Journal of Applied Analysis and Computation
Volume 10, Number 6, December 2020, 2711–2721
Website:http://www.jaac-online.com
DOI:10.11948/20200084

STABILITY AND HOPF BIFURCATION ANALYSIS ON A SPRUCE-BUDWORM MODEL WITH DELAY

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Abstract In this paper, the dynamics of a spruce-budworm model with delay is investigated. We show that there exists Hopf bifurcation at the positive equilibrium as the delay increases. Some sufficient conditions for the existence of Hopf bifurcation are obtained by investigating the associated characteristic equation. By using the theory of normal form and center manifold, explicit expression for determining the direction of Hopf bifurcations and the stability of bifurcating periodic solutions are presented.

Keywords Spruce-budworm, Hopf bifurcation, normal form, center manifold.

MSC(2010) 34K18, 34K20, 34K25.

1. Introduction

The outbreaks of insect pests have significant impacts on the forest ecosystems. Eastern spruce budworm is a destructive insect living in the spruce and fir forest of North America. Every 35-40 years there is an outbreak of these insects, resulting in serious defoliation. Fortunately, trees are hardly killed on account of defoliation, but they need 7 to 10 years to replace their foliage. However, the periodical outbreaks of these insects lasting for about 10 years cause billions of dollars loss to forest industry. Therefore, how to control the growth of budworm and protect spruce and fir forest is of great importance. Birds preying on budworms always have a flexible diet, that is, they search for other food resources when the density of budworm under a certain threshold. C.S. Holling [6] believed that the switching functional response of predators facilitate the budworm outbreaks. To understand the dynamics of spruce budworm population, Ludwing et al [9] proposed a model separating the timescales of slow spruce regrowth versus the fast population dynamics of budworm larve and their predators. For the food limited dynamics of budworms in the absence of predators the so-called logistic model is applied for the budworm population density $N(t)$

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \tag{1.1}$$

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where r and K are the intrinsic rate and the carrying capacity of the population, respectively. May [11] further reduced the model proposed in [9] to the following two differential equations

$$\begin{aligned} \tfrac{dN}{dt} &= rN\left(1-\tfrac{N}{KS}\right)-\beta\tfrac{PN^2}{\eta^2S^2+N^2}, \\ \tfrac{dS}{dt} &= \rho S\left(1-\tfrac{S}{S_{max}}\right)-\delta N, \end{aligned} \tag{1.2}$$

where the variables N and S are the spruce budworm population density and average leaf area per tree, respectively. Refer to [11] for the biological meaning of other parameters. Following [11], Rasmusse et al [14] studied a relaxation oscillation of this system by using singular perturbation theory. Liu et al [8] and Muratori [12] pointed out that the existence of such relaxation implies that the coexistence of predators and prey periodically alternated. Taking into account time delay, Wang and Han [18] proposed a system with distributed delay based on system (1.2). By using geometric singular perturbation theory, they illustrated the existence of the relaxation oscillation and transition of the solution trajectory. Motivated by [18] and taking into the consideration the fact that spruce leaf needs some time to grow, we apply the following system

$$\begin{aligned} \tfrac{dN}{dt} &= rN\left(1-\tfrac{N}{KS}\right)-\beta\tfrac{PN^2}{\eta^2S^2+N^2}, \\ \tfrac{dS}{dt} &= \rho S\left(1-\tfrac{S(t-\tau)}{S_{max}}\right)-\delta N, \end{aligned} \tag{1.3}$$

to describe the dynamics of interactions between a predator and a prey specie with a delay in the prey population $S(t)$ at time t. A number of nonlinear model equations with time delay [2–4, 17, 18] have been investigated in recent decades.

Let $z = \frac{\rho}{P\beta}N$, $y = \frac{K\rho}{P\beta}S$, $\bar{t} = \rho t$, $\bar{\tau} = \rho\tau$. Dropping the bar for conciseness, then (1.3) can be rewritten as

$$\frac{dz}{dt} = f(z,y), \quad \frac{dy}{dt} = y\left(1-\frac{y(t-\tau)}{m}-\frac{z}{y}Q\right), \tag{1.4}$$

