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EXISTENCE RESULTS OF SOLUTIONS FOR ANTI-PERIODIC FRACTIONAL LANGEVIN EQUATION*

Ahmed Salem

Abstract Recently, Khalili and Yadollahzadeh [9] have investigated the uniqueness and existence of solution $u(t)$, $t \in [0,1]$ for a class of nonlocal boundary conditions to fractional Langevin equation. The authors used the boundary condition $u'(0) = 0$ by incorrect method. In the current contribution, we show the correct method for using this condition and study the existence and uniqueness of solution for the same class of equation in slightly different form with anti-periodic and nonlocal integral boundary conditions as well as the boundary condition $u'(0) = 0$. An exemplar is provided to illustrate our results.

Keywords Existence solution, fractional Langevin equation, anti-periodic condition.

MSC(2010) 26A33, 34A12, 34A08.

1. Introduction

Khalili and Yadollahzadeh [9] have considered a fractional Langevin equation:

$$^cD^\beta(^cD^\alpha + \lambda)u(t) = f(t, u(t), D^\alpha u(t)), \qquad t \in [0,1], \tag{1.1}$$

where D^α denotes the Riemann-Liouville fractional derivative, the symbols $^cD^\beta$ and $^cD^\alpha$ are fractional derivatives in the Caputo sense with values of $\beta \in (1,2]$ and $\alpha \in (0,1]$ and $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function with nonlocal boundary conditions:

$$u(0) = 0, \qquad u'(0) = 0, \qquad u(1) = \gamma u(\eta), \tag{1.2}$$

where $0 < \eta < 1$ and $0 < \gamma\eta^{\alpha+1} < 1$.

It is known that the Riemann-Liouville fractional integral has the form [10, 11]:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \qquad \alpha > 0,$$

where $\Gamma(.)$ denotes the Gamma function, provided that the right-hand-side integral exists. Based on this definition, the Riemann-Liouville and the Caputo fractional

Email address: ahmedsalem74@hotmail.com (A. Salem)
Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
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derivatives are defined, respectively, as:

$$D^\alpha f(t) = D^n I^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$
$${}^c D^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $n \in \mathbb{N}$ and $n-1 < \alpha \le n$, provided that the integrals exist.

In view of the relations (2.114) and (2.115) in [11], we can derive, for $\alpha > 0$ and $k \in \mathbb{N}$, the following identities:

$$D^k I^\alpha f(t) = I^{\alpha-k} f(t), \qquad \alpha \ge k,$$
$$D^k I^\alpha f(t) = I^\alpha D^k f(t) + \sum_{j=1}^{k} \frac{t^{\alpha-j}}{\Gamma(1+\alpha-j)} f^{(k-j)}(0), \qquad \alpha < k.$$

In particular, if $k=1$, we get:

$$DI^\alpha f(t) = I^{\alpha-1} f(t), \qquad \alpha \ge 1, \tag{1.3}$$
$$DI^\alpha f(t) = I^\alpha Df(t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} f(0), \qquad \alpha < 1. \tag{1.4}$$

The authors in [9] considered the linear fractional Langevin equation:

$${}^c D^\beta ({}^c D^\alpha + \lambda) u(t) = h(t), \qquad t \in [0,1], \tag{1.5}$$

and gave its solution as:

$$u(t) = I^{\alpha+\beta} h(t) - \lambda I^\alpha u(t) + \frac{t^\alpha}{\Gamma(\alpha+1)} c_0 + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} c_1 + c_2, \tag{1.6}$$

and the derivative of solution as:

$$u'(t) = I^{\alpha+\beta-1} h(t) - \lambda I^{\alpha-1} u(t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} c_0 + \frac{t^\alpha}{\Gamma(\alpha+1)} c_1,$$

where c_0, c_1 and c_2 are constants. It is noting that when $\alpha < 1$, the second fractional integral is undefined and so the relation above is incorrect. To avoid this error, we use the relation (1.4) to rewrite the derivative of solution in the correct form as:

$$u'(t) = I^{\alpha+\beta-1} h(t) - \lambda I^\alpha u'(t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} [c_0 - \lambda u(0)] + \frac{t^\alpha}{\Gamma(\alpha+1)} c_1.$$

Therefore, when applying the boundary condition $u'(0) = 0$, we have to take $u'(t)$ is continuous for all $t \in [0,1]$ and $c_0 = \lambda u(0) = \lambda c_2$, when $\alpha < 1$.

