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BIFURCATION OF LIMIT CYCLES AT A NILPOTENT CRITICAL POINT IN A SEPTIC LYAPUNOV SYSTEM*

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Abstract In this paper, we characterize local behavior of an isolated nilpotent critical point for a class of septic polynomial differential systems, including center conditions and bifurcation of limit cycles. With the help of computer algebra system-MATHEMATICA 12.0, the first 15 quasi-Lyapunov constants are deduced. As a result, necessary and sufficient conditions of such system having a center are obtained. We prove that there exist 16 small amplitude limit cycles created from the third-order nilpotent critical point. And then we give a lower bound of cyclicity of third-order nilpotent critical point for septic Lyapunov systems.

Keywords Third-order nilpotent critical point, center-focus problem, bifurcation of limit cycles, Quasi-Lyapunov constant.

MSC(2010) 34C05, 37G15.

1. Introduction

The phenomenon of limit cycles was first discovered and introduced by Poincaré. Later, he developed a breakthrough qualitative method called Poincaré Map, to determine the existence of limit cycles. Until now, this method is still the most basic and classic tool for investigating stability and bifurcation of periodic orbits. Afterward, many quantitative methodologies were put forward to approximate limit cycles. In recent decades, with the help of computer algebra systems such as Mathematica, Maple etc., symbolic algorithms and programs have been created to overcome the computational complexity in the analysis of bifurcation of limit cycles.

The progress of limit cycle theory is closely related to the celebrated Hilbert's 16th problem, among 23 mathematical problems proposed by Hilbert at 2nd International Congress of Mathematics in 1900. In general, research on Hilbert's 16th problem usually proceeds by the investigation of particular classes of polynomial systems. In this paper, we consider an autonomous planar ordinary differential

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equation having a third-order nilpotent critical point with the form

$$\begin{aligned}\frac{dx}{dt} &= \lambda y + \lambda x^3 - \lambda x^2 y + x^4 y\left(1 - \tfrac{71275\lambda}{378}\right) + a_{12}xy^2 + a_{32}x^3y^2 + a_{03}y^3 + xy^3 \\ &\quad + a_{23}x^2y^3 + \tfrac{5}{8}y^4 + a_{14}xy^4 + a_{05}y^5 + a_{15}xy^5 + a_{06}y^6 - \lambda y(x^2+y^2)^3, \\ \frac{dy}{dt} &= -2\lambda x^3 + \lambda xy^2 + b_{21}x^2y - 2x^3y^2\left(1 - \tfrac{71275\lambda}{378}\right) + b_{03}y^3 + b_{23}x^2y^3 - \tfrac{1}{4}y^4 \\ &\quad + b_{14}xy^4 + b_{05}y^5 - \tfrac{1}{6}a_{15}y^6 + \lambda x(x^2+y^2)^3.\end{aligned} \tag{1.1}$$

Suppose that X and Y are polynomials and that the origin is a monodromic critical point for

$$\begin{aligned}\frac{dx}{dt} &= X(x,y), \\ \frac{dy}{dt} &= Y(x,y).\end{aligned} \tag{1.2}$$

We are concerned with two closely related problems, both of which are significant elements in work on Hilbert's 16th problem. The first is the number of limit cycles bifurcated from a critical point and the rest one is the derivation of necessary and sufficient conditions for a critical point to be a center. Involving extensive use of Computer Algebra, much effort has been devoted over the years to the aforementioned problems. They are intertwined issues: in particular, an understanding of the center conditions is required to resolve the question of bifurcation. The problem of distinguishing between a center and a focus is of independent interest.

In some suitable coordinates, the Lyapunov system with the origin as a nilpotent critical point can be written as

$$\begin{aligned}\frac{dx}{dt} &= y + \sum_{i+j=2}^{\infty} a_{ij}x^iy^j = X(x,y), \\ \frac{dy}{dt} &= \sum_{i+j=2}^{\infty} b_{ij}x^iy^j = Y(x,y).\end{aligned} \tag{1.3}$$

Suppose that the function $y = y(x)$ satisfies $X(x,y) = 0, y(0) = 0$. Lyapunov proved (see for instance [3]) that the origin of system (1.3) is a monodromic critical point (i.e., a center or a focus) if and only if

$$\begin{aligned}&Y(x,y(x)) = \alpha x^{2n+1} + o(x^{2n+1}),\ \alpha < 0, \\ &\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial x}\right]_{y=y(x)} = \beta x^n + o(x^n), \\ &\beta^2 + 4(n+1)\alpha < 0,\end{aligned} \tag{1.4}$$

where n is a positive integer. The monodromy problem for a general nilpotent singularity was solved in [4] and the center problem in [28], see also in [30]. As far as we know, there are essentially three differential ways to obtain Lyapunov constants for a monodromic nondegenerate singular point, which are normal form theory [9], the Poincaré return map [6] and Lyapunov functions [29]. These three ways have been also used to study the center-focus problem of nilpotent critical points. In [1] the monodromy and stability for nilpotent critical points with the method of computing the Poincaré return map was investigated. In [8] the local analytic integrability of nilpotent centers was studied by using Lyapunov functions.

In [28] Moussu investigated the center-focus problem of nilpotent critical points with the normal form theory.

In [32], Takens proved that system (1.3) can be formally transformed into a generalized Liénard system. Furthermore, in [2] it was proved that the generalized Liénard system could be simplified even more by a reparametrization of the time. At the same time, Giacomini, Giné and Llibre in [15, 16] proved that the analytic nilpotent system with a center can be expressed as limit of non-degenerate system with a center. Therefore, any nilpotent center can be detected using the same methods that for a non-degenerate center, for instance the Poincaré-Lyapunov method can be used to find the nilpotent centers. The ideas of the works [15, 16] have been corrected in the [12], which proved that all the nilpotent centers of planar analytic differential systems are limit of centers with purely imaginary eigenvalues, and consequently the Poincaré-Liapunov method to detect centers with purely imaginary eigenvalues can be used to detect nilpotent centers. Han et al. [18] obtained a new bifurcation theorem concerning the limit cycle bifurcation near the nilpotent center. Jiang et al. [21] proved that a centrally symmetric quintic near-Hamiltonian system can have ten limit cycles by using a homoclinic bifurcation method based on stability change. Han et al. [19] proved that there are exact three cases: a center, a cusp or a saddle for polynomial Hamiltonian systems with an isolated nilpotent critical point. Han and Romanovski [20] studied analytic properties of the Poincaré return map and generalized focal values of analytic planar systems with a nilpotent focus or center. Zhang and Chen [33] extended the previous result by analyzing the global phase portraits of polynomial Hamiltonian systems. Many good results have also been obtained (see [10, 11, 17, 27] and so on).