where $\gamma = \frac{r}{\rho}$, $\alpha = \frac{\eta}{K}$, $m = \frac{K\rho S_{max}}{P\beta}$, $Q = \frac{K\delta}{\rho}$ and $f(z,y) = \gamma z\left(1-\frac{z}{y}\right)-\frac{\gamma z^2}{\alpha^2y^2+z^2}$. It has been shown in [14] that system (1.4) has exactly one equilibrium when $\alpha^2 \geq \frac{1}{27}$ and can has one, two or three equilibrium points when $0 < \alpha^2 < \frac{1}{27}$. In this paper, we will analysis the effect of delay τ on the stability of the positive equilibrium of the system and investigate the Hopf bifurcation by normal form and center manifold theory. Here we will focus on the influence of time delay on bifurcations when $\alpha^2 \geq \frac{1}{27}$, however the case when $0 < \alpha^2 < \frac{1}{27}$ will be considered in the future.

2. Stability of equilibrium and existence of Hopf bifurcations

In this section, we focus on the investigation of local stability and Hopf bifurcation criteria of the positive equilibrium $E(z_0, y_0)$ for system (1.4). Letting $z_1 = z(t) - z_0$, $z_2 = y(t) - y_0$, we rewrite system (1.4) by Taylor series expression at $E(z_0, y_0)$ as follows:

$$
\begin{aligned}
z_1^{'}(t) &= \gamma r_1 z_1(t) + \gamma r_2 z_2(t) + \sum_{i,j\geq 2} \tfrac{1}{i!j!} f_{ij} z_1^i(t) z_2^j(t),\\
z_2^{'}(t) &= -Q z_1(t) + x_0 Q z_2(t) - \tfrac{y_0}{m} z_2(t-\tau),
\end{aligned}
\tag{2.1}
$$

where $x_0 = \frac{z_0}{y_0}$, $r_1 = -x_0 + \frac{x_0}{y_0}\frac{x_0^2-\alpha^2}{(\alpha^2+x_0^2)^2}$, $r_2 = x_0^2 + \frac{1}{y_0}\frac{2x_0^2\alpha^2}{(\alpha^2+x_0^2)^2}$. The characteristic equation of the delay system takes the form

$$
J \equiv \begin{vmatrix} \lambda - \gamma r_1 & -\gamma r_2 \\ Q & \lambda - \left(x_0 Q - \frac{y_0}{m} e^{-\lambda\tau}\right) \end{vmatrix} = 0,
\tag{2.2}
$$

i.e.

$$
\lambda^2 - (r_1\gamma + Qx_0)\lambda + r_1\gamma Q x_0 + Q\gamma r_2 + \frac{y_0}{m}(\lambda - r_1\gamma)e^{-\lambda\tau} = 0.
\tag{2.3}
$$

When the time delay $\tau = 0$, the characteristic equation (2.3) becomes

$$
\lambda^2 - \left(r_1\gamma + Qx_0 - \frac{y_0}{m}\right)\lambda + r_1\gamma\left(Qx_0 - \frac{y_0}{m}\right) + Q\gamma r_2 = 0.
\tag{2.4}
$$

Clearly, the roots of (2.4) must have negative real parts when $r_1\gamma + x_0 Q - \frac{y_0}{m} < 0$ and $r_1\gamma\left(x_0 Q - \frac{y_0}{m}\right) + \gamma r_2 Q > 0$. For $\tau > 0$, let $\lambda = i\omega(\omega > 0)$ be a root of (2.3), then we have

$$
-\omega^2 - i(r_1\gamma + Qx_0)\omega + r_1\gamma x_0 Q + Q\gamma r_2 + \frac{y_0}{m}(i\omega - r_1\gamma)e^{-i\omega\tau} = 0.
\tag{2.5}
$$

Separating the real and imaginary parts, we get

$$
\begin{aligned}
&-\omega^2 + r_1\gamma x_0 Q + Q\gamma r_2 + \frac{y_0}{m}(\omega sin\omega\tau - r_1\gamma cos\omega\tau) = 0,\\
&\omega(r_1\gamma + Qx_0) - \frac{y_0}{m}(\omega cos\omega\tau + r_1\gamma sin\omega\tau) = 0,
\end{aligned}
\tag{2.6}
$$

which leads to the following polynomial equation

$$
\omega^4 - [2Q\gamma(r_1x_0+r_2) + \frac{y_0^2}{m^2} - (r_1y+Qx_0)^2]\omega^2 + Q^2\gamma^2(r_1x_0+r_2)^2 - \frac{y_0^2}{m^2}r_1^2\gamma^2 = 0.
\tag{2.7}
$$

