

# DYNAMICAL BEHAVIOR ANALYSIS OF A TWO-DIMENSIONAL DISCRETE PREDATOR-PREY MODEL WITH PREY REFUGE AND FEAR FACTOR

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**Abstract** This paper investigates the dynamics of an improved discrete Leslie-Gower predator-prey model with prey refuge and fear factor. First, a discrete Leslie-Gower predator-prey model with prey refuge and fear factor has been introduced. Then, the existence and stability of fixed points of the model are analyzed. Next, the bifurcation behaviors are discussed, both flip bifurcation and Neimark-Sacker bifurcation have been studied. Finally, some simulations are given to show the effectiveness of the theoretical results.

**Keywords** Discrete predator-prey model, fear factor, fixed points, flip bifurcation, Neimark-Sacker bifurcation.

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## 1. Introduction

Since the significant works about predator-prey model by Lotka [19] and Volterra [31], the dynamical behaviors of predator-prey model have attracted many researchers in the fields of mathematics and biology [12, 21]. Many kinds of improvements for the predator-prey model have been proposed, lots of results have been published in recent years [1, 3, 10, 20, 29, 36]. Among the variety of models, the traditional two-dimensional predator-prey models still getting a lot of attention, in which, interactions between prey and predator species in both population dynamics and mathematical ecology can be described [24, 38].

The Leslie-Gower predator-prey model [20] has been studied by many scholars recently. For example, stability and bifurcations of Leslie-Gower predator-prey model have been investigated in [4, 8, 18]. In [9], the authors have investigated the spatiotemporal dynamics of a Leslie-Gower predator-prey model incorporating a prey refuge subject to the Neumann boundary conditions. A Leslie-Gower model with the influence of the prey refuge can be described as following

$$\begin{cases} \frac{dx(t)}{dt} = x(t)(r - ax(t)) - \frac{(1-m)x(t)y(t)}{b + (1-m)x(t)}, \\ \frac{dy(t)}{dt} = y(t) \left( \mu - \frac{cy(t)}{b + (1-m)x(t)} \right), \end{cases} \quad (1.1)$$

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where  $x(t)$  and  $y(t)$  are prey and predator population densities at time  $t$ , respectively.  $r$ ,  $a$ ,  $b$ ,  $c$  and  $\mu$  are model parameters assumed as only positive values. Specifically,  $r$  is the growth rate of preys,  $\mu$  is the growth rate of predators,  $a$  is the strength of competition among individuals of species,  $b$  is the half-saturation constant (i.e., the substrate concentration at which the rate of product formation is half maximal), and  $c$  is the maximum value per capita reduction of prey due to predator. A portion  $m$  of the prey population is completely protected from predation and the rest portion  $(1-m)x(t)$  is available to the predator, where  $m \in (0, 1)$  is a constant that measures refuge availability. Biologically, a refuge can help prolong a predator-prey interaction by reducing the chance of extinction due to predation.

Most of the results have been considered only the direct impact of predator population on prey species, see for example [11, 37]. However, the indirect impact of predator species on prey species also has significant effects on the population dynamics from many field data (see [5, 7, 22, 23]). For example, only direct killing is observable in nature but all the prey can respond to the perceived predation risk and they show different anti-predator behaviours like new selection of habitat and several psychological changes to protect themselves (see [6, 26–28, 30]).

In 2001, by lots of experiments, Zanette et al. [39] found that there is a 40% reduce on in offspring production of the sparrows on account of fear from the predator. Then in order to show the effects on predator-prey model of fear factor, Wang et al. [33] first introduced the fear factor  $F(k, y)$ , which accounts the cost of anti-predator defense due to fear about predator-prey model. Based on previous research, Wang et al. [34] studied a new model with fear factor as following

$$\begin{cases} \frac{dx(t)}{dt} = \frac{x(t)}{1+ky(t)}(r-ax(t)) - \frac{(1-m)x(t)y(t)}{b+(1-m)x(t)}, \\ \frac{dy(t)}{dt} = y(t) \left( \mu - \frac{cy(t)}{b+(1-m)x(t)} \right), \end{cases} \quad (1.2)$$

where  $k$  refers to the degree of fear, which is due to the anti-predator response of prey.

For populations with overlapping generations, the reproductive process is continuous, so the interactions of prey and middle predator and top predator are usually described by ordinary differential equations. But many species have non-overlapping generations or born in the normal breeding seasons. So the interactions of the species can be described by difference equations or discrete time maps. The discrete predator-prey models show more complex dynamics than the continuous models (see e.g., Irfan. [15]) and also can provide efficient computational models of continuous models for numerical simulations [13]. The first pioneer discrete predator-prey model is known as Nicholson and Bailey model [25], which is the classical model for predator-prey interactions (Kot [16]). This model can generates large oscillations which can drive both species to extinction. Beddington introduced self-limitation to the prey population and proposed a new model (Beddington et al. [2]) to stabilize the model. Though much researches have been seen in the Leslie-Gower predator-prey model, such models are not well studied in the sense that most results are only continuous time cases related. However, a little work has been done for the discrete Leslie-Gower predator-prey model. So, we study the discrete Leslie-Gower predator-prey model in this paper.