There are very few results known for concrete differential systems with monodromic nilpotent critical points. Gasull and Torregrosa in [14] have generalized the scheme of computation of Lyapunov constants for systems of the form

$$\begin{aligned} \dot{x} &= y + \sum_{k \geq n+1} F_k(x, y), \\ \dot{y} &= -x^{2n-1} + \sum_{k \geq 2n} G_k(x, y), \end{aligned} \tag{1.5}$$

where F_k and G_k are $(1, n)$-quasi-homogeneous functions of degree k. Chavarriga, García, and Giné investigated the integrability of centers perturbed by (p, q)-quasi-homogeneous polynomials in [7].

For a given family of polynomial differential equations, in general, the number of Lyapunov constants needed to solve the center-focus problem is also related with the so-called cyclicity of the point, i.e., the number of limit cycles that appear from it by small perturbations of the coefficients of the given differential equation inside the family considered (see [13] for cases where this relation does not exist for the case of nondegenerate centers). Let $N(n)$ be the maximum possible number of limit cycles bifurcating from nilpotent critical points for analytic vector fields of degree n. It was found that $N(3) \geq 2, N(5) \geq 5, N(7) \geq 9$ in [5], $N(3) \geq 3, N(5) \geq 5$ in [1], and for a family of Kukles system with 6 parameters $N(3) \geq 3$ in [2]. Recently, Liu and Li proved $N(3) \geq 8$ in [26]. Recently, Li and Yu studied a class of symmetric systems with nilpotent singular points in [22, 23].

In particular, Sun and Zhao [31] studied the number of isolated zeros of Abelian integrals associated to a class of hyper-elliptic Hamiltonian systems of degree seven with nilpotent singularities and obtained the bounds and sharp bounds. Despite all of these efforts, however, more classifications of centers and the number of bifurcation of limit cycles with nilpotent critical points seem to be very difficult. In this paper, employing the inverse integral factor method introduced in [25], see also in [26], we will prove $N(7) \geq 16$. To the best of our knowledge, our result on the lower bounds of cyclicity of third-order nilpotent critical points for septic systems is new.

This paper will be organized as follows. In Section 2, we state some preliminary knowledge given in [25] which is useful throughout the paper. In Section 3, using the linear recursive formulae in [25] to do direct computation, we obtain the first 15 quasi-Lyapunov constants and the sufficient and necessary conditions of center. This paper is ended with Section 4 in which the 15-order weak focus conditions and the result that there exist 16 limit cycles in the neighborhood of the third-order nilpotent critical point are proved.

2. Preliminary knowledge

When the nilpotent critical point is a focus or a center, it is more difficult to determine whether it is a center or not, because in a neighborhood of the critical point, the method of the Poincaré formal series cannot be used in order to compute Lyapunov constants. Fortunately, Liu and Li [25] found that there always exists a formal inverse integrating factor for third-order nilpotent critical points, but it was not true for other order nilpotent critical points. They gave a new definition of the focal values under the generalized triangle polar coordinates and the method of commutating Lyapunov constants using the inverse integral factors for the third-order nilpotent critical point. The idea of this section comes from [25], see also [26]. Next, we will recall related notions and results.

The origin of system (1.2) is a third-order monodromic critical point if and only if the system is the following form:

$$\begin{aligned} \frac{dx}{dt} &= y + \mu x^2 + \sum_{i+2j=3}^{\infty} a_{ij} x^i y^j = X(x,y), \\ \frac{dy}{dt} &= -2x^3 + 2\mu xy + \sum_{i+2j=4}^{\infty} b_{ij} x^i y^j = Y(x,y). \end{aligned} \tag{2.1}$$

Lemma 2.1. *For the system* (2.1), *one can derive successively the terms of the following series with non-zero convergence radius:*

$$\begin{aligned} u(x,y) &= x + \sum_{\alpha+\beta=2}^{\infty} a'_{\alpha\beta} x^\alpha y^\beta, \\ v(x,y) &= y + \sum_{\alpha+\beta=2}^{\infty} b'_{\alpha\beta} x^\alpha y^\beta, b'_{20} = -\mu, \\ \zeta(x,y) &= 1 + \sum_{\alpha+\beta=1}^{\infty} c'_{\alpha\beta} x^\alpha y^\beta, \end{aligned} \tag{2.2}$$

such that by the transformation

$$u = u(x,y), \quad v = v(x,y), \quad dt = \zeta(x,y) d\tau, \tag{2.3}$$

system (2.1) *is reduced to the following Liénard equations*

$$\begin{aligned} \frac{du}{d\tau} &= v + 2\mu u^2 + \sum_{k=1}^{\infty} A_k u^{4k} + \sum_{k=1}^{\infty} B_k u^{4k+2} + \sum_{k=1}^{\infty} C_k u^{2k+1} = U(u,v), \\ \frac{dv}{d\tau} &= -2(1+\mu^2)u^3 = V(u,v). \end{aligned} \tag{2.4}$$

In addition, the origin of system (2.1) *is a center if and only if for all* k, $C_k = 0$.

Definition 2.1. Write that $B_0 = 2\mu$.

1. If $\mu \neq 0$, then the origin of system (2.1) is called a three-order nilpotent critical point of 0-class.
2. If $\mu = 0$, and there exists a positive integer s, such that $B_0 = B_1 = \cdots = B_{s-1} = 0$, but $B_s \neq 0$, then the origin of system (2.1) is called a three-order nilpotent critical point of s-class.
3. If $\mu = 0$ and for all positive integer s, $B_s = 0$, then the origin of system (2.1) is called a three-order nilpotent critical point of ∞-class.

Theorem 2.1. *For any positive integer* s *and a given real number sequence*

$$\{c_{0\beta}\}, \beta \geq 3, \tag{2.5}$$

one can construct successively the terms with the coefficients $c_{\alpha\beta}$ *satisfying* $\alpha \neq 0$ *of the formal series*

$$M(x,y) = y^2 + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta} x^\alpha y^\beta = \sum_{k=2}^{\infty} M_k(x,y), \tag{2.6}$$

such that

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) M - (s+1)\left(\frac{\partial M}{\partial x} X + \frac{\partial M}{\partial y} Y\right) = \sum_{m=3}^{\infty} \omega_m(s,\mu) x^m, \tag{2.7}$$

where for all k, $M_k(x,y)$ *is a* k*-homogeneous polynomial of* x, y *and* $s\mu = 0$.

It is easy to see that (2.7) is linear with respect to the function M, so we can easily find the following recursive formulae for the calculation of $c_{\alpha\beta}$ and $\omega_m(s,\mu)$.