When $\alpha = 1$, the Langevin equation will be converted to the sequential fractional differential equation which has the solution in the form:

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(\beta)} \int_0^t e^{-\lambda(t-s)} \left(\int_0^s (s-v)^{\beta-1} h(v) dv \right) ds \\ &+ \frac{c_0}{\lambda} \left(1 - e^{-\lambda t}\right) + \frac{c_1}{\lambda^2} \left(\lambda t - 1 + e^{-\lambda t}\right) + c_2 e^{-\lambda t}, \quad t \in [0,1], \end{aligned} \tag{1.7}$$

and the derivative of this solution has the form:

$$\begin{aligned}
u'(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds - \frac{\lambda}{\Gamma(\beta)}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s (s-v)^{\beta-1}h(v)dv\right)ds \\
&\quad + c_0 e^{-\lambda t} + \frac{c_1}{\lambda}\left(1-e^{-\lambda t}\right) - \lambda c_2 e^{-\lambda t} \\
&= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds - \lambda u(t) + c_0 + c_1 t, \quad t\in[0,1]. \qquad (1.8)
\end{aligned}$$

In this case, we also find $c_0 = \lambda u(0) = \lambda c_2$.

The authors in [9] were lucky because they have imposed that $u(0) = 0$ which led to that their main results are correct.

In the present paper, we consider the nonlinear Langevin equation:

$$^cD^\beta(^cD^\alpha + \lambda)u(t) = f(t, u(t), ^cD^\alpha u(t)), \qquad t\in[0,1], \qquad (1.9)$$

subject to anti-periodic and nonlocal integral boundary conditions:

$$u(0) + u(1) = 0, \quad u'(0) = 0, \quad ^cD^\alpha u(1) = \frac{\mu}{\Gamma(\gamma)}\int_0^\eta (\eta-s)^{\gamma-1}u(s)ds, \qquad (1.10)$$

where $0 < \eta < 1,\ \gamma > 0$ and $\mu\in\mathbb{R}$.

The first condition is an anti-periodic boundary condition which appears in several mathematical modeling of a assortment of physical approaches and has recently drawn great attention for many contributors. The interpretation of the third condition is able to be expressed as the linear combination of the fractional derivative of unknown function at the end is proportional to the the Riemann-Liouville fractional integral of unknown function.

It is worth pointing out that the Langevin equations of fractional order have received considerable attentiveness. There is an unusual turnout to study fractional Langevin equations by a large number of authors (for instance, see [1–8, 12–19, 21–24]) due to its multiple applications in different fields of science.

2. Preliminaries

This section is dedicated to complete some fractional calculus concepts mentioned in the introduction. Also, to identify the general form of the solution for the linear Langevin equation of two fractional order (1.5) with the boundary conditions (1.10).

Lemma 2.1. *Let α and β be positive reals. If f is a continuous function, then we have:*

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta}f(t).$$

Lemma 2.2. *Let α be positive real. Then we have:*

$$\begin{aligned}
I^\alpha t^\rho &= \frac{\Gamma(\rho+1)}{\Gamma(\rho+\alpha+1)}t^{\rho+\alpha}, \qquad \rho > -1, \\
^cD^\alpha t^\rho &= \frac{\Gamma(\rho+1)}{\Gamma(\rho-\alpha+1)}t^{\rho-\alpha}, \qquad \rho > -1, \rho \neq 0, 1, \cdots, [\alpha] \\
^cD^\alpha t^\rho &= 0, \qquad \rho = 0, 1, \cdots, [\alpha]
\end{aligned}$$

where $[a]$ is the largest integer less than α.

Lemma 2.3. *Let $n \in \mathbb{N}$ and $n-1 < \alpha \leq n$. If u is a continuous function, then we have:*

$$I^{\alpha}\ {}^{c}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}.$$

Let us now examine the linear fractional Langevin differential equation (1.5) subject to the boundary conditions (1.10):

Lemma 2.4. *If $h \in C[0,1]$, then the unique solution of the boundary value problem (1.5) and (1.10) for $0 < \alpha < 1$ is given by:*

