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FURTHER STUDIES ON LIMIT CYCLE BIFURCATIONS FOR PIECEWISE SMOOTH NEAR-HAMILTONIAN SYSTEMS WITH MULTIPLE PARAMETERS*

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Abstract This paper investigates the limit cycle bifurcations for piecewise smooth near-Hamiltonian systems with multiple parameters. The formulas for the second and third term in expansions of the first order Melnikov function are derived respectively. The main results improve some known conclusions.

Keywords Piecewise Hamiltonian systems, Melnikov function, limit cycle bifurcation.

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1. Introduction and Main Results

Piecewise smooth systems are frequently encountered in practical applications, such as control systems and engineering [1,12,21]. In recent years, there are lots of works on studying the number of limit cycles and their relative positions of nonsmooth dynamical systems on the plane and have obtained many meaningful results [4,5,10]. It is well known that the Melnikov method is a useful tool to determine the number of limit cycles bifurcating from a family of periodic orbits of the unperturbed systems. The authors in [16] established a formula for the first order Melnikov function for planar piecewise smooth systems, which plays an important role in estimating the number of limit cycles, see for instance [13,24]. For high-dimensional piecewise smooth near-integrable systems, the authors of [20] established the Melnikov function theory and gave an expression for the first order Melnikov vector function. We note that the averaging method developed in [7,14,15,18] is another common technique. For some applications of this method see [2,17,19] for example. It was proved in [8] that the averaging method is equivalent to the Melnikov function method for studying the number of limit cycles of planar analytic (or C^∞) near-Hamiltonian systems.

In this paper, we consider a piecewise smooth near-Hamiltonian system with

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multiple parameters of the following form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} H_y^+(x,y,\lambda) + \varepsilon p^+(x,y,\lambda) \\ -H_x^+(x,y,\lambda) + \varepsilon q^+(x,y,\lambda) \end{pmatrix}, & x > 0, \\ \begin{pmatrix} H_y^-(x,y,\lambda) + \varepsilon p^-(x,y,\lambda) \\ -H_x^-(x,y,\lambda) + \varepsilon q^-(x,y,\lambda) \end{pmatrix}, & x \le 0, \end{cases} \tag{1.1}$$

where $H^\pm$, $p^\pm$ and $q^\pm$ are C^∞ functions, λ and ε are both sufficiently small real parameters with $0 < \varepsilon \ll \lambda \ll 1$. Suppose system (1.1) satisfies the following assumptions as in [9, 16, 20]:

(I) There exist an interval $J = (\alpha, \beta)$ and two points $A_\lambda(h) = (0, a(h,\lambda))$ and $B_\lambda(h) = (0, b(h,\lambda))$ such that for $h \in J$,

$$\begin{aligned} &H^+(A_\lambda(h),\lambda) = H^+(B_\lambda(h),\lambda) = h, \\ &H^-(A_\lambda(h),\lambda) = H^-(B_\lambda(h),\lambda), \quad a(h,\lambda) > b(h,\lambda). \end{aligned}$$

(II) The equation $H^+(x,y,\lambda) = h$, $x \ge 0$, defines an orbital arc L_h^+ starting from $A_\lambda(h)$ and ending at $B_\lambda(h)$; the equation $H^-(x,y,\lambda) = H^-(A_\lambda(h),\lambda)$, $x \le 0$, defines an orbital arc L_h^- starting from $B_\lambda(h)$ and ending at $A_\lambda(h)$, such that system $(1.1)|_{\varepsilon=0}$ has a family clockwise oriented periodic orbits $L_h = L_h^+ \cup L_h^-$.

(III) The curves $L_h^\pm, h \in J$ are not tangent to the switch plane $x = 0$ at points $A_\lambda(h)$ and $B_\lambda(h)$. In other words, $H_y^\pm(A_\lambda,\lambda) \neq 0$ and $H_y^\pm(B_\lambda,\lambda) \neq 0$ for each $h \in J$.

Under the conditions (I)–(III), we have the first order Melnikov function of system (1.1) from [9, 16]

$$M(h,\lambda) = \int_{\widehat{A_\lambda B_\lambda}} q^+ dx - p^+ dy + \frac{H_y^+(A_\lambda,\lambda)}{H_y^-(A_\lambda,\lambda)} \int_{\widehat{B_\lambda A_\lambda}} q^- dx - p^- dy. \tag{1.2}$$

Sometimes the system we consider has the following form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} f_1^+(x,y,\lambda) + \varepsilon f_2^+(x,y,\lambda) \\ g_1^+(x,y,\lambda) + \varepsilon g_2^+(x,y,\lambda) \end{pmatrix}, & x \ge 0, \\ \begin{pmatrix} f_1^-(x,y,\lambda) + \varepsilon f_2^-(x,y,\lambda) \\ g_1^-(x,y,\lambda) + \varepsilon g_2^-(x,y,\lambda) \end{pmatrix}, & x < 0, \end{cases}$$

where the functions $f_1^\pm, f_2^\pm, g_1^\pm$ and $g_2^\pm$ are C^∞ functions such that the above unperturbed system has integrating factors μ_1 and μ_2 and first integrals H^+ and H^- respectively for $x \ge 0$ and $x < 0$, satisfying

$$\begin{aligned} \mu_1 f_1^+ = H_y^+, \quad \mu_1 g_1^+ = -H_x^+, \\ \mu_2 f_1^- = H_y^-, \quad \mu_2 g_1^- = -H_x^-. \end{aligned}$$

Then the above differential equation is equivalent to a near-Hamiltonian system of the form (1.1), and the corresponding Melnikov function has the form

$$M(h,\lambda) = \int_{\widehat{A_\lambda B_\lambda}} \mu_1 (g_2^+ dx - f_2^+ dy) + \frac{H_y^+(A_\lambda,\lambda)}{H_y^-(A_\lambda,\lambda)} \int_{\widehat{B_\lambda A_\lambda}} \mu_2 (g_2^- dx - f_2^- dy).$$

In system (1.1), the functions $H^\pm, p^\pm$ and $q^\pm$ depend on another small parameter λ leading to the dependence of the function M on λ. Then for $\lambda > 0$ small

$$M(h,\lambda) = M_0(h) + \lambda M_1(h) + \lambda^2 M_2(h) + O(\lambda^3). \tag{1.3}$$

The function $M(h,\lambda)$ can be used to study not only Poincaré bifurcation (bifurcation of limit cycles from a period annulus) but also Hopf bifurcation and homoclinic and heteroclinic bifurcations. The formulas of $M_1(h)$ and $M_2(h)$ were obtained in [11] for smooth case. If $H^-(A_\lambda(h),\lambda) = h$, the author [22] gave the formulas of $M_1(h)$ and $M_2(h)$, which has some applications, see for example [3, 23].

