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ON AN EXTENDED HARDY-HILBERT'S INEQUALITY IN THE WHOLE PLANE*

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Abstract By introducing independent parameters and applying the weight coefficients, we give an extended Hardy-Hilbert's inequality in the whole plane with a best possible constant factor. Furthermore, the equivalent forms, a few particular cases and the operator expressions are considered.

Keywords Hardy-Hilbert's inequality, parameter, weight coefficient, equivalent form, operator expression.

MSC(2010) 26D15.

1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have the well-known Hardy-Hilbert's inequality as follows (cf. [6]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \tag{1.1}$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. In 1934, Hardy proved the following more accurate inequality of (1.1) (cf. [7]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n-1} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \tag{1.2}$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is still the best possible.

If $f(x), g(x) \geq 0, 0 < \int_0^{\infty} f^p(x)dx < \infty$ and $0 < \int_0^{\infty} g^q(x)dx < \infty$, then we have the integral analogue of (1.1) as follows (cf. [7]):

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty} f^p(x)dx\right)^{1/p}\left(\int_0^{\infty} g^q(x)dx\right)^{1/q}. \tag{1.3}$$

Inequalities (1.1)-(1.3) are important in analysis and its applications (cf. [7], [20]). In 2007, Yang [21] first gave a Hilbert-type integral inequality in the whole plane

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as follows:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{(1+e^{x+y})^{\lambda}}dxdy$$
$$< B(\frac{\lambda}{2},\frac{\lambda}{2})(\int_{-\infty}^{\infty}e^{-\lambda x}f^{2}(x)dx\int_{-\infty}^{\infty}e^{-\lambda y}f^{2}(y)dy)^{\frac{1}{2}}, \tag{1.4}$$

where the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ $(\lambda > 0)$ is the best possible. A lot of generalizations and improvements of inequalities (1.1)-(1.4) were provided by [1,2,4,5,8–10, 14–17,19,22,26,28] and [3].

In 2016, Yang and Chen [27] gave an extension of (1.2) in the whole plane as follows:

$$\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{a_m b_n}{(|m-\xi|+|n-\eta|)^{\lambda}}$$
$$< 2B(\lambda_1,\lambda_2)\left[\sum_{|m|=1}^{\infty}|m-\xi|^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}$$
$$\times\left[\sum_{|n|=1}^{\infty}|n-\eta|^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{1.5}$$

where, the constant factor $2B(\lambda_1,\lambda_2)$ $(0<\lambda_1,\lambda_2\le 1,\lambda_1+\lambda_2=\lambda,\xi,\eta\in[0,\frac{1}{2}])$ is the best possible.

In this article, by introducing independent parameters and applying the weight coefficients, we give a new extension of (1.1) in the whole plane with a best possible constant factor. Furthermore, the equivalent forms, a few particular cases and the operator expressions are considered.

2. A few definitions and lemmas

In what follows, we suppose that $p>1,\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda_1,\ \lambda_2\le 1,\lambda_1+\lambda_2=\lambda$ (≤ 2), $\alpha,\beta\in(0,\pi)$ and

$$k_{\gamma}(\lambda_1)=\frac{2\pi\csc^2\gamma}{\lambda\sin\pi(\lambda_1/\lambda)}\ (\gamma=\alpha,\beta). \tag{2.1}$$

Definition 2.1. For $|x|,|y|>0$, we define

$$k(x,y):=\frac{1}{(|x|+x\cos\alpha)^{\lambda}+(|y|+y\cos\beta)^{\lambda}}. \tag{2.2}$$

Definition 2.2. Define the weight coefficients as follows:

$$\omega(\lambda_2,m):=\sum_{|n|=1}^{\infty}k(m,n)\frac{(|m|+m\cos\alpha)^{\lambda_1}}{(|n|+n\cos\beta)^{1-\lambda_2}},|m|\in\mathbf{N}, \tag{2.3}$$

$$\varpi(\lambda_1,n):=\sum_{|m|=1}^{\infty}k(m,n)\frac{(|n|+n\cos\beta)^{\lambda_2}}{(|m|+m\cos\alpha)^{1-\lambda_1}},|n|\in\mathbf{N}, \tag{2.4}$$

where, $\sum_{|j|=1}^{\infty}\cdots=\sum_{j=-1}^{-\infty}\cdots+\sum_{j=1}^{\infty}\cdots$ $(j=m,n)$.