Motivating by the above discussions, this paper will consider the discrete-time

model (1.2). According to the forward Euler scheme [14], let

$$\frac{dx}{dt} = \frac{x_{t+1} - x_t}{h}, \quad \frac{dy}{dt} = \frac{y_{t+1} - y_t}{h},$$

where  $x_t$  and  $y_t$  are the densities of the prey and predator populations in discrete time (generation)  $t$ . Moreover, let  $h \rightarrow 1$ . We have the equations for the  $(t + 1)$ th generation of the prey and predator populations

$$\begin{cases} x_{t+1} = x_t + \frac{x_t}{1 + ky_t}(r - ax_t) - \frac{(1 - m)x_t y_t}{b + (1 - m)x_t}, \\ y_{t+1} = y_t \left( 1 + \mu - \frac{cy_t}{b + (1 - m)x_t} \right), \end{cases} \quad (1.3)$$

rewriting (1.3) as a map, we obtain the discrete predator-prey model with fear factor

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x + \frac{x}{1 + ky}(r - ax) - \frac{(1 - m)xy}{b + (1 - m)x}, \\ y \left( 1 + \mu - \frac{cy}{b + (1 - m)x} \right), \end{pmatrix}, \quad (1.4)$$

where  $a, b, c, r, \mu, m$  and  $k$  are all positive constants, which have been introduced above.

The contributions of this paper contain the following aspects: (1) the extinction state and coexistence state analysis of predator and prey by the calculation of fixed points, (2) giving the dynamics behavior analysis between predator and prey by bifurcation. Using the biological meaning of the model variables, we only consider model (1.4) in the plane  $\Omega = \{(x, y) : x \geq 0, y \geq 0\}$ .

The rest of this paper is organized as follows. In section 2, we get four fixed points and obtain the stability of every fixed points about map (1.4) by complicated calculation. In section 3, we discuss bifurcations of codimension 1 at every fixed points about map (1.4), including flip bifurcation, Neimark-Sacker bifurcation. Numerical simulations which can prove our conclusions are given in section 4.

## 2. The existence and stability of fixed points

In this section, we present results on the existence and stability of fixed points of the discrete model (1.4). For simplicity, we define

$$\theta := 1 - m \in (0, 1).$$

By simple calculations, one can see that the map (1.4) has at most four fixed points under various conditions:

- (1) The extinction state of total population  $E_0(0, 0)$ ;
- (2) The extinction state of the predator (or prey-only)  $E_1(\frac{r}{a}, 0)$ ;
- (3) The extinction state of the prey (or predator-only)  $E_2(0, \frac{b\mu}{c})$ ;
- (4) The coexistence state of the prey and predator  $E_3(x_1, y_1)$  exists if  $\theta < \frac{cr}{\mu}$  and  $k < \frac{c(cr - \mu\theta)}{b\mu^2\theta}$ , where

$$x_1 = \frac{c(cr - \mu\theta) - \theta\mu^2bk}{k\mu^2\theta^2 + ac^2}, \quad y_1 = \frac{b\mu}{c} + \frac{\mu\theta x_1}{c}.$$

The Jacobian matrix of map (1.4) at any point  $J(x, y)$  is given by

$$J(x, y) = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}, \quad (2.1)$$

where

$$\begin{aligned} j_{11} &= 1 - \frac{2ax - r}{ky + 1} - \frac{\theta by}{(\theta x + b)^2}, \quad j_{12} = \frac{kx(ax - r)}{(ky + 1)^2} - \frac{\theta x}{\theta x + b}, \\ j_{21} &= \frac{c\theta y^2}{(\theta x + b)^2}, \quad j_{22} = 1 + \mu - \frac{2cy}{\theta x + b}. \end{aligned} \quad (2.2)$$

Here the determinant of  $A$  can be expressed as  $|A|$ . Then, the characteristic equation of the Jacobian matrix  $J(x, y)$  can be written as

$$|J - \lambda E| = \lambda^2 - (j_{11} + j_{22})\lambda + j_{11}j_{22} - j_{12}j_{21} = 0. \quad (2.3)$$

We obtain the main results about the locally asymptotical stability about every fixed point of map (1.4) as following.

**Theorem 2.1.** *For the trivial fixed point  $E_0(0, 0)$  and the semitrivial fixed points  $E_1(\frac{r}{a}, 0)$ ,  $E_2(0, \frac{b\mu}{c})$ .*

- (1)  $E_0(0, 0)$  always is a source;
- (2)  $E_1(\frac{r}{a}, 0)$  always is a saddle;
- (3) If  $k = \frac{c^2 r - c\mu\theta}{b\mu^2\theta}$ ,  $E_2(0, \frac{b\mu}{c})$  is non-hyperbolic; if  $k < \frac{c^2 r - c\mu\theta}{b\mu^2\theta}$ ,  $E_2(0, \frac{b\mu}{c})$  is unstable; if  $k > \frac{c^2 r - c\mu\theta}{b\mu^2\theta}$ ,  $E_2(0, \frac{b\mu}{c})$  is stable.

