

SELF-ADJOINT REALIZATION OF A CLASS OF THIRD-ORDER DIFFERENTIAL OPERATORS WITH AN EIGENPARAMETER CONTAINED IN THE BOUNDARY CONDITIONS

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Abstract The present paper deals with a class of third-order differential operators with eigenparameter dependent boundary conditions. Using operator theoretic formulation, the self-adjointness of this operator is proved, the properties of spectrum are investigated, its Green function and the resolvent operator are also obtained.

Keywords Third-order differential operators, eigenparameter dependent boundary conditions, self-adjointness, Green function.

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1. Introduction

Spectral theory of ordinary differential operators with an eigenparameter contained in the boundary conditions has been discussed for a long time. The feature of such problems is that the eigenparameter appears both in the differential equation and boundary conditions. Due to its wide applications in mechanics and mathematical physics, such as electric circuits, mechanical vibrations, acoustic scattering theory etc, [8, 10, 11, 17], more and more researchers have been paid attention to such problems. Moreover, many excellent results of such problems for second-order or fourth-order differential operators have been obtained, such as self-adjointness, asymptotical formula of the eigenvalues and eigenfunctions, oscillation of the eigenfunction, inverse spectral problems and so on, see, for example [1, 2, 4–6, 12, 18–21, 25, 27, 33, 35, 36]. However, little is known for the case of third-order differential operators.

The characterization of self-adjoint boundary conditions is an important part in the differential operators theory. Such problems have been well established for regular or singular differential operators, see [3, 24, 31, 32] and references cited therein. Wang etc characterized the self-adjoint domain of even order differential operators by using real parameter solutions [31] and gave the classification of boundary condi-

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tions, that is, separated, coupled and mixed [30]. Similar results have been obtained by Hao etc for the case of odd differential operators [15].

Third-order differential equations appear in many physical phenomenons, for example, the deflection of a curved beam varying cross section, three-layer beam and so on [13]. Hao etc in [23] proved that there is no strict separated boundary conditions for self-adjoint third-order differential operators, that is to say, there only mixed and coupled boundary conditions exist for third-order self-adjoint differential operators. Uğurlu investigated a class of third-order differential operators with mixed boundary conditions, and gave the dependence of eigenvalues on the data, and then generalized these results to differential operators with interface conditions [28, 29]. There are also many literatures focusing on other issues for the third-order equation, see, for instance, [16, 22, 26, 34].

In the present paper, we study third-order symmetric differential equation

$$\ell y := \frac{1}{w} \{-i[q_0(q_0 y)']' - (p_0 y)'\} + i[q_1 y' + (q_1 y)'] + p_1 y = \lambda y, \text{ on } [a, b], \quad (1.1)$$

with boundary conditions

$$L_1 y := (\alpha_1 \lambda + \tilde{\alpha}_1) y(a) - (\alpha_2 \lambda + \tilde{\alpha}_2) y^{[2]}(a) = 0, \quad (1.2)$$

$$L_2 y := (\beta_1 \lambda + \tilde{\beta}_1) y(b) + (\beta_2 \lambda + \tilde{\beta}_2) y^{[2]}(b) = 0 \quad (1.3)$$

and

$$L_3 y := (\sin \beta + i) y^{[1]}(a) + (i \sin \beta + 1) y^{[1]}(b) = 0, \quad (1.4)$$

where λ is the spectral parameter, q_0, q_1, p_0, p_1, w satisfy the following conditions

$$q_0^{-1}, q_0^{-2}, p_0, q_1, p_1, w \in L^1([a, b], \mathbb{R}), \quad q_0 > 0, \quad w > 0. \quad (1.5)$$

$\alpha_k, \tilde{\alpha}_k, \beta_k, \tilde{\beta}_k (k = 1, 2)$ are arbitrary real numbers and satisfying

$$\rho_1 = \tilde{\alpha}_1 \alpha_2 - \alpha_1 \tilde{\alpha}_2 > 0, \quad \rho_2 = \tilde{\beta}_1 \beta_2 - \beta_1 \tilde{\beta}_2 > 0. \quad (1.6)$$

Here we consider a third-order differential equation with mixed boundary conditions (1.1)-(1.4), where two boundary conditions are of separated, affinely dependent on the eigenparameter and the rest one is of coupled. By using the classical analysis techniques and spectral theory of linear operator, we define a new linear operator T associated with the problem (1.1)-(1.4) in an appropriate Hilbert space \mathcal{H} such that the eigenvalues of the problem (1.1)-(1.4) coincide with those of T . The paper is organized as follows: In Section 2, we investigate some basic notations and preliminaries. In Section 3, we introduce a new Hilbert space and construct an operator T associated with the problem (1.1)-(1.4), and discuss the self-adjointness, the properties of eigenvalues of this operator. The Green function and the resolvent operator are discussed in Section 4.

2. Notations and preliminaries

Let the quasi-derivatives of y be defined as [14]

$$y^{[0]} = y, \quad y^{[1]} = -\frac{1+i}{\sqrt{2}} q_0 y', \quad y^{[2]} = i q_0 (q_0 y)' + p_0 y' - i q_1 y$$

and $H_w = L_w^2[a, b]$ be the weighted Hilbert space consisting of functions y which satisfy $\int_a^b |y|^2 w dx < \infty$ under the inner product $\langle y, z \rangle_w = \int_a^b y \bar{z} w dx$.

