Journal of Applied Analysis and Computation
Volume 10, Number 6, December 2020, 2592–2618
Website:http://www.jaac-online.com
DOI:10.11948/20190430

RANDOM ATTRACTORS FOR NON-AUTONOMOUS FRACTIONAL STOCHASTIC GINZBURG-LANDAU EQUATIONS ON UNBOUNDED DOMAINS*

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Abstract This paper deals with the dynamical behavior of solutions for non-autonomous stochastic fractional Ginzburg-Landau equations driven by additive noise with $\alpha \in (0,1)$. We prove the existence and uniqueness of tempered pullback random attractors for the equations in $L^2(\mathbf{R}^3)$. In addition, we also obtain the upper semicontinuity of random attractors when the intensity of noise approaches zero. The main difficulty here is the noncompactness of Sobolev embeddings on unbounded domains. To solve this, we establish the pullback asymptotic compactness of solutions in $L^2(\mathbf{R}^3)$ by the tail-estimates of solutions.

Keywords Non-autonomous stochastic fractional Ginzburg-Landau equation, random dynamical system, random attractor, additive noise, upper semicontinuity.

MSC(2010) 37L55, 60H15, 35Q56.

1. Introduction

In this paper, we consider the following non-autonomous stochastic fractional Ginzburg-Landau equation with additive noise

$$\frac{\partial u}{\partial t} + (1+i\lambda)(-\Delta)^\alpha u + \gamma u = f(u) + g(t,x) + \delta h(x)\frac{dW}{dt}, \ \ x \in \mathbf{R}^3, t > \tau, \tag{1.1}$$

with initial condition

$$u(\tau, x) = u_\tau(x), \ \ x \in \mathbf{R}^3, \tag{1.2}$$

where $u(t,x)$ is a unknown complex-valued function, i is the imaginary unit, $\lambda \in \mathbf{R}$, $\alpha \in (0,1)$, $\gamma \in \mathbf{R}$, the nonlinear term $f(u)$ is a complex-valued function, $g(t,x) \in L^\infty_{loc}(\mathbf{R}, L^2(\mathbf{R}^3))$, $\delta > 0$, $h \in H^{2\alpha}(\mathbf{R}^3) \bigcap W^{2\alpha,4}(\mathbf{R}^3)$, W is a two-sided real-valued Wiener process on a probability space, $\tau \in \mathbf{R}$.

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*The authors were supported by National Natural Science Foundation of China (No.11871138), joint research project of Laurent Mathematics Center of Sichuan Normal University and National-Local Joint Engineering Laboratory of System Credibility Automatic Verification.

Recently, fractional partial differential equations arise in a wide range of fields within in physics, biology, chemistry, etc., and some classical equations of mathematical physics have been postulated with fractional derivative to better describe complex phenomena, including the fractional Schrödinger equation [10,16,17], fractional Landau-Lifshitz equation [20], fractional Landau-Lifshitz-Maxwell equation [35] and fractional Ginzburg-Landau equation [18,36,45].

When $\alpha \in (0,1)$, we call the operator $(-\Delta)^\alpha$ a fractional Laplacian. There are different definitions for the fractional Laplace operator on bounded domain U, including the integral fractional operator and the spectral fractional operator. These two kinds of fractional operators are distinct, and specially they have different eigenfunctions and eigenvalues as expressed in [40]. When $\alpha = 1$, it becomes the standard Laplace operator $(-\Delta)$. The concept of pullback random attractor, which is a generalization of global attractor in deterministic systems (see [21,37,39,46]), was introduced in [1,6–8,13,38], and characterizes the long-time behavior of random dynamical systems perfectly. The random attractor for stochastic equations have been widely discussed by many authors, see, e.g., [3–5, 14, 22, 31, 32, 42, 43, 49–51, 56, 57, 62] in the autonomous stochastic partial differential equations, and [12, 52–55, 58, 59, 63] in the non-autonomous case. In [4], Bates etc. discussed the random attractor of stochastic reaction-diffusion equation on unbounded domains. In recent years, there are some results on the random attractors for stochastic equations with the fractional Laplacian $(-\Delta)^\alpha$ with $\alpha \in (\frac{1}{2}, 1)$ in [28–30, 41, 44]. However, there are few results in the fractional case of $\alpha \in (0,1)$. In [55], Wang discussed the asymptotic behavior of fractional reaction-diffusion equation with $\alpha \in (0,1)$.

The generalized complex Ginzburg-Landau equation is one of the most important equations in mathematical physics, which can describe turbulent dynamics and has a long history in physics as a generic amplitude equation near the onset of instabilities in fluid mechanical systems, as well as in the theory of phase transitions and superconductivity [2, 9]. The fractional Ginzburg-Landau equation describes the dynamical processes in a medium with fractal dispersion and the fractional generalization of Ginzburg-Landau equation from variational Euler-Lagrange equation for the fractal media is derived in [45]. The global existence and long behavior of Ginzburg-Landau equation were studied in [9, 11, 19, 25, 26]. In fractional case, the well-posedness and dynamical behavior were proved in [27, 36]. For stochastic fractional Ginzburg-Landau equation with $\alpha \in (\frac{1}{2}, 1)$, the existence of the random attractor in L^2 and H^1 were respectively discussed in [28, 41, 44]. We note that in [28], the authors derived the estimates of solutions for $H^{1+\alpha}$ and the tail-estimates of solutions in H^1 instead of H^α. This may be caused by the definition of fractional Laplace operator. In this paper, we will furthermore consider uniform a priori estimates of solutions in H^α and the tail-estimates of solutions in L^2 by introducing another definition of fractional Laplace operator, which is different from [28].

As we know, there are some results on random attractors for stochastic fractional Ginzburg-Landau equation, but few results for the fractional case with $\alpha \in (0,1)$(see [23, 24, 60]). In this paper, motivated by [4, 55], we explore the random attractors for non-autonomous stochastic fractional Ginzburg-Landau equation with additive noise for $\alpha \in (0,1)$. However, there are several difficulties to overcome. Firstly, the fractional Laplace operator $(-\Delta)^\alpha$ is non-local and thus deriving uniform estimates on the solutions of (1.1) is much more involved than the standard Laplacian $-\Delta$. Secondly, the domain is unbounded, so the Sobolev embedding $H^\alpha(\mathbf{R}^3) \hookrightarrow L^2(\mathbf{R}^3)$

with $\alpha \in (0,1)$ is not compact. Thirdly, since the Ginzburg-Landau equation is a complex equation, the condition of nonlinearity and uniform estimates of solutions in L^2 are slightly different from the real equation such as reaction-diffusion equation [55], thus we need to develop some different technologies to solve these problems. Lastly, due to $\alpha \in (0,1)$ instead of $\alpha \in (\frac{1}{2},1)$, the methods in [28] are not suitable, so we have to deal with (1.1) by some different methods. We mention that, in this paper, since considering fractional Laplacian operator with $\alpha \in (0,1)$ and estimates of solutions for H^α, the result of random attractors is a generalization in some sense for the results of [28] with $\alpha \in (\frac{1}{2},1)$.

When we prove the existence and uniqueness of random attractors in $H^\alpha(\mathbf{R}^3)$, the nonlinearity $f(u)$ is special form, i.e.,$f(u) = -(1+i\mu)|u|^2u$ with $\mu \in \mathbf{R}$, which is consistent with the general physical background for the Ginzburg-Landau equation [2, 9]. In addition, we apply equivalent representations of the fractional Laplace operator $(-\Delta)^\alpha$ to derive the uniform estimates on solutions of (1.1) in $H^\alpha(\mathbf{R}^3)$ and carefully treat all terms involved. To overcome the lack of compactness of Sobolev embeddings on unbounded domains, we apply the idea of uniform estimates on the tails of solutions and prove the solutions are asymptotically null when t and x tend to infinity, which is slightly different from the fractional case [28] with $\alpha \in (\frac{1}{2},1)$, and the standard Laplacian case [4] in $H^1(\mathbf{R}^3)$.

This paper is organized as follows. In Section 2, the working function space, some basic concepts related to the non-autonomous random dynamical system, upper semicontinuity of random attractors, the fractional derivative and Sobolev space are introduced. In Section 3, we transform the stochastic equation into a random equation which solutions generate a random dynamical system, then give the existence and uniqueness of solutions for non-autonomous stochastic fractional Ginzburg-Landau equation. In Section 4, we derive uniform estimates for solutions and the pullback asymptotic compactness, then the existence of a pullback random attractor is proved. In Section 5, we establish the upper semicontinuity of random attractors when the coefficient δ approaches zero.

2. Preliminaries

In this section, we first present some basic notions about random attractors and non-autonomous random dynamical systems, which can be found in [1, 6, 38].

Let $(X, \|\cdot\|_X)$ be a separable Hilbert space with the Borel σ-algebra $\mathcal{B}(X)$, $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbf{R}})$ be an ergodic metric dynamical system, $H = L^2(U)$ with the usual scalar product and norm $\{(\cdot,\cdot), \|\cdot\|_2\}$ and $L^p(U)$ be the p-times integrable functions space on D with norm denoted by $\|\cdot\|_p$.

Definition 2.1. A continuous random dynamical system on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbf{R}})$ is a $(\mathcal{B}(\mathbf{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping:

$$\varphi : \mathbf{R}^+ \times \mathbf{R} \times \Omega \times X \to X, (\cdot, \tau, \cdot, \cdot) \mapsto \varphi(\cdot, \tau, \cdot, \cdot)$$

such that the following properties hold:
(1) $\varphi(0, \tau, \omega, \cdot)$ is the identity on X;
(2) $\varphi(t+s, \tau, \omega, \cdot) = \varphi(t, \tau+s, \theta_s\omega, \varphi(s, \tau, \omega, \cdot))$ for all $s, t \geq 0$;
(3) $\varphi(t, \tau, \omega, \cdot) : X \to X$ is continuous for all $t \geq 0$.

Definition 2.2. (1) A set-valued mapping $\{D(\tau,\omega)\} : \Omega \to 2^X, \omega \to D(\tau,\omega)$, is said to be a random set if the mapping $\omega \mapsto d(u, D(\tau,\omega))$ is measurable for any

$u \in X$. If $D(\tau,\omega)$ is also closed (compact) for each $\omega \in \Omega, \{D(\tau,\omega)\}$ is called a random closed (compact) set. A random set $\{D(\tau,\omega)\}$ is said to be bounded if there exist $u_0 \in X$ and a random variable $R(\tau,\omega) > 0$ such that

$$D(\tau,\omega) \subset \{u \in X : \|u - u_0\|_X \leq R(\tau,\omega)\}, \quad for \;\; all \;\; \omega \in \Omega.$$

(2) A random set $\{D(\tau,\omega)\}$ is called tempered provided for P-a.e. $\omega \in \Omega$

$$\lim_{t\to+\infty} e^{-\beta t} d(D(\tau + t, \theta_t \omega)) = 0, \quad for \;\; all \;\; \beta > 0,$$

where $d(D) = sup\{\|b\|_X : b \in D\}$.
(3) A random set $\{B(\tau,\omega)\}$ is said to be a random absorbing set if for any tempered random set $\{D(\tau,\omega)\}$, and P-a.e. $\omega \in\in \Omega$, there exists t_0 such that

$$\varphi(t, \tau - t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega), \quad for \;\; all \;\; t \geq t_0.$$

(4) A random set $\{B_1(\tau,\omega)\}$ is said to be a random attracting set if for any tempered random set $\{D(\tau,\omega)\}$, pull-back attractor and P-a.e. $\omega \in\in \Omega$, we have

$$\lim_{t\to+\infty} d_H(\varphi(t, \tau - t, \theta_{-t}\omega, D(\theta_{-t}\omega)), B_1(\tau - t, \omega)) = 0,$$

where d_H is the Hausdorff semi-distance given by $d_H(E,F) = sup_{u\in E}\, inf_{v\in F} \|u - v\|_X$ for any $E, F \subset X$.
(5) $\mathcal{D}$ is called inclusion-closed if $D = \{D(\tau,\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and if $\tilde{D} = \{\tilde{D}(\tau,\omega)\}_{\omega\in\Omega}$ is a random subset of X with $\tilde{D}(\tau,\omega) \subseteq D(\tau,\omega)$ for all $\omega \in \Omega$ then $\tilde{D} \in \mathcal{D}$.
(6) Let D be a collection of random subsets of X. Then φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for P-a.e. $\omega \in \Omega$, $\{\varphi(t_n, \tau - t_n, \theta_{-t_n}\omega, X_n\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\tau - t_n, \theta_{-t_n}\omega)$ with $\{B(\tau,\omega)\} \in D$.