Lemma 2.1. *We have the following inequalities:*

$$k_\beta(\lambda_1)(1-\theta(\lambda_2,m)) < \omega(\lambda_2,m) < k_\beta(\lambda_1), |m| \in \mathbf{N}, \tag{2.5}$$

where,

$$\begin{aligned}\theta(\lambda_2,m) &:= \frac{\lambda}{\pi}\sin(\frac{\pi\lambda_1}{\lambda})\int_0^{\frac{1+|\cos\beta|}{|m|+m\cos\alpha}} \frac{u^{\lambda_2-1}}{1+u^\lambda}du \\ &= O\left(\frac{1}{(|m|+m\cos\alpha)^{\lambda_2}}\right) \in (0,1). \end{aligned}\tag{2.6}$$

Proof. For $|x|>0$, we set

$$k(x,y) = k^{(1)}(x,y) := \frac{1}{(|x|+x\cos\alpha)^\lambda + [y(\cos\beta-1)]^\lambda}, y<0;$$
$$k(x,y) = k^{(2)}(x,y) := \frac{1}{(|x|+x\cos\alpha)^\lambda + [y(\cos\beta+1)]^\lambda}, y>0,$$

wherefrom,

$$k^{(1)}(x,-y) = \frac{1}{(|x|+x\cos\alpha)^\lambda + [y(1-\cos\beta)]^\lambda}, y>0. \tag{2.7}$$

We find

$$\begin{aligned}\omega(\lambda_2,m) &= \sum_{n=-1}^{-\infty} k^{(1)}(m,n)\frac{(|m|+m\cos\alpha)^{\lambda_1}}{[n(\cos\beta-1)]^{1-\lambda_2}} \\ &\quad+\sum_{n=1}^{\infty} k^{(2)}(m,n)\frac{(|m|+m\cos\alpha)^{\lambda_1}}{[n(1+\cos\beta)]^{1-\lambda_2}} \\ &= \frac{(|m|+m\cos\alpha)^{\lambda_1}}{(1-\cos\beta)^{1-\lambda_2}}\sum_{n=1}^{\infty}\frac{k^{(1)}(m,-n)}{n^{1-\lambda_2}} \\ &\quad+\frac{(|m|+m\cos\alpha)^{\lambda_1}}{(1+\cos\beta)^{1-\lambda_2}}\sum_{n=1}^{\infty}\frac{k^{(2)}(m,n)}{n^{1-\lambda_2}}. \end{aligned}\tag{2.8}$$

For fixed $|m|\in\mathbf{N}, 0<\lambda_2\le 1$, in virtue of $\lambda>0$ and

$$\frac{d}{dy}\frac{k^{(i)}(m,(-1)^i y)}{y^{1-\lambda_2}} < 0 \ (y>0, i=1.2), \tag{2.9}$$

it follows that both $\frac{k^{(1)}(m,-y)}{y^{1-\lambda_2}}$ and $\frac{k^{(2)}(m,y)}{y^{1-\lambda_2}}$ are strict decreasing with respect to $y\in(0,\infty)$. By the decreasingness property, we find

$$\begin{aligned}\omega(\lambda_2,m) &< \frac{(|m|+m\cos\alpha)^{\lambda_1}}{(1-\cos\beta)^{1-\lambda_2}}\int_0^\infty \frac{k^{(1)}(m,-y)}{y^{1-\lambda_2}}dy \\ &\quad+\frac{(m+m\cos\alpha)^{\lambda_1}}{(1+\cos\beta)^{1-\lambda_2}}\int_0^\infty \frac{k^{(2)}(m,y)}{y^{1-\lambda_2}}dy. \end{aligned}\tag{2.10}$$