Our main task in this paper is to remove the condition $H^-(A_\lambda(h),\lambda) = h$ and give expressions of $M_1(h)$ and $M_2(h)$ under the conditions (I)–(III). For the purpose, assume the functions $H^\pm$, $p^\pm$ and $q^\pm$ have the following form for $\lambda > 0$ small

$$\begin{aligned} H^\pm(x,y,\lambda) =& H_0^\pm(x,y) + \lambda H_1^\pm(x,y) + \lambda^2 H_2^\pm(x,y) + O(\lambda^3),\\ p^\pm(x,y,\lambda) =& p_0^\pm(x,y) + \lambda p_1^\pm(x,y) + \lambda^2 p_2^\pm(x,y) + O(\lambda^3),\\ q^\pm(x,y,\lambda) =& q_0^\pm(x,y) + \lambda q_1^\pm(x,y) + \lambda^2 q_2^\pm(x,y) + O(\lambda^3). \end{aligned} \tag{1.4}$$

Then from (1.2), (1.3) and above expansions, it is easy to see that

$$M_0(h) = \int_{\widehat{AB}} q_0^+ dx - p_0^+ dy + \frac{H_{0y}^+(A)}{H_{0y}^-(A)} \int_{\widehat{BA}} q_0^- dx - p_0^- dy, \tag{1.5}$$

where $A = A_\lambda|_{\lambda=0} = (0, a(h,0))$, $B = B_\lambda|_{\lambda=0} = (0, b(h,0))$.

For convenience, we introduce some notations below. Denote

$$\begin{aligned} \Upsilon(r) =& r(0,a(h,\lambda))\frac{\partial a}{\partial \lambda} - r(0,b(h,\lambda))\frac{\partial b}{\partial \lambda},\\ g(h,\lambda) =& (H^-(A_\lambda,\lambda))_\lambda, \qquad G(h,\lambda) = \frac{H_y^+(A_\lambda,\lambda)}{H_y^-(A_\lambda,\lambda)},\\ \Phi(h) =& \Big(H_{0yy}^+(A)\frac{\partial a}{\partial \lambda}(h,0) + H_{1y}^+(A)\Big)H_{0y}^-(A) - \Big(H_{0yy}^-(A)\frac{\partial a}{\partial \lambda}(h,0)\\ &+ H_{1y}^-(A)\Big)H_{0y}^+(A),\\ \psi^\pm(h) =& H_{0yyy}^\pm(A)\Big(\frac{\partial a}{\partial \lambda}\Big)^2(h,0) + 2H_{1yy}^\pm(A)\frac{\partial a}{\partial \lambda}(h,0) + H_{0yy}^\pm(A)\frac{\partial^2 a}{\partial \lambda^2}(h,0)\\ &+ 2H_{2y}^\pm(A). \end{aligned} \tag{1.6}$$

The main results are as follows.

Theorem 1.1. *Under the conditions (I)–(III), we have*

$$\begin{aligned} M_1(h) =& -\int_{\widehat{AB}} H_1^+(p_{0x}^+ + q_{0y}^+)dt + \Upsilon(p_0^+)|_{\lambda=0} + \int_{\widehat{AB}} q_1^+ dx - p_1^+ dy\\ &+ G(h,0)\Big[-\int_{\widehat{BA}}(H_1^- - g(h,0))(p_{0x}^- + q_{0y}^-)dt - \Upsilon(p_0^-)|_{\lambda=0}\\ &+ \int_{\widehat{BA}} q_1^- dx - p_1^- dy\Big] + G_\lambda(h,0)\int_{\widehat{BA}} q_0^- dx - p_0^- dy, \end{aligned} \tag{1.7}$$

where

$$g(h,0) = H_{0y}^-(A)\frac{\partial a}{\partial\lambda}(h,0) + H_1^-(A),$$
$$G(h,0) = \frac{H_{0y}^+(A)}{H_{0y}^-(A)}, \quad G_\lambda(h,0) = \frac{\Phi(h)}{(H_{0y}^-(A))^2}.$$

Theorem 1.2. *Under the conditions (I)–(III), suppose further that there exist a region U and C^∞ functions $\bar{p}_0^\pm(x,y)$ and $\bar{q}_0^\pm(x,y)$ defined on U such that*

$$-H_1^\pm(p_{0x}^\pm + q_{0y}^\pm) = H_{0x}^\pm \bar{p}_0^\pm + H_{0y}^\pm \bar{q}_0^\pm, \quad (x,y) \in U.$$

We have

$$\begin{aligned}
M_2(h) = & -\int_{\widehat{AB}} \varphi_1^+(x,y)dt + \int_{\widehat{AB}} q_2^+ dx - p_2^+ dy + \Delta^+(h) \\
& + G(h,0)\Bigg[-\int_{\widehat{BA}} \varphi_1^-(x,y)dt + \int_{\widehat{BA}} q_2^- dx - p_2^- dy - \Delta^-(h) \\
& + \int_{\widehat{BA}} g(h,0)\Big(\phi(x,y) + \frac{\Psi(x,y,h)}{2(H_{0y}^-)^2}\Big)dt + \frac{1}{2}\int_{\widehat{BA}} g_\lambda(h,0)(p_{0x}^- + q_{0y}^-)dt\Bigg] \\
& + G_\lambda(h,0)\Bigg[-\int_{\widehat{BA}} (H_1^- - g(h,0))(p_{0x}^- + q_{0y}^-)dt - \Upsilon(p_0^-)|_{\lambda=0} \\
& + \int_{\widehat{BA}} q_1^- dx - p_1^- dy\Bigg] + \frac{1}{2}G_{\lambda\lambda}(h,0)\int_{\widehat{BA}} q_0^- dx - p_0^- dy,
\end{aligned} \tag{1.8}$$