Definition 2.3. Let $\mathcal{D}$ be a collection of random subsets of X and $\{\mathcal{A}(\tau,\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then $\{\mathcal{A}(\tau,\omega)\}_{\omega\in\Omega}$ is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for φ if the following conditions are satisfied, for P.a.e.$\omega \in \Omega$,
(1) $\{\mathcal{A}(\tau,\omega)\}$ is compact, and $\omega \to d(\mathcal{X}, \mathcal{A}(\tau,\omega))$ is measurable for every $\mathcal{X} \in X$;
(2) $\{\mathcal{A}(\tau,\omega)\}_{\omega\in\Omega}$ is strictly invariant, i.e., $\varphi(t,\tau,\omega,\mathcal{A}(\tau,\omega)) = \mathcal{A}(t+\tau,\theta_t\omega)$, and for a.e. $\omega \in \Omega$;
(3) $\{\mathcal{A}(\tau,\omega)\}_{\omega\in\Omega}$ attracts all sets in $\mathcal{D}$, i.e., for all $B \in \mathcal{D}$ and a.e. $\omega \in \Omega$, we have

$$\lim_{t\to\infty} d_H(\varphi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau,\omega)) = 0.$$

From [52], we have the existence and uniqueness theorem of random attractors for non-autonomous random dynamical system.

Theorem 2.4. *Let φ be a continuous random dynamical system on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbf{R}})$, If there exists a closed random tempered absorbing set $\{B(\tau,\omega)\}$ of φ and φ is asymptotically compact in X, then $\{A(\tau,\omega)\}$ is a random attractor of φ, where*

$$A(\tau,\omega) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \varphi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega))}, \quad \omega \in \Omega.$$

Moreover,$\{A(\tau,\omega)\}$ is the unique random attractor of φ.

In this following, we give a theorem on upper semicontinuity of random attractors(see [54])

Theorem 2.5. *Let I be an interval of $\mathbf{R}$ and given $a \in I$. Let $\Phi^a(t,\tau,\omega)_{a\in I}$ be a family of continuous RDSs on X over $\mathbf{R}$ and $(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbf{R}})$. Given $a_0 \in I, \Phi^{a_0}(t,\tau)$ is a continuous process over $\mathbf{R}$ independent of $\omega\in\Omega$. Suppose that*

(1) $\Phi^{a_0}(t,\tau)$ has a pullback attractor $A^{a_0}(\tau)_{\tau\in\mathbf{R}}$ with properties:
  - *(a) $A^{a_0}(\tau)$ is compact for $\tau\in\mathbf{R}$;*
  - *(b) $\Phi^{a_0}(t,\tau)A^{a_0}(\tau) = A^{a_0}(t)$ for $t \ge \tau$;*
  - *(c) for any bounded set $B\subset X, \lim_{t\to+\infty} d_H(\Phi^{a_0}(\tau,\tau-t)B, A^{a_0}(\tau)) = 0$;*

(2) $\Phi^{a_0}(t,\tau)$ has a uniform pullback absorbing set $B^{a_0} = \{x\in X : \|x\|_X \le R^{a_0}\}\subset X$ and for each $a\in I$, Φ^a has a $D(X)$-pullback random attractor $A^a(\tau,\omega)\in D(X)$ and a $D(X)$-pullback random absorbing set $K^a(\tau,\omega)\in D(X)$ such that for all $\tau\in\mathbf{R}$ and $\omega\in\Omega$, $\lim sup_{a\to a_0}\|K^a(\tau,\omega)\|_X \le R^{a_0}$;

(3) for every $\tau\in\mathbf{R}$ and $\omega\in\Omega, \bigcup_{a\in I} A^a(\tau,\omega)$ is precompact in X;

(4) for every $t\in\mathbf{R}^+,\tau\in\mathbf{R},\omega\in\Omega, a_n\in I$ with $a_n\to a_0$ and $x_n, x\in X$ with $x_n\to x$, it holds $\lim_{n\to\infty}\Phi^{a_n}(t,\tau-t,\theta_{-t}\omega,x_n) = \Phi^{a_0}(t,\tau-t,x)$.

Then for every $\tau\in\mathbf{R}$ and $\omega\in\Omega$, $d_H(A^a(\tau,\omega), A^{a_0}(\tau))\to 0$ as $a\to a_0$.

At last, we review some concepts and notations of the fractional derivative and fractional Sobolev space(see [34]) for details). Let $\mathcal{S}$ be the Schwartz space of rapidly decaying C^∞ functions on $\mathbf{R}^3$, then for $0<\alpha<1$, the fractional Laplace operator $(-\Delta)^\alpha$ is given by, for $u\in\mathcal{S}$,

$$(-\Delta)^\alpha u(x) = -\frac{1}{2}C(\alpha)\int_{\mathbf{R}^3}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{3+2\alpha}}dy,\ \ x\in\mathbf{R}^3, \tag{2.1}$$

where $C(\alpha)$ is a positive constant depending on α as given by

$$C(\alpha) = (\int_{\mathbf{R}^3}\frac{1-\cos(\xi_1)}{|\xi|^{3+2\alpha}}d\xi)^{-1},\ \ \xi=(\xi_1,\xi_2,\xi_3)\in\mathbf{R}^3. \tag{2.2}$$

In particular, it follows from [34] that for any $u\in\mathcal{S}$,

$$(-\Delta)^\alpha u = \mathcal{F}^{-1}(|\xi|^{2\alpha}(\mathcal{F}u)), \xi\in\mathbf{R}^3, \tag{2.3}$$

where $\mathcal{F}$ is the Fourier transform defined by

$$(\mathcal{F}u)(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbf{R}^3} e^{-ix.\xi}u(x)dx, u\in\mathcal{S},$$

and $\mathcal{F}^{-1}$ is the inverse Fourier transform. Let $H^\alpha(\mathbf{R}^3)$ be the fractional Sobolev space defined by

$$H^\alpha(\mathbf{R}^3) = \{u\in L^2(\mathbf{R}^3) : \int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}}dxdy<\infty\}$$

which is equipped with the norm

$$\|u\|_{H^\alpha(\mathbf{R}^3)} = (\int_{\mathbf{R}^3}|u(x)|^2dx + \int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}}dxdy)^{\frac{1}{2}}.$$

From now on, we write the norm and the inner product of $L^2(\mathbf{R}^3)$ as $\|.\|$ and $(.,.)$, respectively. We also write the Gagliardo semi-norm of $H^\alpha(\mathbf{R}^3)$ as $\|.\|_{\dot{H}^\alpha(\mathbf{R}^3)}$, i.e.,

$$\|u\|_{\dot{H}^\alpha(\mathbf{R}^3)} = \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}} dxdy, \ \ u \in H^\alpha(\mathbf{R}^3).$$

Then for all $u \in H^\alpha(\mathbf{R}^3)$ we have $\|u\|^2_{H^\alpha(\mathbf{R}^3)} = \|u\|^2 + \|u\|^2_{\dot{H}^\alpha(\mathbf{R}^3)}$. Note that $H^\alpha(\mathbf{R}^3)$ is a Hilbert space with inner product given by

$$(u,v)_{H^\alpha(\mathbf{R}^3)} = \int_{\mathbf{R}^3} u(x)\overline{v(x)}dxdy + \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{(u(x)-u(y))(\overline{v(x)}-\overline{v(y)})}{|x-y|^{3+2\alpha}} dxdy,$$
$$u,v \in H^\alpha(\mathbf{R}^3).$$

In terms of (2.3), one can verify (see [34]):

$$\|u\|^2_{H^\alpha(\mathbf{R}^3)} = \|u\|^2_{L^2(\mathbf{R}^3)} + \frac{2}{C(\alpha)}\|(-\Delta)^{\frac{\alpha}{2}}u\|^2_{L^2(\mathbf{R}^3)}, \ \ for \ \ all \ \ u \in H^\alpha(\mathbf{R}^3), \tag{2.4}$$

and hence $(\|u\|^2_{L^2(\mathbf{R}^3)} + \|(-\Delta)^{\frac{\alpha}{2}}u\|^2_{L^2(\mathbf{R}^3)})^{\frac{1}{2}}$ is an equivalent norm of $H^\alpha(\mathbf{R}^3)$.

3. The stochastic fractional Ginzburg-Landau equation with additive noise

In this section, we will give the existence and uniqueness of solutions of problem (1.1)-(1.2) which generates a continuous random dynamical system.

The standard probability space $(\Omega, \mathcal{F}, P)$ will be used in this paper where $\Omega = \{\omega \in C(\mathbf{R},\mathbf{R}) : \omega(0) = 0\}$, and $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P is the Wiener measure on $(\Omega, \mathcal{F})$. Given $t \in \mathbf{R}$, define $\theta_t : \Omega \to \Omega$ by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega.$$

Then $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbf{R}})$ is a parametric dynamical system. Let $y : \Omega \to \mathbf{R}$ be a random variable given by: $y(\omega) = -\gamma\int_{-\infty}^0 e^{\gamma\tau}\omega(\tau)d\tau$ for $\omega \in \Omega$. Then $y(t)$ is the unique stationary solution of the stochastic equation

$$dy + \gamma y dt = dW. \tag{3.1}$$

In addition, it follows from [1], that there exists a θ_t-invariant set of full measure (still denoted by Ω) such that $y(\theta_t\omega)$ is pathwise continuous for each fixed $\omega \in \Omega$ and there exists a tempered function $r(\omega) > 0$ such that

$$|y(\omega)|^2 + |y(\omega)|^4) \le r(\omega), \tag{3.2}$$

where $r(\omega)$ satisfies, for $P.a.e.\omega \in \Omega$,

$$r(\theta_t\omega) \le e^{\frac{\gamma}{2}|t|}r(\omega), \quad t \in \mathbf{R}.$$

From above, we obtain that, for $P - a.e.\omega \in \Omega$,

$$|y(\theta_t\omega)|^2 + |y(\theta_t\omega)|^4) \le e^{\frac{\gamma}{2}|t|}r(\omega), t \in \mathbf{R}. \tag{3.3}$$

We now transform the stochastic equation (1.1) into a pathwise deterministic one by using the random variable z. Put $z(\theta_t\omega) = h(x)y(\theta_t\omega)$. Given $\tau \in \mathbf{R}, t \geq \tau, \omega \in \Omega$ and $u_\tau \in L^2(\mathbf{R}^3)$, if $u = u(t,\tau,\omega,u_\tau)$ is a solution of (1.1)-(1.2), then we introduce a new variable $v = v(t,\tau,\omega,v_\tau)$ by

$$v(t,\tau,\omega,v_\tau) = u(t,\tau,\omega,u_\tau) - \delta z(\theta_t\omega) \ \ with \ \ v_\tau = u_\tau - \delta z(\theta_\tau\omega). \tag{3.4}$$

From (1.1)-(1.2) and (3.4) we obtain, for $t > \tau$,

$$\begin{aligned}&\frac{dv}{dt} + (1+i\lambda)(-\Delta)^\alpha v + \delta(1+i\lambda)(-\Delta)^\alpha z(\theta_t\omega) + \gamma v\\ =&f(u) - (1+i\mu)|u|^2u + g(t,x), \qquad x \in \mathbf{R}^3,\end{aligned} \tag{3.5}$$

and initial condition

$$v(\tau,x) = v_\tau(x), \ \ x \in \mathbf{R}^3, \tag{3.6}$$

where $f(u) = -(1+i\mu)|u|^2u$.