Setting $u = \frac{y(1-\cos\beta)}{|m|+m\cos\alpha}$(resp. $u = \frac{y(1+\cos\beta)}{|m|+m\cos\alpha}$) in the above first (resp. second) integral, by simplifications, we have

$$\begin{aligned}
\omega(\lambda_2, m) &< \left(\frac{1}{1-\cos\beta} + \frac{1}{1+\cos\beta}\right)\int_0^\infty \frac{u^{\lambda_2-1}}{1+u^\lambda}du \\
&= \frac{2}{\lambda\sin^2\beta}\int_0^\infty \frac{v^{(\lambda_2/\lambda)-1}}{1+v}dv \\
&= \frac{2\pi\csc^2\beta}{\lambda\sin\pi(\lambda_2/\lambda)} = \frac{2\pi\csc^2\beta}{\lambda\sin\pi(\lambda_1/\lambda)} = k_\beta(\lambda_1). \qquad (2.11)
\end{aligned}$$

In view of the decreasingness property, we still have

$$\begin{aligned}
\omega(\lambda_2, m) &> \frac{(|m|+m\cos\alpha)^{\lambda_1}}{(1-\cos\beta)^{1-\lambda_2}}\int_1^\infty \frac{k^{(1)}(m,-y)}{y^{1-\lambda_2}}dy \\
&\quad + \frac{(|m|+m\cos\alpha)^{\lambda_1}}{(1+\cos\beta)^{1-\lambda_2}}\int_1^\infty \frac{k^{(2)}(m,y)}{y^{1-\lambda_2}}dy \\
&\geq \left(\frac{1}{1-\cos\beta} + \frac{1}{1+\cos\beta}\right)\int_{\frac{1+|\cos\beta|}{|m|+m\cos\alpha}}^\infty \frac{u^{\lambda_2-1}}{1+u^\lambda}du \\
&= k_\beta(\lambda_1) - 2\csc^2\beta\int_0^{\frac{1+|\cos\beta|}{|m|+m\cos\alpha}} \frac{u^{\lambda_2-1}}{1+u^\lambda}du \\
&= k_\beta(\lambda_1)(1-\theta(\lambda_2, m)) > 0, \qquad (2.12)
\end{aligned}$$

where, $\theta(\lambda_2, m)$ is indicated by (2.6) and $\theta(\lambda_2, m) < 1$. It follows that

$$\begin{aligned}
0 < \theta(\lambda_2, m) &< \frac{\lambda}{\pi}\sin(\frac{\pi\lambda_1}{\lambda})\int_0^{\frac{1+|\cos\beta|}{|m|+m\cos\alpha}} u^{\lambda_2-1}du \\
&= \frac{\lambda}{\pi\lambda_2}\sin(\frac{\pi\lambda_1}{\lambda})\left(\frac{1+|\cos\beta|}{|m|+m\cos\alpha}\right)^{\lambda_2}. \qquad (2.13)
\end{aligned}$$

Hence, both (2.5) and (2.6) are valid. □

In the same way, we still have

Lemma 2.2. *We have the following inequalities:*

$$k_\alpha(\lambda_1)(1-\widetilde{\theta}(\lambda_1, n)) < \varpi(\lambda_1, n) < k_\alpha(\lambda_1), |n| \in \mathbf{N}, \qquad (2.14)$$

where,

$$\begin{aligned}
\widetilde{\theta}(\lambda_1, n) &:= \frac{\lambda}{\pi}\sin(\frac{\pi\lambda_1}{\lambda})\int_0^{\frac{1+|\cos\alpha|}{|n|+n\cos\beta}} \frac{u^{\lambda_1-1}}{1+u^\lambda}du \\
&= O\left(\frac{1}{(|n|+n\cos\beta)^{\lambda_1}}\right) \in (0,1). \qquad (2.15)
\end{aligned}$$

Lemma 2.3. *If $\theta \in (0, \pi), \rho > 0$, then we have*

$$\begin{aligned}
H_\rho(\theta) &:= \sum_{|n|=1}^\infty \frac{1}{(|n|+n\cos\theta)^{1+\rho}} \\
&= \frac{1+o(1)}{\rho}\left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right]. \qquad (\rho \to 0^+) \qquad (2.16)
\end{aligned}$$

Proof. We find

$$H_\rho(\theta) = \sum_{n=-1}^{-\infty} \frac{1}{[n(\cos\theta - 1)]^{1+\rho}} + \sum_{n=1}^{\infty} \frac{1}{[n(\cos\theta + 1)]^{1+\rho}}$$
$$= \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right] \sum_{n=1}^{\infty} \frac{1}{n^{1+\rho}}. \tag{2.17}$$