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s)ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds \\ & + E_\alpha(t) \left(\frac{\mu}{\Gamma(\gamma)} \int_0^\eta (\eta-s)^{\gamma-1} u(s)ds - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)ds \right) \\ & - F_\alpha(t) \left(\frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} h(s)ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} u(s)ds \right), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} E_\alpha(t) &= \frac{(2t^{\alpha+1}-1)\Gamma(\alpha+2) - \lambda(1+\alpha)(1-t)t^\alpha}{\Delta\Gamma(\alpha+2)}, \\ F_\alpha(t) &= \frac{\Gamma(\alpha+2) + \lambda(1+\alpha-2t)t^\alpha}{\Delta}, \\ \Delta &= 2\Gamma(\alpha+2) - \lambda(1-\alpha) \neq 0. \end{aligned}$$

When $\alpha = 1$ and $\lambda \neq 0$, the unique solution is given by:

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\beta)} \int_0^t e^{-\lambda(t-s)} \left(\int_0^s (s-v)^{\beta-1} h(v)dv \right) ds \\ & + \frac{E(t)}{\Gamma(\beta)} \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s (s-v)^{\beta-1} h(v)dv \right) ds \\ & + F(t) \left[\frac{\mu}{\Gamma(\gamma)} \int_0^\eta (\eta-s)^{\gamma-1} u(s)ds - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)ds \right], \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} E(t) &= \frac{2(\lambda t - 1 + e^{-\lambda t}) - \lambda^2}{2\lambda(1-e^{-\lambda})}, \\ F(t) &= \frac{2(\lambda t - 1 + e^{-\lambda t}) - (\lambda - 1 + e^{-\lambda})}{2\lambda(1-e^{-\lambda})}. \end{aligned}$$

Proof. As we mentioned in the introduction, the solution of the equation (1.5) for $\alpha < 1$ has the form (1.6) and with the boundary condition $u'(0) = 0$ we deduced that $c_0 = \lambda c_2$. By using the anti-periodic boundary condition $u(0) + u(1) = 0$, we find that $u(1) = -c_2$ which can be rewritten as:

$$\begin{aligned} \left(2 + \frac{\lambda}{\Gamma(\alpha+1)}\right) c_2 + \frac{1}{\Gamma(\alpha+2)} c_1 = {} & \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} u(s)ds \\ & - \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} h(s)ds. \end{aligned}$$

By operating ${}^cD^\alpha$ on both sides of (1.6) and using the last condition in (1.10), we get:

$$2\lambda c_2 + c_1 = \frac{\mu}{\Gamma(\gamma)} \int_0^\eta (\eta - s)^{\gamma-1} u(s)ds - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)ds.$$

Solving the former two equations to obtain:

$$\begin{aligned} c_1 = {} & \frac{\mu(\Delta + 2\lambda)}{\Delta\Gamma(\gamma)} \int_0^\eta (\eta - s)^{\gamma-1} u(s)ds - \frac{\Delta + 2\lambda}{\Delta\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)ds \\ & + \frac{2\lambda\Gamma(\alpha+2)}{\Delta\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} h(s)ds - \frac{2\lambda^2\Gamma(\alpha+2)}{\Delta\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} u(s)ds, \end{aligned}$$

and

$$\begin{aligned} c_2 = {} & -\frac{\mu}{\Delta\Gamma(\gamma)} \int_0^\eta (\eta - s)^{\gamma-1} u(s)ds + \frac{1}{\Delta\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)ds \\ & - \frac{\Gamma(\alpha+2)}{\Delta\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} h(s)ds + \frac{\lambda\Gamma(\alpha+2)}{\Delta\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} u(s)ds. \end{aligned}$$

Substituting into (1.6) to obtain (2.1).

When $\alpha = 1$ and $\lambda \neq 0$, we have $c_0 = \lambda c_2$ and by using the anti-periodic boundary condition $u(0) + u(1) = 0$, we find that:

$$2c_2 + \frac{c_1}{\lambda^2}\left(\lambda - 1 + e^{-\lambda}\right) = -\frac{1}{\Gamma(\beta)} \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s (s-v)^{\beta-1} h(v)dv \right) ds.$$

From (1.8) and the last condition in (1.10), we get:

$$2\lambda c_2 + c_1 = \frac{\mu}{\Gamma(\gamma)} \int_0^\eta (\eta - s)^{\gamma-1} u(s)ds - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)ds.$$