Next we will first give the existence and uniqueness of solutions for problem (3.5)-(3.6), and then obtain the solutions of (1.1)-(1.2) by the transform (3.4). Recall that V is a Hilbert space given by $V = \{u \in H^\alpha(\mathbf{R}^3)\}$. The dual space of V is denoted by V^*. To give the existence of solutions, we also need the space $H = \{u \in L^2(\mathbf{R}^3)\}$.

By the standard Galerkin method and compactness argument, as shown in [24], we can prove that in the case of a bounded domain with Dirichet boundary conditions, for P.a.e.$\omega \in \Omega$ and for all $v_\tau \in L^2(\mathbf{R}^3)$, equation (3.5) has a unique solution $v(t,\tau,\omega,v_\tau) \in C([0,\infty), L^2(\mathbf{R}^3)) \cap L^2((0,T), H^\alpha(\mathbf{R}^3))$ with $v(\tau,\tau,\omega,v_\tau) = v_\tau$ for every $T > \tau$. This is similar to [9, 36, 55]. Then, following the approach in [33], we take the domain to be a sequence of balls with radii approaching ∞ to deduce the existence of a weak solution of equation (3.5) on $\mathbf{R}^3$. Furthermore, we can get that $v(t,\tau,\omega,v_\tau)$ is unique and continuous with respect to v_τ in $H^\alpha(\mathbf{R}^3)$ for all $t \geq \tau$.

Now by the solution v of (3.5)-(3.6) and the transform (3.4), we get a solution u of the stochastic equation (1.1)-(1.2) which is given by

$$u(t,\tau,\omega,u_\tau) = v + \delta z(\theta_t\omega)$$

with $u_\tau = v_\tau + \delta z(\theta_\tau\omega)$. We note that $u(t,\tau,\omega,u_\tau)$ is both continuous in $t \in [\tau,\infty)$ and in $u_\tau \in H$. Moreover, $u(t,\tau,.,u_\tau) : \Omega \to H$ is measurable. Then we can define a continuous cocycle in H associated with the solutions of problem (1.1)-(1.2). Let $\Phi : \mathbf{R}^+ \times \mathbf{R} \times \Omega \times H \to H$ be a mapping given by, for every $t \in \mathbf{R}^+, \tau \in \mathbf{R}, \omega \in \Omega$ and $u_\tau \in H$,

$$\Phi(t,\tau,\omega,u_\tau) = u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau) = v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau) + \delta z(\theta_t\omega), \tag{3.7}$$

where $v_\tau = u_\tau - \delta z(\theta_\tau\omega)$. In later sections, we will prove the existence and upper semicontinuity of tempered random attractors for Φ in H.

Let $B = \{B(\tau,\omega) : \tau \in \mathbf{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of H. Such a family B is called tempered if for every $c > 0, \tau \in \mathbf{R}$ and $\omega \in \Omega$,

$$\lim_{t\to-\infty} e^{ct}\|B(\tau+t,\theta_t\omega)\| = 0,$$

where the norm $\|B\|$ of set B in H is given by $\|B\| = \sup_{u\in B}\|u\|$. From now on, we will use $\mathcal{D}$ to denote the collection of all tempered families of bounded nonempty subsets of H:

$$\mathcal{D} = \{B = \{B(\tau,\omega) : \tau \in \mathbf{R}, \omega \in \Omega\} : B \ \text{is tempered in} \ H\}.$$

When deriving uniform estimates, for simplicity, we assume that $\gamma > 0$, and also assume that for every $\tau \in \mathbf{R}$,

$$\int_{-\infty}^{0} e^{\gamma s}\|g(s+\tau,.)\|^2 ds < \infty. \tag{3.8}$$

Sometimes, we also assume g is tempered in the following sense: for every $c > 0$,

$$\lim_{r\to\infty} e^{-cr}\int_{-\infty}^{0} e^{\gamma s}\|g(s-r,.)\|^2 ds = 0. \tag{3.9}$$

Note that these conditions do not require g to be bounded in H when $t \to \infty$. Through this paper, c denotes a positive constant which may be different from the context.

4. Random attractors

In this section, we will derive uniform estimates on the solutions of non-autonomous stochastic fractional Ginzburg-Landau equation in H and V. These estimates are necessary for proving the existence of random attractors. Then we prove the existence and uniqueness of pullback random attractors.

We first derive uniform estimates of solutions in H.

Lemma 4.1. *Suppose* (3.8) *holds. Then for every $\delta_0 > 0, r \in \mathbf{R}, \tau \in \mathbf{R}, \omega \in \Omega$ and $B = \{B(\tau,\omega) : \tau \in \mathbf{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T(\tau,\omega,B,r,\delta_0) \geq 0$ such that for all $t \geq T$ and $0 < \delta \leq \delta_0$, the solution v of problem* (3.5)-(3.6) *satisfies*

$$\begin{aligned}
&\|v(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 + \int_{-t}^{r-\tau} e^{\gamma(s-r+\tau)}\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{H^\alpha(\mathbf{R}^3)}ds \\
&+ \int_{-t}^{r-\tau} e^{\gamma(s-r+\tau)}\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^4_{L^4(\mathbf{R}^3)}ds \\
\leq& c + c\int_{-\infty}^{r-\tau} e^{\gamma(s-r+\tau)}(e^{\frac{1}{2}\gamma|s|}r(\omega) + \|g(s+\tau)\|^2)ds,
\end{aligned}$$

where $v_{\tau-t} + \delta z(\theta_{-t}\omega) \in B(\tau-t,\theta_{-t}\omega)$.

Proof. Taking the inner product of (3.5) with v, and taking the real part, we have

$$\frac{d}{dt}\|v\|^2 + C(\alpha)\|v\|^2_{\dot{H}^\alpha} + 2\gamma\|v\|^2 = -2\delta Re((1+i\lambda)(-\Delta)^\alpha z(\theta_t\omega), v) \tag{4.1}$$

$$-2Re(1+i\mu)\int_{\mathbf{R}^3}|u|^2 u\bar{v}dx + 2Re\int_{\mathbf{R}^3} g(t,x)\bar{v}dx. \tag{4.2}$$

We now estimate all terms in (4.1). For the last term, we have

$$2Re\int_{\mathbf{R}^3} g(t,x)\bar{v}dx \leq \frac{\gamma}{2}\|v\|^2 + 2\gamma^{-1}\|g(t,x)\|^2. \tag{4.3}$$

On the other hand, we find

$$-2Re(\delta(1+i\lambda)(-\Delta)^\alpha z(\theta_t\omega), v)$$

$$
\begin{aligned}
&\leq \frac{2\delta^2(1+\lambda^2)}{\gamma}\|(-\Delta)^\alpha z(\theta_t\omega)\|^2 + \frac{\gamma}{2}\|v\|^2 \qquad (4.4)\\
&\quad - 2Re(1+i\mu)\int_{\mathbf{R}^3}|u|^2 u\bar{v}dx\\
&= - 2Re(1+i\mu)\int_{\mathbf{R}^3}|u|^2 u\bar{u}dx + 2Re(1+i\mu)\int_{\mathbf{R}^3}|u|^2 u\delta z(\theta_t\omega)dx\\
&\leq - 2\int_{\mathbf{R}^3}|u|^4dx + 2|(1+i\mu)|\delta\int_{\mathbf{R}^3}|u|^3|z(\theta_t\omega)|dx\\
&\leq - \|u\|_4^4 + c\|z(\theta_t\omega)\|_4^4. \qquad (4.5)
\end{aligned}
$$

It follows from (4.1)-(4.5) that

$$
\begin{aligned}
&\frac{d}{dt}\|v\|^2 + \gamma\|v\|^2 + C(\alpha)\|v\|^2_{\dot{H}^\alpha(\mathbf{R}^3)} + \|u\|^4_{L^4(\mathbf{R}^3)}\\
\leq& c(\|z(\theta_t\omega)\|_4^4 + \|(-\Delta)^\alpha z(\theta_t\omega)\|^2) + 2\gamma^{-1}\|g(t,x)\|^2. \qquad (4.6)
\end{aligned}
$$

Note that $z(\theta_t\omega) = h(x)y(\theta_t\omega)$, $h(x) \in H^{2\alpha}\cap W^{2\alpha,4}$. Therefore, the first term on the right-hand side of (4.6) can be bounded by

$$c(\|z(\theta_t\omega)\|_4^4 + \|z(\theta_t\omega)\|^2) = p_1(\theta_t\omega).$$

By (3.3), we find that for $P - a.e. \omega \in \Omega$

$$p_1(\theta_\tau\omega) \leq ce^{\frac{1}{2}|\tau|\gamma}r(\omega), \quad for \;\; all \;\; \tau \in \mathbf{R}. \qquad (4.7)$$

Multiplying (4.6) with $e^{\gamma t}$, then integrating the inequality on $(\tau-t,\sigma)$ with $\sigma > \tau-t$, we get

$$
\begin{aligned}
&\|v(\sigma,\tau-t,\omega,v_{\tau-t})\|^2 + C(\alpha)\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|v(s,\tau-t,\omega,v_{\tau-t})\|^2_{\dot{H}^\alpha}ds\\
&+ \int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|u(s,\tau-t,\omega,v_{\tau-t})\|^4_{L^4}ds\\
\leq& e^{\gamma(\tau-t-\sigma)}\|v_{\tau-t}\|^2 + \int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}p_1(\theta_{s-\tau}\omega)ds + 2\gamma^{-1}\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|g(s)\|^2ds.
\end{aligned}
$$

Replacing ω by $\theta_{-\tau}\omega$ we have

$$
\begin{aligned}
&\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 + C(\alpha)\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{\dot{H}^\alpha}ds\\
&+ \int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|u(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^4_{L^4}ds\\
\leq& e^{\gamma(\tau-t-\sigma)}\|v_{\tau-t}\|^2 + \int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}p_1(\theta_{s-\tau}\omega)ds\\
&+ 2\gamma^{-1}\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|g(s)\|^2ds.
\end{aligned}
$$