Theorem 2.2. *For* $\alpha \geq 1, \alpha+\beta \geq 3$ *in* (2.6) *and* (2.7), $c_{\alpha\beta}$ *can be uniquely determined by the recursive formula*

$$c_{\alpha\beta} = \frac{1}{(s+1)\alpha}(A_{\alpha-1,\beta+1} + B_{\alpha-1,\beta+1}). \tag{2.8}$$

For $m \geq 1$, $\omega_m(s,\mu)$ *can be uniquely determined by the recursive formula*

$$\omega_m(s,\mu) = A_{m,0} + B_{m,0}, \tag{2.9}$$

where

$$\begin{aligned} A_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} [k - (s+1)(\alpha-k+1)] a_{kj} c_{\alpha-k+1,\beta-j}, \\ B_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} [j - (s+1)(\beta-j+1)] b_{kj} c_{\alpha-k,\beta-j+1}. \end{aligned} \tag{2.10}$$

Notice that in (2.10), *we set*

$$\begin{aligned} &c_{00} = c_{10} = c_{01} = 0, \\ &c_{20} = c_{11} = 0, c_{02} = 1, \\ &c_{\alpha\beta} = 0, \text{ if } \alpha < 0 \text{ or } \beta < 0. \end{aligned} \tag{2.11}$$

If the origin of system (2.1) is s-class or ∞-class, then, by choosing $\{c_{\alpha\beta}\}$, such that

$$\omega_{2k+1}(s, \mu) = 0, k = 1, 2, \cdots, \tag{2.12}$$

we can obtain a solution group of $\{c_{\alpha\beta}\}$ of (2.12), thus, we have

$$\mu_m = \frac{\omega_{2m+4}(s, \mu)}{2m - 4s - 1}. \tag{2.13}$$

Clearly, the recursive formulae by Theorem 2.2 is linear with respect to all $c_{\alpha\beta}$. Therefore, it is convenient to realize the computations of quasi-Lyapunov constants by using computer algebraic system like MATHEMATICA.

3. Quasi-Lyapunov constants and center conditions

According to Theorem 2.1, for system (1.1), we can find a positive integer s and a formal series $M(x, y) = x^4 + y^2 + o(r^4)$, such that (2.7) holds. Applying the recursive formulae presented in Theorem 2.2 to carry out calculations in MATHEMATICA, we have

$$\begin{aligned} &\omega_3 = \omega_4 = \omega_5 = 0, \\ &\omega_6 = \frac{1}{3\lambda}(-1 + 4s)(b_{21} + 3\lambda), \\ &\omega_7 = 3(s + 1)c_{03}, \\ &\omega_8 = -\frac{2}{5\lambda}(-3 + 4s)(a_{12} + 3b_{03}), \\ &\omega_9 = 0, \\ &\omega_{10} = -\frac{2}{7\lambda}(-5 + 4s)(a_{32} + b_{23}), \\ &\omega_{11} = \frac{15}{16\lambda}(s + 1)(4\lambda c_{05} - 1), \\ &\omega_{12} = \frac{4}{15\lambda}(-7 + 4s)(-a_{14} + a_{23} - 5b_{05} + 2b_{14}), \\ &\omega_{13} = 0, \\ &\omega_{14} = \frac{40}{77\lambda^2}(-9 + 4s)(a_{23} + 2b_{14})(\lambda - b_{03}), \\ &\omega_{15} = \frac{5}{32\lambda^2}(s + 1)(7a_{03} - 8a_{06}\lambda - 14\lambda c_{04} + 28\lambda^2 c_{07}), \\ &\omega_{16} = \frac{2}{7371\lambda^2}(-11 + 4s)(a_{23} + 2b_{14})(-756 + 252a_{32} + 142739\lambda), \\ &\omega_{17} = 0, \end{aligned} \tag{3.1}$$

$$\omega_{18} = -\frac{8}{6237\lambda^2}(-13+4s)(a_{23}+2b_{14})(-189+378b_{05}-189b_{14}+35741\lambda),$$
$$\begin{aligned}\omega_{19} = \frac{15}{1024\lambda^3}[&-84a_{03}^2(1+s)+84a_{05}\lambda(1+s)+96a_{03}a_{06}\lambda(1+s)\\ &+49a_{23}\lambda+98b_{14}\lambda-14a_{23}s\lambda-28b_{14}s\lambda+168a_{03}\lambda c_{04}(1+s)\\ &-192a_{06}\lambda^2c_{04}(1+s)-252\lambda^2c_{06}(1+s)+336\lambda^3c_{09}(1+s)],\end{aligned}$$
$$\omega_{20} = \frac{2}{125307\lambda^2}(-15+4s)(a_{23}+2b_{14})(-945+2646a_{03}+232516\lambda),$$
$$\omega_{21} = \frac{1}{84\lambda^2}(s-4)(a_{23}+2b_{14})(-7-48a_{06}+28a_{15}).$$

From (2.13) and (3.1), we obtain the first eight quasi-Lyapunov constants of system (1.1):

$$\begin{aligned}
\lambda_1 &= \tfrac{\omega_6}{1-4s} = \tfrac{1}{3\lambda}(b_{21}+3\lambda),\\
\lambda_2 &= \tfrac{\omega_8}{3-4s} = \tfrac{2}{5\lambda}(a_{12}+3b_{03}),\\
\lambda_3 &= \tfrac{\omega_{10}}{5-4s} = \tfrac{2}{7\lambda}(a_{32}+b_{23}),\\
\lambda_4 &= \tfrac{\omega_{12}}{7-4s} = \tfrac{4}{15\lambda}(a_{14}-a_{23}+5b_{05}-2b_{14}),\\
\lambda_5 &= \tfrac{\omega_{14}}{9-4s} = -\tfrac{40}{77\lambda^2}(2b_{14}+a_{23})(\lambda-b_{03}),\\
\lambda_6 &= \tfrac{\omega_{16}}{11-4s} = -\tfrac{2}{7371\lambda^2}(2b_{14}+a_{23})(-756+252a_{32}+142739\lambda),\\
\lambda_7 &= \tfrac{\omega_{18}}{13-4s} = -\tfrac{8}{6237\lambda^2}(2b_{14}+a_{23})(189-378b_{05}+189b_{14}-35741\lambda),\\
\lambda_8 &= \tfrac{\omega_{18}}{13-4s} = \tfrac{2}{125307\lambda^2}(2b_{14}+a_{23})(-945+2646a_{03}+232516\lambda).
\end{aligned} \tag{3.2}$$

We see from $\omega_7=\omega_9=\omega_{11}=\omega_{13}=\omega_{15}=\omega_{17}=\omega_{19}=\omega_{21}=0$ that

$$\begin{aligned}
&c_{03}=0,\ c_{05}=\tfrac{1}{4\lambda},\ c_{07}=\tfrac{-7a_{03}+8a_{06}\lambda+14\lambda c_{04}}{28\lambda^2},\\
&c_{09}=-\tfrac{1}{336\lambda^3(1+s)}[-84a_{03}^2(1+s)+84a_{05}\lambda(1+s)+96a_{03}a_{06}\lambda(1+s)\\
&\qquad +49a_{23}\lambda+98b_{14}\lambda-14a_{23}s\lambda-28b_{14}s\lambda+168a_{03}\lambda c_{04}(1+s)\\
&\qquad -192a_{06}\lambda^2c_{04}(1+s)-252\lambda^2c_{06}(1+s)],\\
&s=4.
\end{aligned} \tag{3.3}$$

Furthermore, taking $s=4$, we obtain the following conclusion.

