2-\phi_M^2}{\phi_L^2-a}}, k\right)$. The left arch is the limit curve of the family of periodic orbits of system (1.5) enclosing the equilibrium point $(-\phi_1, 0)$, which gives rise to a upper periodic peakon solution (see Fig. 4 (a)) of equation (1.1) with the parametric representation

$$\phi(\xi) = -\left(\phi_L^2 - \frac{\phi_L^2-\phi_M^2}{\mathrm{dn}^2(\Omega_1\xi,k)}\right)^{\frac{1}{2}}, \quad \xi \in [-\xi_{01}, \xi_{01}], \tag{3.6}$$

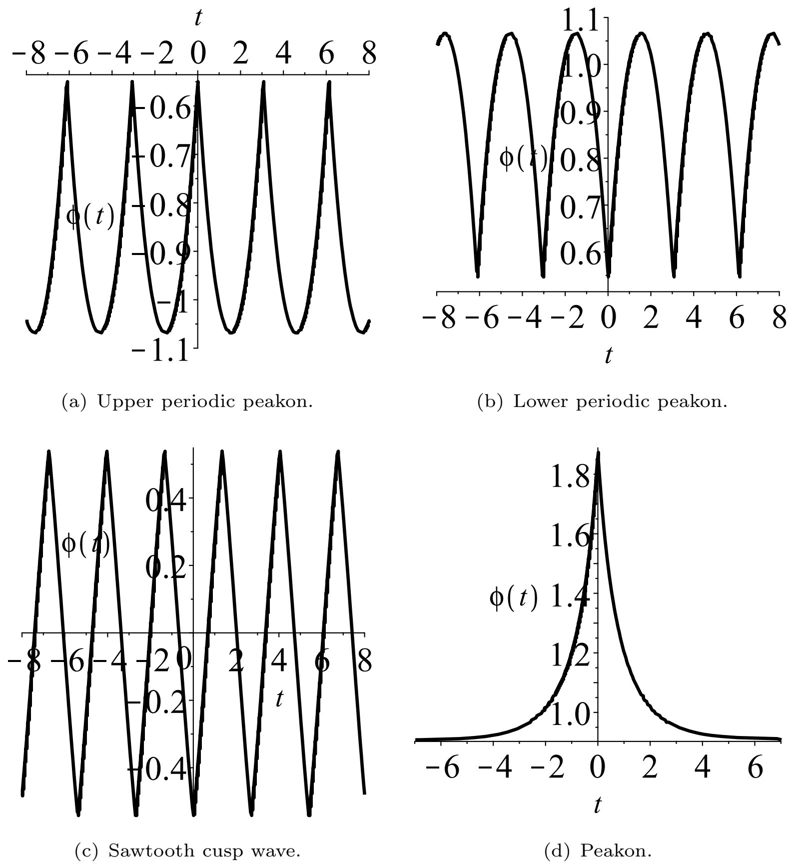
The middle two curves are the limit curves of the family of periodic orbits of system (1.5) enclosing the equilibrium point $O(0,0)$, they give rise to a sawtooth cusp wave solution (see Fig. 4 (c)) of equation (1.1).

(ii) Corresponding to the level curves defined by $H(\phi, y) = h_s$ in Fig. 3 (b), there exists an oval orbit tangent to two singular straight lines $\phi = \mp\sqrt{a}$ at the points $(\mp\phi_1, 0)$. In addition, there are two open orbits passing through the $\phi-$axis at the points $(\mp\phi_L, 0)$. In this case, we have $G(\phi) = (\phi_L - \phi^2)(\phi_1 - \phi^2)^3$. Thus, for the oval orbit, (10) reduces to $\sqrt{p}\,\xi = \int_{\psi}^{\psi_1} \frac{d\psi}{\sqrt{(\psi_L-\psi)(\psi_1-\psi)\psi}}$, where $\psi_L = \phi_L^2$ and $\psi_1 = \phi_1^2$. It follows the parametric representation of a periodic solution of equation (1.1):

$$\begin{aligned} \phi(\xi) &= \left(\phi_L^2 - \tfrac{\phi_L^2-\phi_1^2}{\mathrm{dn}^2(\Omega_2\xi,k)}\right)^{\frac{1}{2}}, \quad \xi \in [-T_2, T_2], \\ \phi(\xi) &= -\left(\phi_L^2 - \tfrac{\phi_L^2-\phi_1^2}{\mathrm{dn}^2(\Omega_2\xi,k)}\right)^{\frac{1}{2}}, \quad \xi \in [T_2, 3T_2], \end{aligned} \tag{3.7}$$

where $\Omega_2 = \frac{1}{2}\phi_L\sqrt{p}, k^2 = \frac{\phi_1^2}{\phi_L^2}, T_2 = \frac{K(k)}{\Omega_2}$. $k^2 = \frac{\phi_1^2}{\phi_L^2}$.