Denote by L_{\max} the maximal operator with the domain

$$D_{\max} = \{y \in L_w^2[a, b] \mid y^{[0]}, y^{[1]}, y^{[2]} \in AC[a, b], \ell y \in L_w^2[a, b]\},$$

and the rule

$$L_{\max} y = \ell y, \quad y \in D_{\max}.$$

Then for arbitrary $y, z \in D_{\max}$, integration by parts yields Lagrange identity

$$\langle L_{\max} y, z \rangle_w - \langle y, L_{\max} z \rangle_w = [y, \bar{z}]_a^b,$$

where

$$\begin{aligned} [y, z]_a^b &= [y, z](b) - [y, z](a), \\ [y, z](x) &= y(x)z^{[2]}(x) - y^{[2]}(x)z(x) + iy^{[1]}(x)z^{[1]}(x). \end{aligned}$$

By the definition of quasi-derivatives, we can transfer the equation (1.1) to the following first-order system

$$Y' + QY = \lambda WY, \quad (2.1)$$

where

$$Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ y^{[2]} \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -w & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & \frac{\sqrt{2}}{(1+i)q_0} & 0 \\ \frac{(1+i)q_1}{\sqrt{2}q_0} & -i\frac{p_0}{q_0} & \frac{\sqrt{2}}{(1+i)q_0} \\ -p_1 & \frac{(1+i)q_1}{\sqrt{2}q_0} & 0 \end{pmatrix}.$$

Then by (1.5), (2.1), we have the following result.

Theorem 2.1 ([28]). *There exists an unique solution for the equation (1.1) with initial conditions $y^{[j]}(c, \lambda) = c_j(\lambda)$, where $c \in [a, b]$, $c_j(\lambda)$ are entire functions of λ . Moreover, $y^{[j]}(\cdot, \lambda)$ are entire functions of λ , $j = 0, 1, 2$.*

3. Operator theoretic formulation and self-adjointness

Motivated by Friedman [9], Mukhtarov [20, 25] and Fulton [11], we can construct a new Hilbert space $\mathcal{H} = L_w^2[a, b] \oplus \mathbb{C}^2$ under a suitable inner product by combining the parameters in the boundary conditions. With this goal, the inner product is defined by

$$\langle Y, Z \rangle = \int_a^b y \bar{z} w dx + \frac{1}{\rho_1} y_1 \bar{z}_1 + \frac{1}{\rho_2} y_2 \bar{z}_2, \quad (3.1)$$

where $Y = (y(x), y_1, y_2)^T$, $Z = (z(x), z_1, z_2)^T \in \mathcal{H}$.

We shall use the following notations:

$$\begin{aligned} M_1(y) &= \alpha_1 y(a) - \alpha_2 y^{[2]}(a), \quad M_2(y) = \beta_1 y(b) + \beta_2 y^{[2]}(b), \\ N_1(y) &= \tilde{\alpha}_2 y^{[2]}(a) - \tilde{\alpha}_1 y(a), \quad N_2(y) = -[\tilde{\beta}_1 y(b) + \tilde{\beta}_2 y^{[2]}(b)]. \end{aligned} \quad (3.2)$$

Define the operator T in the Hilbert space \mathcal{H} with domain

$$D(T) = \{Y = (y(x), y_1, y_2)^T \in \mathcal{H} \mid L_3 y = 0, y_1 = M_1(y), y_2 = M_2(y), y \in D_{\max}\}, \quad (3.3)$$

and

$$Y = (y(x), M_1(y), M_2(y))^T \in D(T), \quad TY = (\ell y, N_1(y), N_2(y))^T. \quad (3.4)$$

Then we get that the eigenvalue problem of BVP (1.1)-(1.4) is transferred to the spectra problem of the operator T .

Considering the operator T we have the following properties.

Lemma 3.1. *BVP problem (1.1)-(1.4) has the same eigenvalues with the operator T , and the eigenfunctions of BVP problem (1.1)-(1.4) are the first component of the corresponding eigenvectors of the operator T .*

Proof. For arbitrary $Y = (y(x), y_1, y_2)^T \in D(T)$, by (3.3) and (3.4), we have

$$TY = (\ell y, N_1(y), N_2(y))^T = \lambda(y(x), M_1(y), M_2(y))^T.$$

Comparing this with BVP problem (1.1)-(1.4) yields the conclusions. \square

Lemma 3.2. *$D(T)$ is dense in \mathcal{H} .*

Proof. Let $F = (f(x), f_1, f_2) \in \mathcal{H}$, $F \perp D(T)$. Since $C_0^\infty \oplus 0 \oplus 0 \subset D(T)$ ($0 \in \mathbb{C}$), for arbitrary $V = \{v(x), 0, 0\} \in C_0^\infty \oplus 0 \oplus 0$, we have

$$\langle F, V \rangle = \int_a^b f \bar{v} w dx = 0.$$

Because C_0^∞ is dense in $L_w^2[a, b]$, therefore $f(x) = 0$, that is, $F = (0, f_1, f_2)$. For any $Y = (y(x), y_1, 0) \in D(T)$, we have

$$\langle F, Y \rangle = \frac{1}{\rho_1} f_1 \bar{y}_1 = 0$$

by the inner product in \mathcal{H} . Through the arbitrariness of y_1 , then $f_1 = 0$. Moreover, for all $Z = (z(t), z_1, z_2) \in D(T)$, we have

$$\langle F, Z \rangle = \frac{1}{\rho_2} f_2 \bar{z}_2 = 0.$$

By the arbitrariness of z_2 , we have $f_2 = 0$. Hence $F = (0, 0, 0)$, and the proof is completed. \square