Solving the former two equations to obtain:

$$\begin{aligned} c_1 = {} & \frac{\lambda}{1 - e^{-\lambda}} \left[\frac{\lambda}{\Gamma(\beta)} \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s (s-v)^{\beta-1} h(v)dv \right) ds \right. \\ & \left. + \frac{\mu}{\Gamma(\gamma)} \int_0^\eta (\eta - s)^{\gamma-1} u(s)ds - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)ds \right], \end{aligned}$$

and

$$\begin{aligned} c_2 = {} & -\frac{\lambda}{2(1 - e^{-\lambda})\Gamma(\beta)} \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s (s-v)^{\beta-1} h(v)dv \right) ds \\ & - \frac{\lambda - 1 + e^{-\lambda}}{2\lambda(1 - e^{-\lambda})} \left[\frac{\mu}{\Gamma(\gamma)} \int_0^\eta (\eta - s)^{\gamma-1} u(s)ds - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)ds \right]. \end{aligned}$$

Substituting into (1.7) to obtain (2.2). □

To facilitate writing, we assume that:

$$E_1 = \max_{t \in [0,1]} |E_\alpha(t)| = \frac{\Gamma(\alpha+2) + |\lambda|(1+\alpha)}{|\Delta|\Gamma(\alpha+2)}, \tag{2.3}$$

$$F_1 = \max_{t\in[0,1]} |F_\alpha(t)| = \frac{\Gamma(\alpha+2) + |\lambda|(1+\alpha)}{|\Delta|}. \tag{2.4}$$

Also, from the increasingly of the function $t \mapsto \lambda t - 1 + e^{-\lambda t}$ on $(0, 1)$ for all $0 \neq \lambda \in \mathbb{R}$, we can introduce that:

$$E = \max_{t\in[0,1]} |E(t)| = \frac{|2(\lambda - 1 + e^{-\lambda}) - \lambda^2|}{2\lambda(1 - e^{-\lambda})}, \tag{2.5}$$

$$F = \max_{t\in[0,1]} |F(t)| = \frac{\lambda - 1 + e^{-\lambda}}{2\lambda(1 - e^{-\lambda})}. \tag{2.6}$$

According to Lemma 2.2, we can deduce that:

$$\begin{aligned}
{}^cD^\alpha E_\alpha(t) &= \frac{2t\Gamma(\alpha+2) - \lambda[1 - (1+\alpha)t]}{\Delta}, \\
{}^cD^\alpha F_\alpha(t) &= \frac{\lambda(1-2t)\Gamma(\alpha+2)}{\Delta}, \\
E'(t) = F'(t) &= \frac{1 - e^{-\lambda t}}{1 - e^{-\lambda}}.
\end{aligned}$$

It is not difficult to show that:

$$\max_{t\in[0,1]} |{}^cD^\alpha E_\alpha(t)| = \frac{2\Gamma(\alpha+2) + |\lambda|(2+\alpha)}{|\Delta|} = E_2, \tag{2.7}$$

$$\max_{t\in[0,1]} |{}^cD^\alpha F_\alpha(t)| = \frac{|\lambda|\Gamma(\alpha+2)}{|\Delta|} = F_2, \tag{2.8}$$

$$\max_{t\in[0,1]} |E'(t)| = \max_{t\in[0,1]} |F'(t)| = 1. \tag{2.9}$$

3. Basic Constructions

In the present section, we investigate the existence of solution for the nonlinear fractional Langevin equation (1.9) subject to the anti-periodic and nonlocal integral boundary conditions (1.10). In order to do this, we apply the same fixed point theorems used in [9]. However, we do not use the technique of the green function due to its length.

Let the space $C[0, 1]$ be the space of all continuous functions defined on the interval $[0, 1]$. Define the space

$$\mathbb{E} = \{v | v \in C[0, 1], {}^c D^\rho v \in C[0, 1], 0 < \rho \leq 1\}$$

equipped with the norm

$$\|v\|_{\mathbb{E}} = \max_{t\in[0,1]} |v(t)| + \max_{t\in[0,1]} |{}^cD^\rho v(t)|.$$

It is worth pointing out that Su [20] proved that $\mathbb{E}$ is a Banach space equipped with the former norm.

As an indispensable part of the basic needs to complete our investigations, we assume the assumptions:

($\mathfrak{R}_1$) The function $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous.

($\mathfrak{R}_2$) There exist nonnegative function $\omega \in L_1[0,1]$ such that:

$$|f(t,u,v)| \le \omega(t) + c|u|^{\delta_1} + k|v|^{\delta_2}, \qquad \forall (t,u,v) \in ([0,1],\mathbb{R},\mathbb{R}),$$

where c and k are positive constants and $0 < \delta_i < 1, i = 1,2$.