where

$$\begin{aligned}
\varphi_1^\pm(x,y) = & \frac{1}{2}\big[(H_1^\pm \bar{p}_0^\pm)_x + (H_1^\pm \bar{q}_0^\pm)_y\big] + H_2^\pm(p_{0x}^\pm + q_{0y}^\pm) + H_1^\pm(p_{1x}^\pm + q_{1y}^\pm), \\
\Delta^\pm(h) = & \frac{1}{2}\Upsilon(\bar{p}_0^\pm)|_{\lambda=0} + \frac{1}{2}(\Upsilon(p_0^\pm))_\lambda|_{\lambda=0} + \Upsilon(p_1^\pm)|_{\lambda=0}, \\
\phi(x,y) = & p_{1x}^- + q_{1y}^- + \frac{1}{2}\bar{p}_{0x}^- + \frac{1}{2}\bar{q}_{0y}^-, \\
\Psi(x,y,h) = & H_{0y}^-(p_{0xy}^- + q_{0yy}^-)(-H_1^- + g(h,0)) - (p_{0x}^- + q_{0y}^-)\big(H_{0yy}^-(-H_1^- \\
& + g(h,0)) + H_{0y}^- H_{1y}^-\big), \\
g_\lambda(h,0) = & H_{0yy}^\pm(A)\Big(\frac{\partial a}{\partial\lambda}\Big)^2(h,0) + 2H_{1y}^\pm(A)\frac{\partial a}{\partial\lambda}(h,0) + H_{0y}^\pm(A)\frac{\partial^2 a}{\partial\lambda^2}(h,0) \\
& + 2H_2^\pm(A), \\
G_{\lambda\lambda}(h,0) = & \frac{1}{(H_{0y}^-(A))^3}\Big[\big(\psi^+(h)H_{0y}^-(A) - \psi^-(h)H_{0y}^+(A)\big)H_{0y}^-(A) - 2\Big(H_{1y}^-(A) \\
& + H_{0yy}^-(A)\frac{\partial a}{\partial\lambda}(h,0)\Big)\Phi(h)\Big].
\end{aligned}$$

In the next two sections, we provide proofs of the above theorems and present an example showing an application of our main results, respectively.

2. Proof of main results

Enlightened by the idea in [22], we first present a preliminary lemma, which will be used in deducing the expressions of $M_1(h)$ and $M_2(h)$.

Lemma 2.1. *Suppose that*

$$\hat{M}^+(h,\lambda) = \int_{\widehat{A_\lambda B_\lambda}} \hat{q}^+ dx - \hat{p}^+ dy, \quad \hat{M}^-(h,\lambda) = \int_{\widehat{B_\lambda A_\lambda}} \hat{q}^- dx - \hat{p}^- dy,$$

where $\hat{p}^\pm$ and $\hat{q}^\pm$ are C^∞ functions in (x,y) and independent of λ. Then

$$\begin{aligned} \hat{M}^+_\lambda(h,\lambda) &= -\int_{\widehat{A_\lambda B_\lambda}} H^+_\lambda(\hat{p}^+_x + \hat{q}^+_y)dt + \Upsilon(\hat{p}^+), \\ \hat{M}^-_\lambda(h,\lambda) &= -\int_{\widehat{B_\lambda A_\lambda}} (H^-_\lambda - g(h,\lambda))(\hat{p}^-_x + \hat{q}^-_y)dt - \Upsilon(\hat{p}^-). \end{aligned} \tag{2.1}$$

Proof. The first formula in (2.1) was obtained in [22], thus we omit its proof here and only prove the second one in (2.1). By using Green formula twice, we have

$$\begin{aligned} \hat{M}^-(h,\lambda) &= \int_{\widehat{B_\lambda A_\lambda}} \hat{q}^- dx - \hat{p}^- dy \\ &= \int_{\widehat{B_\lambda A_\lambda} \cup \overrightarrow{A_\lambda B_\lambda}} \hat{q}^- dx - \hat{p}^- dy - \int_{\overrightarrow{A_\lambda B_\lambda}} \hat{q}^- dx - \hat{p}^- dy \\ &= \iint_{int.(\widehat{B_\lambda A_\lambda} \cup \overrightarrow{A_\lambda B_\lambda})} (\hat{p}^-_x + \hat{q}^-_y)dxdy + \int_{\overrightarrow{A_\lambda B_\lambda}} \hat{p}^-(0,y)dy \\ &= \int_{\widehat{B_\lambda A_\lambda}} \breve{q}^-(x,y)dx + \int_{\overrightarrow{A_\lambda B_\lambda}} \hat{p}^-(0,y)dy, \end{aligned} \tag{2.2}$$

where

$$\breve{q}^-(x,y) = \int_0^y \hat{p}^-_x(x,y)dy + \hat{q}^-(x,y) - \hat{q}^-(x,0),$$

with $\breve{q}^-_y = \hat{p}^-_x + \hat{q}^-_y$.

Denote the most left point of the orbit $\widehat{B_\lambda A_\lambda}$ by $\bar{C}_\lambda(h) = (\bar{c}(h,\lambda), \tilde{c}(h,\lambda))$. For the sake of simplicity, assume that $\widehat{B_\lambda A_\lambda}$ can be represented as $y = y_1^-(x,h,\lambda)$ and $y = y_2^-(x,h,\lambda)$ for $\bar{c}(h,\lambda) \le x \le 0$ with $y_1^-(x,h,\lambda) \le y_2^-(x,h,\lambda)$. Then from (2.2) we have

$$\hat{M}^-(h,\lambda) = \int_0^{\bar{c}(h,\lambda)} (\breve{q}^-(x, y_1^-(x,h,\lambda)) - \breve{q}^-(x, y_2^-(x,h,\lambda)))dx + \int_{a(h,\lambda)}^{b(h,\lambda)} \hat{p}^-(0,y)dy.$$

A direct computation shows that

$$\begin{aligned} \hat{M}^-_\lambda(h,\lambda) = \int_0^{\bar{c}(h,\lambda)} &\left(\breve{q}^-_y(x, y_1^-(x,h,\lambda))\frac{\partial y_1^-}{\partial\lambda} - \breve{q}^-_y(x, y_2^-(x,h,\lambda))\frac{\partial y_2^-}{\partial\lambda}\right)dx \\ &+ (\breve{q}^-(\bar{c}(h,\lambda), y_1^-(\bar{c}(h,\lambda),h,\lambda)) - \breve{q}^-(\bar{c}(h,\lambda), y_2^-(\bar{c}(h,\lambda),h,\lambda)))\frac{\partial\bar{c}}{\partial\lambda} \\ &+ \hat{p}^-(0,b(h,\lambda))\frac{\partial b}{\partial\lambda} - \hat{p}^-(0,a(h,\lambda))\frac{\partial a}{\partial\lambda}. \end{aligned} \tag{2.3}$$