After change of variables, we obtain

$$\|v(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 + C(\alpha)\int_{-t}^{r-\tau} e^{\gamma(s-r+\tau)}\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{\dot{H}^\alpha}ds$$

$$
\begin{aligned}
&+\int_{-t}^{r-\tau} e^{\gamma(s-r+\tau)}\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|_{L^4}^4 ds\\
\le& e^{\gamma(\tau-t-r)}\|v_{\tau-t}\|^2+\int_{-t}^{r-\tau} e^{\gamma(s-r+\tau)}p_1(\theta_s\omega)ds\\
&+2\gamma^{-1}\int_{-t}^{r-\tau} e^{\gamma(s-r+\tau)}\|g(s+\tau)\|^2 ds. \qquad (4.8)
\end{aligned}
$$

Now we estimate the first term on the right-hand side of (4.8). Since $B=\{B(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}$ is tempered, $v_{\tau-t}\in B(\tau-t,\theta_{-t}\omega)$, there exists $T=T(\tau,\omega,B,r,\delta_0)>0$ such that for all $t\ge T$, $0<\delta<\delta_0$,

$$
e^{\gamma(\tau-t-r)}\|v_{\tau-t}\|^2\le 1. \qquad (4.9)
$$

For the second and last term we have

$$
\begin{aligned}
&\int_{-t}^{r-\tau} e^{\gamma(s-r+\tau)}p_1(\theta_s\omega)ds\le\int_{-\infty}^{r-\tau} e^{\gamma(s-r+\tau)}p_1(\theta_s\omega)ds,\\
&\int_{-t}^{r-\tau} e^{\gamma(s-r+\tau)}\|g(s+\tau)\|^2 ds\le\int_{-\infty}^{r-\tau} e^{\gamma(s-r+\tau)}\|g(s+\tau)\|^2 ds. \qquad (4.10)
\end{aligned}
$$

By (4.8)-(4.10) we obtain, for all $t\ge T$,

$$
\begin{aligned}
&\|v(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2+C(\alpha)\int_{-t}^{r-\tau} e^{\gamma(s-r+\tau)}\|v(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|_{\dot H^\alpha}^2 ds\\
&+\int_{-t}^{r-\tau} e^{\gamma(s-r+\tau)}\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|_{L^4}^4 ds\\
\le& 1+\int_{-\infty}^{r-\tau} e^{\gamma(s-r+\tau)}(ce^{\frac12|s|\gamma}r(\omega)+2\gamma^{-1}\|g(s+\tau)\|^2)ds. \qquad (4.11)
\end{aligned}
$$

From (4.11), the desired estimates follow immediately. □

Based on Lemma 4.1, we claim the solution operator of problem (3.5)-(3.6) has a random pullback absorbing set in H as stated below.

Lemma 4.2. *Suppose* (3.8) *holds. Let* $B_1=\{B_1(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}$ *be a random set given by*

$$
B_1(\tau,\omega)=\{v\in H:\|v\|^2\le R_1(\tau,\omega)\},
$$

where $R_1(\tau,\omega)$ *is defined by*

$$
R_1(\tau,\omega)=c+c\int_{-\infty}^{0}(e^{\frac12\gamma s}r(\omega)+e^{\gamma s}\|g(s+\tau)\|^2)ds. \qquad (4.12)
$$

Then for every $\tau\in\mathbf{R},\omega\in\Omega\}$ *and* $B=\{B(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}\in\mathcal{D}$, *there exists* $T=T(\tau,\omega,B,\delta)>0$ *such that the solution* v *of* (3.5)-(3.6) *with* $v_{\tau-t}+\delta z(\theta_{-t}\omega)\in B(\tau-t,\theta_{-t}\omega)$ *satisfies, for all* $t\ge T$,

$$
v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\in B_1(\tau,\omega). \qquad (4.13)
$$

In addition, the random variable R_1 *as in* (4.12) *is tempered, i.e., for any* $c>0$,

$$
\lim_{t\to\infty}e^{-ct}R_1(\tau-t,\theta_{-t}\omega)=0. \qquad (4.14)
$$

Proof. As a special case of Lemma 4.1 with $r = \tau$, we obtain (4.13) immediately. We now prove (4.13). From (4.12) we have

$$R_1(\tau - t, \omega) = c + c \int_{-\infty}^{0} (e^{\frac{1}{2}\gamma s} r(\omega) + e^{\gamma s} \|g(s + \tau - t)\|^2) ds. \tag{4.15}$$

Note that $r(\omega)$ is a tempered function. By (3.9) we get

$$\begin{aligned}
&\limsup_{t\to\infty} e^{-ct} R_1(\tau - t, \omega) \\
\le &\limsup_{t\to\infty} e^{-ct} \int_{-\infty}^{0} (e^{\frac{1}{2}\gamma s} r(\omega) + e^{\gamma s} \|g(s + \tau - t)\|^2) ds \\
\le &\limsup_{r\to\infty} e^{-c\tau} e^{-cr} \int_{-\infty}^{0} (e^{\frac{1}{2}\gamma s} r(\omega) + e^{\gamma s} \|g(s - r)\|^2) ds = 0.
\end{aligned}$$

□

We now derive uniform estimates of solutions in V.

Lemma 4.3. *Suppose* (3.8) *holds. Then for every* $\delta_0 > 0, \sigma \in \mathbf{R}, \tau \in \mathbf{R}, \omega \in \Omega$ *and* $B = \{B(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega\} \in \mathcal{B}$, *there exists* $T(\tau, \omega, B, \sigma, \delta_0) \ge 0$ *such that for all* $t \ge T$ *and* $0 < \delta \le \delta_0$, *the solution* v *of problem* (3.5)-(3.6) *satisfies*

$$\begin{aligned}
&\|v(\sigma, \tau - t, \theta_{-\tau}\omega, v_{\tau - t})\|^2_{H^\alpha(\mathbf{R}^3)} ds \\
\le &c + c \int_{-\infty}^{\sigma - \tau} e^{\gamma(s - \sigma + \tau)} (e^{\frac{1}{2}|s|\gamma} r(\omega) + \|g(s + \tau)\|^2) ds,
\end{aligned}$$

where $v_{\tau - t} + \delta z(\theta_{-t}\omega) \in B(\tau - t, \theta_{-t}\omega)$.

Proof. Taking the inner product of (3.5) with $(-\Delta)^\alpha v$, and taking the real part, we have

$$\begin{aligned}
&\frac{d}{dt} \|(-\Delta)^{\frac{\alpha}{2}} v\|^2 + 2\|(-\Delta)^\alpha v\|^2 + 2\gamma \|(-\Delta)^{\frac{\alpha}{2}} v\|^2 \\
= &- 2Re((1 + i\mu)|u|^2 u, (-\Delta)^\alpha v) + 2Re(g(t, x), (-\Delta)^\alpha v) \\
&- 2Re(\delta(1 + i\lambda)(-\Delta)^\alpha z(\theta_t\omega), (-\Delta)^\alpha v).
\end{aligned} \tag{4.16}$$

We now estimate the terms on the right-hand side of (4.16). For the nonlinear term, from Taylor's formula and Lemma 4.1, we have

$$\begin{aligned}
&- 2Re(1 + i\mu)(|u|^2 u, (-\Delta)^\alpha v) \\
= &-C(\alpha) Re(1 + i\mu)(|u|^2 u, u)_{\dot{H}^\alpha(\mathbf{R}^3)} + 2Re(1 + i\mu)(|u|^2 u, (-\Delta)^\alpha \delta z(\theta_t\omega)) \\
\le &-C(\alpha) Re(1 + i\mu) \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{(|u(x)|^2 u(x) - |u(y)|^2 u(y))(\bar{u}(x) - \bar{u}(y))}{|x - y|^{3 + 2\alpha}} dxdy \\
&+ 2\delta |1 + i\mu| \int_{\mathbf{R}^3} |u|^3 |(-\Delta)^\alpha z(\theta_t\omega)| dx \\
\le &c(\|(-\Delta)^{\frac{\alpha}{2}} u\|^2 + \|u\|^4) + c\|(-\Delta)^\alpha z(\theta_t\omega)\|_4^4.
\end{aligned} \tag{4.17}$$

For the second term, we obtain

$$2Re(g(t, x), (-\Delta)^\alpha v) \le \frac{1}{2} \|(-\Delta)^\alpha v\|^2 + 2\|g(t)\|^2. \tag{4.18}$$

For the last term, we have

$$\begin{aligned}&-2Re(\delta(1+i\lambda)(-\Delta)^{\alpha}z(\theta_t\omega)),(-\Delta)^{\alpha}v)\\ \leq&\frac{1}{2}\|(-\Delta)^{\alpha}v\|^2+2\delta^2(1+\lambda^2)\|(-\Delta)^{\alpha}z(\theta_t\omega)\|^2.\end{aligned}\tag{4.19}$$

It follows from (4.16)-(4.19) that

$$\begin{aligned}&\frac{d}{dt}\|(-\Delta)^{\frac{\alpha}{2}}v\|^2+\|(-\Delta)^{\alpha}v\|^2+\gamma\|(-\Delta)^{\frac{\alpha}{2}}v\|^2\\ \leq&c(\|(-\Delta)^{\frac{\alpha}{2}}u\|^2+\|u\|_4^4)+c(\|(-\Delta)^{\alpha}z(\theta_t\omega)\|^2+\|(-\Delta)^{\alpha}z(\theta_t\omega)\|_4^4)+2\|g(t)\|^2.\end{aligned}\tag{4.20}$$

Note that $z(\theta_t\omega)=h(x)y(\theta_t\omega)$, $h(x)\in H^{2\alpha}\cap W^{2\alpha,4}$. Therefore, the second term on the right-hand side of (4.20) can be bounded by

$$c(\|z(\theta_t\omega)\|_4^4+\|z(\theta_t\omega)\|^2)=p_2(\theta_t\omega).$$

By (3.3), we find that for $P-a.e.\omega\in\Omega$

$$p_2(\theta_\tau\omega)\leq ce^{\frac{1}{2}|\tau|\gamma}r(\omega),\ \text{ for all }\ \tau\in\mathbf{R}.\tag{4.21}$$

Given $t\in\mathbf{R}^+,\tau\in\mathbf{R}$ and $\omega\in\Omega$, let $\sigma\in(\tau-1,\tau)$ and $\sigma\in(\tau-2,\tau-1)$. Multiplying (4.20) with $e^{\gamma t}$,first integrating with respect to t on (r,σ) we obtain

$$\begin{aligned}&\|(-\Delta)^{\frac{\alpha}{2}}v(\sigma,\tau-t,\omega,v_{\tau-t})\|^2\\ \leq&e^{\gamma(r-\sigma)}\|(-\Delta)^{\frac{\alpha}{2}}v(r,\tau-t,\omega,v_{\tau-t})\|^2dr\\ &+\int_r^\sigma e^{\gamma(\varsigma-\sigma)}(c(\|(-\Delta)^{\frac{\alpha}{2}}u(\varsigma,\tau-t,\omega,u_{\tau-t})\|^2+c\|u(\varsigma,\tau-t,\omega,u_{\tau-t})\|_4^4)\\ &+ce^{\frac{1}{2}|\varsigma|\gamma}r(\omega))+2\|g(\zeta+\tau)\|^2)d\varsigma dr.\end{aligned}\tag{4.22}$$