Then we obtain

$$H_\rho(\theta) = \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right] \left(1 + \sum_{n=2}^{\infty} \frac{1}{n^{1+\rho}}\right)$$
$$< \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right] \left(1 + \int_1^{\infty} \frac{dy}{y^{1+\rho}}\right)$$
$$= \frac{1+\rho}{\rho} \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right],$$
$$H_\rho(\theta) = \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right] \sum_{n=1}^{\infty} \frac{1}{n^{1+\rho}}$$
$$> \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right] \int_1^{\infty} \frac{dy}{y^{1+\rho}}$$
$$= \frac{1}{\rho} \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right].$$

Hence, for $\rho \to 0^+$, we prove that (2.16) is valid. □

3. Main results and operator expressions

Theorem 3.1. *Suppose that* $a_m, b_n \geq 0$ $(|m|, |n| \in \mathbf{N})$,

$$0 < \sum_{|m|=1}^{\infty} (|m| + m\cos\alpha)^{p(1-\lambda_1)-1} a_m^p < \infty,$$
$$0 < \sum_{|n|=1}^{\infty} (|n| + n\cos\beta)^{q(1-\lambda_2)-1} b_n^q < \infty,$$
$$k(\lambda_1) := k_\beta^{1/p}(\lambda_1) k_\alpha^{1/q}(\lambda_1) = \frac{2\pi}{\lambda \sin\pi(\lambda_1/\lambda)} \csc^{2/p}\beta \csc^{2/q}\alpha. \tag{3.1}$$

We have the following equivalent inequalities:

$$I := \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} k(m,n) a_m b_n$$
$$< k(\lambda_1) \left[\sum_{|m|=1}^{\infty} (|m| + m\cos\alpha)^{p(1-\lambda_1)-1} a_m^p\right]^{1/p}$$
$$\times \left[\sum_{|n|=1}^{\infty} (|n| + n\cos\beta)^{q(1-\lambda_2)-1} b_n^q\right]^{1/q}, \tag{3.2}$$

$$J := \left[\sum_{|n|=1}^{\infty} (|n| + n\cos\beta)^{p\lambda_2 - 1} \left(\sum_{|m|=1}^{\infty} k(m,n) a_m \right)^p \right]^{\frac{1}{p}}$$
$$< k(\lambda_1) \left[\sum_{|m|=1}^{\infty} (|m| + m\cos\alpha)^{p(1-\lambda_1)-1} a_m^p \right]^{1/p}. \tag{3.3}$$

Proof. By Hölder's inequality with weight (cf. [18]) and (2.4), we find

$$\left(\sum_{|m|=1}^{\infty} k(m,n) a_m \right)^p$$
$$= \left\{ \sum_{|m|=1}^{\infty} k(m,n) \left[\frac{(|m| + m\cos\alpha)^{(1-\lambda_1)/q}}{(|n| + n\cos\beta)^{(1-\lambda_2)/p}} a_m \right] \right.$$
$$\left. \times \left[\frac{(|n| + n\cos\beta)^{(1-\lambda_2)/p}}{(|m| + m\cos\alpha)^{(1-\lambda_1)/q}} \right] \right\}^p$$
$$\leq \sum_{|m|=1}^{\infty} k(m,n) \frac{(|m| + m\cos\alpha)^{(1-\lambda_1)p/q}}{(|n| + n\cos\beta)^{1-\lambda_2}} a_m^p$$
$$\times \left[\sum_{|m|=1}^{\infty} k(m,n) \frac{(|n| + n\cos\beta)^{(1-\lambda_2)q/p}}{(|m| + m\cos\alpha)^{1-\lambda_1}} \right]^{p-1}$$
$$= \frac{(\varpi(\lambda_1, n))^{p-1}}{(|n| + n\cos\beta)^{p\lambda_2 - 1}} \sum_{|m|=1}^{\infty} k(m,n) \frac{(|m| + m\cos\alpha)^{(1-\lambda_1)p/q}}{(|n| + n\cos\beta)^{1-\lambda_2}} a_m^p. \tag{3.4}$$