Obviously,

$$(\breve{q}^-(x,y_1^-(x,h,\lambda))-\breve{q}^-(x,y_2^-(x,h,\lambda)))|_{x=\bar{c}(h,\lambda)}=0. \tag{2.4}$$

Further noting that $H^-(x,y,\lambda)=H^-(A_\lambda(h),\lambda)$ along $\widehat{B_\lambda A_\lambda}$, we can obtain by (1.6)

$$\frac{\partial y}{\partial\lambda}=-\frac{H_\lambda^-(x,y,\lambda)-(H^-(A_\lambda,\lambda))_\lambda}{H_y^-(x,y,\lambda)}=-\frac{H_\lambda^-(x,y,\lambda)-g(h,\lambda)}{H_y^-(x,y,\lambda)} \tag{2.5}$$

for $y=y_1^-$ or y_2^-.

Thus, substituting (2.4) and (2.5) into (2.3) yields that

$$\begin{aligned}\hat{M}_\lambda^-(h,\lambda)=&\int_0^{\bar{c}(h,\lambda)}-\Big(\breve{q}_y^-(x,y_1^-(x,h,\lambda))\frac{H_\lambda^-(x,y_1^-(x,h,\lambda),\lambda)-g(h,\lambda)}{H_y^-(x,y_1^-(x,h,\lambda),\lambda)}\\&-\breve{q}_y^-(x,y_2^-(x,h,\lambda))\frac{H_\lambda^-(x,y_2^-(x,h,\lambda),\lambda)-g(h,\lambda)}{H_y^-(x,y_2^-(x,h,\lambda),\lambda)}\Big)dx-\Upsilon(\hat{p}^-)\\=&\int_{\widehat{B_\lambda A_\lambda}}-\breve{q}_y^-(x,y)\frac{H_\lambda^-(x,y,\lambda)-g(h,\lambda)}{H_y^-(x,y,\lambda)}dx-\Upsilon(\hat{p}^-).\end{aligned}$$

On the other hand, noting that

$$dx=H_y^-dt \tag{2.6}$$

along the orbit $\widehat{B_\lambda A_\lambda}$ and that $\breve{q}_y^-=\hat{p}_x^-+\hat{q}_y^-$, we have

$$\begin{aligned}\hat{M}_\lambda^-(h,\lambda)&=\int_{\widehat{B_\lambda A_\lambda}}-\breve{q}_y^-(x,y)(H_\lambda^--g(h,\lambda))dt-\Upsilon(\hat{p}^-)\\&=-\int_{\widehat{B_\lambda A_\lambda}}(H_\lambda^--g(h,\lambda))(\hat{p}_x^-+\hat{q}_y^-)dt-\Upsilon(\hat{p}^-).\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.1. According to (1.2), (1.4) and (1.6), we have

$$\begin{aligned}M(h,\lambda)=&\int_{\widehat{A_\lambda B_\lambda}}(q_0^++\lambda q_1^++\lambda^2q_2^++O(\lambda^3))dx-(p_0^++\lambda p_1^++\lambda^2p_2^++O(\lambda^3))dy\\&+G(h,\lambda)\int_{\widehat{B_\lambda A_\lambda}}(q_0^-+\lambda q_1^-+\lambda^2q_2^-+O(\lambda^3))dx-(p_0^-+\lambda p_1^-+\lambda^2p_2^-\\&+O(\lambda^3))dy\end{aligned}$$

which implies that

$$\begin{aligned}M_\lambda(h,\lambda)=&I_{0\lambda}^++I_1^++\lambda I_{1\lambda}^++2\lambda I_2^++O(\lambda^2)+G(h,\lambda)(I_{0\lambda}^-+I_1^-+\lambda I_{1\lambda}^-\\&+2\lambda I_2^-+O(\lambda^2))+G_\lambda(h,\lambda)(I_0^-+\lambda I_1^-+O(\lambda^2))\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}M_{\lambda\lambda}(h,\lambda)=&I_{0\lambda\lambda}^++2I_{1\lambda}^++2I_2^++O(\lambda)+G(h,\lambda)(I_{0\lambda\lambda}^-+2I_{1\lambda}^-+2I_2^-+O(\lambda))\\&+G_\lambda(h,\lambda)(2I_{0\lambda}^-+2I_1^-+O(\lambda))+G_{\lambda\lambda}(h,\lambda)(I_0^-+O(\lambda)),\end{aligned} \tag{2.8}$$

where

$$I_i^+(h,\lambda)=\int_{\widehat{A_\lambda B_\lambda}} q_i^+dx-p_i^+dy,\quad I_i^-(h,\lambda)=\int_{\widehat{B_\lambda A_\lambda}} q_i^-dx-p_i^-dy \tag{2.9}$$

for $i=0,1,2$.

It follows from (1.3) and (2.7) that

$$\begin{aligned}M_1(h)=M_\lambda(h,0)=&I_{0\lambda}^+(h,0)+I_1^+(h,0)+G(h,0)(I_{0\lambda}^-(h,0)+I_1^-(h,0))\\&+G_\lambda(h,0)I_0^-(h,0).\end{aligned} \tag{2.10}$$

Further, using Lemma 2.1 we have for $i=0,1,2$

$$\begin{aligned}I_{i\lambda}^+(h,\lambda)&=-\int_{\widehat{A_\lambda B_\lambda}} H_\lambda^+(p_{ix}^++q_{iy}^+)dt+\Upsilon(p_i^+),\\I_{i\lambda}^-(h,\lambda)&=-\int_{\widehat{B_\lambda A_\lambda}}(H_\lambda^--g(h,\lambda))(p_{ix}^-+q_{iy}^-)dt-\Upsilon(p_i^-).\end{aligned} \tag{2.11}$$

Thus substituting (2.9) and (2.11) into (2.10) and then taking $\lambda=0$ give the conclusion of Theorem 1.1. □

In order to deduce the formula for $M_2(h)$, we first prove the following helpful lemmas.