(iii) Corresponding to the level curves defined by $H(\phi, y) = h_s$ in Fig. 3 (c), there exist a closed orbit intersecting the $\phi-$axsis at two points $(\mp\phi_l, 0)$ and two arches enclosing the equilibrium points $(\mp\phi_2, 0)$ and intersecting the $\phi-$axis at the points $(\mp\phi_m, 0)$, respectively. In this case, we have $G(\phi) = (a - \phi^2)^2(\phi^2 - \phi_m^2)(\phi^2 - \phi_l^2)$.



(a) Upper periodic peakon. (b) Lower periodic peakon.

(c) Sawtooth cusp wave. (d) Peakon.

Figure 4. The wave profiles given by the functions $\phi(\xi)$ of system (1.5)

For the periodic orbit, (3.3) becomes $\sqrt{p}\,\xi = \int_{\psi}^{\psi_l} \frac{d\psi}{\sqrt{(\psi_m-\psi)(\psi_l-\psi)\psi}}$ where $\psi_m = \phi_m^2$ and $\psi_l = \phi_l^2$. Hence, we obtain the following periodic solution of equation (1.1):

$$\begin{aligned} \phi(\xi) &= \left(\phi_m^2 - \frac{\phi_m^2 - \phi_l^2}{\mathrm{dn}^2(\Omega_3\xi, k)}\right)^{\frac{1}{2}}, \quad \xi \in [-T_3, T_3], \\ \phi(\xi) &= -\left(\phi_m^2 - \frac{\phi_m^2 - \phi_l^2}{\mathrm{dn}^2(\Omega_3\xi, k)}\right)^{\frac{1}{2}}, \quad \xi \in [T_3, 3T_3], \end{aligned} \tag{3.8}$$

where $\Omega_3 = \frac{1}{2}\phi_m\sqrt{p}, k^2 = \frac{\phi_l^2}{\phi_m^2}, T_3 = \frac{K(k)}{\Omega_3}$.

For the right arch orbit, (3.3) becomes $\sqrt{p}\,\xi = \int_{\psi_m}^{\psi} \frac{d\psi}{\sqrt{(\psi-\psi_m)(\psi-\psi_l)\psi}}$. It gives the following lower periodic peakon solution:

$$\phi(\xi) = \left(\phi_m^2 + (\phi_m^2 - \phi_l^2)\mathrm{tn}^2(\Omega_3\xi, k)\right)^{\frac{1}{2}}, \quad \xi \in (-\xi_{02}, \xi_{02}), \tag{3.9}$$

where $\xi_{02} = \mathrm{tn}^{-1}\left(\sqrt{\frac{a-\phi_m^2}{\phi_m^2-\phi_l^2}}, k\right)$.

For the left arch orbit, we have

$$\phi(\xi) = -\left(\phi_m^2 + (\phi_m^2 - \phi_l^2)\mathrm{tn}^2(\Omega_3\xi, k)\right)^{\frac{1}{2}}, \quad \xi \in (-\xi_{02}, \xi_{02}). \tag{3.10}$$

It gives rise to an upper periodic peakon solution.

(iv) Corresponding to the level curves defined by $H(\phi, y) = h_s$ in Fig. 3 (d), there exist two heteroclinic orbits connecting the equilibrium points $(-\phi_1, 0)$ and $(\phi_1, 0)$, and two curve triangles enclosing the equilibrium points $(-\phi_2, 0)$ and $(\phi_2, 0)$, respectively. Now, we have $G(\phi) = (a - \phi^2)^2(\phi_1^2 - \phi^2)^2$. For the heteroclinic orbits, (3.3) has the form $\sqrt{p}\,\xi = \int_0^{\psi} \frac{d\psi}{(\psi_1 - \psi)\sqrt{\psi}}$. It gives rise to the following kink and anti-kink solutions of equation (1.1):

$$\phi(\xi) = \pm\phi_1 \tanh\left(\frac{1}{2}\sqrt{p}\phi_1\xi\right). \tag{3.11}$$