($\mathfrak{R}_3$) The function f satisfies the Lipschitz condition:

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \le \mathcal{L}(|u_1-u_2| + |v_1-v_2|), \qquad \forall t \in [0,1],$$

where $\mathcal{L}$ is a positive constant and $u_1,u_2,v_1,v_2 \in \mathbb{R}$.

Now, let the operator $\mathcal{P}_\alpha : \mathbb{E} \to \mathbb{E}$ be defined, for $\alpha < 1$, as:

$$\begin{aligned}\mathcal{P}_\alpha u(t) = &\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} f(s,u(s),{}^c D^\alpha u(s))ds \\ &- \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds + E_\alpha(t)\left(\frac{\mu}{\Gamma(\gamma)}\int_0^\eta (\eta-s)^{\gamma-1}u(s)ds\right. \\ &\left.-\frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,u(s),{}^c D^\alpha u(s))ds\right) \\ &+ F_\alpha(t)\left(\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}u(s)ds\right. \\ &\left.-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1} f(s,u(s),{}^c D^\alpha u(s))ds\right),\end{aligned} \tag{3.1}$$

and the operator $\mathcal{P} : \mathbb{E} \to \mathbb{E}$ be defined as:

$$\begin{aligned}\mathcal{P}u(t) = &\frac{1}{\Gamma(\beta)}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s (s-v)^{\beta-1} f(v,u(v),u'(v))dv\right)ds \\ &+ \frac{E(t)}{\Gamma(\beta)}\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s (s-v)^{\beta-1} f(v,u(v),u'(v))dv\right)ds \\ &+ F(t)\left[\frac{\mu}{\Gamma(\gamma)}\int_0^\eta (\eta-s)^{\gamma-1}u(s)ds\right. \\ &\left.-\frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,u(s),u'(s))ds\right].\end{aligned} \tag{3.2}$$

For the sake of the readers' convenience, we provide the following constants:

$$Q = Q_1 + Q_2 \qquad \text{and} \qquad R = R_1 + R_2, \tag{3.3}$$

where

$$\begin{aligned} Q_1 &= \frac{1+F_1}{\Gamma(\alpha+\beta+1)} + \frac{E_1}{\Gamma(\beta+1)}, & R_1 &= \frac{|\lambda|(1+F_1)}{\Gamma(\alpha+1)} + \frac{|\mu|E_1\eta^\gamma}{\Gamma(\gamma+1)}, \\ Q_2 &= \frac{1+E_2}{\Gamma(\beta+1)} + \frac{F_2}{\Gamma(\alpha+\beta+1)}, & R_2 &= |\lambda| + \frac{|\mu|E_2\eta^\gamma}{\Gamma(\gamma+1)} + \frac{|\lambda|F_2}{\Gamma(\alpha+1)}, \end{aligned}$$

and

$$\Phi = \Phi_1 + \Phi_2 \qquad \text{and} \qquad \Psi = \Psi_1 + \Psi_2, \tag{3.4}$$

where

$$\Phi_1 = \frac{(1+E)(1-e^{-\lambda})}{\lambda\Gamma(\beta+1)} + \frac{F}{\Gamma(\beta+1)}, \qquad \Psi_1 = \frac{F|\mu|\eta^\gamma}{\Gamma(\gamma+1)},$$
$$\Phi_2 = \frac{(1+|\lambda|)(1-e^{-\lambda})}{\lambda\Gamma(\beta+1)} + \frac{2}{\Gamma(\beta+1)}, \qquad \Psi_2 = \frac{|\mu|\eta^\gamma}{\Gamma(\gamma+1)}.$$

Consider the closed balls:

$$S_r = \{u \in \mathbb{E} |\ \|u\|_{\mathbb{E}} \le r\}, \tag{3.5}$$
$$S_\ell = \{u \in \mathbb{E} |\ \|u\|_{\mathbb{E}} \le \ell\}, \tag{3.6}$$

with radii, respectively,

$$r \ge \max\left\{4Q\|\omega\|_{\mathbb{E}}, (4cQ)^{\frac{1}{1-\delta_1}}, (4kQ)^{\frac{1}{1-\delta_2}}, 4rR\right\},$$
$$\ell \ge \max\left\{4\Phi\|\omega\|_{\mathbb{E}}, (4c\Phi)^{\frac{1}{1-\delta_1}}, (4k\Phi)^{\frac{1}{1-\delta_2}}, 4\ell\Psi\right\}.$$