For the case when

$$
r_1\gamma\left(Qx_0 - \frac{y_0}{m}\right) + Q\gamma r_2 = Q\gamma(r_1x_0 + r_2) - r_1\gamma\frac{y_0}{m} > 0,
$$

we can see easily that equation (2.7) has only one positive root

$$
\omega_0 = \sqrt{\frac{b + \sqrt{b^2 - 4c}}{2}},
$$

where $b = 2Q\gamma(r_1x_0 + r_2) + \frac{y_0^2}{m^2} - (r_1\gamma + Qx_0)^2$ and $c = Q^2\gamma^2(r_1x_0 + r_2)^2 - \frac{y_0^2}{m^2}r_1^2\gamma^2$. Substituting the value of ω_0 in (2.6) yields

$$
\tau_j = \frac{1}{\omega_0}\left(arccos\left(\frac{\omega_0^2 Qx_0 + Q\gamma^2 r_1(r_1x_0 + r_2)}{\frac{y_0}{m}(r_1^2\gamma^2 + \omega_0^2)}\right) + 2j\pi\right),\ j = 0, 1, 2, \cdots.
$$

Hence the characteristic equation (2.3) has a pair of purely imaginary roots $\pm i\omega_0$ when $\tau = \tau_j$.

Let $\lambda(\tau) = r(\tau) + i\omega(\tau)$ be a root of (2.3) such that $r(\tau_j) = 0$ and $\omega(\tau_j) = \omega_0$. Substituting $\lambda(\tau) = r(\tau) + i\omega(\tau)$ into (2.3) and differentiating the resulting expression with respect to τ, we get

$$2\lambda\frac{d\lambda}{d\tau} - (r_1\gamma + Qx_0)\frac{d\lambda}{d\tau} + \frac{y_0}{m}e^{-\lambda\tau}\frac{d\lambda}{d\tau} + \frac{y_0}{m}(\lambda - r_1\gamma)e^{-\lambda\tau}(-\lambda - \tau\frac{d\lambda}{d\tau}) = 0.$$

Consequently, it holds

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda - (r_1\gamma + Qx_0))\, e^{\lambda\tau} + \frac{y_0}{m}(1 - (\lambda - r_1\gamma)\tau)}{\frac{y_0}{m}\lambda(\lambda - r_1\gamma)}\ . \tag{2.8}$$

Inserting $\tau = \tau_j$ into (2.8) and careful computing, one has

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_j} = \frac{m^2\sqrt{b^2 - 4c}}{y_0^2(\omega_0^2 + r_1^2\gamma^2)} > 0, \tag{2.9}$$

which implies that the transversality condition holds, so Hopf bifurcation occurs at $\tau = \tau_0$.

3. Direction and stability of Hopf bifurcations

In the previous section, we have obtained the conditions which guarantee that spruce-budworm model with delay (1.3) undergo Hopf bifurcation at some critical values of τ. We now apply the center manifold theory and normal form method proposed by Hassard et al [5] to study the stability of the bifurcated periodic solutions and the direction of these Hopf bifurcations.

Rescale time by $t \to \frac{t}{\tau}$ to normalize the delay and let $\tau = \tau^* + \mu,\ \mu \in \mathbb{R}$ and $\tau^* \in \{\tau_j\}$, then $\mu = 0$ is the Hopf bifurcation value and system (2.1) can be rewritten as the following functional differential equation

$$\dot{x}(t) = L_\mu(x_t) + F(\mu, x_t), \tag{3.1}$$

where $x_t(\theta) = x(t+\theta) \in \mathbb{C} = C([-1,0], \mathbb{R}^2)$, and $L_\mu : \mathbb{C} \to \mathbb{R}^2$, $F : \mathbb{R} \times \mathbb{C} \to \mathbb{R}^2$ are given by

$$L_\mu(\phi) = (\tau^* + \mu)\begin{pmatrix} \gamma r_1\varphi_1(0) + \gamma r_2\varphi_2(0) \\ \frac{1}{m}\varphi_1(0) + Qx_0\varphi_2(0) - \frac{y_0}{m}\varphi_2(-1) \end{pmatrix} \tag{3.2}$$

and

$$F(\mu, \phi) = (\tau^* + \mu)\begin{pmatrix} \sum\limits_{i+j\geq 2} \frac{1}{i!j!}F_{i,j}\varphi_1^i(0)\varphi_1^j(0) \\ 0 \end{pmatrix}. \tag{3.3}$$