**Proof.** The Jacobian matrix at fixed point  $E_0(0, 0)$  is

$$J(0, 0) = \begin{pmatrix} r + 1 & 0 \\ 0 & \mu + 1 \end{pmatrix}, \quad (2.4)$$

then, let

$$|J(0, 0) - \lambda E| = (r + 1 - \lambda)(\mu + 1 - \lambda) = 0,$$

obviously, it has eigenvalues  $\lambda_1 = r + 1 > 1$ ,  $\lambda_2 = \mu + 1 > 1$ , so we can obtain  $E_0(0, 0)$  is a source.

The Jacobian matrix at fixed point  $E_1(\frac{r}{a}, 0)$  is

$$J\left(\frac{r}{a}, 0\right) = \begin{pmatrix} 1 - r - \frac{-\theta r}{ab + r\theta} \\ 0 & 1 + \mu \end{pmatrix}, \quad (2.5)$$

then

$$\left|J\left(\frac{r}{a}, 0\right) - \lambda E\right| = (1 - r - \lambda)(1 + \mu - \lambda),$$

obviously, it has eigenvalues  $\lambda_1 = 1 - r < 1$ ,  $\lambda_2 = \mu + 1 > 1$ , so we can obtain  $E_1(\frac{r}{a}, 0)$  always is a saddle.

The Jacobian matrix at fixed point  $E_2(0, \frac{b\mu}{c})$  is

$$J\left(0, \frac{b\mu}{c}\right) = \begin{pmatrix} \frac{-bk\mu^2\theta + bck\mu + c^2r - c\mu\theta + c^2}{(bk\mu + c)c} & 0 \\ \frac{\mu^2\theta}{c} & 1 - \mu \end{pmatrix}, \tag{2.6}$$

then

$$\left|J\left(0, \frac{b\mu}{c}\right) - \lambda E\right| = \left(\frac{-bk\mu^2\theta + bck\mu + c^2r - c\mu\theta + c^2}{(bk\mu + c)c} - \lambda\right)(1 - \mu - \lambda),$$

obviously, it has eigenvalues

$$\lambda_1 = \frac{-bk\mu^2\theta + bck\mu + c^2r - c\mu\theta + c^2}{(bk\mu + c)c}, \quad \lambda_2 = 1 - \mu < 1,$$

then if  $\lambda_1 < 1$ ,  $E_2$  is stable; if  $\lambda_1 = 1$ ,  $E_2$  is non-hyperbolic; if  $\lambda_1 > 1$ ,  $E_2$  is unstable. So we can obtain  $E_2(0, \frac{b\mu}{c})$  is non-hyperbolic when  $k = \frac{c^2r - c\mu\theta}{b\mu^2\theta}$ ; if  $k < \frac{c^2r - c\mu\theta}{b\mu^2\theta}$ ,  $E_2(0, \frac{b\mu}{c})$  is unstable; if  $k > \frac{c^2r - c\mu\theta}{b\mu^2\theta}$ ,  $E_2(0, \frac{b\mu}{c})$  is stable.  $\square$

To study the stability for the map (1.4) about  $E_3$ , we need the following Lemma (see [38]).

**Lemma 2.1.** *Let  $F(\lambda) = \lambda^2 + Q\lambda + S$ , where  $Q$  and  $S$  are two real constants. Suppose that  $F(1) > 0$ ,  $\lambda_1$  and  $\lambda_2$  are two roots of  $F(\lambda) = 0$ . Then*

- (1)  $-1 < \lambda_{1,2} < 1$ , if and only if  $-2 < Q < 2$ ,  $Q^2 \geq 4S$  and  $F(-1) > 0$ ;
- (2)  $\lambda_{1,2} > 1$ , if and only if  $Q < -2$  and  $Q^2 \geq 4S$ ;
- (3)  $\lambda_{1,2} < -1$ , if and only if  $Q > 2$ ,  $Q^2 \geq 4S$  and  $F(-1) > 0$ ;
- (4)  $\lambda_1 < -1$ ,  $-1 < \lambda_2 < 1$ , if and only if  $F(-1) < 0$ ;
- (5)  $\lambda_1 = -1$ ,  $\lambda_2 \neq -1$ , if and only if  $Q \neq 2$ ,  $F(-1) = 0$ ;
- (6)  $\lambda_{1,2} = -1$ , if and only if  $Q = 2$ ,  $F(-1) = 0$ .

**Theorem 2.2.** *For map (1.4) about  $E_3$ .*

- (i) *If  $-2 < C_1 < 2$ ,  $C_1^2 \geq 4C_2$ , however, we promise  $\lambda_1$  and  $\lambda_2$  exist first. The fixed point  $E_3(x_1, y_1)$  is locally asymptotically stable, if (a) or (b) holds*

$$\begin{aligned} \text{(a)} \quad & b\delta(k)\mu^2\theta^3 > bc\mu^2\delta(k)\theta, \\ & k > \frac{c(\delta(k)\mu^3\theta^3 - cr\delta(k)\mu^2\theta^2 - c\mu^2s(k)\theta + c^2r\mu s(k) + 2\gamma(k))}{(b\mu^3\theta(-\delta(k)\mu\theta^2 + cs(k)))}; \\ \text{(b)} \quad & b\delta(k)\mu^2\theta^3 < bc\mu^2\delta(k)\theta, \\ & k < \frac{c(\delta(k)\mu^3\theta^3 - cr\delta(k)\mu^2\theta^2 - c\mu^2s(k)\theta + c^2r\mu s(k) + 2\gamma(k))}{(b\mu^3\theta(-\delta(k)\mu\theta^2 + cs(k)))}; \end{aligned}$$