Proposition 3.1. *For system* (1.1), *one can determine successively the terms of the formal series* $M(x,y)=x^4+y^2+o(r^4)$, *such that*

$$\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)M-2\left(\frac{\partial M}{\partial x}X+\frac{\partial M}{\partial y}Y\right)=\sum_{m=1}^{14}\lambda_m[(2m-5)x^{2m+4}+o(r^{32})], \tag{3.4}$$

where λ_m *is the* m*-th quasi-Lyapunov constant at the origin of system* (1.1), $m=1,2,\cdots,14$.

Theorem 3.1. *For system* (1.1), *the first 15 quasi-Lyapunov constants at the origin are given by*

$$
\begin{aligned}
\lambda_1 &= \frac{1}{3\lambda}(b_{21} + 3\lambda),\\
\lambda_2 &= \frac{2}{5\lambda}(a_{12} + 3b_{03}),\\
\lambda_3 &= \frac{2}{7\lambda}(a_{32} + b_{23}),\\
\lambda_4 &= \frac{4}{15\lambda}(a_{14} - a_{23} + 5b_{05} - 2b_{14}),\\
\lambda_5 &= -\frac{40}{77\lambda^2}(2b_{14} + a_{23})(\lambda - b_{03}),\\
\lambda_6 &= -\frac{2}{7371\lambda^2}(2b_{14} + a_{23})(-756 + 252a_{32} + 142739\lambda),\\
\lambda_7 &= -\frac{8}{6237\lambda^2}(2b_{14} + a_{23})(189 - 378b_{05} + 189b_{14} - 35516\lambda),\\
\lambda_8 &= \frac{2}{125307\lambda^2}(2b_{14} + a_{23})(-945 + 2646a_{03} + 232516\lambda),\\
\lambda_9 &= -\frac{1}{358435\lambda^2}(2b_{14} + a_{23})(-34020 + 4116a_{23} + 1372b_{14} + 5024095\lambda),\\
\lambda_{10} &= \frac{1}{386822709\lambda^3}(2b_{14} + a_{23})(-714420 - 593011314\lambda - 14002632a_{05}\lambda\\
&\quad + 90016920a_{23}\lambda + 103381799653\lambda^2),\\
\lambda_{11} &= \frac{1}{13088182338\lambda^3}(2b_{14} + a_{23})(1122889635 + 2669835168a_{06} - 1557403848a_{15}\\
&\quad - 84015792a_{23} - 74423819736\lambda + 100018800a_{23}\lambda - 4304736163454\lambda^2),\\
\lambda_{12} &= -\frac{1}{8189802497575573056\lambda^4}(2b_{14} + a_{23})(161058273264 - 111993387549309\lambda\\
&\quad - 369366355822464a_{06}\lambda + 215463707563104a_{15}\lambda + 29167309199814912\lambda^2\\
&\quad + 110900918757781152a_{06}\lambda^2 - 64692202608705672a_{15}\lambda^2\\
&\quad + 898784309039302518\lambda^3 - 182814242015394566756\lambda^4),\\
\lambda_{13} &= -\frac{1}{673359632081887091362 56\lambda^5(-46116+13846163\lambda)}(2b_{14}+a_{23})\\
&\quad \times(-46025300634106752+46534212594321669000\lambda-51139222926785280a_{15}\lambda\\
&\quad - 1690489698969044423 1987\lambda^2 - 5488281440605475568a_{15}\lambda^2\\
&\quad + 2321597608028417543158026\lambda^3 + 30634368862645041258 24a_{15}\lambda^3\\
&\quad - 1142417373226676682076601 10\lambda^4 - 285381993806159335515360a_{15}\lambda^4\\
&\quad +2443151138824686797146507940\lambda^5+58047178124728082836365120a_{15}\lambda^5), \qquad (3.5)\\
\lambda_{14} &= \frac{1}{112099336873609474265933389132 8\lambda^6(-46116 + 13846163\lambda)^2 f(\lambda)}\\
&\quad \times (2b_{14} + a_{23})(31534409734073413224180419837546011238 40\\
&\quad - 5183764525638507229200544728006410110039296\lambda\\
&\quad + 15986101338338855841402609051562086875198913 12\lambda^2\\
&\quad + 8968868335329690807932487942168689698646834724 81\lambda^3\\
&\quad - 918576192072616403257175610954436756895783155287879\lambda^4\\
&\quad + 34610965409225870773424414814172251684558004131990852 0\lambda^5\\
&\quad - 683943780459306818373519234608632370244835337003989630 74\lambda^6
\end{aligned}
$$