By the Reisz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \text{ for } \phi \in C. \tag{3.4}$$

In fact, we can choose

$$\eta(\theta,\mu)=(\tau^*+\mu)\begin{pmatrix}\gamma r_1 & \gamma r_2\\ -Q & Qx_0\end{pmatrix}\delta(\theta)-(\tau^*+\mu)\begin{pmatrix}0 & 0\\ 0 & -\frac{y_0}{m}\end{pmatrix}\delta(\theta+1), \quad (3.5)$$

where

$$\delta(\theta)=\begin{cases}0, & \theta\neq 0,\\ 1, & \theta=0.\end{cases}$$

For $\phi\in C^1([-1,0],\mathbb{R}^2)$, define

$$A(\mu)\phi=\begin{cases}\frac{d\phi}{d\theta}, & \theta\in[-1,0),\\ \int_{-1}^{0}[d\eta(s,\mu)]\phi(s), & \theta=0,\end{cases} \quad (3.6)$$

and

$$R(\mu)\phi=\begin{cases}0, & \theta\in[-1,0),\\ F(\mu,\phi), & \theta=0.\end{cases} \quad (3.7)$$

Then (3.1) can be rewritten as

$$\dot{x}(t)=A(\mu)x_t+R(\mu)x_t. \quad (3.8)$$

For $\psi\in C^1([0,1],(\mathbb{R}^2)^*)$, define

$$A^*\psi(s)=\begin{cases}-\frac{d\psi}{ds}, & s\in(0,1],\\ \int_{-1}^{0}[d\eta^T(t,0)]\psi(-t), & s=0.\end{cases} \quad (3.9)$$

and the adjoint bilinear form on $C^*\times C$ as follows:

$$<\psi(s),\phi(\theta)>=\bar{\psi}(0)\phi(0)-\int_{-1}^{0}\int_{\xi}^{\theta}\bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \quad (3.10)$$

where $\eta(\theta)=\eta(\theta,0)$. Then A^* and $A(0)$ are adjoint operators, and $\pm i\tau^*\omega_0$ are eigenvalues of $A(0)$. It is clear that they are also eigenvalues of A^* .

Suppose that $q(\theta)=q(0)e^{i\omega_0\tau^*\theta}$ is an eigenvector of $A(0)$ corresponding to $i\omega_0\tau^*$, where $q(0)=(q_1,q_2)^T$. From (3.4), (3.5) and (3.6), we get

$$\tau^*\begin{pmatrix}i\omega_0-\gamma r_1 & -\gamma r_2\\ Q & i\omega_0-\left(Qx_0-\frac{y_0}{m}e^{-i\omega_0\tau^*}\right)\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}. \quad (3.11)$$

By direct computation, we get

$$q_1=1,\ q_2=\frac{i\omega_0-\gamma r_1}{\gamma r_2}.$$

It is easy to verify that $q^* = D(q_1^*, q_2^*)e^{-i\omega_0\tau^*\theta}$ is an eigenvector of A^* corresponding to $-i\omega_0\tau^*$ which satisfies $< q^*, q >= 1$, $< q^*, \bar{q} >= 0$. Here $q_1^* = 1$, $q_2^* = \frac{\gamma r_1 - i\omega_0}{Q}$ and

$$\overline{D} = \frac{Q\gamma r_2}{Q\gamma r_2 - (\omega_0^2 + \gamma^2 r_1^2) - \tau^* e^{-i\omega_0\tau^*}(\omega_0^2 + \gamma^2 r_1^2)(Qx_0 - \frac{y_0}{m}e^{-i\omega_0\tau^*})}.$$

Using the same notations as in literature [5, 16, 20], we now compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let x_t be the solution of (3.1) with $\mu = 0$ and $z(t) =< q^*, x_t >$,

$$W(t, \theta) = x_t - z(t)q(\theta) - \bar{z}(t)\bar{q}(\theta) = x_t(\theta) - 2Re\{z(t)q(\theta)\}.$$

On the center manifold C_0, we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta),$$

where

$$W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots, \tag{3.12}$$

$z(t)$ and $\bar{z}(t)$ are local coordinates of center manifold C_0 in the direction of q and q^*, respectively. According the center manifold theory, we see that $W(0, 0, \theta) = 0$ and $W'(0, 0, \theta) = 0$.