- (ii) *If  $-2 < C_1 < 2$ ,  $C_1^2 \geq 4C_2$ , however, we promise  $\lambda_1$  and  $\lambda_2$  exist first. The fixed point  $E_3(x_1, y_1)$  is non-hyperbolic, if*

$$k = \frac{c(\delta(k)\mu^3\theta^3 - cr\delta(k)\mu^2\theta^2 - c\mu^2s(k)\theta + c^2r\mu s(k) + 2\gamma(k))}{(b\mu^3\theta(-\delta(k)\mu\theta^2 + cs(k)))},$$

where

$$\begin{aligned} C_1 &= j_{11} + j_{22}, \quad C_2 = -j_{12}j_{21} + j_{11}j_{22}, \\ \delta(k) &= ac + k\mu(2abc + \theta(2cr - \mu\theta)), \\ \gamma(k) &= c^2(abc + \theta m_0)(ac + k\mu(ab + r\theta)), \\ s(k) &= \mu^2\theta^2(ab + r\theta)k - ac(abc + \theta m_1) < -\frac{c(ab - r\theta)(abc + \theta m_0)}{b}. \end{aligned} \quad (2.7)$$

**Proof.** The Jacobian matrix at fixed point  $E_3(x_1, y_1)$  is

$$J(x_1, y_1) = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}, \quad (2.8)$$

let

$$m_0 = cr - \mu\theta, \quad m_1 = cr - 2\mu\theta, \quad m_2 = c\mu + cr - 2\mu\theta, \quad (2.9)$$

where

$$\begin{aligned} j_{11} &= \frac{c(cr - \mu\theta) - \theta\mu^2bk}{\gamma(k)}s(k) + 1, \\ j_{12} &= -\frac{(c(cr - \mu\theta) - \theta\mu^2bk)\theta}{\gamma(k)}\delta(k) < 0, \\ j_{21} &= \frac{\mu^2\theta}{c}, \quad j_{22} = 1 - \mu, \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} \delta(k) &= ac + k\mu(2abc + \theta(2cr - \mu\theta)), \\ \gamma(k) &= c^2(abc + \theta m_0)(ac + k\mu(ab + r\theta)), \\ s(k) &= \mu^2\theta^2(ab + r\theta)k - ac(abc + \theta m_1) < -\frac{c(ab - r\theta)(abc + \theta m_0)}{b}, \end{aligned} \quad (2.11)$$

let

$$F(\lambda) = |J(x_1, y_1) - \lambda E| = \lambda^2 - C_1\lambda + C_2, \quad (2.12)$$

where

$$C_1 = j_{11} + j_{22}, \quad C_2 = -j_{12}j_{21} + j_{11}j_{22}, \quad (2.13)$$

then, we obtain

$$F(-1) = \frac{(b\delta(k)\mu^4\theta^3 - bc\mu^3s(k)\theta)k - c^2\delta(k)\mu^2r\theta^2 + c\delta(k)\mu^3\theta^3 + c^3\mu rs(k) - c^2\mu^2s(k)\theta + 2c\gamma(k)}{\gamma(k)c}. \quad (2.14)$$

According to Lemma 2.1, we know the fixed point is locally asymptotically stable if and only if  $-2 < C_1 < 2$ ,  $C_1^2 \geq C_2$  and  $F(-1) > 0$  and the fixed point is non-hyperbolic if and only if  $F(-1) = 0$ . Then by calculation about (2.14), we get Theorem 2. This completes our proof.  $\square$

### 3. Flip bifurcation and Neimark-Sacker bifurcation

This section deals with the study of the flip bifurcation about the map (1.4) at  $E_2$  and the Neimark-Sacker bifurcation about the map (1.4) at the positive fixed point  $E_3$ .

First, we give the explanation of the flip bifurcation and the Neimark-sacker bifurcation, respectively [17]. For the system with parameters as following

$$x \mapsto f(x, \mu), \tag{3.1}$$

where  $x \in \mathbb{R}^n$ ,  $\mu \in \mathbb{R}^m$ ,  $f$  is a smooth function of  $x$  and  $\mu$ .

**Definition 3.1.** When the parameter  $\mu$  in system (3.1) changes continuously, if an eigenvalue of the jacobian matrix of fixed point  $x_0$  is  $\lambda = -1$ , when  $\mu = \mu_0$ , thus the map is going to produce flip bifurcation.

Flip bifurcation in the discrete system: in the process of changing the system parameters, iteration is carried out from any initial value. As the number of iterations increasing, the system switches to a new behavior twice as long as the previous period without bifurcation. In continuous systems, the flip bifurcation is often called the double periodic bifurcation.