Then integrating with respect to r on $(\tau-2,\tau-1)$, we get

$$\begin{aligned}&\|(-\Delta)^{\frac{\alpha}{2}}v(\sigma,\tau-t,\omega,v_{\tau-t})\|^2\\ \leq&\int_{\tau-2}^{\tau-1}e^{\gamma(r-\sigma)}\|(-\Delta)^{\frac{\alpha}{2}}v(r,\tau-t,\omega,v_{\tau-t})\|^2dr\\ &+\int_{\tau-2}^{\tau-1}\int_r^\sigma e^{\gamma(\varsigma-\sigma)}(c(\|(-\Delta)^{\frac{\alpha}{2}}u(\varsigma,\tau-t,\omega,u_{\tau-t})\|^2+c\|u(\varsigma,\tau-t,\omega,u_{\tau-t})\|_4^4)\\ &+ce^{\frac{1}{2}|\varsigma|\gamma}r(\omega))+2\|g(\zeta+\tau)\|^2)d\varsigma dr.\end{aligned}\tag{4.23}$$

Replacing ω by $\theta_{-\tau}\omega$ we have

$$\begin{aligned}&\|(-\Delta)^{\frac{\alpha}{2}}v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2\\ \leq&\int_{\tau-2}^{\tau-1}e^{\gamma(r-\sigma)}\|(-\Delta)^{\frac{\alpha}{2}}v(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2dr\\ &+\int_{\tau-2}^{\tau-1}\int_r^\sigma e^{\gamma(\varsigma-\sigma)}(c(\|(-\Delta)^{\frac{\alpha}{2}}u(\varsigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2+c\|u(\varsigma)\|_4^4)\\ &+ce^{\frac{1}{2}|\varsigma|\gamma}r(\omega))+2\|g(\zeta+\tau)\|^2)d\varsigma dr\end{aligned}$$

$$
\begin{aligned}
&\le \int_{-2}^{-1} e^{\gamma(r-\sigma+\tau)} \|(-\Delta)^{\frac{\alpha}{2}} v(r+\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 dr \\
&\quad + \int_{-2}^{-1} \int_{r-\tau}^{\sigma-\tau} e^{\gamma(\varsigma-\sigma+\tau)} (c(\|(-\Delta)^{\frac{\alpha}{2}} u(\varsigma+\tau, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 + c\|u(\varsigma+\tau)\|_4^4) \\
&\quad + ce^{\frac{1}{2}|\varsigma|\gamma} r(\omega) + 2\|g(\varsigma+\tau)\|^2) d\varsigma dr \\
&\le \int_{-2}^{-1} e^{\gamma(r-\sigma+\tau)} \|(-\Delta)^{\frac{\alpha}{2}} v(r+\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 dr \\
&\quad + \int_{-2}^{\sigma-\tau} e^{\gamma(\varsigma-\sigma+\tau)} (c\|(-\Delta)^{\frac{\alpha}{2}} u(\varsigma+\tau, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 + c\|u(\varsigma+\tau)\|_4^4 \\
&\quad + ce^{\frac{1}{2}|\varsigma|\gamma} r(\omega) + 2\|g(\varsigma+\tau)\|^2) d\varsigma. \qquad (4.24)
\end{aligned}
$$

Let T be the constant in Lemma 4.1, and $T_0 = max\{2, T\}$. Note that $u = v + \delta z(\theta_{-t}\omega)$ and (3.3), from $\sigma \in (\tau-1, \tau)$ and Lemma 4.1, then for all $t \ge T_0$, we obtain

$$
\begin{aligned}
&\|(-\Delta)^{\frac{\alpha}{2}} v(\sigma, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 \\
\le &(1+c) \int_{-2}^{\sigma-\tau} e^{\gamma(r-\sigma+\tau)} \|(-\Delta)^{\frac{\alpha}{2}} v(r+\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 dr \\
&+ \int_{-2}^{\sigma-\tau} e^{\gamma(s-\sigma+\tau)} (ce^{\frac{1}{2}|\varsigma|\gamma} r(\omega) + 2\|g(\varsigma+\tau)\|^2) d\varsigma \\
\le &c + c\int_{-\infty}^{\sigma-\tau} e^{\gamma(s-\sigma+\tau)} (e^{\frac{1}{2}|s|\gamma} r(\omega) + \|g(s+\tau)\|^2) ds. \qquad (4.25)
\end{aligned}
$$

□

Based on Lemma 4.3, we claim the solution operator of problem (3.5)-(3.6) has a random pullback absorbing set in V as stated below.

Lemma 4.4. *Suppose* (3.8) *holds. Let $B_2 = \{B_2(\tau,\omega) : \tau \in \mathbf{R}, \omega \in \Omega\}$ be a random set given by*

$$B_2(\tau,\omega) = \{v \in H : \|v\|_{H^\alpha}^2 \le R_2(\tau,\omega)\},$$

where $R_2(\tau,\omega)$ is defined by

$$R_2(\tau,\omega) = c + c\int_{-\infty}^{0} (e^{\frac{1}{2}\gamma s} r(\omega) + e^{\gamma s}\|g(s+\tau)\|^2) ds. \qquad (4.26)$$

Then for every $\tau \in \mathbf{R}, \omega \in \Omega\}$ and $B = \{B(\tau,\omega) : \tau \in \mathbf{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,B,\delta) > 0$ such that the solution v of (3.5)-(3.6) *with $v_{\tau-t} + \delta z(\theta_{-t}\omega) \in B(\tau-t, \theta_{-t}\omega)$ satisfies, for all $t \ge T$,*

$$v(\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t}) \in B_1(\tau,\omega). \qquad (4.27)$$

In addition, the random variable R_2 as in (4.26) *is tempered, i.e., for any $c > 0$,*

$$\lim_{t\to\infty} e^{-ct} R_2(\tau-t, \theta_{-t}\omega) = 0. \qquad (4.28)$$

Proof. As a special case of Lemma 4.3 with $\sigma = \tau$, we obtain (4.27) immediately. We now prove (4.27). From (4.26) we have

$$R_2(\tau-t,\omega) = c + c\int_{-\infty}^{0} (e^{\frac{1}{2}\gamma s} r(\omega) + e^{\gamma s}(\|g(s+\tau-t)\|^2) ds. \qquad (4.29)$$

Note that $r(\omega)$ is a tempered function. By (3.9) we get

$$\begin{aligned}&\limsup_{t\to\infty} e^{-ct}R_2(\tau-t,\omega)\\ \le&\limsup_{t\to\infty} e^{-ct}\int_{-\infty}^0 (e^{\frac{1}{2}\gamma s}r(\omega)+e^{\gamma s}(\|g(s+\tau-t)\|^2)ds\\ \le&\limsup_{r\to\infty} e^{-c\tau}e^{-cr}\int_{-\infty}^0 (e^{\frac{1}{2}\gamma s}r(\omega)+e^{\gamma s}(\|g(s-r)\|^2)ds=0.\end{aligned}$$

□

In this following, we will derive the uniform priori estimates on the tail of solutions. Firstly, we introduce a smooth function $\rho(s)$ defined for $0\le s<\infty$ such that $0\le\rho(s)\le 1$ for $s\ge 0$ and

$$\rho(s)=\begin{cases}0, & \text{if } 0\le s<\frac{1}{2},\\ 1, & \text{if } s\ge 1.\end{cases} \tag{4.30}$$

Note that there exists a positive constant c such that $|\rho'(s)|\le c$ for all $s\ge 0$. From the definition of the cut-off function $\rho(s)$ and fractional Laplace operator $(-\Delta)^\alpha$, we can easily get the following properties of $\rho(s)$.

Lemma 4.5 ([15]). *Let $\rho(s)$ be the smooth function defined by* (4.30) *and $\alpha\in(0,1)$. For every $x,y\in\mathbf{R}^3$ and $l\in\mathbf{N}$, then we have*

$$\int_{\mathbf{R}^3}\frac{|\rho(\frac{|x|}{l})-\rho(\frac{|y|}{l})|^2}{|x-y|^{3+2\alpha}}dx\le\frac{\nu_1}{l^{2\alpha}}, \tag{4.31}$$

where ν_1 is a positive constant,$\rho_l(.)=\rho(\frac{|.|}{l})$.

We now give uniform estimates on the tails of solutions in $L^2(\mathbf{R}^3)$.

Lemma 4.6. *Suppose* (3.8) *holds. Then for every $\varepsilon>0,\tau\in\mathbf{R},\omega\in\Omega$ and $B=\{B(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}\in\mathcal{D}$, there exists $T(\tau,\omega,B,\delta,\varepsilon)\ge 0$ and $L=L(\tau,\omega,\varepsilon)\ge 1$ such that for all $t\ge T$ and $l\ge L$, the solution v of problem* (3.5)-(3.6) *satisfies*

$$\int_{|x|\ge l}|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2dx\le\varepsilon,$$

and

$$\int_{-t}^0 e^{\gamma\zeta}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\rho(\frac{|x|}{l})(v(\zeta+\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}))^2}{|x-y|^{3+2\alpha}}dxdyd\zeta\le\varepsilon,$$

where ρ is defined by (4.30).

Proof. Taking the inner product of (3.5) with $\rho(\frac{|x|}{l})v$ and taking the real part, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|v|^2dx+2\gamma\int_{\mathbf{R}^n}\rho(\frac{|x|}{l})|v|^2dx\\ =&-2Re(1+i\lambda)\int_{\mathbf{R}^3}(-\Delta)^\alpha v\rho(\frac{|x|}{l})\bar{v}dx\end{aligned}$$

$$
\begin{aligned}
&- 2\delta Re(1+i\lambda)\int_{\mathbf{R}^3}(-\Delta)^\alpha z(\theta_t\omega)\rho(\frac{|x|}{l})\bar{v}dx\\
&+ 2Re\int_{\mathbf{R}^3} f(t,x,v+\delta z(\theta_t\omega))\rho(\frac{|x|}{l})\bar{v}dx\\
&+ 2Re\int_{\mathbf{R}^3} g(t,x)\rho(\frac{|x|}{l})\bar{v}dx. \qquad (4.32)
\end{aligned}
$$

Now we estimate each term on the right side of (4.32). For the first term, from (4.31), we get

$$
\begin{aligned}
&- 2Re(1+i\lambda)\int_{\mathbf{R}^3}(-\Delta)^\alpha v\rho(\frac{|x|}{l})\bar{v}dx\\
&= -C(\alpha)Re(1+i\lambda)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{(\rho(\frac{|x|}{l})v(x)-\rho(\frac{|y|}{l})v(y))(v(x)-v(y))}{|x-y|^{3+2\alpha}}dxdy\\
&= -C(\alpha)Re(1+i\lambda)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{(\rho(\frac{|x|}{l})(v(x)-v(y))^2}{|x-y|^{3+2\alpha}}dxdy\\
&\quad - C(\alpha)Re(1+i\lambda)\int_{\mathbf{R}^3}\frac{(\rho(\frac{|x|}{l})-\rho(\frac{|y|}{l}))(v(x)-v(y))v(y)}{|x-y|^{3+2\alpha}}dxdy\\
&\le -C(\alpha)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{(\rho(\frac{|x|}{l})(v(x)-v(y))^2}{|x-y|^{3+2\alpha}}dxdy\\
&\quad + \sqrt{1+\lambda^2}C(\alpha)\|v\|(\int_{\mathbf{R}^3}(\int_{\mathbf{R}^3}\frac{|(\rho(\frac{|x|}{l})-\rho(\frac{|y|}{l}))||v(x)-v(y)|}{|x-y|^{3+2\alpha}}dx)^2dy)^{\frac{1}{2}}\\
&\le -C(\alpha)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{(\rho(\frac{|x|}{l})(v(x)-v(y))^2}{|x-y|^{3+2\alpha}}dxdy\\
&\quad + \sqrt{1+\lambda^2}C(\alpha)\|v\|(\int_{\mathbf{R}^3}(\int_{\mathbf{R}^3}\frac{|(\rho(\frac{|x|}{l})-\rho(\frac{|y|}{l})|^2}{|x-y|^{3+2\alpha}}dx\int_{\mathbf{R}^3}\frac{|v(x)-v(y)|^2}{|x-y|^{3+2\alpha}}dx)dy)^{\frac{1}{2}}\\
&\le -C(\alpha)\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{(\rho(\frac{|x|}{l})(v(x)-v(y))^2}{|x-y|^{3+2\alpha}}dxdy + \sqrt{1+\lambda^2}C(\alpha)cl^{-\alpha}\|v\|^2_{H^\alpha(\mathbf{R}^3)}. \qquad (4.33)
\end{aligned}
$$