Lemma 3.3. *The operator T is symmetric.*

Proof. For any $U, V \in D(T)$, integration by parts yields

$$\begin{aligned} \langle TU, V \rangle - \langle U, TV \rangle &= [u, \bar{v}](b) - [u, \bar{v}](a) + \frac{1}{\rho_1} N_1(u) \overline{M_1(v)} \\ &\quad - \frac{1}{\rho_1} M_1(u) \overline{N_1(v)} + \frac{1}{\rho_2} N_2(u) \overline{M_2(v)} - \frac{1}{\rho_2} M_2(u) \overline{N_2(v)}. \end{aligned} \quad (3.5)$$

By the boundary conditions (1.2)-(1.4) we have

$$u^{[2]}(a)\bar{v}(a) - u(a)\bar{v}^{[2]}(a) = \frac{1}{\rho_1}M_1(u)\overline{N_1(v)} - \frac{1}{\rho_1}N_1(u)\overline{M_1(v)}, \quad (3.6)$$

$$u^{[2]}(b)\bar{v}(b) - u(b)\bar{v}^{[2]}(b) = \frac{1}{\rho_2}N_2(u)\overline{M_2(v)} - \frac{1}{\rho_2}M_2(u)\overline{N_2(v)}, \quad (3.7)$$

$$u^{[1]}(b)\bar{v}^{[1]}(b) - u^{[1]}(a)\bar{v}^{[1]}(a) = 0. \quad (3.8)$$

Inserting (3.6)-(3.8) into (3.5), we have

$$\langle TU, V \rangle - \langle U, TV \rangle = 0.$$

Therefore, the operator T is symmetric. \square

Theorem 3.1. T is a selfadjoint operator in \mathcal{H} .

Proof. Since T is symmetric, it suffices to prove that for any $Y = (y(x), y_1, y_2) \in D(T)$ and some $Z \in D(T^*)$, $U \in \mathcal{H}$ satisfying $\langle TY, Z \rangle = \langle Y, U \rangle$, then $Z \in D(T)$ and $TZ = U$, where $Z = (z(x), z_1, z_2)$, $U = (u(x), u_1, u_2)$, i.e.,

- (i) $z^{[j]}(x) \in AC[a, b]$, $j = 0, 1, 2$, $\ell z \in H_w$;
- (ii) $z_1 = \alpha_1 z(a) - \alpha_2 z^{[2]}(a)$, $z_2 = \beta_1 z(b) + \beta_2 z^{[2]}(b)$;
- (iii) $L_3 z = 0$;
- (iv) $u(x) = \ell z$;
- (v) $u_1 = \tilde{\alpha}_2 z^{[2]}(a) - \tilde{\alpha}_1 z(a)$, $u_2 = -[\tilde{\beta}_1 z(b) + \tilde{\beta}_2 z^{[2]}(b)]$.

Assume that for any $V = \{v(x), 0, 0\} \in \mathbb{C}_0^\infty \oplus 0 \oplus 0 \in D(T)$ satisfying

$$\int_a^b (\ell v)\bar{z}w dx = \int_a^b v\bar{u}w dx,$$

that is, $\langle \ell v, z \rangle_w = \langle v, u \rangle_w$. By the classical differential operator theory [7], we have (i) and (iv) hold. By (3.1), (3.4) and (iv) we get that for all $Y = (y(x), y_1, y_2) \in D(T)$, $\langle TY, Z \rangle = \langle Y, U \rangle$ turns to

$$\langle \ell y, z \rangle_w - \langle y, \ell z \rangle_w = \frac{1}{\rho_1}[M_1(y)\bar{u}_1 - N_1(y)\bar{z}_1] + \frac{1}{\rho_2}[M_2(y)\bar{u}_2 - N_2(y)\bar{z}_2].$$

In light of

$$\langle \ell y, z \rangle_w = \langle y, \ell z \rangle_w + [y, z]_a^b,$$

hence

$$\frac{1}{\rho_1}[M_1(y)\bar{u}_1 - N_1(y)\bar{z}_1] + \frac{1}{\rho_2}[M_2(y)\bar{u}_2 - N_2(y)\bar{z}_2] = [y, z]_a^b. \quad (3.9)$$

Using Naimark Patching Lemma, there exists a $Y = (y(x), y_1, y_2) \in D(T)$ such that

$$y(b) = y^{[1]}(b) = y^{[2]}(b) = 0, \quad y(a) = \alpha_2, \quad y^{[1]}(a) = 0, \quad y^{[2]}(a) = \alpha_1.$$

Substituting this into (3.9) yields $z_1 = \alpha_1 z(a) - \alpha_2 z^{[2]}(a)$. Similarly, there exists a $Y = (y(x), y_1, y_2) \in D(T)$ such that

$$y(a) = y^{[1]}(a) = y^{[2]}(a) = 0, \quad y(b) = \beta_2, \quad g^{[1]}(b) = 0, \quad y^{[2]}(b) = -\beta_1.$$

Then by (3.9), we have $z_2 = \beta_1 z(b) + \beta_2 z^{[2]}(b)$. Therefore, (ii) holds. Using similar methods, one can prove (v) is true.