By (2.14), it follows that

$$J < k_\alpha^{1/q}(\lambda_1) \left[\sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} k(m,n) \frac{(|n| + n\cos\beta)^{(1-\lambda_2)q/p}}{(|m| + m\cos\alpha)^{1-\lambda_1}} a_m^p \right]^{\frac{1}{p}}$$
$$= k_\alpha^{1/q}(\lambda_1) \left[\sum_{|m|=1}^{\infty} \sum_{|n|=1}^{\infty} k(m,n) \frac{(|n| + n\cos\beta)^{(1-\lambda_2)q/p}}{(|m| + m\cos\alpha)^{1-\lambda_1}} a_m^p \right]^{\frac{1}{p}}$$
$$= k_\alpha^{1/q}(\lambda_1) \left[\sum_{|m|=1}^{\infty} \omega(\lambda_2, m) (|m| + m\cos\alpha)^{p(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}}. \tag{3.5}$$

By (2.5) and (3.5), we have (3.3).

Using Hölder's inequality again, we have

$$I = \sum_{|n|=1}^{\infty} \left[(|n| + n\cos\beta)^{\lambda_2 - 1/p} \sum_{|m|=1}^{\infty} k(m,n) a_m \right] (|n| + n\cos\beta)^{(1/p)-\lambda_2} b_n$$
$$\leq J \left[\sum_{|n|=1}^{\infty} (|n| + n\cos\beta)^{q(1-\lambda_2)-1} b_n^q \right]^{\frac{1}{q}}, \tag{3.6}$$

and then we have (3.2) by using (3.3).

On the other hand, assuming that (3.2) is valid, we set

$$b_n := (|n| + n\cos\beta)^{p\lambda_2 - 1}\left(\sum_{|m|=1}^{\infty} k(m,n)a_m\right)^{p-1}, \quad |n| \in \mathbf{N}. \tag{3.7}$$

and find

$$J = \left[\sum_{|n|=1}^{\infty} (|n| + n\cos\beta)^{q(1-\lambda_2)-1} b_n^q\right]^{1/p}. \tag{3.8}$$

By (3.5), it follows that $J < \infty$. If $J = 0$, then (3.5) is trivially valid. If $0 < J < \infty$, then we have

$$\begin{aligned}
0 &< \sum_{|n|=1}^{\infty} (|n| + n\cos\beta)^{q(1-\lambda_2)-1} b_n^q = J^p = I \\
&< k(\lambda_1)\left[\sum_{|m|=1}^{\infty} (|m| + m\cos\alpha)^{p(1-\lambda_1)-1} a_m^p\right]^{1/p} \\
&\quad \times \left[\sum_{|n|=1}^{\infty} (|n| + n\cos\beta)^{q(1-\lambda_2)-1} b_n^q\right]^{1/q}, \\
J &= \left[\sum_{|n|=1}^{\infty} (|n| + n\cos\beta)^{q(1-\lambda_2)-1} b_n^q\right]^{1/p} \\
&< k(\lambda_1)\left[\sum_{|m|=1}^{\infty} (|m| + m\cos\alpha)^{p(1-\lambda_1)-1} a_m^p\right]^{1/p},
\end{aligned} \tag{3.9}$$

Hence (3.3) is valid, which is equivalent to (3.2). □

Theorem 3.2. *With regards to the assumptions of Theorem 3.1, the constant factor $k(\lambda_1)$ is the best possible in (3.2) and (3.3).*

Proof. For $0 < \varepsilon < q\lambda_2$, we set $\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}, \widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}$ $(\in (0,1))$, and

$$\begin{aligned}
\widetilde{a}_m &:= (|m| + m\cos\alpha)^{\lambda_1 - \varepsilon/p - 1} = (|m| + m\cos\alpha)^{\widetilde{\lambda}_1 - \varepsilon - 1} \quad (|m| \in \mathbf{N}), \\
\widetilde{b}_n &:= (|n| + n\cos\beta)^{\lambda_2 - \varepsilon/q - 1} = (|n| + n\cos\beta)^{\widetilde{\lambda}_2 - 1} \quad (|n| \in \mathbf{N}).
\end{aligned}$$