Lemma 3.1. *Assume that the assumption ($\mathcal{R}_2$) holds, then the operator $\mathcal{P}_\alpha : \mathbb{E} \to \mathbb{E}$ defined in (3.1), satisfies $\mathcal{P}_\alpha S_r \subseteq S_r$ where S_r defined in (3.5).*

Proof. Taking $u \in S_r$ and using ($\mathcal{R}_2$) yield:

$$\begin{aligned}
&|\mathcal{P}_\alpha u(t)| \\
\le& \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |f(s, u(s), {}^c D^\alpha u(s))| ds \\
&+ \frac{|\lambda|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u(s)| ds + |E_\alpha(t)| \left(\frac{|\mu|}{\Gamma(\gamma)} \int_0^\eta (\eta-s)^{\gamma-1} |u(s)| ds \right. \\
&\left. - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} |f(s, u(s), {}^c D^\alpha u(s))| ds \right) \\
&+ |F_\alpha(t)| \left(\frac{|\lambda|}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |u(s)| ds \right. \\
&\left. + \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} |f(s, u(s), {}^c D^\alpha u(s))| ds \right) \\
\le& \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} (\omega(s) + c|u(s)|^{\delta_1} + k|{}^c D^\alpha u(s)|^{\delta_2}) ds \\
&+ \frac{|\lambda|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u(s)| ds + |E_\alpha(t)| \left(\frac{|\mu|}{\Gamma(\gamma)} \int_0^\eta (\eta-s)^{\gamma-1} |u(s)| ds \right. \\
&\left. + \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} (\omega(s) + c|u(s)|^{\delta_1} + k|{}^c D^\alpha u(s)|^{\delta_2}) ds \right) \\
&+ |F_\alpha(t)| \left(\frac{|\lambda|}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |u(s)| ds \right. \\
&\left. + \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} (\omega(s) + c|u(s)|^{\delta_1} + k|{}^c D^\alpha u(s)|^{\delta_2}) ds \right) \\
\le& \frac{\|\omega\|_{\mathbb{E}} + cr^{\delta_1} + kr^{\delta_2}}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} ds + \frac{|\lambda| \|u\|_{\mathbb{E}}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds
\end{aligned}$$

$$+\max_{t\in[0,1]}|E_\alpha(t)|\left(\frac{|\mu|\|u\|_{\mathbb{E}}}{\Gamma(\gamma)}\int_0^\eta(\eta-s)^{\gamma-1}|ds+\frac{\|\omega\|_{\mathbb{E}}+cr^{\delta_1}+kr^{\delta_2}}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}ds\right)$$
$$+\max_{t\in[0,1]}|F_\alpha(t)|\left(\frac{\|\omega\|_{\mathbb{E}}+cr^{\delta_1}+kr^{\delta_2}}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}ds+\frac{|\lambda|\|u\|_{\mathbb{E}}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}ds\right)$$
$$\leq\left(\frac{1+F_1}{\Gamma(\alpha+\beta+1)}+\frac{E_1}{\Gamma(\beta+1)}\right)(\|\omega\|_{\mathbb{E}}+cr^{\delta_1}+kr^{\delta_2})+\left(\frac{|\lambda|(1+F_1)}{\Gamma(\alpha+1)}+\frac{|\mu|E_1\eta^\gamma}{\Gamma(\gamma+1)}\right)\|u\|_{\mathbb{E}}$$
$$\leq Q_1(\|\omega\|_{\mathbb{E}}+cr^{\delta_1}+kr^{\delta_2})+R_1r.$$

It is easy to see that:

$$\begin{aligned}&{}^cD^\alpha\mathcal{P}_\alpha u(t)\\
=&\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}f(s,u(s),{}^cD^\alpha u(s))ds-\lambda u(t)\\
&+{}^cD^\alpha E_\alpha(t)\left(\frac{\mu}{\Gamma(\gamma)}\int_0^\eta(\eta-s)^{\gamma-1}u(s)ds-\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}f(s,u(s),{}^cD^\alpha u(s))ds\right)\\
&-{}^cD^\alpha F_\alpha(t)\left(\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f(s,u(s),{}^cD^\alpha u(s))ds-\frac{\lambda}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}u(s)ds\right).\end{aligned}$$