For two boundary curves of the two curve triangles, (3.3) becomes that $\sqrt{p}\xi = \int_{\psi}^{a} \frac{d\psi}{(\psi - \psi_1)\sqrt{\psi}}$. We obtain the following peakon (see Fig.4 (d)) and unti-peakon solutions of equation (1.1):

$$\phi(\xi) = \pm\phi_1\left(\frac{e^{\omega_0|\xi|} + q_0 e^{-\omega_0|\xi|}}{e^{\omega_0|\xi|} - q_0 e^{-\omega_0|\xi|}}\right) \equiv \pm\phi_1 \mathrm{ctnh}_{q_0}(\omega_0|\xi|), \tag{3.12}$$

where $\omega_0 = \frac{1}{2}\sqrt{p}\phi_1, q_0 = \frac{\sqrt{a} - \phi_1}{\sqrt{a} + \phi_1}$, and $\mathrm{ctnh}_{q_0}(\omega_0|\xi|)$ is a generalized hyperbolic function.

3.2. Assume that $p < 0, q > 0, r < 0$ and $\Delta_1 > 0$ (see Fig. 2(a)–(g)).

When $f(\sqrt{a}) > 0$, the level curves defined by $H(\phi, y) = h_s$ in Fig. 2 can be shown in Fig. 5 (a)–(e).

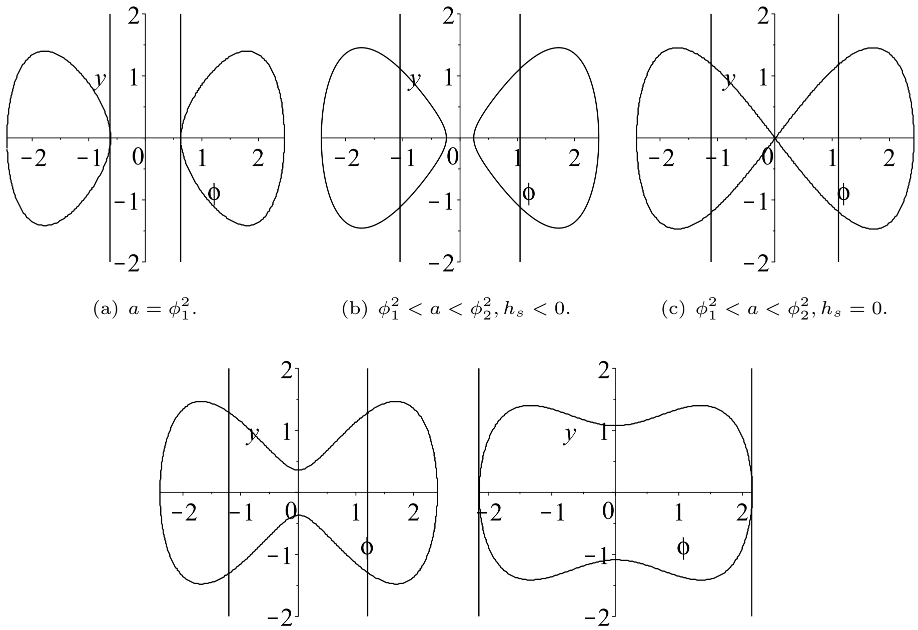
(i) Corresponding to the level curves defined by $H(\phi, y) = h_s$ in Fig. 5 (a), there exist two oval orbits contacting to the singular straight lines $\phi = \mp\sqrt{a}$ and intersecting the $\phi-$axis at the points $(\mp\phi_M, 0)$, respectively. In this case, we have $G(\phi) = (\phi^2 - a)^3(\phi_M^2 - \phi^2)$. Now, (1.1) becomes $\sqrt{|p|}\,\xi = \int_{\psi}^{\psi_M} \frac{d\psi}{\sqrt{(\psi_M - \psi)(\psi - a)\psi}}$. It implies the two periodic solutions of equation (1.1):

$$\phi(\xi) = \mp\left(\phi_M^2 - (\phi_M^2 - a)\mathrm{sn}^2(\Omega_4\xi, k)\right)^{\frac{1}{2}}, \tag{3.13}$$

where $\Omega_4 = \frac{1}{2}\sqrt{|p|}\phi_M$ and $k^2 = \frac{\phi_M^2 - a}{\phi_M^2}$.