For $\mu = 0$ and a solution x_t of equation (3.1) on C_0 , we have

$$\begin{aligned} \dot{z}(t) &= < q^*, \dot{x}(t) >=< q^*, L_\mu(x_t) + F(\mu, x_t) > \\ &= < q^*, i\omega_0\tau^* x_t > + < q^*, F(0, W + 2Re\{(z(t)q(\theta)\}) > \\ &= i\omega_0\tau^* z_t + \bar{q}^*(0)F(0, W(z, \bar{z}, 0)) + 2Re\{(z(t)q(0)\}) \\ &\stackrel{\text{def}}{=} i\omega_0\tau^* z_t + \bar{q}^*(0)F_0(z, \bar{z}). \end{aligned} \tag{3.13}$$

Consider the formal Taylor expansion of $F_0(z, \bar{z})$ as follows

$$F_0(z, \bar{z}) = F_{z^2}\frac{z^2}{2} + F_{\bar{z}}\frac{\bar{z}^2}{2} + F_{z\bar{z}}z\bar{z} + F_{z^2\bar{z}}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.14}$$

We rewrite (3.13) as

$$\dot{z}(t) = i\omega_0\tau^* z_t + g(z, \bar{z}), \tag{3.15}$$

where

$$g(z, \bar{z}) = \bar{q}^*(0)F_0(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{12}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.16}$$

Comparing (3.13) with (3.15) gives

$$\begin{aligned} g_{20} =& \bar{D}\tau^*(f_{20} + 2f_{11}q_2 + f_{02}q_2^2), \\ g_{11} =& \bar{D}\tau^*(f_{20} + f_{11}(q_2 + \bar{q}_2) + f_{02}q_2\bar{q}_2), \\ g_{02} =& \bar{D}\tau^*(f_{20} + 2f_{11}\bar{q}_2 + f_{02}\bar{q}_2), \\ g_{12} =& \bar{D}\tau^*[f_{20}(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)) + f_{11}(W_{20}^{(2)}(0) + \bar{q}_2 W_{20}^{(1)}(0) + 2q_2 W_{11}^{(1)}(0) \\ &+ 2W_{11}^{(2)}(0)) + f_{02}(\bar{q}_2 W_{20}^{(2)}(0) + 2q_2 W_{11}^{(2)}(0))]. \end{aligned}$$

In order to deduce g_{12}, we now compute $W_{20}(\theta)$ and $W_{11}(\theta)$. We know from (3.8) that

$$\dot{W} = \dot{x}_t - \dot{z}(t)q(\theta) - \dot{\bar{z}}(t)\bar{q}(\theta) = \begin{cases} AW - gq(\theta) - \bar{g}\bar{q}(\theta), & \theta \in [-1, 0), \\ AW - gq(0) - \bar{g}\bar{q}(0) + F(\mu, \phi), & \theta = 0. \end{cases} \quad (3.17)$$

Differentiating (3.12) with respect to t gives

$$\begin{aligned} \dot{W} &= W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}} \\ &= [W_{20}(\theta)z + W_{11}(\theta)\bar{z}]\dot{z} + [W_{11}(\theta)z + W_{02}(\theta)\bar{z}]\dot{\bar{z}} + \cdots \\ &= [W_{20}(\theta)z + W_{11}(\theta)\bar{z}](i\omega_0\tau^* z + g(z, \bar{z})) \\ &\quad + [W_{11}(\theta)z + W_{02}(\theta)\bar{z}](i\omega_0\tau^*\bar{z} + \bar{g}(z, \bar{z})) + \cdots \end{aligned} \quad (3.18)$$

Substituting (3.16) and (3.18) into (3.17) and comparing the coefficients, we have

$$(2i\omega_0\tau^* - A)W_{20}(\theta) = \begin{cases} -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), & \theta \in [-1, 0), \\ -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + F_{z^2}, & \theta = 0, \end{cases} \quad (3.19)$$

and

$$(i\omega_0\tau^* - A)W_{11}(\theta) = \begin{cases} -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta), & \theta \in [-1, 0), \\ -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + F_{z\bar{z}}, & \theta = 0. \end{cases} \quad (3.20)$$

Based on (2.19) and (3.20), for $\theta \in [-1, 0)$,

$$W_{20}'(\theta) = 2i\omega_0\tau^* W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