$$
\begin{aligned}
&+ 7615128586518718845757734511546928216282100011070384370800\lambda^{7}\\
&- 5382139404186675460866486830210121416405625897187330286704 08\lambda^{8}\\
&+ 2871621204616511339100194141547208705333664415427815315882 6912\lambda^{9}\\
&- 1079351287105638131380982282679868880887040622326920461860661600\lambda^{10}\\
&+ 1634592512876704923670413341142788060980205830225945673512178 3360\lambda^{11}),
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{15} = {} & \frac{1}{107683234451717662486078882328544768\lambda^6(-46116+13846163\lambda)^2 f(\lambda)^2}\\
&\times(2b_{14}+a_{23})(-109731709877768837055569915740085083061572786196720959488 00\\
&+228977492626822339819927327259495190547319799578362114889318 40\lambda\\
&-249145799862468856649713383178984419464100736859802281056801121 28\lambda^{2}\\
&+167033447220310304632197621822677449514446622123947781884162324502 40\lambda^{3}\\
&-756211305142301954719837948145687812398625698223238433539828816360\\
&\quad 3235\lambda^{4}\\
&+242069950036323319444644553507637267259962860131958718918214176832\\
&\quad 3760314\lambda^{5}\\
&-558209459425573106712981933918883843679241502544509844593888595958\\
&\quad 539650464\lambda^{6}\\
&+920866987294863221177230561699056835597380064578210573099979564626\\
&\quad 15180052713\lambda^{7}\\
&-106511675203037111188897006048014240136319734900122467882776065793\\
&\quad 85507453408964\lambda^{8}\\
&+924433806393526910710340057644352467612106386055522305470062223423\\
&\quad 341554286688096\lambda^{9}\\
&-900813570945239322859566710400741930390411994899130724501084049737\\
&\quad 61531224165823312\lambda^{10}\\
&+120535763388059624480881262588231322282254211698481879178636917151\\
&\quad 1014062337934977621 2\lambda^{11}\\
&-131633788866496819091010251484151404294789178202456060270100681644\\
&\quad 633082227485387102464 0\lambda^{12}\\
&+ 864285950404097088986855730687317460943549196610249338070731046470\\
&\quad 94043062918786115404720\lambda^{13}\\
&- 301466532129644584160462373960013221779506198079878490513324896862\\
&\quad 1262451019107171371742400\lambda^{14}\\
&+ 487591010089300016329868301917556947929491239344630165949485702999\\
&\quad 191580253698333343166668800\lambda^{15}),
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
f(\lambda) = {} & -805291366320 - 86424184942767\lambda + 48240061827042456\lambda^{2}\\
& - 4493921545196512590\lambda^{3} + 9140712100769728337 80\lambda^{4}.
\end{aligned}
\tag{3.7}
$$

In the above expression of λ_k, we have already let $\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} =$

$0, k = 2, \cdots, 15$.

From Theorem 3.1, we obtain the following assertion.

Proposition 3.2. *The first 15 quasi-Lyapunov constants at the origin of system* (1.1) *are zero if and only if the following condition is satisfied:*

$$b_{21} = -3\lambda, a_{12} = -3b_{03}, b_{23} = -a_{32}, a_{14} = a_{23} - 5b_{05} + 2b_{14}, a_{23} = -2b_{14}. \quad (3.8)$$

When the condition in Proposition 3.2 holds, system (1.1) can be brought to

$$\begin{aligned} \frac{dx}{dt} &= \lambda y + \lambda x^3 - \lambda x^2 y - 3b_{03}xy^2 + a_{03}y^3 + a_{32}x^3y^2 + xy^3 + \tfrac{5}{8}y^4 - 2b_{14}x^2y^3 \\ &\quad - 5b_{05}xy^4 + (1 - \tfrac{71275\lambda}{378})x^4y + a_{15}xy^5 + a_{05}y^5 + a_{06}y^6 - \lambda y(x^2+y^2)^3, \\ \frac{dy}{dt} &= -2\lambda x^3 - 3\lambda x^2 y + \lambda xy^2 + b_{03}y^3 + b_{14}xy^4 - a_{32}x^2y^3 + b_{05}y^5 \\ &\quad - 2(1 - \tfrac{71275\lambda}{378})x^3y^2 - \tfrac{1}{4}y^4 - \tfrac{1}{6}a_{15}y^6 + \lambda x(x^2+y^2)^3, \end{aligned} \quad (3.9)$$

who has an analytic first integral

$$\begin{aligned} F(x,y) &= \tfrac{1}{2}\lambda x^4 + \tfrac{1}{2}\lambda y^2 + \tfrac{1}{4}a_{03}y^4 + \tfrac{1}{8}y^5 + \tfrac{1}{6}a_{05}y^6 + \tfrac{1}{7}a_{06}y^7 - xy^3(b_{03} + b_{05}y^2) \\ &\quad + \tfrac{1}{4}xy^4 - \tfrac{1}{2}x^2y^2(b_{14}y^2 + \lambda) + \tfrac{1}{756}x^4y^2(378 + 252y - 71275\lambda) \\ &\quad + x^3(\tfrac{1}{3}a_{32}y^3 + \lambda y) + \tfrac{1}{6}a_{15}xy^6 - \tfrac{1}{8}\lambda(x^2+y^2)^4. \end{aligned} \quad (3.10)$$

Therefore, Proposition 3.2 implies that

Proposition 3.3. *The origin of system* (3.9) *is a center.*

We see from Propositions 3.2 and 3.3 that

Theorem 3.2. *The origin of system* (1.1) *is a center if and only if the first 15 quasi-Lyapunov constants are zero, that is, the condition in Proposition 3.2 is satisfied.*

4. Multiple bifurcation of limit cycles

In the previous section, we have derived the expressions of the first 15 quasi-Lyapunov constants of system (1.1). In this section we will prove that when the third-order nilpotent critical point $O(0,0)$ is a 15-order weak focus. As a result, the perturbed system of (1.1) can generate 16 limit cycles enclosing an elementary node at the origin of perturbation system (1.1).

Under the fact

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = \lambda_{13} = \lambda_{14} = 0, \lambda_{15} \neq 0, \quad (4.1)$$

we obtain

Theorem 4.1. *The origin of system* (1.1) *is a 15-order weak focus if and only if*

$$
\begin{aligned}
&b_{21} = -3\lambda, a_{12} = -3b_{03},\\
&b_{23} = -a_{32}, a_{14} = a_{23} - 5b_{05} + 2b_{14}, b_{03} = \lambda,\\
&a_{32} = \tfrac{1}{252}(756 - 143639\lambda), b_{05} = \tfrac{1}{378}(189 + 189b_{14} - 35741\lambda),\\
&a_{03} = \tfrac{1}{22646}(945 - 232516\lambda), b_{14} = \tfrac{1}{1372}(34020 - 4116a_{23} - 5064595\lambda),\\
&a_{05} = \tfrac{1}{14002632\lambda}(-714420 - 593011314\lambda + 90016920a_{23}\lambda + 103381799653\lambda^2),\\
&a_{23} = \tfrac{1}{4000752(-21+25\lambda)}(-1122889635 - 2669835168a_{06} + 1557403848a_{15}\\
&\quad + 74423819736\lambda + 4304736163454\lambda^2),\\
&a_{06} = \tfrac{1}{8009505504\lambda(-46116+13846163\lambda)}(-161058273264 + 111993387549309\lambda\\
&\quad - 215463707563104a_{15}\lambda - 29167309199814912\lambda^2 + 64692202608705672a_{15}\lambda^2\\
&\quad - 898784309039302518\lambda^3 + 182814242015394566756\lambda^4),\\
&a_{15} = \tfrac{1}{63504\lambda f(\lambda)}(460253006341067520 - 46534212594321669000\lambda\\
&\quad + 1690489698969044423198 7\lambda^2 - 2321597608028417543158026\lambda^3\\
&\quad + 114241737322667668207660110\lambda^4 - 2443151138824686797146507940\lambda^5),\\
&\lambda \approx A_i, \quad i = 1, \cdots, 5,\\
&A_1 = -0.00162852,\ A_2 = -0.00162852,\ A_3 = 0.00337192,\\
&A_4 = 0.00471939,\ A_5 = 0.0332452.
\end{aligned}
\tag{4.2}
$$