The Neimark-Sacker bifurcation is to study the limit cycle in the neighborhood of fixed point. When the linearized matrix  $Df(x_0, \mu_0)$  of fixed point  $x_0$  has a pair of complex eigenvalues with module 1. As long as the parameter can change the stability of the fixed point, the limit cycle can often be generated. In continuous systems, the Neimark-Sacker bifurcation is often called the hopf bifurcation.

**Theorem 3.1.** Map (1.4) undergoes a flip bifurcation around  $E_2$  when  $k = \frac{c^2r - c\mu\theta}{b\mu^2\theta}$ .

**Proof.** In order to analyze flip bifurcation at the fixed point  $E_2$ , let  $k$  as bifurcation parameter. Let  $u = x$ ,  $v = y - \frac{b\mu}{c}$  and  $w = k - \frac{c^2r - c\mu\theta}{b\mu^2\theta}$ , we transform the fixed point  $E_2(0, \frac{b\mu}{c})$  to the origin and expand it. Thus, the map (1.4) becomes

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} u - \frac{\mu\theta(ab - r\theta)}{crb}u^2 + \frac{\theta(\mu\theta - 2cr)}{uv} - \frac{b\theta^2\mu^3}{c^3r}uw + O(|u, v, w|^3) \\ \frac{\mu^2\theta}{c}u + (1 - \mu)v - \frac{\mu^2\theta^2crb}{bc}u^2 - \frac{c}{b}v^2 + \frac{2\mu\theta}{b}uv + O(|u, v, w|^3) \end{pmatrix}, \tag{3.2}$$

where  $w$  is the new variable and is sufficient small.

Consider the following map

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \rightarrow \begin{pmatrix} u - \frac{\mu\theta(ab - r\theta)}{crb}u^2 + \frac{\theta(\mu\theta - 2cr)}{uv} - \frac{b\theta^2\mu^3}{c^3r}uw + O(|u, v, w|^3) \\ \frac{\mu^2\theta}{c}u + (1 - \mu)v - \frac{\mu^2\theta^2crb}{bc}u^2 - \frac{c}{b}v^2 + \frac{2\mu\theta}{b}uv + O(|u, v, w|^3) \\ w \end{pmatrix}, \tag{3.3}$$

linearizing map (3.3) at  $(0, 0, 0)$ , we obtain the associated Jacobian matrix

$$J = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\mu^2\theta}{c} & 1 - \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.4)$$

let

$$T = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\mu\theta}{c} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.5)$$

and using the transformation

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = T \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \quad (3.6)$$

then the map (1.4) becomes

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} f_1(X, Y, Z) \\ f_2(X, Y, Z) \\ 0 \end{pmatrix}, \quad (3.7)$$

where

$$\begin{aligned} f_1(X, Y, Z) &= -\frac{\mu\theta(abc - 3cr\theta + \mu\theta^2)}{c^2rb}X^2 - \frac{\theta(2cr - \mu\theta)}{crb}XY - \frac{b\theta^2\mu^2}{c^3r}XZ, \\ f_2(X, Y, Z) &= -\frac{c}{b}Y^2. \end{aligned} \quad (3.8)$$

By the center manifold theory, the stability of  $(X, Y) = (0, 0)$  near  $Z = 0$  can be determined by studying a one-parameter family of reduced equations on a center manifold, which can be represented as follows

$$\{W^c(0) = (X, Y, Z) \in R^3 \mid Y = h(X, Z), h(0, 0) = 0, Dh(0, 0) = 0\},$$

for  $X$  and  $Z$  sufficiently small.

We assume that  $h(X, Z)$  takes the form

$$h(X, Z) = h_1X^2 + h_2XZ + h_3Z^2 + O(|X, Z|^3), \quad (3.9)$$

then  $h(X, Z)$  must satisfy

$$h(X + f_1(X, h(X, Z), Z), Z) - (1 - \mu)h(X, Z) - f_2(X, h(X, Z), Z) = 0. \quad (3.10)$$

By calculate, we obtain

$$h_1 = h_2 = h_3 = 0, \tag{3.11}$$

the map restricted to the center manifold is given by

$$X \rightarrow \tilde{f}(X, Z) = X - \frac{\mu\theta(abc - 3cr\theta + \mu\theta^2)}{c^2rb} X^2 - \frac{b\theta^2\mu^2}{c^3r} XZ. \tag{3.12}$$

It is easy to see that

$$\begin{aligned} \frac{\partial \tilde{f}}{\partial X}(0, 0) &= 1, \quad -3 \left( \frac{\partial^2 \tilde{f}}{\partial X^2}(0, 0) \right)^2 - 2 \frac{\partial^3 \tilde{f}}{\partial X^3}(0, 0) = \frac{6\mu\theta(abc - 3cr\theta + \mu\theta^2)}{c^2rb}, \\ \frac{\partial^2 \tilde{f}}{\partial Z \partial X}(0, 0) &= \frac{-b\theta^2\mu^2}{c^3r}. \end{aligned}$$

Therefore, by [17], map (1.4) undergoes a flip bifurcation at the fixed point  $E_2$  when  $k = \frac{c^2r - c\mu\theta}{b\mu^2\theta}$ . □

To study the Neimark-Sacker for map (1.4) about  $E_3$ , we need the following explicit criterion (see e.g., Wen [32], Yao [35]).