It is clear that the solution of above equation is

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau^*}q(\theta)e^{i\omega_0\tau^*\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau^*}\bar{q}(\theta)e^{-i\omega_0\tau^*\theta} + E_1 e^{2i\omega_0\tau^*\theta}. \quad (3.21)$$

For $\theta = 0$, we see from (3.19) that

$$\int_{-1}^{0} d_\theta\eta(0, \theta)W_{20} = 2i\omega_0\tau^* W_{20}(0) + g_{20}q(0) + \bar{g}_{02}\bar{q}(0) - F_{z^2}. \quad (3.22)$$

Substituting (3.21) into (3.22), we have

$$\left((2i\omega_0\tau^* I - \int_{-1}^{0} d_\theta\eta(\theta)e^{2i\omega_0\tau^*\theta}\right)E_1 = 2\tau^* \begin{pmatrix} f_{20} + 2f_{11}q_2 + f_{02}q^2 \\ 0 \end{pmatrix}, \quad (3.23)$$

namely,

$$\begin{pmatrix} 2i\omega_0 - \gamma r_1 & -\gamma r_2 \\ Q & 2i\omega_0 - x_0 Q + \frac{y_0}{m}e^{-2i\omega_0\tau^*} \end{pmatrix} E_1 = 2\begin{pmatrix} f_{20} + 2f_{11}q_2 + f_{02}q^2 \\ 0 \end{pmatrix}. \quad (3.24)$$

So we have

$$E_1^{(1)} = \frac{2}{M_1}\begin{vmatrix} f_{20} + 2f_{11}q_2 + f_{02}q^2 & -\gamma r_2 \\ 0 & 2i\omega_0 - x_0 Q + \frac{y_0}{m}e^{-2i\omega_0\tau^*} \end{vmatrix},$$

$$E_1^{(2)} = \frac{2}{M_1}\begin{vmatrix} 2i\omega_0 - \gamma r_1 & 2f_{11}q_2 - \gamma f_{20} + f_{02}q_2^2 \\ Q & 0 \end{vmatrix},$$

with

$$M_1 = \begin{vmatrix} 2i\omega_0 - \gamma r_1 & -\gamma r_2 \\ Q & 2i\omega_0 - x_0 Q + \frac{y_0}{m}e^{-2i\omega_0\tau^*} \end{vmatrix}.$$

Similarly, we have

$$E_2^{(1)} = \frac{2}{M_2}\begin{vmatrix} f_{20} + f_{11}(q_2 + \bar{q}_2) + f_{02}q_2\bar{q}_2 & -\gamma r_2 \\ 0 & i\omega_0 - x_0 Q + \frac{y_0}{m}e^{i\omega_0\tau^*} \end{vmatrix},$$

$$E_2^{(2)} = \frac{2}{M_2}\begin{vmatrix} i\omega_0 - \gamma r_1 & f_{20} + f_{11}(q_2 + \bar{q}_2) + f_{02}q_2\bar{q}_{22} \\ Q & 0 \end{vmatrix},$$

where

$$M_2 = \begin{vmatrix} i\omega_0 - \gamma r_1 & -\gamma r_2 \\ Q & i\omega_0 - x_0 Q + \frac{y_0}{m}e^{i\omega_0\tau^*} \end{vmatrix}.$$

Thus, g_{12} is determined by the parameters and delay. The following quantities can be derived accordingly

$$C_1(0) = \frac{i}{2\omega_0\tau^*}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{12}}{2}, \qquad \mu_2 = -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau^*)\}},$$
$$\beta_2 = 2Re\{C_1(0)\}, \qquad T_2 = -\frac{Im(C_1(0) + \mu_2\lambda'(\tau^*))}{\tau^*\omega_0}.$$

According to the results in Hassard [5], one knows that the sign of μ_2 determines the direction of the Hopf bifurcation, the sign of β_2 determines the stability of the bifurcating periodic solutions and the sign of T_2 determines the monotonicity of the period of the bifurcating periodic solutions. Thus we have the following conclusions.

Theorem 3.1. *The following statements hold for system (1.3).*

(1) The Hopf bifurcation is supercritical (subcritical) if $\mu_2 > 0$ $(\mu_2 < 0)$*;*

(2) The bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0$ $(\beta_2 > 0)$*;*

(3) The periodic of bifurcating periodic solutions increase (decrease) if $T_2 > 0$ $(T_2 < 0)$.