Proof. By setting

$$
\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = \lambda_{13} = 0, \tag{4.3}
$$

we obtain the relations of $b_{21},\ a_{12},\ b_{23},\ a_{14}, b_{03},\ a_{32},\ b_{05},\ a_{03},\ b_{14},\ a_{05},\ a_{23},\ a_{06},\ a_{15}$. Solving the equation $\lambda_{14} = 0$, we get the following 11 solutions

$$
\begin{aligned}
&\lambda_1 \approx -0.00162852,\ \lambda_2 \approx -0.00162852,\ \lambda_3 \approx 0.00337192,\ \lambda_4 \approx 0.00471939,\\
&\lambda_5 \approx 0.0332452,\ \lambda_6 \approx 0.000296049 - 0.0160406i,\ \lambda_7 \approx 0.000296049 + 0.0160406,\\
&\lambda_8 \approx 0.00178134 - 0.00228599i,\ \lambda_9 \approx 0.00178134 + 0.00228599i,\\
&\lambda_{10} \approx 0.0106331 + 0.001286i,\ \lambda_{11} \approx 0.0106331 - 0.001286i,
\end{aligned}
\tag{4.4}
$$

Notice that $\lambda \in \mathbb{R}$, we choose $\lambda \approx A_i\ i = 1, \cdots, 5$ and a simple calculation gives

$$
Resultant[\lambda_{14}, \lambda_{15}, \lambda] \approx 3.569182365662753 \times 10^{1628} \neq 0. \tag{4.5}
$$

Thus $\lambda_{15} \neq 0$, the origin of system (1.1) is a 15-order weak focus. □

We next study the perturbed system of (1.1) as follows:

$$\begin{aligned}\frac{dx}{dt} =&\delta(\varepsilon)x+\lambda(\varepsilon)y+\lambda(\varepsilon)x^3-\lambda(\varepsilon)x^2y+x^4y\left(1-\frac{71275\lambda(\varepsilon)}{378}\right)+a_{12}(\varepsilon)xy^2\\&+a_{32}(\varepsilon)x^3y^2+a_{03}(\varepsilon)y^3+xy^3+a_{23}(\varepsilon)x^2y^3+\frac{5}{8}y^4+a_{14}(\varepsilon)xy^4+a_{05}(\varepsilon)y^5\\&+a_{15}(\varepsilon)xy^5+a_{06}(\varepsilon)y^6-\lambda(\varepsilon)y(x^2+y^2)^3,\\ \frac{dy}{dt} =&-2\lambda(\varepsilon)x^3+\lambda(\varepsilon)xy^2+b_{21}(\varepsilon)x^2y-2x^3y^2\left(1-\frac{71275\lambda(\varepsilon)}{378}\right)+b_{03}(\varepsilon)y^3\\&+b_{23}(\varepsilon)x^2y^3-\frac{1}{4}y^4+b_{14}(\varepsilon)xy^4+b_{05}(\varepsilon)y^5-\frac{1}{6}a_{15}(\varepsilon)y^6+\lambda(\varepsilon)x(x^2+y^2)^3,\end{aligned} \tag{4.6}$$

Remember that $\lambda \approx A_i,\ i=1,\cdots,5$ are the simple zeros of $\lambda_{14}=0$. Therefore, when one of four conditions in (4.2) holds, we have:

If $\lambda \approx A_1$, then

$$\frac{\partial(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12},\lambda_{13},\lambda_{14})}{\partial(b_{21},a_{12},b_{23},a_{14},b_{03},a_{32},b_{05},a_{03},b_{14},a_{05},a_{23},a_{06},a_{15},\lambda)} \approx 1.16971\times10^{72}; \tag{4.7}$$

If $\lambda \approx A_2$, then

$$\frac{\partial(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12},\lambda_{13},\lambda_{14})}{\partial(b_{21},a_{12},b_{23},a_{14},b_{03},a_{32},b_{05},a_{03},b_{14},a_{05},a_{23},a_{06},a_{15},\lambda)} \approx -3.11372\times10^{80}; \tag{4.8}$$

If $\lambda \approx A_3$, then

$$\frac{\partial(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12},\lambda_{13},\lambda_{14})}{\partial(b_{21},a_{12},b_{23},a_{14},b_{03},a_{32},b_{05},a_{03},b_{14},a_{05},a_{23},a_{06},a_{15},\lambda)} \approx 5.2377\times10^{77}; \tag{4.9}$$

If $\lambda \approx A_4$, then

$$\frac{\partial(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12},\lambda_{13},\lambda_{14})}{\partial(b_{21},a_{12},b_{23},a_{14},b_{03},a_{32},b_{05},a_{03},b_{14},a_{05},a_{23},a_{06},a_{15},\lambda)} \approx -1.01994\times10^{55}; \tag{4.10}$$

If $\lambda \approx A_5$, then

$$\frac{\partial(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12},\lambda_{13},\lambda_{14})}{\partial(b_{21},a_{12},b_{23},a_{14},b_{03},a_{32},b_{05},a_{03},b_{14},a_{05},a_{23},a_{06},a_{15},\lambda)} \approx 9.41132\times10^{33}. \tag{4.11}$$

In fact, if we denote

$$J=\frac{\partial(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12},\lambda_{13},\lambda_{14})}{\partial(b_{21},a_{12},b_{23},a_{14},b_{03},a_{32},b_{05},a_{03},b_{14},a_{05},a_{23},a_{06},a_{15},\lambda)}, \tag{4.12}$$

then

$$Resultant[\lambda_{14},J,\lambda]\approx 2.032752172806839\times10^{4741}\neq 0, \tag{4.13}$$

it is to see that $J\neq 0$ when $\lambda\approx A_i,\ i=1,\cdots,5$.