For the second term, we have

$$
\begin{aligned}
&- 2\delta Re(1+i\lambda)\int_{\mathbf{R}^3}(-\Delta)^\alpha z(\theta_t\omega)\rho(\frac{|x|}{l})\bar{v}dx\\
&\le \frac{2\delta^2(1+\lambda^2)}{\gamma}\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|(-\Delta)^\alpha z(\theta_t\omega)|^2dx + \frac{\gamma}{2}\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|v|^2dx. \qquad (4.34)
\end{aligned}
$$

For the last term, we get

$$
2Re\int_{\mathbf{R}^3} g(t,x)\rho(\frac{|x|}{l})\bar{v}dx \le \frac{\gamma}{2}\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|v|^2dx + \frac{2}{\gamma}\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|g(t,x)|^2dx. \qquad (4.35)
$$

For the nonlinear term, we find

$$
2Re\int_{\mathbf{R}^3} f(t,x,v+\delta z(\theta_t\omega))\rho(\frac{|x|}{l})\bar{v}dx
$$

$$
\begin{aligned}
&= 2Re\int_{\mathbf{R}^3} f(t,x,u)\rho(\frac{|x|}{k})\bar{u}dx - 2\delta Re\int_{\mathbf{R}^3} f(t,x,u)\rho(\frac{|x|}{l})z(\theta_t\omega)dx\\
&\leq -2\int_{\mathbf{R}^3}\rho(\frac{|x|}{k})|u|^4dx + 2\delta\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|f(t,x,u)||z(\theta_t\omega)|dx\\
&\leq -2\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|u|^4dx + 2\delta|1+i\mu|\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|u|^3|z(\theta_t\omega)|dx\\
&\leq -\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|u|^4dx + c\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|z(\theta_t\omega)|^4dx. \qquad (4.36)
\end{aligned}
$$

It follows from (4.32)-(4.36) that for all $l \geq L_1(\omega) \geq 1$

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|v|^2dx + \gamma\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|v|^2dx + C(\alpha)\int_{\mathbf{R}^3}\frac{\rho(\frac{|x|}{l})(v(x)-v(y))^2}{|x-y|^{3+2\alpha}}dxdy\\
&\leq cl^{-\alpha}\|v\|^2_{H^\alpha} + \int_{\mathbf{R}^3}\rho(\frac{|x|}{l})(2\delta|z(\theta_t\omega)|^4 + \frac{2}{\gamma}|(-\Delta)^\alpha z(\theta_t\omega)|^2)dx + \frac{2}{\gamma}\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|g(t)|^2dx\\
&\leq c\|v\|^2_{H^\alpha} + c\int_{|x|\geq\frac{1}{2}l}(|z(\theta_t\omega)|^4 + |(-\Delta)^\alpha z(\theta_t\omega)|^2)dx + c\int_{|x|\geq\frac{1}{2}l}|g(t,x)|^2dx. \qquad (4.37)
\end{aligned}
$$

Given $t \in \mathbf{R}^+, \tau \in \mathbf{R}$ and $\omega \in \Omega$, multiplying (4) with $e^{\gamma t}$, then integrating the result on $(\tau - t, \tau)$, we get for all $l \geq L_1(\omega)$

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|v(\tau,\tau-t,\omega,v_{\tau-t})|^2dx\\
&+ C(\alpha)\int_{\tau-t}^{\tau}e^{\gamma(\zeta-\tau)}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\rho(\frac{|x|}{l})(v(\zeta,\tau-t,\omega,v_{\tau-t})(x)-v(y))^2}{|x-y|^{3+2\alpha}}dxdyd\zeta\\
\leq& e^{-\gamma t}\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|v_{\tau-t}|^2dx + c\int_{\tau-t}^{\tau}e^{\gamma(s-\tau)}\|v(s,\tau-t,\omega,v_{\tau-t})\|^2_{H^\alpha}ds\\
&+ c\int_{\tau-t}^{\tau}\int_{|x|\geq\frac{1}{2}l}e^{\gamma(s-\tau)}|g(s,x)|^2dxds\\
&+ c\int_{\tau-t}^{\tau}\int_{|x|\geq\frac{1}{2}l}e^{\gamma(s-\tau)}(|z(\theta_s\omega)|^4 + |(-\Delta)^\alpha z(\theta_s\omega)|^2)dxds.
\end{aligned}
$$

Then replacing ω by $\theta_{-\tau}\omega$ we have for all $l \geq L_1(\omega)$

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2dx\\
&+ C(\alpha)\int_{\tau-t}^{\tau}e^{\gamma(\zeta-\tau)}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\rho(\frac{|x|}{l})(v(\zeta,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})(x)-v(y))^2}{|x-y|^{3+2\alpha}}dxdyd\zeta\\
\leq& e^{-\gamma t}\int_{\mathbf{R}^3}\rho(\frac{|x|}{l})|v_{\tau-t}|^2dx + c\int_{\tau-t}^{\tau}e^{\gamma(s-\tau)}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2_{H^\alpha}ds\\
&+ c\int_{\tau-t}^{\tau}\int_{|x|\geq\frac{1}{2}l}e^{\gamma(s-\tau)}|g(s,x)|^2dxds\\
&+ c\int_{\tau-t}^{\tau}\int_{|x|\geq\frac{1}{2}l}e^{\gamma(s-\tau)}(|z(\theta_s\omega)|^4 + |(-\Delta)^\alpha z(\theta_s\omega)|^2)dxds.
\end{aligned}
$$

After change of variables, for all $l \geq L_1(\omega) \geq 1$ we obtain

$$\int_{\mathbf{R}^3} \rho(\frac{|x|}{l})|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})|^2 dx \tag{4.38}$$

$$+ C(\alpha) \int_{-t}^{0} e^{\gamma\zeta} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\rho(\frac{|x|}{l})(v(x) - v(y))^2}{|x - y|^{3+2\alpha}} dxdyd\zeta$$

$$\leq e^{-\gamma t} \int_{\mathbf{R}^3} \rho(\frac{|x|}{l})|v_{\tau-t}|^2 dx + c \int_{-t}^{0} e^{\gamma(s-\tau)} \|v(s + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|_{H^\alpha}^2 ds$$

$$+ c \int_{-t}^{0} \int_{|x| \geq \frac{1}{2}l} e^{\gamma(s-\tau)} |g(s + \tau, x)|^2 dxds$$

$$+ c \int_{-t}^{0} \int_{|x| \geq \frac{1}{2}l} e^{\gamma(s-\tau)} (|z(\theta_s\omega)|^4 + |(-\Delta)^\alpha z(\theta_s\omega)|^2) dxds. \tag{4.39}$$

Now we estimate the first term on the right-hand side of (4.38). Since $B = \{B(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega\}$ is tempered, $v_{\tau-t} \in B(\tau - t, \theta_{-t}\omega)$, there exists $T_1 = T_1(\tau, \omega, B, \varepsilon) > 0$ such that for all $t \geq T_1$,

$$e^{-\gamma t} \int_{\mathbf{R}^3} \rho(\frac{|x|}{l})|v_{\tau-t}|^2 dx \leq \varepsilon. \tag{4.40}$$

For the second term, from Lemma 4.1 with $\sigma = \tau$, there exists $T_2 = T_2(\tau, \omega, B, \varepsilon) > 0$ such that for all $t \geq T_2$, we have

$$c \int_{-t}^{0} e^{\gamma(s-\tau)} \|v(s + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|_{H^\alpha}^2 ds \leq \varepsilon C(\tau, \omega). \tag{4.41}$$

For the third term, from (3.8)-(3.9) we find

$$c \int_{-\infty}^{0} \int_{\mathbf{R}^3} e^{\gamma s} |g(s + \tau, x)|^2 dxds < \infty. \tag{4.42}$$

Hence, there exists $L_2 = L_2(\tau, \omega, \varepsilon) \geq L_1$ such that for all $l \geq L_2$,

$$c \int_{-\infty}^{0} \int_{|x| \geq \frac{1}{2}l} e^{\gamma s} |g(s + \tau, x)|^2 dxds < \varepsilon. \tag{4.43}$$

For the last term, from (3.3) we have

$$c \int_{-\infty}^{0} \int_{\mathbf{R}^3} e^{\gamma s} (|z(\theta_s\omega)|^4 + |(-\Delta)^\alpha z(\theta_s\omega)|^2) dxds < \infty. \tag{4.44}$$

Hence, there exists $L_3 = L_3(\tau, \omega, \varepsilon) \geq L_2$ such that for all $l \geq L_3$,

$$c \int_{-\infty}^{0} \int_{|x| \geq \frac{1}{2}l} e^{\gamma s} (|z(\theta_s\omega)|^4 + |(-\Delta)^\alpha z(\theta_s\omega)|^2) dxds < \varepsilon. \tag{4.45}$$

By (4.38)-(4.45) we obtain, for all $l \geq L_3$ and $t \geq T_2$,

$$\int_{\mathbf{R}^3} \rho(\frac{|x|}{l})|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})|^2 dx$$

$$+C(\alpha)\int_{-t}^{0}e^{\gamma\zeta}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\rho(\frac{|x|}{l})(v(x)-v(y))^2}{|x-y|^{3+2\alpha}}dxdyd\zeta$$
$$\leq\varepsilon(3+C(\tau,\omega)). \tag{4.46}$$

From (4.46), the desired estimates follow immediately. □

In this following, we give the existence of tempered pullback absorbing set in H, and the asymptotic compactness of (1.1)-(1.2) in H.

Lemma 4.7. *Suppose* (3.8) *holds. Given* $\delta>0,\tau\in\mathbf{R}$ *and* $\omega\in\Omega$, *let*

$$K_\delta(\tau,\omega)=\{u\in H:\|u\|^2\leq c(r(\omega)+R_1(\tau,\omega))\},$$

where $R_1(\tau,\omega)$ *is the same number as in* (4.12). *Then* K_1 *is a closed measurable tempered pullback absorbing set of cocycle* Φ *in* H.