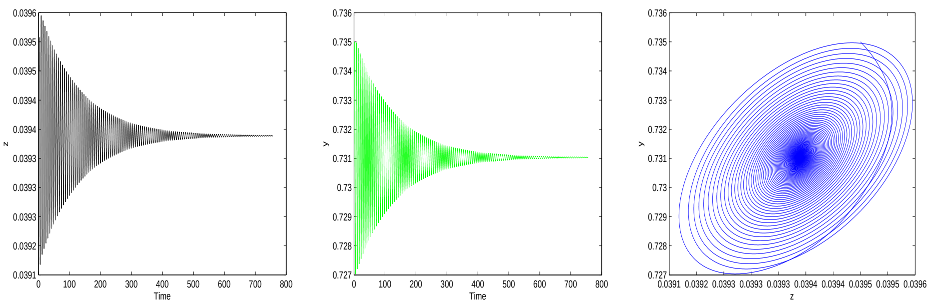
4. Numerical Simulations

In this section, we perform some numerical simulations of system (1.4) to testify the analytical results proved in the previous sections.

Set $\alpha^2 = 0.075$, $m = 0.8$, $\gamma = 0.6$, $Q = 1.6$. By direct computation, we get $\tau_0 \approx 1.5422$ and the equilibrium E is $(0.039388, 0.731032)$.

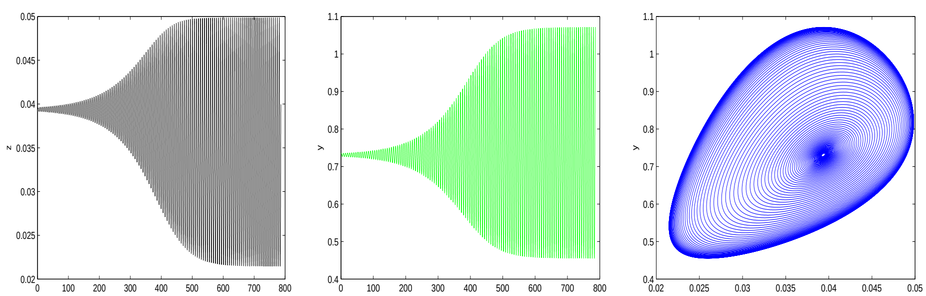
(i) Take $\tau_0 = 1.51$. According the previous analysis, the equilibrium should be stable. Starting from the initial value $(z(0), y(0)) = (0.0395335, 0.7351731)$, we get Figure 1. The simulation results coincide exactly with the our conclusion.

(ii) Take $\tau_0 = 1.57$. According the previous analysis, the equilibrium should be unstable and there exists a periodic solution bifurcating from the equilibrium. Starting from the same initial value as (i), we get Figure 2. It implies that the Hopf bifurcation associated with the critical value $\tau_0 \approx 1.5422$ is supercritical, the bifurcating periodic solution is stable, and the equilibrium E is unstable. All these results are consistent with our analysis.



(a) The graph of z(t). (b) The graph of y(t). (c) The orbit of system (1.4).

Figure 1. Behavior of system (1.4) with $\tau = 1.51$. when $\tau < \tau_0 \approx 1.5422$, the positive equilibrium E is asymptotically stable.



(a) The graph of z(t). (b) The graph of y(t). (c) The orbit of system (1.4).

Figure 2. Behavior of system (1.4) with $\tau = 1.57$. when $\tau > \tau_0 \approx 1.5422$, the positive equilibrium E is unstable and there exists a bifurcated periodic solution.

5. Conclusions

In this paper, we have discussed the dynamics of a spruce-budworm model with delay, which is based on system (1.2). Firstly, we studied the effects of the discrete time delay τ on the stability of positive equilibrium of system (1.4). Next, we investigated the existence of Hopf bifurcation, the bifurcating direction and stability of the bifurcating periodic solutions by the normal form and center manifold theorems. Just as we have pointed out in Introduction that system (1.3) possibly has one, two or three equilibrium for the case when $\alpha^2 < \frac{1}{27}$, which implies that it may have more abundant dynamics. In fact, the dynamics of a system might be impacted by other factors. It has been shown that Allee effect exists in many natural populations, such as birds, insects, mammals, marine invertebrates and plants [7, 15, 21]. Allee effect has attracted a significant amount of attention from both mathematicians and ecologists in recent decades. Many interesting results on Allee effect have been obtained in various model equations [1, 10, 13, 19]. The impact of Allee effect on the system investigated in this paper will be considered in our next work.

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