**Lemma 3.1.** *Considering an  $n$ -dimensional discrete dynamical system  $X_{s+1} = f_r(X_s)$ , where  $r \in \mathbb{R}$  is a bifurcation parameter. Let  $X^*$  be a fixed point of  $f_r$  and the characteristic polynomial for Jacobian matrix  $J(X^*) = (c_{ij})_{n \times n}$  of  $n$ -dimensional map  $f_r(X_s)$  is given by:*

$$P_\lambda = \lambda^n + c_1\lambda^{n-1} + \dots + c_{n-1}\lambda + c_n,$$

where  $c_i = c_i(r, u)$ ,  $i = 1, 2, 3, \dots, n$  and  $u$  is control parameter or another parameter to be determined. Let  $\Delta_0^\pm(r, u) = 1, \Delta_1^\pm(r, u), \dots, \Delta_n^\pm(r, u)$  be a sequence of determinants defined by  $\Delta_i^\pm(r, u) = \det(M_1 \pm M_2)$ ,  $i = 1, 2, 3, \dots, n$ , where

$$M_1 = \begin{pmatrix} 1 & c_1 & c_2 & \dots & c_{i-1} \\ 0 & 1 & c_1 & \dots & c_{i-2} \\ 0 & 0 & 1 & \dots & c_{i-3} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, M_2 = \begin{pmatrix} c_{n-i+1} & c_{n-i+2} & \dots & c_{n-1} & c_n \\ c_{n-i+2} & c_{n-i+3} & \dots & c_n & 0 \\ \dots & \dots & \dots & \dots & \dots \\ c_{n-1} & c_n & \dots & 0 & 0 \\ c_n & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.13}$$

Moreover, the following conditions hold:

- H1** *Eigenvalue assignment:*  $\Delta_{n-1}^-(r_0, u) = 0, \Delta_{n-1}^+(r_0, u) > 0,$   
 $P_{r_0}(1) > 0, (-1)^n P_{r_0}(-1) > 0, \Delta_{n-1}^\pm(r_0, u) > 0, i = n - 3, n - 5, \dots, 1(\text{or } 2),$   
 when  $n$  is even or odd, respectively.
- H2** *Transversality condition:*  $\left[ \frac{d(\Delta_{n-1}^-(r, u))}{dr} \right]_{r=r_0} \neq 0.$
- H3** *Non-resonance condition:*  $\cos(2\pi/m) \neq \psi$ , or resonance condition  $\cos(2\pi/m) = \psi$ , where  $m = 3, 4, 5, \dots$ , and

$$\psi = -1 + 0.5P_{r_0}(1) \Delta_{n-3}^-(r_0, u) / \Delta_{n-2}^+(r_0, u).$$

Then, Neimark-Sacker bifurcation occurs at  $r_0$ .

The following result shows that map (1.4) undergoes Neimark-Sacker bifurcation if we choose  $k$  as bifurcation parameter.

**Theorem 3.2.** *The map (1.4) undergoes Neimark-Sacker bifurcation at fixed point  $E_3$ , if the following conditions hold*

$$\begin{aligned} 1 - C_1 + C_2 &> 0, \\ 1 + C_1 + C_2 &> 0, \end{aligned} \tag{3.14}$$

where  $C_1, C_2$  are given in (2.13).

**Proof.** According to Lemma 3.1, two-dimensional map, we have the characteristic equation (2.12) about map (1.4) at  $E_3$ . Then we obtain the following equations and inequalities

$$\begin{aligned} \Delta_1^-(k) &= 1 - 1 = 0, \\ \Delta_1^+(k) &= 1 + 1 > 0, \\ F_k(1) &= 1 - C_1 + C_2 > 0, \\ (-1)^2 F_k(-1) &= 1 + C_1 + C_2 > 0, \end{aligned} \tag{3.15}$$

can promise Neimark-Sacker bifurcation occurs at  $E_3$ . Then we get Theorem 3.2.  $\square$

### 4. Numerical Simulations at positive fixed point

In this section, we provide numerical simulations for the result Theorem 3.2 that we obtained in section 3. We take  $a = 0.2, b = 0.2, c = 0.3, r = 2, \mu = 0.4, \theta = 0.4$ , and  $k \in [0, 2]$  in map (1.4) with the initial conditions  $(x_0, y_0) = (3.4, 2.1)$ . When  $k$  is taken as a bifurcation parameter, then around at  $k = 0.64$ , the unique positive fixed point  $(x^*, y^*) = (3.600744532, 2.187063751)$  becomes unstable and map (1.4) undergoes Neimark-Sacker bifurcation. The characteristic polynomial evaluated at this point is given by:

$$F(\lambda) = \lambda^2 - 1.837949217\lambda + 0.8505669801,$$

moreover we have

$$\begin{aligned} F_k(1) &= 1 - C_1 + C_2 = 3.688516197 > 0, \\ (-1)^2 F_k(-1) &= 1 + C_1 + C_2 = 0.0126177631 > 0. \end{aligned}$$