From the statement mentioned above, Theorem 2.1 follows that

Theorem 4.2. *If the origin of system* (1.1) *is a 15-order weak focus, for* $0<\delta\ll 1$, *making a small perturbation to the coefficients of system* (1.1), *then, for system* (4.6), *in a small neighborhood of the origin, there exist exactly 15 small amplitude limit cycles enclosing the origin* $O(0,0)$, *which is an elementary node.*

Now, we consider another perturbation which was called to be double bifurcation, according to theorem 9 in [24], considering the perturbed system

$$\begin{aligned}\frac{dx}{dt} =& x(x^2-\varepsilon^2)\lambda+\left[1+x(-\lambda x+x^3\left(1-\frac{71275}{378}\right)-\lambda x^5)\right]y+f_1(x,y)y^2=\tilde{X}(x,y),\\ \frac{dy}{dt} =& [4\delta\varepsilon-2\varepsilon^2\lambda]y-(x^2-\varepsilon^2)\left[2x\left(\lambda-\frac{\lambda}{2}x^4\right)-b_{21}y\right]+f_2(x,y)y^2=\tilde{Y}(x,y).\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}f_1(x,y) =& a_{12}x+a_{32}x^3+a_{03}y+xy+a_{23}x^2y+\tfrac{5}{8}y^2+a_{14}xy^2+a_{05}y^3+a_{15}xy^3\\ &+a_{06}y^4-\lambda y(3x^4+3x^2y^2+y^4),\\ f_2(x,y) =& \lambda x-2x^3\left(1-\tfrac{71275\lambda}{378}\right)+b_{03}y+b_{23}x^2y-\tfrac{1}{4}y^2+b_{14}xy^2+b_{05}y^3-\tfrac{1}{6}a_{15}y^4\\ &+\lambda x(3x^4+3x^2y^2+y^4).\end{aligned} \tag{4.15}$$

The following theorem can be obtained directly.

Theorem 4.3. *If the origin of system* (1.1) *is a 15-order weak focus, making a double perturbation to the system* (1.1)*, then, in a small neighborhood of the origin, there exist exactly 16 small amplitude limit cycles enclosing the origin* $O(0,0)$ *with the scheme* $14 \supset (1\bigcup 1)$.

By double perturbation, the nilpotent origin can be broken into two element focus and a element saddle. If the origin of system (1.1) is a 15-order weak focus, there can exist 14 limit cycles enclosing the origin. At the same time, there are two limit cycle enclosing the two element focus respectively. The scheme $14 \supset (1\bigcup 1)$ can be drawn as in Figure 1.

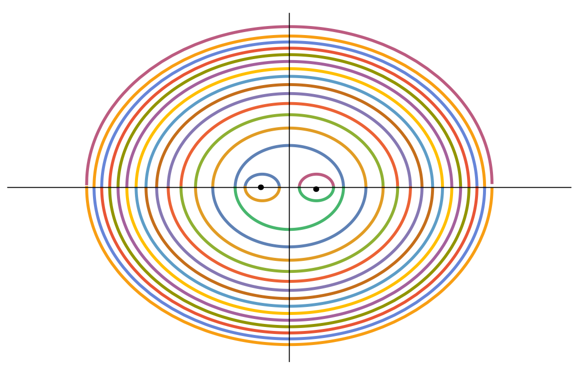

Figure 1. The scheme $14 \supset (1\bigcup 1)$ of limit cycle.

Appendix A

Detailed recursive MATHEMATICA code to compute the quasi-Lyapunov constants at the origin of system (1.1):

$$c[0,0]=0, c[1,0]=0, c[0,1]=0, c[2,0]=0, c[1,1]=0, c[0,2]=1;$$

when $k < 0$, or $j < 0$, $c[k, j] = 0$; else

$$
\begin{aligned}
c[k,j] = & \frac{-1}{1512k(1+s)\lambda}(-3024\lambda c[-8+k,2+j] - 1512j\lambda c[-8+k,2+j] \\
& - 3024s\lambda c[-8+k,2+j] - 1512js\lambda c[-8+k,2+j] - 9072\lambda c[-6+k,j] \\
& - 4536j\lambda c[-6+k,j] + 1512k\lambda c[-6+k,j] - 9072s\lambda c[-6+k,j] \\
& - 4536js\lambda c[-6+k,j] + 1512ks\lambda c[-6+k,j] - 9072\lambda c[-4+k,-2+j] \\
& - 4536j\lambda c[-4+k,-2+j] + 4536k\lambda c[-4+k,-2+j] \\
& - 9072s\lambda c[-4+k,-2+j] - 4536js\lambda c[-4+k,-2+j] \\
& + 4536ks\lambda c[-4+k,-2+j] + 6048c[-4+k,j] + 3024jc[-4+k,j] \\
& - 1512kc[-4+k,j] + 6048sc[-4+k,j] + 3024jsc[-4+k,j] \\
& - 1512ksc[-4+k,j] - 1147600\lambda c[-4+k,j] - 573800j\lambda c[-4+k,j] \\
& + 286900k\lambda c[-4+k,j] - 1147600s\lambda c[-4+k,j] - 573800js\lambda c[-4+k,j] \\
& + 286900ks\lambda c[-4+k,j] + 6048\lambda c[-4+k,2+j] + 3024j\lambda c[-4+k,2+j] \\
& + 6048s\lambda c[-4+k,2+j] + 3024js\lambda c[-4+k,2+j] \\
& + 9072a_{32}c[-3+k,-1+j] + 6048b_{23}c[-3+k,-1+j] \\
& - 1512b_{23}jc[-3+k,-1+j] - 1512a_{32}kc[-3+k,-1+j] \\
& + 4536a_{32}sc[-3+k,-1+j] + 1512b_{23}sc[-3+k,-1+j] \\
& - 1512b_{23}jsc[-3+k,-1+j] - 1512a_{32}ksc[-3+k,-1+j] \\
& - 1512b_{21}jc[-3+k,1+j] - 1512b_{21}sc[-3+k,1+j] \\
& - 1512b_{21}jsc[-3+k,1+j] + 9072\lambda c[-3+k,1+j] - 1512k\lambda c[-3+k,1+j] \\
& + 4536s\lambda c[-3+k,1+j] - 1512ks\lambda c[-3+k,1+j] - 3024\lambda c[-2+k,-4+j] \\
& - 1512j\lambda c[-2+k,-4+j] + 4536k\lambda c[-2+k,-4+j] - 3024s\lambda c[-2+k,-4+j] \\
& - 1512js\lambda c[-2+k,-4+j] + 4536ks\lambda c[-2+k,-4+j] \\
& + 6048a_{23}c[-2+k,-2+j] + 9072b_{14}c[-2+k,-2+j] \\
& - 1512b_{14}jc[-2+k,-2+j] - 1512a_{23}kc[-2+k,-2+j] \\
& + 3024a_{23}sc[-2+k,-2+j] + 3024b_{14}sc[-2+k,-2+j] \\
& - 1512b_{14}jsc[-2+k,-2+j] - 1512a_{23}ksc[-2+k,-2+j] - 3024\lambda c[-2+k,j] \\
& - 1512j\lambda c[-2+k,j] + 1512k\lambda c[-2+k,j] - 3024s\lambda c[-2+k,j] \\
& - 1512js\lambda c[-2+k,j] + 1512ks\lambda c[-2+k,j] + 504a_{15}c[-1+k,-4+j] \\
& + 252a_{15}jc[-1+k,-4+j] - 1512a_{15}kc[-1+k,-4+j] \\
& + 504a_{15}sc[-1+k,-4+j] + 252a_{15}jsc[-1+k,-4+j] \\
& - 1512a_{15}ksc[-1+k,-4+j] + 3024a_{14}c[-1+k,-3+j] \\
& + 12096b_{05}c[-1+k,-3+j] - 1512b_{05}jc[-1+k,-3+j] \\
& - 1512a_{14}kc[-1+k,-3+j] + 1512a_{14}sc[-1+k,-3+j] \\
& + 4536b_{05}sc[-1+k,-3+j] - 1512b_{05}jsc[-1+k,-3+j] \\
& - 1512a_{14}ksc[-1+k,-3+j] + 756c[-1+k,-2+j] \\
& + 378jc[-1+k,-2+j] - 1512kc[-1+k,-2+j] + 756sc[-1+k,-2+j] \\
& + 378jsc[-1+k,-2+j] - 1512ksc[-1+k,-2+j] + 3024a_{12}c[-1+k,-1+j] \\
& + 6048b_{03}c[-1+k,-1+j] - 1512b_{03}jc[-1+k,-1+j]
\end{aligned}
$$