Proof. We first prove that K_δ absorbs every member B of $\mathcal{D}$. By (3.7) we have

$$u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})=v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})+\delta z(\theta_{-\tau}\omega). \tag{4.47}$$

If $u_{\tau-t}\in B(\tau-t,\theta_{-\tau}\omega)$, then by (4.47) we get $v_{\tau-t}+\delta z(\theta_{-\tau}\omega)\in D(\tau-t,\theta_{-\tau}\omega)$ which together with Lemma 4.2 implies that there exists $T=T(\tau,\omega,B,\delta)>0$ such that for all $t\geq T$,

$$v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\in B_1(\tau,\omega), \tag{4.48}$$

where $B_1(\tau,\omega)$ is the same as in (4.13). It follows from (4.47)-(4.48) and (4.13)-(4.14) that for all $t\geq T$,

$$\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\leq c(r(\omega)+R_1(\tau,\omega)). \tag{4.49}$$

On the other hand, by (3.7) we have

$$\Phi(t,\tau-t,\theta_{-t}\omega,u_{\tau-t})=u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}), \tag{4.50}$$

which along with (4.50) shows that $\Phi(t,\tau-t,\theta_{-t}\omega,u_{\tau-t})\in K_\delta$ for all $t\geq T$, and hence K_δ absorbs all elements of $\mathcal{D}$. We now prove K_δ is tempered, i.e., $K_\delta\in\mathcal{D}$. Note that $r(\omega)$ is tempered, by (4.47) we find that for every $c>0$

$$\lim_{t\to\infty}e^{-ct}\|K_\delta(\tau-t,\theta_{-t}\omega)\|=c\lim_{t\to\infty}e^{-ct}(r(\omega)+R_1(\tau,\omega))^{\frac{1}{2}}=0,$$

which implies that $K_\delta\in\mathcal{D}$. Note that $R_1(\tau,\omega)$ is measurable in $\omega\in\Omega$ and so in $K_\delta(\tau,\omega)$, which completes the proof. □

Next we first give the $\mathcal{D}$-pullback asymptotic compactness of the solutions of problem (3.5)-(3.6) in $L^2(\mathbf{R}^3)$.

Lemma 4.8. *Suppose* (3.8) *holds. Then for every* $\tau\in\mathbf{R},\omega\in\Omega$ *and* $B=\{B(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}\in\mathcal{D}$, *the sequence* $\{v(\tau,\tau-t_n,\theta_{-\tau}\omega,v_{0,n})\}_{n=1}^{\infty}$ *of solutions of problem* (3.5)-(3.6) *has a convergent subsequence in* $L^2(\mathbf{R}^2)$ *provided* $t_n\to\infty$, *and* $v_{0,n}\in B(\tau-t_n,\theta_{-t_n}\omega)$.

Proof. Let $t_n\to\infty$, $B=\{B(\tau,\omega)\}\in\mathcal{D}$ and $v_{0,n}\in B(\tau-t_n,\theta_{-t_n}\omega)$. Applying Lemma 4.1, for all $\omega\in\Omega$, we have $\{v(\tau,\tau-t_n,\theta_{-\tau}\omega,v_{0,n})\}_{n=1}^{\infty}$ is bounded in $L^2(\mathbf{R}^3)$. Therefore, there exists $\eta\in L^2(\mathbf{R}^3)$ and a subsequence, still denoted by $\{v(\tau,\tau-t_n,\theta_{-\tau}\omega,v_{0,n})\}_{n=1}^{\infty}$, such that

$$v(\tau,\tau-t_n,\theta_{-\tau}\omega,v_{0,n})\to\eta\ \text{ weakly in }\ L^2(\mathbf{R}^3),\text{for some }\ \eta=\eta(\omega)\in L^2(\mathbf{R}^3).$$

By Lemma 4.6, there is $T^* = T_B^*(\omega)$ and $l^* = l^*(\omega, \varepsilon)$ such that for all $t \geq T^*$,

$$\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_0)\|^2_{L^2(|x| \geq l^*)} \leq \varepsilon. \tag{4.51}$$

Since $t_n \to \infty$, there exists $N_1 = N_1(B, \omega, \varepsilon)$ such that $t_n \geq T_B^*$ for all $n \geq N_1$. By (4.51), for all $n \geq N_1$, one has

$$\|v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})\|^2_{L^2(|x| \geq l^*)} \leq \varepsilon.$$

By Lemma 4.2 and 4.4, there exists $T_{1_B} = T_{1_B}(\omega)$ such that for all $t \geq T_{1_B}$,

$$\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_0)\|^2_{H^\alpha(R^3)} \leq c(R_2(\tau, \omega) + r(\omega)). \tag{4.52}$$

Let $N_2 = N_2(B, \omega, \varepsilon)$ be large enough such that $t_n \geq T_{1_B}$ and $n \geq N_2$. It follows (4.52) that, for all $n \geq N_2$,

$$\|v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n})\|^2_{H^\alpha(\mathbf{R}^3)} \leq c(R_2(\tau, \omega) + r(\omega)), \tag{4.53}$$

and so a further subsequence converges weakly in $H^\alpha(\mathbf{R}^3)$, again to η. Thus, $\eta \in H^\alpha(\mathbf{R}^3)$. Let $\bar{B}_{l^*} = \{X \in \mathbf{R}^3 : |x| \leq l^*\}$. By the compactness of the embedding $H^\alpha(\bar{B}_{l^*}) \hookrightarrow L^2(\bar{B}_{l^*})$, together with (4.53), we obtain, up to a subsequence depending on l^*,

$$v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}) \to \eta \quad \text{strongly in } \ L^2(\bar{B}_{l^*}),$$

which implies that for given $\varepsilon > 0$, there exists $N_3 = N_3(B, \omega, \varepsilon)$ such that for all $n \geq N_3$,

$$\|v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}) - \eta\|^2_{L^2(\bar{B}_{l^*})} \leq \varepsilon. \tag{4.54}$$

Since $\eta \in L^2(\mathbf{R}^3)$, there exists $l^{**} = l^{**}(\omega) > 0$ such that

$$\int_{|x| \geq l^{**}} |\eta(x)|^2 dx \leq \varepsilon. \tag{4.55}$$

Let $l' = max\{l^*, l^{**}\}$ and $N' = \max\{N_1, N_3\}$. Then, from (4.53)-(4.55), we obtain that for all $n \geq N'$,

$$\begin{aligned}
\|v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}) - \eta\|^2_{L^2(R^3)} \leq & \|v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}) - \eta\|^2_{L^2(B_{l'})} \\
& + \|v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}) - \eta\|^2_{L^2(B^c_{l'})} \\
\leq & 5\varepsilon,
\end{aligned}$$

which implies $v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{0,n}) \to \eta$ strongly in $L^2(\mathbf{R}^3)$. This completes the proof. □

From Lemma 4.8, we immediately get the $\mathcal{D}$-pullback asymptotic compactness of the solutions of problem (1.1)-(1.2) in $L^2(\mathbf{R}^n)$.

Lemma 4.9. *Suppose* (3.8) *holds, for every* $\tau \in \mathbf{R}, \omega \in \Omega$ *and* $B = \{B(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega\}$, *the sequence* $\{\Phi(t_n, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})\}_{n=1}^\infty$ *of solutions of problem* (1.1)-(1.2) *has a convergent subsequence in* $L^2(\mathbf{R}^n)$ *provided* $t_n \to \infty$, *and* $u_{0,n} \in B(\tau - t_n, \theta_{-t_n}\omega)$.

Proof. By (3.4) and (3.7) we obtain

$$\begin{aligned}\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n}) &= u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{\tau - t_n}) \\ &= v(\tau, \tau - t_n, \theta_{-\tau}\omega, v_{\tau - t_n}) + \delta z(\omega) \\ &\text{with } u_{\tau - t_n} = v_{\tau - t_n} + \delta z(\theta_{-t_n}\omega)\end{aligned}$$

which along with Lemma 4.8 implies Lemma 4.9 directly. □

Now we give the existence of tempered pullback random attractors of Φ in H.

Theorem 4.10. *Suppose* (3.8) *and* $\alpha \in (\frac{3}{4}, 1)$ *hold. Then the cocycle* Φ *of problem* (1.1)-(1.2) *has a unique* $\mathcal{D}$*-pullback random attractor* $\mathcal{A}_\delta = \{\mathcal{A}_\delta(\tau, \omega) : \tau \in \mathbf{R}, \omega \in \Omega\} \in \mathcal{D}$ *in* H*.*

Proof. From [52, 54], based on Lemma 4.7, Lemma 4.9 and Theorem 2.4, the existence and uniqueness of the $\mathcal{D}$-pullback attractor $\mathcal{A}_\delta$ follows immediately. □

5. Upper semicontinuity of random attractors

In this section, we discuss the limiting behavior of random pullback attractors $\mathcal{A}_\delta$ of fractional stochastic Ginzburg-Landau equation (1.1) as the intensity of noise $\delta \to 0$. Throughout this section, we assume $\delta \in [0, 1]$, and write the cocycle of problem (1.1)-(1.2) as Φ_δ to indicate its dependence on δ. Then has a tempered pullback attractor $\mathcal{A}_\delta$ by Theorem 4.9, and has a tempered pullback absorbing set K_δ by Lemma 4.7. Given $\tau \in \mathbf{R}, \omega \in \Omega$, let

$$\tilde{R}(\tau, \omega) = c + c \int_{-\infty}^{0} (e^{\frac{1}{2}\gamma s} r(\omega) + e^{\gamma s}(\|g(s + \tau)\|^2) ds, \tag{5.1}$$

and

$$K(\tau, \omega) = \{u \in H : \|u\|^2 \leq c(r(\omega) + \tilde{R}(\tau, \omega))\}. \tag{5.2}$$

Thus we have

$$\|K(\tau - t, \theta_{-t}\omega)\|^2 \leq c(r(\omega) + \tilde{R}(\tau - t, \omega)),$$

and

$$\|\Phi(\tau - t, \theta_{-t}\omega)\|^2 \leq c(r(\omega) + \tilde{R}(\tau - t, \omega)). \tag{5.3}$$

By (3.7) and $r(\omega)$ is tempered, for every $\varepsilon > 0, \tau \in \mathbf{R}$ and $\omega \in \Omega$, there exists $T = T(\tau, \omega, \varepsilon) > 0$ such that for all $t > T$,

$$e^{-ct}\|K(\tau - t, \theta_{-t}\omega)\|^2 \leq \varepsilon. \tag{5.4}$$

From (5.4), we now prove the following uniform estimates on the tails of functions in random attractors.

Lemma 5.1. *Suppose* (3.8) *holds, then for every* $\varepsilon > 0$*,*$\tau \in \mathbf{R}$ *and* $\omega \in \Omega$*, there exists* $L = L(\tau, \omega, \varepsilon) \geq 1$ *such that for all* $l \geq L$*,*

$$\int_{|x| \geq l} |\xi(x)|^2 dx \leq \varepsilon \;\; for \;\; all \;\; \xi \in \bigcup_{0 < \delta \leq 1} \mathcal{A}_\delta(\tau, \omega).$$

Proof. From the proof of Lemma 4.1 and 4.3, by (5.4), we can verify that for every $\varepsilon > 0$, $\tau \in \mathbf{R}$ and $\omega \in \Omega$, there exists $T = T(\tau, \omega, \varepsilon) > 0$ and $L = L(\tau, \omega, \varepsilon) \geq 1$ such that for all $t > T, l \geq L$ and for all $\delta \in (0, 1]$, the solution u_δ of (1.1)-(1.2) satisfies

$$\int_{|x|\geq l} |u_\delta(\tau, \tau - t, \theta_{-t}\omega, u_{\delta,\tau-t})|^2 dx \leq \varepsilon, \tag{5.5}$$

where $u_{\delta,\tau-t} \in K(\tau - t, \theta_{-t}\omega)$ with K given by (5.2). By (5.2)-(5.5) we have

$$\bigcup_{0<\delta\leq 1} \mathcal{A}_\delta(\tau, \omega) \subseteq \bigcup_{0<\delta\leq 1} K_\delta(\tau, \omega) \subseteq K(\tau, \omega). \tag{5.6}$$

