

SOLUTIONS FOR THE KIRCHHOFF TYPE EQUATIONS WITH FRACTIONAL LAPLACIAN*

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Abstract Due to the singularity and nonlocality of the fractional Laplacian, the classical tools such as Sturm comparison, Wronskians, Picard–Lindelöf iteration, and shooting arguments (which are all purely local concepts) are not applicable when analyzing solutions in the setting of the nonlocal operator $(-\Delta)^s$. Furthermore, the nonlocal term of the Kirchhoff type equations will also cause some mathematical difficulties. The present work is motivated by the method of semi-classical problems which show that the existence of solutions of the Kirchhoff type equations are equivalent to the corresponding associated fractional differential and algebraic system. In such case, the existence of the fractional Kirchhoff equation can be obtained by using the corresponding fractional elliptic equation. Therefore some qualitative properties of solutions for the associated problems can be inherited. In particular, the classical uniqueness results can be applied to this equation.

Keywords Kirchhoff equation, fractional Laplacian, equivalence, uniqueness.

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1. Introduction

In this note, we will consider the Kirchhoff type equation with fractional Laplacian of the form

$$M\left([u]_s^2\right)(-\Delta)^s u + \lambda f(u) = \mu g(u) \text{ in } H^s(\mathbb{R}^N) \quad (1.1)$$

and its corresponding associated equation

$$(-\Delta)^s u + \lambda f(u) = \mu g(u) \text{ in } H^s(\mathbb{R}^N) \quad (1.2)$$

where

$$[u]_s = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2} \quad (1.3)$$

is the so-called Gagliardo (semi)norm of u (see Di Nezza, Palatucci and Valdinoci [26]), $s \in (0, 1)$, $N > 2s$ is a positive integer, $\lambda, \mu \in \mathbb{R}$, $M : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous

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function whose properties will be introduced later, and the functions f and g are continuous in \mathbb{R} .

Fiscella and Valdinoci in [10] first proposed a stationary Kirchhoff variational model as follows:

$$\left(m_0 + 2b[u]_s^2\right) (-\Delta)^s u = f \text{ in } \mathbb{R}^N, \quad (1.4)$$

in bounded domains of \mathbb{R}^N . In model (1.4) the authors took into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string, see Appendix A in [10] for more details. Recently, the results of [10] had been extended in [3] to the non-degenerate case. Other related problems were also considered in [9, 29, 32, 36]. When the fractional Laplacian $(-\Delta)^s$ is replaced by the p -Laplacian, some problems were also established by some authors, for example, see Caponi and Pucci [4], Pucci, Xiang and Zhang [27].

When $s = 1$, equation (1.4) is reduced to the standard Kirchhoff type equation which was first proposed by Kirchhoff in [17] to describe the transversal oscillations of a stretched string. The boundary problems then attracted several researchers mainly after the work of Lions [22], where a functional analysis approach was proposed to attack it. For more mathematical and physical background, we refer readers to [2, 3, 7, 13–15, 20–23, 26, 33–35], and the references therein.

The symbol $(-\Delta)^s$ with $s \in (0, 1)$ is called the fractional Laplacian which can be defined by

$$\begin{aligned} (-\Delta)^s u(x) &= C(N, s) P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy \\ &= -\frac{C(N, s)}{2} \int_{\mathbb{R}^N} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{N+2s}} dy \end{aligned} \quad (1.5)$$

for $x \in \mathbb{R}^N$, where $C(N, s)$ is a dimensional constant that depends on N and s , see Di Nezza, Palatucci and Valdinoci [26]. Due to the singularity and nonlocality of the kernel, it is evident that the theory of ordinary differential equations (ODE) itself does not provide any means to establish such results. In particular, classical tools such as Sturm comparison, Wronskians, Picard–Lindelöf iteration, and shooting arguments (which are all purely local concepts) are not applicable when analyzing radial solutions in the setting of the nonlocal operator $(-\Delta)^s$, see Frank and Lenzmann [11], Frank, Lenzmann and Silvestre [12], Moroz and Van Schaftingen [25] and Di Nezza, Palatucci and Valdinoci [26]. Furthermore, the nonlocal term M will also cause some mathematical difficulties, for example, see [5–9, 13–15, 19, 20, 22, 23, 26, 27, 29, 33–36] and the references therein.