By (2.5) and (2.16), we find

$$\begin{aligned}
\widetilde{I}_1 := &\left[\sum_{|m|=1}^{\infty} (|m| + m\cos\alpha)^{p(1-\lambda_1)-1} \widetilde{a}_m^p\right]^{1/p} \\
&\times \left[\sum_{|n|=1}^{\infty} (|n| + n\cos\beta)^{q(1-\lambda_2)-1} \widetilde{b}_n^q\right]^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&= \left[\sum_{|m|=1}^{\infty} \frac{1}{(|m| + m\cos\alpha)^{1+\varepsilon}}\right]^{\frac{1}{p}} \left[\sum_{|n|=1}^{\infty} \frac{1}{(|n| + n\cos\beta)^{1+\varepsilon}}\right]^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}}\right]^{1/p} (1+o_1(1))^{\frac{1}{p}} \\
&\quad \times \left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}} + \frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right]^{1/q} (1+o_2(1))^{\frac{1}{q}}, \\
\widetilde{I} &:= \sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} k(m,n)\widetilde{a}_m\widetilde{b}_n \\
&= \sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} k(m,n)\frac{(|m| + m\cos\alpha)^{\widetilde{\lambda}_1-\varepsilon-1}}{(|n| + n\cos\beta)^{1-\widetilde{\lambda}_2}} \\
&= \sum_{|m|=1}^{\infty} \frac{\varpi(\widetilde{\lambda}_2, m)}{(|m| + m\cos\alpha)^{1+\varepsilon}} \geq k_\beta(\widetilde{\lambda}_1)\sum_{|m|=1}^{\infty} \frac{1-\theta(\widetilde{\lambda}_2, m)}{(|m| + m\cos\alpha)^{1+\varepsilon}} \\
&= k_\beta(\widetilde{\lambda}_1)\left\{\sum_{|m|=1}^{\infty} \frac{1}{(|m| + m\cos\alpha)^{1+\varepsilon}}\right. \\
&\quad \left. - \sum_{|m|=1}^{\infty} \frac{1}{O\left((|m| + m\cos\alpha)^{\frac{\varepsilon}{p}+\lambda_2+1}\right)}\right\} \\
&= \frac{k_\beta(\widetilde{\lambda}_1)}{\varepsilon}\left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}}\right] \\
&\quad \times [(1+o_1(1)) - \varepsilon O(1)]. \qquad (3.10)
\end{aligned}
$$

If there exists a positive number $k \leq k(\lambda_1)$, such that (3.2) is still valid when replacing $k(\lambda_1)$ by k, then in particular, we have

$$
\varepsilon\widetilde{I} = \varepsilon\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} k(m,n)\widetilde{a}_m\widetilde{b}_n < \varepsilon k\widetilde{I}_1.
$$

We obtain from the above results that

$$
\begin{aligned}
&k_\beta(\lambda_1 + \frac{\varepsilon}{q})\left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}}\right] \\
&\times[(1+o_1(1)) - \varepsilon O(1)] \\
< {}& k\left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}}\right]^{1/p} \\
&\times\left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}} + \frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right]^{1/q} \\
&\times(1+o_1(1))^{\frac{1}{p}}(1+o_2(1))^{\frac{1}{q}}, \qquad (3.11)
\end{aligned}
$$

and then it follows that

$$
\frac{4\pi}{\lambda\sin\pi(\lambda_1/\lambda)}\csc^2\beta\csc^2\alpha \leq 2k\csc^{2/p}\alpha\csc^{2/q}\beta \ (\varepsilon \to 0^+), \qquad (3.12)
$$

namely, $k(\lambda_1) = \frac{2\pi}{\lambda \sin \pi(\lambda_1/\lambda)} \csc^{2/p} \beta \csc^{2/q} \alpha \leq k$. Hence $k = k(\lambda_1)$ is the best value of (3.2).

The constant factor $k(\lambda_1)$ in (3.3) is still the best possible. Otherwise, we would reach a contradiction by (3.6) that the constant factor in (3.2) is not the best value. □