Whence, as above, we can find that:

$$\begin{aligned}|{}^cD^\alpha\mathcal{P}_\alpha u(t)|\leq&\frac{\|\omega\|_{\mathbb{E}}+cr^{\delta_1}+kr^{\delta_2}}{\Gamma(\beta+1)}+|\lambda|r\\
&+E_2\left(\frac{|\mu|\eta^\gamma r}{\Gamma(\gamma+1)}+\frac{\|\omega\|_{\mathbb{E}}+cr^{\delta_1}+kr^{\delta_2}}{\Gamma(\beta+1)}\right)\\
&+F_2\left(\frac{\|\omega\|_{\mathbb{E}}+cr^{\delta_1}+kr^{\delta_2}}{\Gamma(\alpha+\beta+1)}+\frac{|\lambda|r}{\Gamma(\alpha+1)}\right)\\
=&Q_2(\|\omega\|_{\mathbb{E}}+cr^{\delta_1}+kr^{\delta_2})+R_2r.\end{aligned}$$

These lead to:

$$\begin{aligned}\|\mathcal{P}_\alpha u\|_{\mathbb{E}}&=\max_{t\in[0,1]}|\mathcal{P}_\alpha u(t)|+\max_{t\in[0,1]}|{}^cD^\alpha\mathcal{P}_\alpha u(t)|\\
&\leq Q(\|\omega\|_{\mathbb{E}}+cr^{\delta_1}+kr^{\delta_2})+Rr\leq r,\end{aligned}$$

where Q and R are defined as in (3.3), which means that the operator $\mathcal{P}_\alpha:\mathbb{E}\to\mathbb{E}$ satisfies $\mathcal{P}_\alpha S_r\subseteq S_r$. □

Lemma 3.2. *Assume that the assumption ($\mathcal{R}_2$) holds, then the operator $\mathcal{P}:\mathbb{E}\to\mathbb{E}$ defined in (3.2), satisfies $\mathcal{P}S_\ell\subseteq S_\ell$ where S_ℓ defined in (3.6).*

Proof. Taking $u\in S_\ell$ and using ($\mathcal{R}_3$) yield:

$$\begin{aligned}|\mathcal{P}u(t)|\leq&\frac{1}{\Gamma(\beta)}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s(s-v)^{\beta-1}|f(v,u(v),u'(v))|dv\right)ds\\
&+\frac{|E(t)|}{\Gamma(\beta)}\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s(s-v)^{\beta-1}|f(v,u(v),u'(v))|dv\right)ds\\
&+|F(t)|\left(\frac{|\mu|}{\Gamma(\gamma)}\int_0^\eta(\eta-s)^{\gamma-1}|u(s)|ds\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}|f(s,u(s),u'(s))|ds\Bigg)\\
\leq&\frac{\|\omega\|_{\mathbb{E}}+c\ell^{\delta_1}+k\ell^{\delta_2}}{\Gamma(\beta+1)}\left(\int_0^t s^{\beta}e^{-\lambda(t-s)}ds+|E(t)|\int_0^1 s^{\beta}e^{-\lambda(1-s)}ds\right)\\
&+|F(t)|\left(\frac{|\mu|\eta^{\gamma}\|u\|_{\mathbb{E}}}{\Gamma(\gamma+1)}+\frac{\|\omega\|_{\mathbb{E}}+c\ell^{\delta_1}+k\ell^{\delta_2}}{\Gamma(\beta+1)}\right).
\end{aligned}
$$

It is well-known that:

$$
\int_0^t s^{\beta}e^{-\lambda(t-s)}ds\leq t^{\beta}\int_0^t e^{-\lambda(t-s)}ds=\frac{1-e^{-\lambda t}}{\lambda}t^{\beta}\leq\frac{1-e^{-\lambda}}{\lambda},
$$

and also

$$
\int_0^1 s^{\beta}e^{-\lambda(1-s)}ds\leq\int_0^1 e^{-\lambda(1-s)}ds=\frac{1-e^{-\lambda}}{\lambda},
$$

which yield:

$$
\begin{aligned}
|\mathcal{P}u(t)|\leq&\frac{\|\omega\|_{\mathbb{E}}+c\ell^{\delta_1}+k\ell^{\delta_2}}{\Gamma(\beta+1)}\frac{(1+E)(1-e^{-\lambda})}{\lambda}+F\left(\frac{|\mu|\eta^{\gamma}\ell}{\Gamma(\gamma+1)}+\frac{\|\omega\|_{\mathbb{E}}+c\ell^{\delta_1}+k\ell^{\delta_2}}{\Gamma(\beta+1)}\right)\\
=&\Phi_1(\|\omega\|_{\mathbb{E}}+c\ell^{\delta_1}+k\ell^{\delta_2})+\Psi_1\ell.
\end{aligned}
$$