The present work is motivated by the method of semi-classical problems (see also [19], [14] and [31]). For example, we consider the semi-classical problem of the form

$$\varepsilon^{2s} (-\Delta)^s u + \lambda f(u) = \mu g(u) \text{ in } H^s(\mathbb{R}^N), \quad (1.6)$$

where $\varepsilon > 0$ is a small parameter, typically related to the Planck constant. If u_ε is a solution of (1.6) and $a \in \mathbb{R}^N$, then the function $v_\varepsilon(y) = u_\varepsilon(a + \varepsilon y)$ solves the related equation

$$(-\Delta)^s v_\varepsilon + \lambda f(v_\varepsilon) = \mu g(v_\varepsilon) \text{ in } \mathbb{R}^N. \quad (1.7)$$

If v is a solution of (1.7), then the function

$$u_\varepsilon(x) = v\left(\frac{x - a}{\varepsilon}\right)$$

is a solution of (1.6). See Moroz and Van Schaftingen [25] (see also Van Schaftingen and Xia [28]).

In this note, we mainly prove the following facts: If $Q(x)$ is a solution of (1.2), then

$$u(x) = Q\left(\frac{x}{c_{[Q]_s^*}^*} + t\right)$$

is a solution of (1.1) for any $t \in \mathbb{R}^N$ when the algebraic equation (2.3) (be defined later) has a positive root $c_{[Q]_s^*}^*$. Notice that $u(x)$ is expressed by Q . Thus, its qualitative properties are similar with Q . In this case, if $Q(x)$ is a unique solution of (1.2) then $c_{[Q]_s^*}^*$ is unique which implies that $u(x)$ is also unique. If $v(x)$ is the second solution of (1.1), in view of Theorem 2.1 (in the next section), we can obtain the corresponding second solution of (1.2), which is a contradiction. By using this fact, we can obtain some existence results from the known results of (1.2). On the other hand, our methods are also valid for the other pseudo-differential operators, for example, the operator $(-\Delta + m^2)^{s/2}$, see Frank, Lenzmann and Silvestre [12].

2. Main Results

First of all, we assume that $u \in H^s(\mathbb{R}^N)$ is a nonzero solution of (1.1) and denote $M([u]_s^2) = c^{2s}$. Then, we have

$$c^{2s}(-\Delta)^s u + \lambda f(u) = \mu g(u). \quad (2.1)$$

For any $a \in \mathbb{R}^N$, in view of (1.7), we know that the function $v_c(y) = u_c(a + cy)$ satisfies

$$(-\Delta)^s v_c + \lambda f(v_c) = \mu g(v_c) \text{ in } \mathbb{R}^N. \quad (2.2)$$

Now, we assume that $Q(x)$ is a solution of (1.2) and denote

$$u(x) = Q\left(\frac{x}{c} + t\right).$$

Notice that

$$\begin{aligned} [Q]_s^2 &= [u(c(x-t))]_s^2 \\ &= c^{2s-N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(c(x-t)) - u(c(y-t))|^2}{|c(x-t) - c(y-t)|^{N+2s}} d[c(x-t)] d[c(y-t)] \\ &= c^{2s-N} [u]_s^2, \end{aligned}$$

therefore, we get that

$$M([u]_s^2) = M([Q]_s^2 c^{N-2s}) = c^{2s}.$$

Assume that the algebraic equation

$$M([Q]_s^2 c^{N-2s}) = c^{2s} \quad (2.3)$$

admits a positive root $c_{[Q]_s}^*$. Then,

$$u(x) = Q\left(\frac{x}{c_{[Q]_s}^*} + t\right)$$

for some $t \in \mathbb{R}^N$.