Lemma 2.2. *Assume*

$$I^+(h,\lambda)=-\int_{\widehat{A_\lambda B_\lambda}} H_\lambda^+(\hat{p}_x^++\hat{q}_y^+)dt,\quad I^-(h,\lambda)=-\int_{\widehat{B_\lambda A_\lambda}} H_\lambda^-(\hat{p}_x^-+\hat{q}_y^-)dt,$$

where $\hat{p}^\pm$ and $\hat{p}^\pm$ are given in Lemma 2.1. Further suppose that there exist a region U and C^∞ functions $\tilde{p}^\pm(x,y)$ and $\tilde{q}^\pm(x,y)$ defined on U such that

$$-H_1^\pm(\hat{p}_x^\pm+\hat{q}_y^\pm)=H_{0x}^\pm\tilde{p}^\pm+H_{0y}^\pm\tilde{q}^\pm,\quad (x,y)\in U. \tag{2.12}$$

Then

$$\begin{aligned}I_\lambda^+(h,0)&=-\int_{\widehat{AB}}[(H_1^+\tilde{p}^+)_x+(H_1^+\tilde{q}^+)_y]dt-2\int_{\widehat{AB}} H_2^+(\hat{p}_x^++\hat{q}_y^+)dt+\Upsilon(\tilde{p}^+)|_{\lambda=0},\\I_\lambda^-(h,0)&=-\int_{\widehat{BA}}[(H_1^-\tilde{p}^-)_x+(H_1^-\tilde{q}^-)_y]dt+\int_{\widehat{BA}} g(h,0)(\tilde{p}_x^-+\tilde{q}_y^-)dt\\&\quad-2\int_{\widehat{BA}} H_2^-(\hat{p}_x^-+\hat{q}_y^-)dt-\Upsilon(\tilde{p}^-)|_{\lambda=0}.\end{aligned} \tag{2.13}$$

Proof. The first formula in (2.13) can be found in [22], thus we just prove the second one. From (1.4) and (2.12), $I^-(h,\lambda)$ can be written in an equivalent way as

$$\begin{aligned}I^-(h,\lambda)&=-\int_{\widehat{B_\lambda A_\lambda}} H_\lambda^-(\hat{p}_x^-+\hat{q}_y^-)dt\\&=-\int_{\widehat{B_\lambda A_\lambda}} H_1^-(\hat{p}_x^-+\hat{q}_y^-)dt-2\lambda\int_{\widehat{B_\lambda A_\lambda}} H_2^-(\hat{p}_x^-+\hat{q}_y^-)dt+O(\lambda^2)\\&=\int_{\widehat{B_\lambda A_\lambda}}(H_{0x}^-\tilde{p}^-+H_{0y}^-\tilde{q}^-)dt-2\lambda\int_{\widehat{B_\lambda A_\lambda}} H_2^-(\hat{p}_x^-+\hat{q}_y^-)dt+O(\lambda^2)\\&=\int_{\widehat{B_\lambda A_\lambda}}(H_x^-\tilde{p}^-+H_y^-\tilde{q}^-)dt-\lambda\int_{\widehat{B_\lambda A_\lambda}}(H_{1x}^-\tilde{p}^-+H_{1y}^-\tilde{q}^-)dt\\&\quad-2\lambda\int_{\widehat{B_\lambda A_\lambda}} H_2^-(\hat{p}_x^-+\hat{q}_y^-)dt+O(\lambda^2).\end{aligned}$$

Thus, we obtain

$$I_\lambda^-(h,0) = \left[\int_{\widehat{B_\lambda A_\lambda}} (H_x^-\tilde{p}^- + H_y^-\tilde{q}^-)dt\right]_\lambda \Bigg|_{\lambda=0} - \int_{\widehat{BA}} (H_{1x}^-\tilde{p}^- + H_{1y}^-\tilde{q}^-)dt$$
$$- 2\int_{\widehat{BA}} H_2^-(\hat{p}_x^- + \hat{q}_y^-)dt. \tag{2.14}$$

By (2.12) and using Lemma 2.1 we have

$$\left[\int_{\widehat{B_\lambda A_\lambda}} (H_x^-\tilde{p}^- + H_y^-\tilde{q}^-)dt\right]_\lambda = \left[\int_{\widehat{B_\lambda A_\lambda}} \tilde{q}^- dx - \tilde{p}^- dy\right]_\lambda$$
$$= -\int_{\widehat{B_\lambda A_\lambda}} (H_\lambda^- - g(h,\lambda))(\tilde{p}_x^- + \tilde{q}_y^-)dt - \Upsilon(\tilde{p}^-). \tag{2.15}$$

Taking $\lambda = 0$ in (2.15) and substituting the result into (2.14) give the second formula of (2.13). This ends the proof. □

Lemma 2.3. *Let*

$$J(h,\lambda) = \int_{\widehat{B_\lambda A_\lambda}} g(h,\lambda)(p_{0x}^- + q_{0y}^-)dt,$$

where g is given in (1.6). *Then*

$$J_\lambda(h,0) = \int_{\widehat{BA}} g_\lambda(h,0)(p_{0x}^- + q_{0y}^-)dt + \int_{\widehat{BA}} \frac{g(h,0)}{(H_{0y}^-)^2}\big[H_{0y}^-(p_{0xy}^- + q_{0yy}^-)(-H_1^-$$
$$+ g(h,0)) - (p_{0x}^- + q_{0y}^-)(H_{0yy}^-(-H_1^- + g(h,0)) + H_{0y}^-H_{1y}^-)\big]dt. \tag{2.16}$$

Proof. By the definition of the orbit $\widehat{B_\lambda A_\lambda}$, we can rewrite $J(h,\lambda)$ as

$$J(h,\lambda) = \int_{\widehat{B_\lambda A_\lambda}} g(h,\lambda)\frac{p_{0x}^- + q_{0y}^-}{H_y^-}dx$$
$$= \int_0^{\bar{c}(h,\lambda)} g(h,\lambda)(J_1(x,h,\lambda) - J_2(x,h,\lambda))dx,$$

where

$$J_i(x,h,\lambda) = \frac{p_{0x}^-(x,y_i^-(x,h,\lambda)) + q_{0y}^-(x,y_i^-(x,h,\lambda))}{H_y^-(x,y_i^-(x,h,\lambda),\lambda)}, \quad i = 1,2.$$