Choosing $Y = (y(x), y_1, y_2) \in D(T)$ such that

$$y(a) = y^{[2]}(a) = y(b) = y^{[2]}(b) = 0, \quad y^{[1]}(a) = i \sin \beta - 1, \quad y^{[1]}(b) = \sin \beta - i.$$

Then by (3.9), (iii) holds. Hence, the operator T is selfadjoint. □

By the self-adjointness of the operator T , we have the following conclusions.

Corollary 3.1. *The eigenvalues of T are real-valued.*

Corollary 3.2. *Let λ_1 and λ_2 be two different eigenvalues of T , $Y_1 = (y_1(x), y_{11}, y_{12})$ and $Y_2 = (y_2(x), y_{21}, y_{22})$ be the corresponding eigenfunctions respectively, then $y_1(x)$ and $y_2(x)$ are orthogonal in the sense of*

$$\int_a^b y_1 \overline{y_2} w dx + \frac{1}{\rho_1} M_1(y_1) \overline{M_1(y_2)} + \frac{1}{\rho_2} M_2(y_1) \overline{M_2(y_2)} = 0.$$

Let

$$A_\lambda = \begin{pmatrix} \alpha_1 \lambda + \tilde{\alpha}_1 & 0 & -(\alpha_2 \lambda + \tilde{\alpha}_2) \\ 0 & i + \sin \beta & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 + i \sin \beta & 0 \\ \beta_1 \lambda + \tilde{\beta}_1 & 0 & -(\beta_2 \lambda + \tilde{\beta}_2) \end{pmatrix}.$$

Then the boundary conditions (1.2)-(1.4) of the BVP (1.1)-(1.4) can be rewritten in the following matrix form

$$A_\lambda Y(a) + B_\lambda Y(b) = 0, \tag{3.10}$$

where $Y(x) = (y(x), y^{[1]}(x), y^{[2]}(x))^T$.

Let $\phi_1(x, \lambda), \phi_2(x, \lambda), \phi_3(x, \lambda)$ be the solutions of equation (1.1) satisfying

$$\begin{pmatrix} \phi_1^{[0]}(a, \lambda) & \phi_2^{[0]}(a, \lambda) & \phi_3^{[0]}(a, \lambda) \\ \phi_1^{[1]}(a, \lambda) & \phi_2^{[1]}(a, \lambda) & \phi_3^{[1]}(a, \lambda) \\ \phi_1^{[2]}(a, \lambda) & \phi_2^{[2]}(a, \lambda) & \phi_3^{[2]}(a, \lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.11}$$

and

$$\Phi(x, \lambda) = \begin{pmatrix} \phi_1^{[0]}(x, \lambda) & \phi_2^{[0]}(x, \lambda) & \phi_3^{[0]}(x, \lambda) \\ \phi_1^{[1]}(x, \lambda) & \phi_2^{[1]}(x, \lambda) & \phi_3^{[1]}(x, \lambda) \\ \phi_1^{[2]}(x, \lambda) & \phi_2^{[2]}(x, \lambda) & \phi_3^{[2]}(x, \lambda) \end{pmatrix}. \tag{3.12}$$

Lemma 3.4. *A complex number λ is an eigenvalue of the problem (1.1)-(1.4) if and only if*

$$\Delta(\lambda) = \det[A_\lambda + B_\lambda \Phi(b, \lambda)] = 0.$$

Proof. Suppose λ_0 is an eigenvalue of the problem (1.1)-(1.4) and $v_0(x)$ is the corresponding eigenfunction. Then we have

$$v_0(x) = c_1\phi_1(x, \lambda_0) + c_2\phi_2(x, \lambda_0) + c_3\phi_3(x, \lambda_0), \quad x \in [a, b],$$

where at least one of coefficients c_i ($i = 1, 2, 3$) is not zero. Inserting $v_0(x)$ into (1.2)-(1.4) yields

$$(A_\lambda + B_\lambda\Phi(b, \lambda_0))(c_1, c_2, c_3)^T = 0 \quad (3.13)$$

by (3.11) and (3.12). Since c_1, c_2, c_3 are not all zero, $\det(A_\lambda + B_\lambda\Phi(b, \lambda_0)) = 0$.

On the contrary, if $\det(A_\lambda + B_\lambda\Phi(b, \lambda_0)) = 0$, then the system of the linear equations (3.13) for the constants c_i ($i = 1, 2, 3$) has non-zero solution (c'_1, c'_2, c'_3) . Let

$$v(x) = c'_1\phi_1(x, \lambda_0) + c'_2\phi_2(x, \lambda_0) + c'_3\phi_3(x, \lambda_0), \quad x \in [a, b],$$

then $v(x)$ is the non-trivial solution of equation $Tv = \lambda v$ satisfying the conditions (1.2)-(1.4) which implies λ_0 is an eigenvalue of the problem (1.1)-(1.4). \square

Theorem 3.2. *The eigenvalues of T are discrete and have no finite point of accumulation. Moreover, the multiplicity of each eigenvalue at most 3.*

Proof. The zeros of $\Delta(\lambda)$ are the eigenvalues of operator T by Lemma 3.4, and all the eigenvalues of T are real by the self-adjointness of T , that is to say, for any $\lambda \in \mathbb{C}$ with its imaginary part not vanishing, then $\Delta(\lambda) \neq 0$. Therefore, by the distribution of zeros of entire functions, the first part holds. The second conclusion follows from the fact that there at most 3 linearly independent solutions exist for the equation (1.1). \square