Theorem 2.1. *Let $u(x)$ be a solution of (1.1), then $Q(x) := u(c(x-t))$ is a solution of (1.2) for any $t \in \mathbb{R}^N$, where $c = M^{1/2s}([u]_s^2)$.*

Theorem 2.2. *Assume that $Q(x)$ is a solution of (1.2) and that the algebraic equation (2.3) admits a positive root $c_{[Q]_s}^*$, then*

$$u(x) \in \left\{ Q\left(\frac{x}{c_{[Q]_s}^*} + t\right), t \in \mathbb{R}^N \right\}$$

is a solution of (1.1).

In view of Theorems 2.1 and 2.2, we have immediately the following result.

Corollary 2.1. *The existence of solution for (1.1) is equivalent to the existence of solution for the pseudo-differential and algebraic system*

$$\begin{cases} (-\Delta)^s u + \lambda f(u) = \mu g(u), & \text{in } \mathbb{R}^N \times \mathbb{R}_+, \\ \delta = M([u]_s^2 \delta^{\frac{N}{2s}-1}), \end{cases}$$

It is well known that it is very difficult to obtain the uniqueness of the solution for PDEs. However, the following result is clear.

Corollary 2.2. *Assume that $Q(x)$ is the unique solution of (1.2) and that the algebraic equation (2.3) exists a unique positive root $c_{[Q]_s}^*$, then*

$$u(x) \in \left\{ Q\left(\frac{x}{c_{[Q]_s}^*} + t\right), t \in \mathbb{R}^N \right\}$$

is a unique solution of (1.1) up to translations.

Example 2.1. Recently, Frank, Lenzmann and Silvestre [12] considered the non-degeneracy, regularity estimates and uniqueness results for ground state solutions of the nonlinear equation

$$(-\Delta)^s u + u = |u|^\alpha u \text{ in } H^s(\mathbb{R}^N) \quad (2.4)$$

involving the fractional Laplacian $(-\Delta)^s$, $N \geq 1$, $0 < \alpha < \alpha_*$,

$$\alpha_* = \begin{cases} \frac{4s}{N-2s} & \text{for } 0 < s < \frac{N}{2}, \\ \infty & \text{for } s \geq \frac{N}{2}, \end{cases}$$

settled conjecture by Kenig et al. [16] and Weinstein [30] for any dimension $N > 1$, and generalized the classical uniqueness result by Amick and Toland [1] on the

uniqueness of solitary waves for the Benjamin–Ono equation and the case $N = 1$ dimension in Frank and Lenzmann [11]. In the local case for $s = 1$, the uniqueness and nondegeneracy of ground states for problem (2.4) was established in a celebrated paper by Kwong [18] (see also Coffman [5] and McLeod [24]).

Now, we consider the algebraic equation

$$m_0 + 2b [Q]_s^2 \delta^{N/2s-1} = \delta. \quad (2.5)$$

If $2s < N < 4s$, clearly, (2.5) exists a unique positive root.

Corollary 2.3. *When $m_0, b > 0$, $N/4 < s < N/2$ ($N = 1$ or 2) and $1 < \alpha < \alpha_*$, then equation*

$$\left(m_0 + 2b [u]_s^2\right) (-\Delta)^s u + u = |u|^\alpha u \quad (2.6)$$

exists a unique ground state solution up to translations.

Remark 2.1. For equation (2.6), Corollary 2.3 is new.

Example 2.2. In [1], Amick and Toland proved the uniqueness (up to translations) of the nontrivial solution $Q \in H^{1/2}(\mathbb{R})$ of

$$(-\Delta)^{1/2} Q + Q - Q^2 = 0 \text{ in } \mathbb{R}.$$

In fact, the unique family of solutions is

$$Q(x) = \frac{2}{1 + (x - x_0)^2}$$

with $x_0 \in \mathbb{R}$. Unfortunately, Corollary 2.3 is not valid because $s = 1/2$.

Remark 2.2. A very special Kirchhoff function M is given by $a + bmt^{m-1}$, $a, b \geq 0$, $a + b > 0$, $m \geq 1$ and $t \geq 0$, see Pucci, Xiang and Zhang [27]. The similar results can also be obtained.

Remark 2.3. Our methods are also valid for other pseudo-differential operators, for example, the operator $(-\Delta + m^2)^{s/2}$, see Frank, Lenzmann and Silvestre [12].

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