From (2.5) we can obtain

$$J_{i\lambda}(x,h,\lambda) = \frac{1}{(H_y^-(x,y_i^-(x,h,\lambda),\lambda))^2}\Big[\big(p_{0xy}^-(x,y_i^-(x,h,\lambda)) + q_{0yy}^-(x,y_i^-(x,h,\lambda))\big)$$
$$\times \frac{\partial y_i}{\partial \lambda}H_y^-(x,y_i^-(x,h,\lambda),\lambda) - \big(p_{0x}^-(x,y_i^-(x,h,\lambda)) + q_{0y}^-(x,y_i^-(x,h,\lambda))\big)$$
$$\times \Big(H_{yy}^-(x,y_i^-(x,h,\lambda),\lambda)\frac{\partial y_i}{\partial \lambda} + H_{y\lambda}^-(x,y_i^-(x,h,\lambda),\lambda)\Big)\Big]$$

$$
\begin{aligned}
=&\frac{1}{(H_y^-(x,y_i^-(x,h,\lambda),\lambda))^3}\Big[\big(p_{0xy}^-(x,y_i^-(x,h,\lambda))+q_{0yy}^-(x,y_i^-(x,h,\lambda))\big)\\
&\times\big(-H_\lambda^-(x,y_i^-(x,h,\lambda),\lambda)+g(h,\lambda)\big)H_y^-(x,y_i^-(x,h,\lambda),\lambda)\\
&-\big(p_{0x}^-(x,y_i^-(x,h,\lambda))+q_{0y}^-(x,y_i^-(x,h,\lambda))\big)\big(H_{yy}^-(x,y_i^-(x,h,\lambda),\lambda)\\
&\times(-H_\lambda^-(x,y_i^-(x,h,\lambda),\lambda)+g(h,\lambda))+H_{y\lambda}^-(x,y_i^-(x,h,\lambda),\lambda)\\
&\times H_y^-(x,y_i^-(x,h,\lambda),\lambda)\big)\Big]. \qquad (2.17)
\end{aligned}
$$

Since

$$(y_1^-(x,h,\lambda)-y_2^-(x,h,\lambda))|_{x=\bar{c}(h,\lambda)}=0,$$

we have

$$(J_1(x,h,\lambda)-J_2(x,h,\lambda))|_{x=\bar{c}(h,\lambda)}=0. \qquad (2.18)$$

Hence, by (2.18) together with some calculations, we obtain from (2.17)

$$
\begin{aligned}
J_\lambda(h,\lambda)=&\int_0^{\bar{c}(h,\lambda)}\Bigg[g(h,\lambda)\Bigg(\frac{p_{0x}^-(x,y_1^-(x,h,\lambda))+q_{0y}^-(x,y_1^-(x,h,\lambda))}{H_y^-(x,y_1^-(x,h,\lambda),\lambda)}\\
&-\frac{p_{0x}^-(x,y_2^-(x,h,\lambda))+q_{0y}^-(x,y_2^-(x,h,\lambda))}{H_y^-(x,y_2^-(x,h,\lambda),\lambda)}\Bigg)\Bigg]_\lambda dx\\
=&\int_{\widehat{B_\lambda A_\lambda}}g_\lambda(h,\lambda)(p_{0x}^-+q_{0y}^-)dt+\int_{\widehat{B_\lambda A_\lambda}}\frac{g(h,\lambda)}{(H_y^-)^3}\big[(p_{0xy}^-+q_{0yy}^-)\cdot\\
&\big(-H_\lambda^-+g(h,\lambda)\big)H_y^--\big(p_{0x}^-+q_{0y}^-\big)\big(H_{yy}^-(-H_\lambda^-+g(h,\lambda))\\
&+H_{y\lambda}^-H_y^-\big)\big]dx. \qquad (2.19)
\end{aligned}
$$

Taking $\lambda=0$ in (2.19) and combining (1.4) and (2.6) give (2.16). □

Proof of Theorem 1.2. By (1.3) and (2.8) we obtain

$$
\begin{aligned}
M_2(h)=\frac{1}{2}M_{\lambda\lambda}(h,0)=&\frac{1}{2}I_{0\lambda\lambda}^+(h,0)+I_{1\lambda}^+(h,0)+I_2^+(h,0)+G(h,0)\Big(\frac{1}{2}I_{0\lambda\lambda}^-(h,0)\\
&+I_{1\lambda}^-(h,0)+I_2^-(h,0)\Big)+G_\lambda(h,0)(I_{0\lambda}^-(h,0)+I_1^-(h,0))\\
&+\frac{1}{2}G_{\lambda\lambda}(h,0)I_0^-(h,0). \qquad (2.20)
\end{aligned}
$$

Using Lemma 2.2 and (2.11), one can see

$$
\begin{aligned}
I_{0\lambda\lambda}^+(h,0)=&-\int_{\widehat{AB}}[(H_1^+\bar{p}_0^+)_x+(H_1^+\bar{q}_0^+)_y]dt-2\int_{\widehat{AB}}H_2^+(p_{0x}^++q_{0y}^+)dt+\Upsilon(\bar{p}_0^+)|_{\lambda=0}\\
&+(\Upsilon(p_0^+))_\lambda|_{\lambda=0},\\
I_{0\lambda\lambda}^-(h,0)=&-\int_{\widehat{BA}}[(H_1^-\bar{p}_0^-)_x+(H_1^-\bar{q}_0^-)_y]dt+\int_{\widehat{BA}}g(h,0)(\bar{p}_{0x}^-+\bar{q}_{0y}^-)dt\\
&-2\int_{\widehat{BA}}H_2^-(p_{0x}^-+q_{0y}^-)dt-\Upsilon(\bar{p}_0^-)|_{\lambda=0}-(\Upsilon(p_0^-))_\lambda|_{\lambda=0}+J_\lambda(h,0),
\end{aligned}
\qquad (2.21)
$$

where $J(h,\lambda)$ is defined in Lemma 2.3.