4. Green's function

In this section, we discuss the Green's function of BVP (1.1)-(1.4) when λ is not an eigenvalue of the problem (1.1)-(1.4). To this end, consider the following operator equation

$$(T - \lambda I)Y = F, \quad F = (f(x), f_1, f_2) \in \mathcal{H}. \quad (4.1)$$

By the definition of the operator T , the equation (4.1) can be transferred to the following inhomogeneous boundary value problems

$$-i[q_0(q_0y')]' - (p_0y')' + i[q_1y' + (q_1y)'] + (p_1 - \lambda w)y = fw, \quad (4.2)$$

$$\mathcal{L}_1(y) := (\alpha_1\lambda + \tilde{\alpha}_1)y(a) - (\alpha_2\lambda + \tilde{\alpha}_2)y^{[2]}(a) = -f_1, \quad (4.3)$$

$$\mathcal{L}_2(y) := (\beta_1\lambda + \tilde{\beta}_1)y(b) + (\beta_2\lambda + \tilde{\beta}_2)y^{[2]}(b) = -f_2, \quad (4.4)$$

$$\mathcal{L}_3(y) := (\sin \beta + i)y^{[1]}(a) + (i \sin \beta + 1)y^{[1]}(b) = 0. \quad (4.5)$$

Let $\phi_1(x, \lambda), \phi_2(x, \lambda), \phi_3(x, \lambda)$ be the solutions of homogeneous equation (1.1) satisfying the initial conditions

$$\Phi(a, \lambda) = \begin{pmatrix} \phi_1^{[0]}(a, \lambda) & \phi_2^{[0]}(a, \lambda) & \phi_3^{[0]}(a, \lambda) \\ \phi_1^{[1]}(a, \lambda) & \phi_2^{[1]}(a, \lambda) & \phi_3^{[1]}(a, \lambda) \\ \phi_1^{[2]}(a, \lambda) & \phi_2^{[2]}(a, \lambda) & \phi_3^{[2]}(a, \lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the general solution of the non-homogeneous differential equation (4.2) can be represented as

$$y(x, \lambda) = C_1(x, \lambda)\phi_1(x, \lambda) + C_2(x, \lambda)\phi_2(x, \lambda) + C_3(x, \lambda)\phi_3(x, \lambda), \quad x \in [a, b], \quad (4.6)$$

by the method of variation of constants, where $C_1(x, \lambda), C_2(x, \lambda), C_3(x, \lambda)$ satisfy the following conditions

$$\begin{cases} C_1'(x, \lambda)\phi_1(x, \lambda) + C_2'(x, \lambda)\phi_2(x, \lambda) + C_3'(x, \lambda)\phi_3(x, \lambda) = 0, \\ C_1'(x, \lambda)\phi_1'(x, \lambda) + C_2'(x, \lambda)\phi_2'(x, \lambda) + C_3'(x, \lambda)\phi_3'(x, \lambda) = 0, \\ C_1'(x, \lambda)\phi_1''(x, \lambda) + C_2'(x, \lambda)\phi_2''(x, \lambda) + C_3'(x, \lambda)\phi_3''(x, \lambda) = f(x)w(x). \end{cases} \quad (4.7)$$

Since λ is not an eigenvalue, linear system of (4.7) has an unique solution, and hence

$$\begin{aligned} \omega(\lambda) &= W(\phi_1(x, \lambda), \phi_2(x, \lambda), \phi_3(x, \lambda)) \\ &= \begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) & \phi_3(x, \lambda) \\ \phi_1'(x, \lambda) & \phi_2'(x, \lambda) & \phi_3'(x, \lambda) \\ \phi_1''(x, \lambda) & \phi_2''(x, \lambda) & \phi_3''(x, \lambda) \end{vmatrix} \neq 0, \end{aligned}$$

and

$$\begin{cases} C_1(x, \lambda) = \int_a^x \frac{f(t)w(t)}{\omega(\lambda)} [\phi_2\phi_3](t, \lambda) dt + C_1, \\ C_2(x, \lambda) = \int_a^x \frac{f(t)w(t)}{\omega(\lambda)} [\phi_3\phi_1](t, \lambda) dt + C_2, \\ C_3(x, \lambda) = \int_a^x \frac{f(t)w(t)}{\omega(\lambda)} [\phi_1\phi_2](t, \lambda) dt + C_3, \end{cases} \quad (4.8)$$

where C_1, C_2, C_3 are arbitrary constants. Inserting $C_1(x, \lambda), C_2(x, \lambda), C_3(x, \lambda)$ into (4.6), one gets that the general solution of (4.2) has the following representation

$$\begin{aligned} y(x, \lambda) &= \phi_1(x, \lambda) \int_a^x \frac{f(t)w(t)}{\omega(\lambda)} [\phi_2\phi_3](t, \lambda) dt + C_1\phi_1(x, \lambda) \\ &\quad + \phi_2(x, \lambda) \int_a^x \frac{f(t)w(t)}{\omega(\lambda)} [\phi_3\phi_1](t, \lambda) dt + C_2\phi_2(x, \lambda) \\ &\quad + \phi_3(x, \lambda) \int_a^x \frac{f(t)w(t)}{\omega(\lambda)} [\phi_1\phi_2](t, \lambda) dt + C_3\phi_3(x, \lambda) \\ &= \int_a^b K(x, t, \lambda) f(t)w(t) dt + C_1\phi_1(x, \lambda) + C_2\phi_2(x, \lambda) \\ &\quad + C_3\phi_3(x, \lambda), \quad x \in [a, b], \end{aligned} \quad (4.9)$$

where

$$K(x, t, \lambda) = \begin{cases} \frac{S(x, t, \lambda)}{\omega(\lambda)}, & a \leq t \leq x \leq b, \\ 0, & a \leq x \leq t \leq b, \end{cases} \quad (4.10)$$

$$S(x, t, \lambda) = \begin{vmatrix} \phi_1(t, \lambda) & \phi_2(t, \lambda) & \phi_3(t, \lambda) \\ \phi_1'(t, \lambda) & \phi_2'(t, \lambda) & \phi_3'(t, \lambda) \\ \phi_1(x, \lambda) & \phi_2(x, \lambda) & \phi_3(x, \lambda) \end{vmatrix}. \quad (4.11)$$