Let $\xi \in \mathcal{A}_\delta(\tau, \omega)$ for some $\delta \in (0, 1]$. By the invariance of $\mathcal{A}_\delta$, there exits $\zeta \in \mathcal{A}_\delta(\tau - T, \theta_{-T}\omega)$ such that $\xi = u_\delta(\tau, \tau - T, \theta_{-t}\omega, \zeta)$, which with (5.5) and (5.6) together implies that for all $l \geq L$,

$$\int_{|x|\geq l} |\xi(x)|^2 dx = \int_{|x|\geq l} |u_\delta(\tau, \tau - T, \theta_{-t}\omega, \zeta)|^2 dx \leq \varepsilon.$$

The proof is completed. $\square$

The limiting equation of (1.1) with $\delta = 0$ is given by

$$\frac{du}{dt} + (1 + i\lambda)(-\Delta)^\alpha u + \gamma u = -(1 + i\mu)|u|^2 u + g(t, x), \;\; x \in \mathbf{R}^3, t > \tau, \tag{5.7}$$

and initial condition

$$u(\tau, x) = u_\tau(x), \;\; x \in \mathbf{R}^3. \tag{5.8}$$

Similar to (1.1)-(1.2), one can prove that problem (5.7)-(5.8) generates a continuous cocycle Φ_0 in H. Moreover, has a unique tempered pullback attractor $\mathcal{A}_0 = \{\mathcal{A}_0(\tau), \tau \in \mathbf{R}\}$ in H and has a tempered pullback absorbing set $K_0 = \{K_0(\tau), \tau \in \mathbf{R}\}$ where $K_0(\tau)$ is given by

$$K_0(\tau) = \{u \in H : \|u\|^2 \leq R_0(\tau)\}, \tag{5.9}$$

and

$$R_0(\tau) = c + c\int_{-\infty}^{0} e^{\gamma s}\|g(s + \tau)\|^2 ds. \tag{5.10}$$

By Lemma 4.7 and (5.9)-(5.10) we get, for all $\tau \in \mathbf{R}$ and $\omega \in \Omega$,

$$\limsup_{\delta\to 0} \|K_\delta(\tau, \omega)\| \leq \|K_0(\tau)\|, \tag{5.11}$$

which will be used for proving the upper semicontinuity of $\mathcal{A}_\delta$. We also need the convergence of solutions of (1.1)-(1.2) as $\delta \to 0$.

Lemma 5.2. *Let $u_\delta(t, \tau, \omega, u_{\delta,\tau})$ and $u(t, \tau, u_\tau)$ be the solutions of* (1.1)-(1.2) *and* (5.7)-(5.8) *with initial data $u_{\delta,\tau}$ and u_τ, respectively. If $\lim_{\delta\to 0} u_{\delta,\tau} = u_\tau$, then for any $t \geq \tau$ and $\omega \in \Omega$,*

$$\lim_{\delta\to 0} u(t, \tau, \omega, u_{\delta,\tau}) = u(t, \tau, u_\tau).$$

Proof. Let v_δ be the solution of (3.5)-(3.6) and $W = v_\delta - u$. Then from (3.5) and (5.7) we have

$$\begin{aligned}&\frac{dW}{dt} + (1+i\lambda)(-\Delta)^\alpha W + \rho W\\ =&-\delta(1+i\lambda)(-\Delta)^\alpha z(\theta_t\omega) - (1+i\mu)|v_\delta + \delta z(\theta_t\omega)|^2(v_\delta + \delta z(\theta_t\omega)) + (1+i\mu)|u|^2 u).\end{aligned}$$

Multiplying the above equation with W and taking the real part, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|W\|^2 + \frac{1}{2}C(\alpha)\|W\|^2_{\dot H^\alpha} + \rho\|W\|^2\\ =&-Re\delta(1+i\lambda)((-\Delta)^\alpha z(\theta_t\omega), W)\\ &+ Re\int_{\mathbf{R}^3}(-(1+i\mu)|v_\delta + \delta z(\theta_t\omega)|^2(v_\delta + \delta z(\theta_t\omega)) + (1+i\mu)|u|^2u))\overline{W}dx. \end{aligned} \tag{5.12}$$

Now we will estimate the terms on the right-hand side of (5.12). For the first term, we have

$$-Re\delta(1+i\lambda)((-\Delta)^\alpha z(\theta_t\omega), W) \le \frac{\gamma}{4}\|W\|^2 + \frac{\delta^2(1+\lambda^2)}{\gamma}\|(-\Delta)^\alpha z(\theta_t\omega)\|^2. \tag{5.13}$$

For the nonlinear term we obtain

$$\begin{aligned}&-Re(1+i\mu)\int_{\mathbf{R}^3}(|v_\delta + \delta z(\theta_t\omega)|^2(v_\delta + \delta z(\theta_t\omega)) - |u|^2u)\overline{W}dx\\ =&-Re(1+i\mu)\int_{\mathbf{R}^3}(|v_\delta + \delta z(\theta_t\omega)|^2(v_\delta + \delta z(\theta_t\omega)) - |u+\delta z(\theta_t\omega)|^2(u + \delta z(\theta_t\omega)))\overline{W}dx\\ &-Re(1+i\mu)\int_{\mathbf{R}^3}(|u + \delta z(\theta_t\omega)|^2(u + \delta z(\theta_t\omega)) - |u|^2u)\overline{W}dx,\end{aligned} \tag{5.14}$$

After simple calculations, we find from (5.14) that for any $\tau \in \mathbf{R}, \omega \in \Omega, T > 0$ and $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\tau, \omega, T, \varepsilon) > 0$ such that for all $\delta \in (0, \delta_0)$ and $t \in [\tau, \tau + T]$,

$$\begin{aligned}&-Re(1+i\mu)\int_{\mathbf{R}^3}(|v_\delta + \delta z(\theta_t\omega)|^2(v_\delta + \delta z(\theta_t\omega)) - |u|^2u)\overline{W}dx\\ \le& c\|W\|^2 + c\varepsilon + c\varepsilon\int_{\mathbf{R}^3}(|u|^4 + |v_\delta|^4)dx.\end{aligned} \tag{5.15}$$

It follows from (5.13) and (5.15) that there exists $\delta_1 > 0$ such that for all $\delta \in (0, \delta_1)$ and $t \in [\tau, \tau + T]$,

$$\frac{d}{dt}\|W\|^2 \le c\|W\|^2 + c\varepsilon + c\varepsilon\int_{\mathbf{R}^3}(|u|^4 + |v_\delta|^4)dx. \tag{5.16}$$

Solving (5.16) on $[\tau, \tau + T]$ we have, for all $\delta \in (0, \delta_1)$ and $t \in [\tau, \tau + T]$,

$$\|W(t)\|^2 \le c\|W(\tau)\|^2 + c\varepsilon + c\varepsilon\int_\tau^t(\|u(s,\tau,u_\tau)\|^4_{L^4(\mathbf{R}^3)} + \|v_\delta(s,\tau,\omega,v_{\delta,\tau})\|^4_{L^4(\mathbf{R}^3)})dx. \tag{5.17}$$

On the other hand, by (4.11) we find that for all $\delta \in (0, 1)$ and $t \in [\tau, \tau + T]$,

$$\|v_\delta(t,\tau,\omega,v_{\delta,\tau})\|^2 + \int_\tau^t \|v_\delta(s,\tau,\omega,v_{\delta,\tau})\|^4_{L^4(\mathbf{R}^3)})ds \le c + c\|v_{\delta,\tau}\|^2. \tag{5.18}$$

Similarly, we also get for all $t \in [\tau, \tau + T]$,

$$\|u(t,\tau,u_\tau)\|^2 + \int_\tau^t \|u(s,\tau,u_\tau)\|^4_{L^4(\mathbf{R}^3)})ds \le c + c\|u_\tau\|^2. \tag{5.19}$$

Let $\delta_2 = \min\{1, \delta_1\}$. By (5.18)-(5.19) we have, for all $\delta \in (0, \delta_2)$ and $t \in [\tau, \tau + T]$,

$$\|v_\delta(t,\tau,\omega,v_{\delta,\tau}) - u(t,\tau,u_\tau)\|^2 \le c\|v_{\delta,\tau} - u_\tau\|^2 + c\varepsilon + c\varepsilon(\|v_{\delta,\tau}\|^2 + \|u_\tau\|^2). \tag{5.20}$$

Due to $v_{\delta,\tau} = u_{\delta,\tau} - \delta z(\theta_\tau\omega)$, we have from (5.20) that if $\lim_{\delta\to 0} v_{\delta,\tau} = u_\tau$ then for all $t \in [\tau, \tau + T]$,

$$\lim_{\delta\to 0} v_\delta(t,\tau,\omega,v_{\delta,\tau}) = u(t,\tau,u_\tau)$$

which along with (3.4) indicates $\lim_{\delta\to 0} u_\delta(t,\tau,\omega,v_{\delta,\tau}) = u(t,\tau,u_\tau)$. □

Lemma 5.3. *Suppose* (3.8) *holds. Let* $\tau \in \mathbf{R}$ *and* $\omega \in \Omega$ *be fixed. If* $\delta_n \to 0$ *and* $u_n \in \mathcal{A}_{\delta_n}(\tau,\omega)$, *then the sequence* $\{u_n\}_{n=1}^\infty$ *is precompact in* H.

Proof. The proof is standard based on argument of [55]. Let K be the family of subsets of $L^2(\mathbf{R}^3)$ given by (5.2). Then it follows from Lemma 5.1 that there exist $T = (\tau,\omega,\varepsilon) \ge 1$ and $l = (\tau,\omega,\varepsilon) \ge 1$ such that for all $t \ge T$ and $\delta \in (0, 1]$

$$\int_{|x|\ge l} |u_\delta(\tau, \tau - t, \theta_{-t}\omega, u_{\delta,\tau-t})(x)|^2 dx \le \frac{1}{2}\varepsilon, \tag{5.21}$$

where $u_{\delta,\tau-t} \in K(\tau - t, \theta_{-t}\omega)$. By (5.6) and (5.21), we get from the invariance of $\mathcal{A}_\delta$ that, for each $\tau \in \mathbf{R}$ and $\omega \in \Omega$,

$$\int_{|x|\ge l} |u(x)|^2 dx \le \frac{1}{2}\varepsilon, \text{ for all } u \in \mathcal{A}_\delta(\tau,\omega) \text{ with } 0 < \delta \le 1. \tag{5.22}$$

By (5.3), we see that the set $\bigcup_{0<\delta\le 1} \mathcal{A}_\delta(\tau,\omega)$ is bounded in $H^\alpha(U)$ with $U = \{x \in \mathbf{R}^3 : |x| < l\}$. Then the compactness of embedding $H^\alpha(U) \hookrightarrow L^2(U)$ implies that the set $\bigcup_{0<\delta\le 1} \mathcal{A}_\delta(\tau,\omega)$ has a finite covering of balls of radii less than $\frac{1}{2}\varepsilon$ in $L^2(U)$, which along with (5.22) completes the proof. □

Now we are in position to present the upper semicontinuity of random attractors $\mathcal{A}_\delta$ as $\delta \to 0$.

Theorem 5.4. *Assume that* (3.8) *holds. Then for every* $\tau \in R$ *and* $\omega \in \Omega$,

$$\lim_{\delta\to 0} dist_H(A_\delta(\tau,\omega), A_0(\tau)) = 0. \tag{5.23}$$

Proof. This is an direct consequence of Theorem 3.2 in [52] from (5.11), Lemma 5.2 and 5.3. □

Acknowledgement

The authors would like to thank the reviewers for their helpful comments.

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