(ii) Corresponding to the level curves defined by $H(\phi, y) = h_s$ in Fig. 5 (b), there exist two oval orbits passing through the singular straight lines $\phi = \mp\sqrt{a}$ and enclosing the equilibrium points $(\mp\phi_1, 0)$ and $(\mp\phi_2, 0)$, respectively. Meanwhile, the two oval orbits intersect the $\phi-$axis at four points $(\mp\phi_m, 0)$ and $(\mp\phi_M, 0)$. Now, we have $G(\phi) = (\phi_M^2 - \phi^2)(a - \phi^2)^2(\phi^2 - \phi_m^2)$ and the integral (3.3) has the form $\sqrt{|p|}\,\xi = \int_{\psi}^{\psi_M} \frac{d\psi}{\sqrt{(\psi_M - \psi)(\psi - \psi_m)\psi}}$. It gives rise to two periodic solutions of equation (1.1):

$$\phi(\xi) = \mp\left(\phi_M^2 - (\phi_M^2 - \phi_m^2)\mathrm{sn}^2(\Omega_4\xi, k)\right)^{\frac{1}{2}}, \tag{3.14}$$



(a) $a = \phi_1^2$. (b) $\phi_1^2 < a < \phi_2^2, h_s < 0$. (c) $\phi_1^2 < a < \phi_2^2, h_s = 0$.

(d) $\phi_1^2 < a < \phi_2^2, h_s > 0$. (e) $a = \phi_2^2$

Figure 5. The level curves defined by $H(\phi, y) = h_s$ of system (1.5).

where $\Omega_4 = \frac{1}{2}\sqrt{|p|}\phi_M, k^2 = \frac{\phi_M^2 - \phi_m^2}{\phi_M^2}$.

Consider the arch in the right side of the singular straight line $\phi = \sqrt{a}$ which is the limit curve of a family of periodic orbits enclosing the equilibrium point $(\phi_2, 0)$. We see from (3.14) that the arch curve has the following parametric representation:

$$\phi(\xi) = \left(\phi_M^2 - (\phi_M^2 - \phi_m^2)\mathrm{sn}^2(\Omega_4\xi, k)\right)^{\frac{1}{2}}, \quad \xi \in (-\xi_{03}, \xi_{03}), \tag{3.15}$$

where $\xi_{03} = \frac{1}{\Omega_4}\mathrm{sn}^{-1}\left(\sqrt{\frac{\phi_M^2 - a}{\phi_M^2 - \phi_m^2}}, k\right)$. (3.15) defines an anti-periodic peakon solution of equation (1.1) (similar to Fig. 3 (b)).

Similarly, corresponding to the arch in the left side of the singular straight line $\phi = -\sqrt{a}$, the the parametric representation

$$\phi(\xi) = \left(\phi_M^2 - (\phi_M^2 - \phi_m^2)\mathrm{sn}^2(\Omega_4\xi, k)\right)^{\frac{1}{2}}, \quad \xi \in \left(\frac{K(k)}{\Omega_4} - \xi_{03}, \frac{K(k)}{\Omega_4} + \xi_{03}\right) \tag{3.16}$$

gives rise to a periodic peakon solution of equation (1.1) (similar to Fig. 3 (a)).

(iii) Corresponding to the level curves defined by $H(\phi, y) = h_s$ in Fig. 5 (c), there exist two homoclinic orbits to the origin $O(0,0)$, which passing through two singular straight lines $\phi = \mp\sqrt{a}$ and enclosing the equilibrium points $(\mp\phi_1, 0)$ and $(\mp\phi_2, 0)$, respectively. In this case, $G(\phi) = (\phi_M^2 - \phi)(\phi^2 - a)^2\phi^2$. For these two homoclinic orbits, (3.3) becomes $\sqrt{|p|}\xi = \int_\psi^{\psi_M} \frac{d\psi}{\psi\sqrt{\psi_M - \psi}}$. Thus, we obtain the following two solitary wave solutions of equation (1.1):

$$\phi(\xi) = \pm\phi_M \mathrm{sech}\,(\omega_1\xi), \tag{3.17}$$

where $\omega_1 = \frac{1}{2}\sqrt{|p|}\phi_M$.