Substituting the general solution $y = y(x, \lambda)$ into (4.3)-(4.5), one gets that

$$\begin{aligned} & C_1 \mathcal{L}_1(\phi_1(x, \lambda)) + C_2 \mathcal{L}_1(\phi_2(x, \lambda)) + C_3 \mathcal{L}_1(\phi_3(x, \lambda)) \\ &= - \int_a^b f(t)w(t)\mathcal{L}_1(K(x, t, \lambda))dt - f_1, \end{aligned} \quad (4.12)$$

$$\begin{aligned} & C_1 \mathcal{L}_2(\phi_1(x, \lambda)) + C_2 \mathcal{L}_2(\phi_2(x, \lambda)) + C_3 \mathcal{L}_2(\phi_3(x, \lambda)) \\ &= - \int_a^b f(t)w(t)\mathcal{L}_2(K(x, t, \lambda))dt - f_2, \end{aligned} \quad (4.13)$$

$$\begin{aligned} & C_1 \mathcal{L}_3(\phi_1(x, \lambda)) + C_2 \mathcal{L}_3(\phi_2(x, \lambda)) + C_3 \mathcal{L}_3(\phi_3(x, \lambda)) \\ &= - \int_a^b f(t)w(t)\mathcal{L}_3(K(x, t, \lambda))dt. \end{aligned} \quad (4.14)$$

The determinant of coefficients of C_1, C_2, C_3 satisfies

$$\begin{vmatrix} \mathcal{L}_1(\phi_1(x, \lambda)) & \mathcal{L}_1(\phi_2(x, \lambda)) & \mathcal{L}_1(\phi_3(x, \lambda)) \\ \mathcal{L}_2(\phi_1(x, \lambda)) & \mathcal{L}_2(\phi_2(x, \lambda)) & \mathcal{L}_2(\phi_3(x, \lambda)) \\ \mathcal{L}_3(\phi_1(x, \lambda)) & \mathcal{L}_3(\phi_2(x, \lambda)) & \mathcal{L}_3(\phi_3(x, \lambda)) \end{vmatrix} = \det(A_\lambda + B_\lambda \Phi(b, \lambda)) = \Delta(\lambda) \neq 0. \quad (4.15)$$

Therefore,

$$C_1 = \frac{\Gamma_1(\lambda) + \Theta_1(\lambda)}{\Delta(\lambda)}, C_2 = \frac{\Gamma_2(\lambda) + \Theta_2(\lambda)}{\Delta(\lambda)}, C_3 = \frac{\Gamma_3(\lambda) + \Theta_3(\lambda)}{\Delta(\lambda)},$$

where

$$\begin{aligned} \Gamma_1(\lambda) &= \begin{vmatrix} - \int_a^b f(t)w(t)\mathcal{L}_1(K(x, t, \lambda))dt & \mathcal{L}_1(\phi_2(x, \lambda)) & \mathcal{L}_1(\phi_3(x, \lambda)) \\ - \int_a^b f(t)w(t)\mathcal{L}_2(K(x, t, \lambda))dt & \mathcal{L}_2(\phi_2(x, \lambda)) & \mathcal{L}_2(\phi_3(x, \lambda)) \\ - \int_a^b f(t)w(t)\mathcal{L}_3(K(x, t, \lambda))dt & \mathcal{L}_3(\phi_2(x, \lambda)) & \mathcal{L}_3(\phi_3(x, \lambda)) \end{vmatrix}, \\ \Theta_1(\lambda) &= \begin{vmatrix} -f_1 & \mathcal{L}_1(\phi_2(x, \lambda)) & \mathcal{L}_1(\phi_3(x, \lambda)) \\ -f_2 & \mathcal{L}_2(\phi_2(x, \lambda)) & \mathcal{L}_2(\phi_3(x, \lambda)) \\ 0 & \mathcal{L}_3(\phi_2(x, \lambda)) & \mathcal{L}_3(\phi_3(x, \lambda)) \end{vmatrix}. \end{aligned}$$