$$
\begin{aligned}
&- 1512a_{12}kc[-1+k,-1+j] + 1512a_{12}sc[-1+k,-1+j]\\
&+ 1512b_{03}sc[-1+k,-1+j] - 1512b_{03}jsc[-1+k,-1+j]\\
&- 1512a_{12}ksc[-1+k,-1+j] + 1512k\lambda c[k,-6+j] + 1512ks\lambda c[k,-6+j]\\
&- 1512a_{06}kc[k,-5+j] - 1512a_{06}ksc[k,-5+j] - 1512a_{05}kc[k,-4+j]\\
&- 1512a_{05}ksc[k,-4+j] - 945kc[k,-3+j] - 945ksc[k,-3+j]\\
&- 1512a_{03}kc[k,-2+j] - 1512a_{03}ksc[k,-2+j]),\\
\omega_m =& -\frac{1}{1512\lambda}(1512\lambda c[-7+m,1] + 1512s\lambda c[-7+m,1] + 3024\lambda c[-5+m,-1]\\
&- 1512m\lambda c[-5+m,-1] + 3024s\lambda c[-5+m,-1] - 1512ms\lambda c[-5+m,-1]\\
&- 4536m\lambda c[-3+m,-3] - 4536ms\lambda c[-3+m,-3] - 1512c[-3+m,-1]\\
&+ 1512mc[-3+m,-1] - 1512sc[-3+m,-1] + 1512msc[-3+m,-1]\\
&+ 286900\lambda c[-3+m,-1] - 286900m\lambda c[-3+m,-1] + 286900s\lambda c[-3+m,-1]\\
&- 286900ms\lambda c[-3+m,-1] - 3024\lambda c[-3+m,1] - 3024s\lambda c[-3+m,1]\\
&- 7560a_{32}c[-2+m,-2] - 7560b_{23}c[-2+m,-2] + 1512a_{32}mc[-2+m,-2]\\
&- 3024a_{32}sc[-2+m,-2] - 3024b_{23}sc[-2+m,-2] + 1512a_{32}msc[-2+m,-2]\\
&- 1512b_{21}c[-2+m,0] - 7560\lambda c[-2+m,0] + 1512m\lambda c[-2+m,0]\\
&- 3024s\lambda c[-2+m,0] + 1512ms\lambda c[-2+m,0] - 3024\lambda c[-1+m,-5]\\
&- 4536m\lambda c[-1+m,-5] - 3024s\lambda c[-1+m,-5] - 4536ms\lambda c[-1+m,-5]\\
&- 4536a_{23}c[-1+m,-3] - 10584b_{14}c[-1+m,-3] + 1512a_{23}mc[-1+m,-3]\\
&- 1512a_{23}sc[-1+m,-3] - 4536b_{14}sc[-1+m,-3] + 1512a_{23}msc[-1+m,-3]\\
&- 1512m\lambda c[-1+m,-1] - 1512ms\lambda c[-1+m,-1] + 1260a_{15}c[m,-5]\\
&+ 1512a_{15}mc[m,-5] + 1260a_{15}sc[m,-5] + 1512a_{15}msc[m,-5]\\
&- 1512a_{14}c[m,-4] - 13608b_{05}c[m,-4] + 1512a_{14}mc[m,-4]\\
&- 6048b_{05}sc[m,-4] + 1512a_{14}msc[m,-4] + 1134c[m,-3]\\
&+ 1512mc[m,-3] + 1134sc[m,-3] + 1512msc[m,-3] - 1512a_{12}c[m,-2]\\
&- 7560b_{03}c[m,-2] + 1512a_{12}mc[m,-2] - 3024b_{03}sc[m,-2]\\
&+ 1512a_{12}msc[m,-2] - 1512\lambda c[1+m,-7] - 1512m\lambda c[1+m,-7]\\
&- 1512s\lambda c[1+m,-7] - 1512ms\lambda c[1+m,-7] + 1512a_{06}c[1+m,-6]\\
&+ 1512a_{06}mc[1+m,-6] + 1512a_{06}sc[1+m,-6] + 1512a_{06}msc[1+m,-6]\\
&+ 1512a_{05}c[1+m,-5] + 1512a_{05}mc[1+m,-5] + 1512a_{05}sc[1+m,-5]\\
&+ 1512a_{05}msc[1+m,-5] + 945c[1+m,-4] + 945mc[1+m,-4]\\
&+ 945sc[1+m,-4] + 945msc[1+m,-4] + 1512a_{03}c[1+m,-3]\\
&+ 1512a_{03}mc[1+m,-3] + 1512a_{03}sc[1+m,-3] + 1512a_{03}msc[1+m,-3]\\
&+ 1512\lambda c[1+m,-1] + 1512m\lambda c[1+m,-1]\\
&+ 1512s\lambda c[1+m,-1] + 1512ms\lambda c[1+m,-1]),\\
\lambda_m =& \frac{\omega_{2m+4}}{2m-4s-1}.
\end{aligned}
$$

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