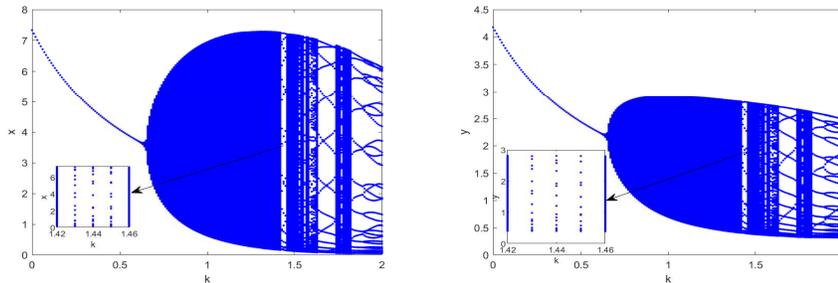


Figure 1. Bifurcation for map (1.4) at  $E_3$  with initial values (3.4,2.1).

According to Theorem 3.2, the conditions of Neimark-Sacker bifurcation are satisfied. The bifurcation diagram in the  $(k, x)$ -plane and  $(k, y)$ -plane for the above parameters are given in Fig. 1, and the corresponding phase portraits and the time responses for the state  $x(t)$  and  $y(t)$  are shown in Fig. 2 and Fig. 3, respectively. Furthermore, according to the Fig. 1, when  $k \in [1.42, 1.46]$ , the period of the orbit is interesting, which can be seen in the Fig. 4 and Fig. 5.

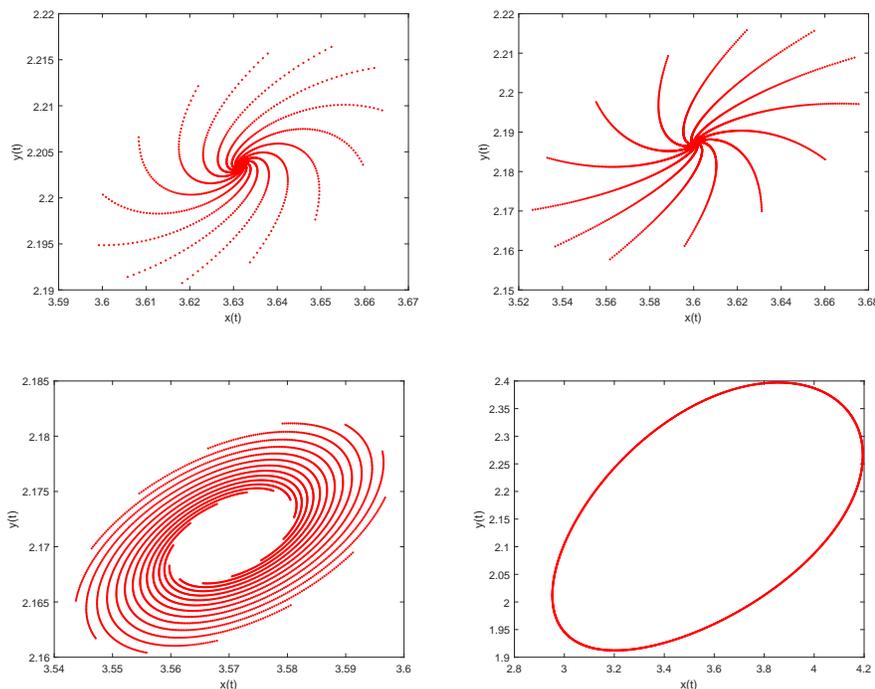


Figure 2. Phase portraits corresponding to Fig. 1, (a)  $k=0.63$ ; (b)  $k=0.64$ ; (c)  $k=0.65$ ; (d)  $k=0.66$ .