Similarly, we can get $\Gamma_2(\lambda), \Gamma_3(\lambda)$ and $\Theta_2(\lambda), \Theta_3(\lambda)$. Hence the general solution has

the following form

$$\begin{aligned}
 y(x, \lambda) &= \int_a^b K(x, t, \lambda) f(t) w(t) dt \\
 &\quad + \frac{1}{\Delta(\lambda)} (\Gamma_1(\lambda) \phi_1(x, \lambda) + \Gamma_2(\lambda) \phi_2(x, \lambda) + \Gamma_3(\lambda) \phi_3(x, \lambda)) \\
 &\quad + \frac{1}{\Delta(\lambda)} (\Theta_1(\lambda) \phi_1(x, \lambda) + \Theta_2(\lambda) \phi_2(x, \lambda) + \Theta_3(\lambda) \phi_3(x, \lambda)) \\
 &= \int_a^b G(x, t, \lambda) f(t) w(t) dt + \frac{1}{\Delta(\lambda)} \Theta(x, t, \lambda),
 \end{aligned} \tag{4.16}$$

where

$$\begin{aligned}
 G(x, t, \lambda) &= K(x, t, \lambda) - \frac{1}{\Delta(\lambda)} \tilde{K}(x, t, \lambda), \\
 \tilde{K}(x, t, \lambda) &= \begin{vmatrix} \mathcal{L}_1(\phi_1(x, \lambda)) & \mathcal{L}_1(\phi_2(x, \lambda)) & \mathcal{L}_1(\phi_3(x, \lambda)) & \mathcal{L}_1(K(x, t, \lambda)) \\ \mathcal{L}_2(\phi_1(x, \lambda)) & \mathcal{L}_2(\phi_2(x, \lambda)) & \mathcal{L}_2(\phi_3(x, \lambda)) & \mathcal{L}_2(K(x, t, \lambda)) \\ \mathcal{L}_3(\phi_1(x, \lambda)) & \mathcal{L}_3(\phi_2(x, \lambda)) & \mathcal{L}_3(\phi_3(x, \lambda)) & \mathcal{L}_3(K(x, t, \lambda)) \\ \phi_1(x, \lambda) & \phi_2(x, \lambda) & \phi_3(x, \lambda) & 0 \end{vmatrix}, \\
 \Theta(x, t, \lambda) &= \begin{vmatrix} \mathcal{L}_1(\phi_1(x, \lambda)) & \mathcal{L}_1(\phi_2(x, \lambda)) & \mathcal{L}_1(\phi_3(x, \lambda)) & -f_1 \\ \mathcal{L}_2(\phi_1(x, \lambda)) & \mathcal{L}_2(\phi_2(x, \lambda)) & \mathcal{L}_2(\phi_3(x, \lambda)) & -f_2 \\ \mathcal{L}_3(\phi_1(x, \lambda)) & \mathcal{L}_3(\phi_2(x, \lambda)) & \mathcal{L}_3(\phi_3(x, \lambda)) & 0 \\ \phi_1(x, \lambda) & \phi_2(x, \lambda) & \phi_3(x, \lambda) & 0 \end{vmatrix}.
 \end{aligned} \tag{4.17}$$

In conclusion, for any $F = (f(x), f_1, f_2) \in \mathcal{H}$, there exists a unique $Y \in D(T)$, $Y = (y(x), M_1(y), M_2(y))$ satisfying $(T - \lambda I)Y = F$.

By the definition of \mathcal{H} , the components of Y are determined by the first one, i.e., in order to find Y , we only need to find its first component $y(x)$, and $y(x)$ is determined by (4.16).

Definition 4.1. The integral kernel $G(x, t, \lambda)$ in (4.17) is called Green’s function of the operator T .

Remark 4.1. It is different from the usual boundary value problems, when the eigenparameter appears in the boundary conditions, the solution $y(x)$ is not only determined by $\int_a^b G(x, t, \lambda) f(t) w(t) dt$ but also $\frac{1}{\Delta(\lambda)} \Theta(x, t, \lambda)$.

Theorem 4.1. *If λ is not an eigenvalue of the operator T , then for any $F = (f(x), f_1, f_2) \in \mathcal{H}$, there exists a unique solution $Y = (y(x), M_1(y), M_2(y))$ of equation $(T - \lambda I)Y = F$ satisfying*

$$y(x, \lambda) = \int_a^b G(x, t, \lambda) f(t) w(t) dt + \frac{1}{\Delta(\lambda)} \Theta(x, t, \lambda).$$

The operator $(T - \lambda I)^{-1}$ is defined in the whole space by Theorem 4.1. It follows from the facts T is symmetric and Closed Graph Theorem that $(T - \lambda I)^{-1}$

is bounded. Therefore, λ is a regular point of T provided that it is not an eigenvalue of T .

Theorem 4.2. *The operator T has only point spectrum, that is to say, $\sigma(T) = \sigma_p(T)$.*

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References

- [1] B. P. Allahverdiev, *A nonself-adjoint 1D singular Hamiltonian system with an eigenparameter in the boundary condition*, Potential Anal., 2013, 38, 1031–1045.
- [2] E. Bairamov and E. Uğurlu, *Spectral analysis of eigenparameter dependent boundary value transmission problems*, J. Math. Anal. Appl. 2014, 413, 482–494.
- [3] Q. Bao, J. Sun, X. Hao and A. Zettl, *New canonical forms of self-adjoint boundary conditions for regular differential operators of order four*, Journal of Applied Analysis and Computation, 2019, 9(6), 2190–2211.
- [4] P. A. Binding, P. J. Browne, and B. A. Watson, *Equivalence of inverse Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter*, J. Math. Anal. Appl., 2004, 291, 246–261.
- [5] P. Binding and B. Çurgus, *Riesz bases of root vectors of indefinite Sturm-Liouville problems with eigenparameter dependent boundary conditions II*, Integr. Equ. Oper. Theory, 2008, 99, 1–27.
- [6] O. Boyko, O. Martynyuk and V. Pivovarchik, *Ambarzumian theorem for non-selfadjoint boundary value problems*, Journal of Operator Theory, 2018, 79, 1213–1223.
- [7] Z. Cao, *Ordinary Differential Operator*, Science Press, Beijing, (in Chinese), 2016 .
- [8] L. Collatz, *Eigenwertaufgaben Mit Technischen Anwendungen*. Akad. Verlagsgesellschaft Geest and Portig, Leipzig, 1963.
- [9] B. Friedman, *Principles and Techniques of Applied Mathematics*, New York, 1956.
- [10] W. Feller, *The parabolic differential equations and the associated semigroups of transforms*, Ann. of Math., 1952, 55, 468–519.
- [11] C. T. Fulton, *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1977, 77, 293–308.