Setting $\varphi(m) := (|m| + m \cos \alpha)^{p(1-\lambda_1)-1} \quad (|m| \in \mathbf{N})$, and

$$\psi(n) := (|n| + n \cos \beta)^{q(1-\lambda_2)-1} \quad (|n| \in \mathbf{N}),$$

wherefrom, $\psi^{1-p}(n) = (|n| + n \cos \beta)^{p\lambda_2 - 1}$, we define the real weighted normed function spaces as follows:

$$l_{p,\varphi} := \left\{ a = \{a_m\}_{|m|=1}^{\infty}; \|a\|_{p,\varphi} = \left(\sum_{|m|=1}^{\infty} \varphi(m) \, |a_m|^p \right)^{1/p} < \infty \right\},$$

$$l_{q,\psi} := \left\{ b = \{b_n\}_{|n|=1}^{\infty}; \|b\|_{q,\psi} = \left(\sum_{|n|=1}^{\infty} \psi(n) \, |b_n|^q \right)^{1/q} < \infty \right\},$$

$$l_{p,\psi^{1-p}} := \left\{ c = \{c_n\}_{|n|=1}^{\infty}; \|c\|_{p,\psi^{1-p}} = \left(\sum_{|n|=1}^{\infty} \psi^{1-p}(n) \, |c_n|^p \right)^{1/p} < \infty \right\}.$$

For $a = \{a_m\}_{|m|=1}^{\infty} \in l_{p,\varphi}$, putting $c_n = \sum_{|m|=1}^{\infty} k(m,n) a_m$ and $c = \{c_n\}_{|n|=1}^{\infty}$, it follows by (3.3) that $\|c\|_{p,\psi^{1-p}} < k(\lambda_1) \|a\|_{p,\varphi}$, namely $c \in l_{p,\psi^{1-p}}$.

Definition 3.1. Define an extended Hardy-Hilbert's operator $T : l_{p,\varphi} \to l_{p,\psi^{1-p}}$ as follows: For $a_m \geq 0, a = \{a_m\}_{|m|=1}^{\infty} \in l_{p,\varphi}$, there exists a unique representation $Ta = c \in l_{p,\psi^{1-p}}$. We also define the following formal inner product of Ta and $b = \{b_n\}_{|n|=1}^{\infty} \in l_{q,\psi}$ $(b_n \geq 0)$ as follows:

$$(Ta, b) := \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} k(m,n) a_m b_n. \tag{3.13}$$

Hence, we may rewrite (3.2) and (3.3) in the following operator expressions:

$$(Ta, b) < k(\lambda_1) \|a\|_{p,\varphi} \|b\|_{q,\psi}, \tag{3.14}$$

$$\|Ta\|_{p,\psi^{1-p}} < k(\lambda_1) \|a\|_{p,\varphi}. \tag{3.15}$$

It follows that the operator T is bounded with

$$\|T\| := \sup_{a(\neq \theta) \in l_{p,\varphi}} \frac{\|Ta\|_{p,\psi^{1-p}}}{\|a\|_{p,\varphi}} \leq k(\lambda_1). \tag{3.16}$$

Since the constant factor $k(\lambda_1)$ in (3.3) is the best possible, we have

$$\|T\| = k(\lambda_1) = \frac{2\pi}{\lambda \sin \pi(\lambda_1/\lambda)} \csc^{2/p} \beta \csc^{2/q} \alpha. \tag{3.17}$$

By the above result, we may get the following corollary, which contains some known results.

Remark 3.1. (i) If $\alpha = \beta = \frac{\pi}{2}$ in (3.2), then we have the following inequality:

$$\sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{a_m b_n}{|m|^{\lambda} + |n|^{\lambda}} < \frac{2\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})} \left[\sum_{|m|=1}^{\infty} |m|^{p(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}} \left[\sum_{|n|=1}^{\infty} |n|^{q(1-\lambda_2)-1} b_n^q \right]^{\frac{1}{q}}. \quad (3.18)$$

It is obvious that (3.2) is an extension of (3.18).

(ii) If $a_{-m} = a_m$ and $b_{-n} = b_n$ $(m, n \in \mathbf{N})$, then (3.18) reduces to

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^{\lambda} + n^{\lambda}} < \frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})} \left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p \right]^{1/p} \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q \right]^{1/q}. \quad (3.19)$$

For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, (3.19) reduces to (1.1), and then inequality (3.2) is an extension of (1.1) with parameters.

4. Conclusions

In this paper, by introducing independent parameters and applying the weight coefficients, we give an extended Hardy-Hilbert's inequality in the whole plane with a best possible constant factor in Theorem 1-2. Furthermore, the equivalent forms, a few particular cases and the operator expressions are considered. The method of real analysis is very important, which is the key to prove the equivalent inequalities with the best possible constant factor. The lemmas and theorems provide an extensive account of this type of inequalities.

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