We notice that for two curve triangles enclosing the equilibrium points $(\mp\phi_1, 0)$, respectively, (3.3) becomes $\sqrt{|p|}\,\xi = \int_{\psi}^{\sqrt{a}} \frac{d\psi}{\psi\sqrt{\psi_M - \psi}}$. Hence, it gives rise to a peakon and an anti-peakon solutions of equation (1.1) having the parametric representations:

$$\phi(\xi) = \pm\phi_M \left(1 - \left(\frac{e^{\omega_1\xi} + q_1 e^{-\omega_1\xi}}{e^{\omega_1\xi} - q_1 e^{-\omega_1\xi}}\right)^2\right)^{\frac{1}{2}} \equiv \pm\phi_M\sqrt{|q_1|}\mathrm{csch}_{q_1}(\omega_1\xi), \tag{3.18}$$

where $q_1 = \frac{\sqrt{\phi_M^2 - a} - \phi_M}{\sqrt{\phi_M^2 - a} + \phi_M}$ and $\mathrm{csch}_{q_1}(\omega_1|\xi|)$ is a generalized hyperbolic function.

The two arches enclosing the equilibrium points $(\mp\phi_2, 0)$, respectively, give rise to two periodic peakon solutions of equation (1.1) having the parametric representations:

$$\phi(\xi) = \pm\phi_M \mathrm{sech}\,(\omega_1\xi)\,, \quad \xi \in (-\xi_{04}, \xi_{04}), \tag{3.19}$$

where $\xi_{04} = \frac{2}{\sqrt{|p|}\phi_M}\mathrm{sech}^{-1}\left(\frac{\sqrt{a}}{\phi_M}\right)$.

(iv) Corresponding to the level curves defined by $H(\phi, y) = h_s$ in Fig. 5 (d), there exists a closed orbit passing through two singular straight lines $\phi = \mp\sqrt{a}$ and enclosing five equilibrium points $O(0,0), (\mp\phi_1, 0)$ and $(\mp\phi_2, 0)$. Now, we have $G(\phi) = (\phi_M^2 - \phi)(\phi^2 - a)^2(\phi^2 + \phi_i^2)$. The formula (3.3) now becomes $\sqrt{|p|}\,\xi = \int_{\psi}^{\psi_M} \frac{d\psi}{\sqrt{(\psi_M - \psi)\psi(\psi + \psi_i)}}$. It gives rise to the following periodic solution of equation (1.1):

$$\phi(\xi) = \phi_M \mathrm{cn}(\Omega_5\xi, k), \tag{3.20}$$

where $\Omega_5 = \frac{1}{2}\sqrt{|p|}(\phi_M^2 + \phi_i^2)$ and $k^2 = \frac{\phi_M^2}{\phi_M^2 + \phi_i^2}$.

The arch enclosing the equilibrium points $(\phi_2, 0)$ gives rise to an anti-peakon solution which have the parametric representation:

$$\phi(\xi) = \phi_M \mathrm{cn}(\Omega_5\xi, k), \quad \xi \in (-\xi_{05}, \xi_{05}), \tag{3.21}$$

where $\xi_{05} = \frac{1}{\Omega_5}\mathrm{cn}^{-1}\left(\frac{a}{\phi_M}, k\right)$.

The curve quadrilateral enclosing the equilibrium points $O(0,0)$ gives rise to a sawtooth cusp wave solution as follows:

$$\phi(\xi) = \phi_M \mathrm{cn}(\Omega_5\xi, k), \quad \xi \in \left(-\frac{2K(k)}{\Omega_5} + \xi_{05}, -\frac{3K(k)}{\Omega_5} + \xi_{05}\right),\ \left(\xi_{05}, \frac{K(k)}{\Omega_5} + \xi_{05}\right). \tag{3.22}$$

(v) Corresponding to the level curves defined by $H(\phi, y) = h_s$ in Fig. 5 (e), there exists a closed orbits contacting to two singular straight lines $\phi = \mp\sqrt{a}$ and eclosing five equilibrium points $O(0,0), (\mp\phi_1, 0)$ and $(\mp\phi_2, 0)$. Now, we have $G(\phi) = (a - \phi^2)^3(\phi^2 + \phi_i^2)$. Formula (3.3) becomes that $\sqrt{|p|}\xi = \int_{\psi}^{a} \frac{d\psi}{\sqrt{(a-\psi)\psi(\psi + \psi_i)}}$. It gives rise to the following periodic solution of equation (1.1):

$$\phi(\xi) = \sqrt{a}\mathrm{cn}(\Omega_6\xi, k), \tag{3.23}$$

where $\Omega_6 = \frac{1}{2}\sqrt{|p|}(a + \phi_i^2)$ and $k^2 = \frac{a}{a + \phi_i^2}$.

4. Acknowledgements

The authors would like to express their sincere gratitude to Prof. Jibin Li for his kindly help.

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