Substituting (2.9), (2.11) and (2.21) into (2.20) and combining (2.16) we can derive the expression (1.8) of $M_2(h)$. This completes the proof. □

3. Application

Consider the following piecewise smooth system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + \lambda(1-x)^2 + \varepsilon(p_0^+(x,y) + \lambda p_1^+(x,y)) \\ -x + 2\lambda y(1-x) + \varepsilon(q_0^+(x,y) + \lambda q_1^+(x,y)) \end{pmatrix}, & x > 0, \\ \begin{pmatrix} y + \lambda(1+2y) + 2\lambda^2 + \varepsilon(p_0^-(x,y) + \lambda p_1^-(x,y)) \\ -x + \varepsilon(q_0^-(x,y) + \lambda q_1^-(x,y)) \end{pmatrix}, & x \le 0, \end{cases} \tag{3.1}$$

where $0 < \varepsilon \ll \lambda \ll 1$, $p_0^\pm, p_1^\pm, q_0^\pm$ and $q_1^\pm$ are polynomials of degree 2 with the form

$$p_0^\pm(x,y) = \sum_{i+j=0}^{2} a_{ij}^\pm x^i y^j, \quad p_1^\pm(x,y) = \sum_{i+j=0}^{2} b_{ij}^\pm x^i y^j,$$
$$q_0^\pm(x,y) = \sum_{i+j=0}^{2} c_{ij}^\pm x^i y^j, \quad q_1^\pm(x,y) = \sum_{i+j=0}^{2} d_{ij}^\pm x^i y^j.$$

Let $\lambda = 0$. Then system (3.1) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + \varepsilon p_0^+(x,y) \\ -x + \varepsilon q_0^+(x,y) \end{pmatrix}, & x > 0, \\ \begin{pmatrix} y + \varepsilon p_0^-(x,y) \\ -x + \varepsilon q_0^-(x,y) \end{pmatrix}, & x \le 0, \end{cases} \tag{3.2}$$

which has been investigated in Liu and Han [16]. By Proposition 3.1 in [16], system (3.2) has at most 2 limit cycles bifurcating from the unperturbed period annulus by using the first order Melnikov function.

Notice that for $\varepsilon = 0$ system (3.1) has Hamiltonian functions H^+ and H^- respectively, where

$$\begin{aligned} H^+(x,y,\lambda) =& \frac{1}{2}(x^2+y^2) + \lambda(1-x)^2 y, \quad x > 0, \\ H^-(x,y,\lambda) =& \frac{1}{2}(x^2+y^2) + \lambda(y+y^2) + 2\lambda^2 y \quad x \le 0. \end{aligned} \tag{3.3}$$

By (3.3) we have

$$A_\lambda(h) = (0, -\lambda + \sqrt{\lambda^2 + 2h}), \quad B_\lambda(h) = (0, -\lambda - \sqrt{\lambda^2 + 2h}). \tag{3.4}$$

In this case, one has

$$H^-(A_\lambda(h), \lambda) = H^-(B_\lambda(h), \lambda) = h + 2h\lambda \neq h$$

as $\lambda \neq 0$. Hence, the formula of $M_1(h)$ and $M_2(h)$ given in the paper [22] cannot be used to study system (3.1).

Taking $x = \sqrt{2h}\sin\theta, y = \sqrt{2h}\cos\theta$, we obtain the expression of $M_0(h)$ for system (3.1) from (1.5)

$$\begin{aligned} M_0(h) =& \int_0^\pi \sqrt{2h}(q_0^+(\sqrt{2h}\sin\theta, \sqrt{2h}\cos\theta)\cos\theta + p_0^+(\sqrt{2h}\sin\theta, \sqrt{2h}\cos\theta)\sin\theta)d\theta \\ &+ \int_\pi^{2\pi} \sqrt{2h}(q_0^-(\sqrt{2h}\sin\theta, \sqrt{2h}\cos\theta)\cos\theta + p_0^-(\sqrt{2h}\sin\theta, \sqrt{2h}\cos\theta)\sin\theta)d\theta \\ =& \sqrt{h}(A + B\sqrt{h} + Ch), \end{aligned}$$

where

$$\begin{aligned}A =&2\sqrt{2}(a_{00}^{+}-a_{00}^{-}),\\ B =&\pi(a_{10}^{+}+a_{10}^{-}+c_{01}^{+}+c_{01}^{-}),\\ C =&\frac{4}{3}\sqrt{2}(-a_{02}^{-}-2a_{20}^{-}-c_{11}^{-}+a_{02}^{+}+2a_{20}^{+}+c_{11}^{+}).\end{aligned}$$

Apparently, the function $M_0(h)$ has at most 2 isolated positive zeros for $h > 0$, as it was in [16].

Let $M_0 = 0$ or equivalently

$$\begin{aligned}a_{00}^{+} =&a_{00}^{-},\\ a_{10}^{+} =&-a_{10}^{-}-c_{01}^{+}-c_{01}^{-},\\ a_{02}^{+} =&a_{02}^{-}+2a_{20}^{-}-2a_{20}^{+}-c_{11}^{+}+c_{11}^{-}.\end{aligned} \tag{3.5}$$

Proposition 3.1. *Assume that* (3.5) *holds. Under proper perturbations, system* (3.1) *can have* 3 *small limit cycles around the origin for* $0 < \varepsilon \ll \lambda \ll 1$.

Proof. From (1.6) and (3.3)–(3.4), we obtain

$$\begin{aligned}\Upsilon(p_0^{+}) =&p_0^{+}(0,a(h,\lambda))\frac{\partial a}{\partial\lambda}-p_0^{+}(0,b(h,\lambda))\frac{\partial b}{\partial\lambda}\\ =&\frac{2(4a_{02}^{+}\lambda^3-2a_{01}^{+}\lambda^2+6a_{02}^{+}h\lambda+a_{00}^{+}\lambda-2a_{01}^{+}h)}{\sqrt{\lambda^2+2h}},\\ \Upsilon(p_0^{-}) =&p_0^{-}(0,a(h,\lambda))\frac{\partial a}{\partial\lambda}-p_0^{-}(0,b(h,\lambda))\frac{\partial b}{\partial\lambda}\\ =&\frac{2(4a_{02}^{-}\lambda^3-2a_{01}^{-}\lambda^2+6a_{02}^{-}h\lambda+a_{00}^{-}\lambda-2a_{01}^{-}h)}{\sqrt{\lambda^2+2h}},\\ g(h,\lambda) =&2h,\quad G(h,\lambda)=\frac{1}{2\lambda+1}.\end{aligned} \tag{3.6}$$