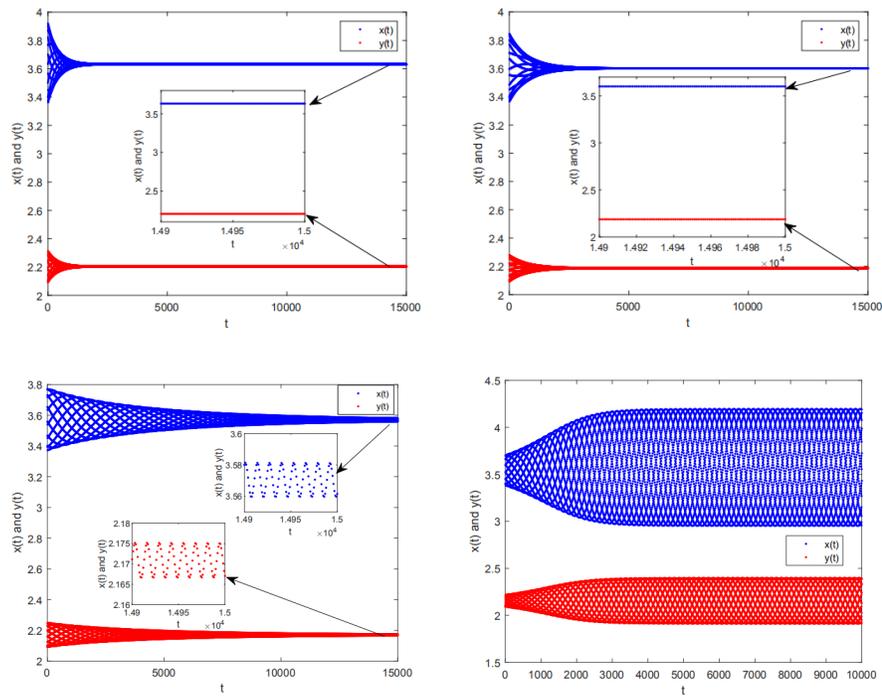
### 5. Conclusion

In this paper, we study the complex behaviors of Leslie-Gower predator-prey model with prey refuge and fear factor as a discrete-time dynamical system in the closed first quadrant  $\mathbb{R}^+$ . The results show that the model (1.4) exhibits a very rich dynamic behavior.

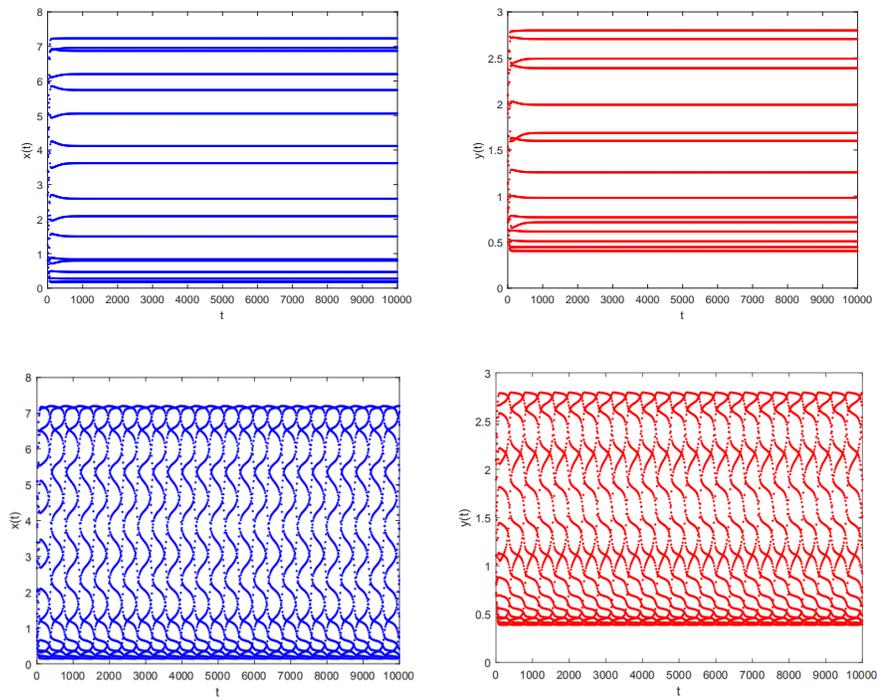
Firstly, we obtain four fixed points existence with various conditions by complex calculation and have the main results about locally asymptotical stability at  $E_0, E_1, E_2, E_3$  (see Theorem 2.1 and Theorem 2.2).

Secondly, we carry out the bifurcation analysis about model (1.4) at  $E_2$  and  $E_3$ . We have map (1.4) undergoes a flip bifurcation around  $E_2$  when  $k = \frac{c^2 r - c\mu\theta}{b\mu^2\theta}$ . And we know the map (1.4) undergoes Neimark-Sacker bifurcation at fixed point  $E_3$  with many conditions.

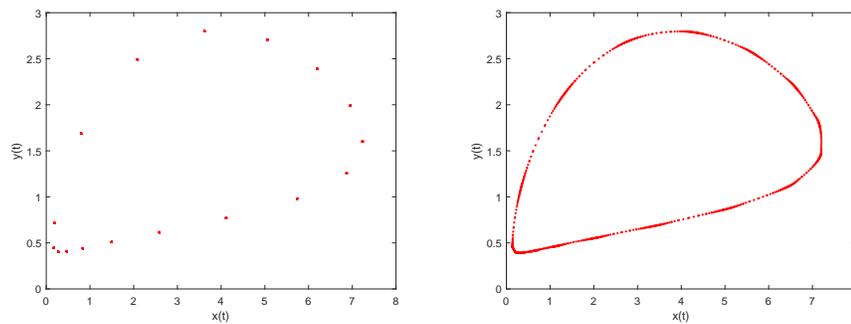
Finally, we prove the Theorem 3.2 by numerical simulations. Through digital simulation, we estimate the value of  $k$ . After theoretical calculation, the Theorem 3.2 is correct.



**Figure 3.** The time responses for the state  $x(t)$  and  $y(t)$  corresponding to Fig. 1, (a)  $k=0.63$ ; (b)  $k=0.64$ ; (c)  $k=0.65$ ; (d)  $k=0.66$ .



**Figure 4.** The time responses for the state  $x(t)$  and  $y(t)$ , (a)  $k=1.43$ ; (b)  $k=1.43$ ; (c)  $k=1.46$ ; (d)  $k=1.46$ .



**Figure 5.** Phase portraits corresponding to Fig. 4, (a)  $k=1.43$ (a period-17 orbit); (b)  $k=1.46$ .

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