- [12] C. Gao, X. Li and R. Ma, *Eigenvalues of a linear fourth-order differential operator with squared spectral parameter in a boundary condition*, Mediterranean Journal of Mathematics, 2018,15(3), 107.
- [13] M. Gregus, *Third Order Linear Differential Equations (Mathematics and its Applications)*, Reidel, Dordrecht, 1987.
- [14] D. B. Hinton, *Deficiency indices of odd-order differential operators*, Rocky Moun. J. Math. 1978, 8, 627–640.
- [15] X. Hao, M. Zhang, J. Sun and A. Zettl, *Characterization of domains of self-adjoint ordinary differential operators of any order, even or odd*, Electronic Journal of Qualitative Theory of Differential Equations, 2017, 2017, 1–19.
- [16] J. Kim, *Oscillatory Properties of Linear Third-Order Differential Equations*, Proceedings of the American Mathematical Society, 1970, 26(2), 286–293.
- [17] R. E. Kraft and W. R. Wells, *Adjointness properties for differential systems with eigenvaluedependent boundary conditions, with application to flow-duct acoustics*, J. Acoust. Soc. Am., 1977, 61, 913–22.
- [18] K. Li, J. Sun, X. Hao and Q. Bao, *Spectral analysis for discontinuous non-self-adjoint singular Dirac operators with eigenparameter dependent boundary condition*, J. Math. Anal. Appl., 2017, 453, 304–316.
- [19] K. Li, M. Zhang, J. Cai and Z. Zheng, *Completeness theorem for eigenparameter dependent dissipative Dirac operator with general transfer conditions*, Journal of Function Spaces, 2020, 2020, 1–8.
- [20] O. S. Mukhtarov and K. Aydemir, *Minimization principle and generalized Fourier series for discontinuous Sturm-Liouville systems in direct sum spaces*, Journal of Applied Analysis and Computation, 2018, 8(5), 1511–1523.
- [21] O. S. Mukhtarov and K. Aydemir, *Eigenfunction expansion for Sturm-Liouville problems with transmission conditions at one interior point*, Acta Mathematica Scientia, 2015, 35(3), 639–649.
- [22] R. Ma and Y. Lu, *Disconjugacy and extremal solutions of nonlinear third-order equations*, Communications on Pure and Applied Analysis, 2014,13, 1223–1236.
- [23] T. Niu, X. L. Hao, J. Sun and K. Li, *Canonical forms of self-adjoint boundary conditions for regular differential operators of order three*, Operator and Matrices, 2020,14, 207–220.
- [24] M. A. Naimark, *Linear Differential Operators*, Ungar, New York, 1968.
- [25] H. Olğar and O. Sh. Mukhtarov, *Weak eigenfunctions of two-interval Sturm-Liouville problems together with interaction conditions*, Journal of Mathematical Physics, 2017, 58(4), 042201–1–042201–13.
- [26] W. C. Troy, *Solution of third order differential equations relevant to draining and coating ows*, SIAM J. Math. Anal., 1993, 24, 155–171.
- [27] C. Tretter, *Boundary eigenvalue problems for differential equations $N\eta = \lambda P\eta$ with λ -polynomial boundary conditions*, J. Differential Equations, 2001, 170, 408–471.
- [28] E. Uğurlu, *Regular third-order boundary value problems*, Applied Mathematics and Computation, 2019, 343, 247–257.

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- [29] E. Uğurlu, *Third-order boundary value transmission problems*, Turkish Journal of Mathematics, 2019, 43, 1518–1532.
- [30] A. P. Wang, J. Sun and A. Zettl, *The classification of self-adjoint boundary conditions: separated, coupled, and mixed*, Journal of Functional Analysis, 2008, 255, 1554–1573.
- [31] A. P. Wang, J. Sun and A. Zettl, *Characterization of domains of self-adjoint ordinary differential operators*, Journal of Differential Equations, 2009, 246, 1600–1622.
- [32] J. Weidmann, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics, Berlin: Springer, 1987.
- [33] J. Walter, *Regular eigenvalue problems with eigenvalue parameter in the boundary condition*, Math. Z., 1973, 133, 301–312.
- [34] Q. Yao and Y. Feng, *The existence of solutions for a third order two-point boundary value problem*, Appl. Math. Lett., 2002, 15, 227–232.
- [35] M. Zhang, J. Sun and A. Zettl, *The spectrum of singular Sturm-Liouville problems with eigenparameter dependent boundary conditions and its approximation*, Results in Mathematics, 2013, 63, 1311–1330.
- [36] X. Zhang and J. Sun, *Green function of fourth-order differential operator with eigenparameter-dependent boundary and transmission conditions*, Acta Mathematicae Applicatae Sinica, English Series, 2017, 33(2), 311–326.