By simple computations, it follows that

$$\begin{aligned}\int_{\widehat{AB}} q_1^{+}dx-p_1^{+}dy =& \int_0^{\pi}\sqrt{2h}(q_1^{+}(\sqrt{2h}\sin\theta,\sqrt{2h}\cos\theta)\cos\theta\\ &+p_1^{+}(\sqrt{2h}\sin\theta,\sqrt{2h}\cos\theta)\sin\theta)d\theta\\ =&\frac{4}{3}\sqrt{2}(b_{02}^{+}+2b_{20}^{+}+d_{11}^{+})h^{\frac{3}{2}}+\pi(b_{10}^{+}+d_{01}^{+})h+2\sqrt{2h}b_{00}^{+},\\ \int_{\widehat{AB}} H_1^{+}(p_{0x}^{+}+q_{0y}^{+})dt =& \int_0^{\pi}\sqrt{2h}(1-\sqrt{2h}\sin\theta)^2(p_{0x}^{+}(\sqrt{2h}\sin\theta,\sqrt{2h}\cos\theta)\\ &+q_{0y}^{+}(\sqrt{2h}\sin\theta,\sqrt{2h}\cos\theta))\cos\theta d\theta\\ =&(\frac{1}{2}a_{11}^{+}\pi+c_{02}^{+}\pi)h^2-\frac{8}{3}\sqrt{2}(2c_{02}^{+}a_{11}^{+})h^{\frac{3}{2}}+\pi(a_{11}^{+}+2c_{02}^{+})h.\end{aligned} \tag{3.7}$$

Similarly, one has

$$\begin{aligned}\int_{\widehat{BA}} q_1^- dx - p_1^- dy =& \int_{\pi}^{2\pi} \sqrt{2h}(q_1^-(\sqrt{2h}\sin\theta, \sqrt{2h}\cos\theta)\cos\theta \\ &+ p_1^-(\sqrt{2h}\sin\theta, \sqrt{2h}\cos\theta)\sin\theta)d\theta \\ =&\frac{4}{3}\sqrt{2}(-b_{02}^- - 2b_{20}^- - d_{11}^-)h^{\frac{3}{2}} + \pi(b_{10}^- + d_{01}^-)h - 2\sqrt{2h}b_{00}^-,\end{aligned}$$

$$\begin{aligned}\int_{\widehat{BA}} (H_1^- - g(h,0))(p_{0x}^- + q_{0y}^-)dt \quad (3.8)\\ =& \int_{\pi}^{2\pi} \sqrt{2h}\cos\theta(1 + \sqrt{2h}\cos\theta - \frac{2h}{\sqrt{2h}\cos\theta}) \\ &(p_{0x}^-(\sqrt{2h}\sin\theta, \sqrt{2h}\cos\theta) + q_{0y}^-(\sqrt{2h}\sin\theta, \sqrt{2h}\cos\theta))d\theta \\ =&\frac{8}{3}\sqrt{2}(2a_{20}^- + c_{11}^-)h^{\frac{3}{2}} + \pi(-a_{10}^- + a_{11}^- - c_{01}^- + 2c_{02}^-)h.\end{aligned}$$

Then, substituting (3.5)–(3.8) into (1.7) together with some calculations we obtain for system (3.1)

$$\begin{aligned}M_1(h) =& -\int_{\widehat{AB}} H_1^+(p_{0x}^+ + q_{0y}^+)dt + \Upsilon(p_0^+)|_{\lambda=0} + \int_{\widehat{AB}} q_1^+ dx - p_1^+ dy \\ &+ G(h,0)\Bigg[-\int_{\widehat{BA}} (H_1^- - g(h,0))(p_{0x}^- + q_{0y}^-)dt - \Upsilon(p_0^-)|_{\lambda=0} \\ &+ \int_{\widehat{BA}} q_1^- dx - p_1^- dy\Bigg] + G_\lambda(h,0)\int_{\widehat{BA}} q_0^- dx - p_0^- dy \\ =&h^{\frac{1}{2}}(V_0 + V_1 h^{\frac{1}{2}} + V_2 h + V_3 h^{\frac{3}{2}}),\end{aligned}$$

where

$$\begin{aligned}V_0 =&2\sqrt{2}(2a_{00}^- - a_{01}^+ + a_{01}^- + b_{00}^+ - b_{00}^-), \\ V_1 =&\pi(-a_{10}^- - a_{11}^+ - a_{11}^- + b_{10}^+ + b_{10}^- - c_{01}^- - 2c_{02}^+ - 2c_{02}^- + d_{01}^+ + d_{01}^-), \\ V_2 =&\frac{4}{3}\sqrt{2}(2a_{11}^+ + 4c_{02}^+ - b_{02}^- + b_{02}^+ + 2b_{20}^+ + 2a_{02}^- + d_{11}^+ - d_{11}^- - 2b_{20}^-), \\ V_3 =&\frac{\pi}{2}(-2c_{02}^+ - a_{11}^+).\end{aligned}$$

It is obvious that $M_1(h)$ has at most 3 isolated positive zeros. Next, we prove that system (3.1) can have 3 limit cycles near the origin.

Let

$$\delta = (a_{00}^-, a_{11}^+, d_{11}^+), \quad \delta_0 = (0,0,0),$$

and fix

$$\begin{aligned}&c_{02}^+ \neq 0, \quad a_{01}^+ = a_{01}^- + b_{00}^+ - b_{00}^-, \\ &a_{10}^- = -a_{11}^- + b_{10}^+ + b_{10}^- - c_{01}^- - 2c_{02}^+ - 2c_{02}^- + d_{01}^+ + d_{01}^-, \\ &d_{11}^- = 4c_{02}^+ - b_{02}^- + b_{02}^+ + 2b_{20}^+ + 2a_{02}^- - 2b_{20}^-.\end{aligned}$$

Through direct computations we obtain

$$V_i(\delta_0) = 0, i = 0, 1, 2, \quad V_3(\delta_0) = -\pi c_{02}^{+} \neq 0,$$

$$\det \frac{\partial(V_0, V_1, V_2)}{\partial(a_{00}^{-}, a_{11}^{+}, d_{11}^{+})}(\delta_0) = \begin{vmatrix} 4\sqrt{2} & 0 & 0 \\ 0 & -\pi & 0 \\ 0 & \frac{8}{3}\sqrt{2} & \frac{4}{3}\sqrt{2} \end{vmatrix} = -\frac{32}{3}\pi.$$

Similar to the proof of Corollary 2.4.1 in [6], one gets that V_0, V_1, V_2 and V_3 can be taken as free parameters. Hence we can vary δ near δ_0 such that

$$0 < |V_0| \ll |V_1| \ll |V_2| \ll |V_3| \ll 1, \quad V_i V_{i+1} < 0, i = 0, 1, 2,$$

which ensures that $M_1(h)$ has 3 isolated positive zeros for $h > 0$. This completes the proof. □

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