It is easy, by using the Leibnitz integral rule, to see that:

$$
\begin{aligned}
\frac{d}{dt}(\mathcal{P}u(t))=&\frac{-\lambda}{\Gamma(\beta)}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s(s-v)^{\beta-1}f(v,u(v),u'(v))dv\right)ds\\
&+\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}f(s,u(s),u'(s))ds\\
&+\frac{E'(t)}{\Gamma(\beta)}\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s(s-v)^{\beta-1}f(v,u(v),u'(v))dv\right)ds\\
&+F'(t)\left[\frac{\mu}{\Gamma(\gamma)}\int_0^{\eta}(\eta-s)^{\gamma-1}u(s)ds\right.\\
&\left.-\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}f(s,u(s),u'(s))ds\right].
\end{aligned}
$$

Whence, as above, we can find that:

$$
\begin{aligned}
\left|\frac{d}{dt}(\mathcal{P}u(t))\right|\leq&\frac{\|\omega\|_{\mathbb{E}}+c\ell^{\delta_1}+k\ell^{\delta_2}}{\Gamma(\beta+1)}\frac{(1+|\lambda|)(1-e^{-\lambda})}{\lambda}+\frac{|\mu|\eta^{\gamma}\ell}{\Gamma(\gamma+1)}\\
&+\frac{2(\|\omega\|_{\mathbb{E}}+c\ell^{\delta_1}+k\ell^{\delta_2})}{\Gamma(\beta+1)}\\
=&\Phi_2(\|\omega\|_{\mathbb{E}}+c\ell^{\delta_1}+k\ell^{\delta_2})+\Psi_2\ell.
\end{aligned}
$$

These lead to:

$$
\begin{aligned}
\|\mathcal{P}u\|_{\mathbb{E}}=&\max_{t\in[0,1]}|\mathcal{P}u(t)|+\max_{t\in[0,1]}|D\mathcal{P}u(t)|\\
\leq&\Phi(\|\omega\|_{\mathbb{E}}+c\ell^{\delta_1}+k\ell^{\delta_2})+\Psi\ell\leq\ell,
\end{aligned}
$$

where Φ and Ψ are defined as in (3.4), which means that the operator $\mathcal{P}:\mathbb{E}\to\mathbb{E}$ satisfies $\mathcal{P}S_{\ell}\subseteq S_{\ell}$. $\square$

Lemma 3.3. *Assume that the assumption ($\mathcal{R}_3$) holds, then the operator $\mathcal{P}_\alpha : \mathbb{E} \to \mathbb{E}$ defined in (3.1), satisfies $\mathcal{P}_\alpha \mathcal{B}_r \subseteq \mathcal{B}_r$ where:*

$$\mathcal{B}_r = \{u \in \mathbb{E} | \ \|u\|_{\mathbb{E}} \leq r\}, \tag{3.7}$$

with radius

$$r \geq |1 - Q\mathcal{L} - R|^{-1} QM,$$

where Q and R are defined in (3.3) and $M = \max_{t\in[0,1]} |f(t,0,0)|$.

Proof. In view of Lipschitz condition in the assumption ($\mathcal{R}_3$), we can find that:

$$\begin{aligned}
|f(t, u(t),{}^c D^\alpha u(t))| &= |f(t, u(t),{}^c D^\alpha u(t)) - f(t,0,0) + f(t,0,0)| \\
&\leq |f(t, u(t),{}^c D^\alpha u(t)) - f(t,0,0)| + |f(t,0,0)| \\
&\leq \mathcal{L}(|u(t)| + |{}^c D^\alpha u(t)|) + M \\
&\leq \mathcal{L}\|u\|_{\mathbb{E}} + M \leq \mathcal{L}r + M.
\end{aligned}$$

Similarity, as in Lemma 3.1, we can complete the proof. □

Also, we can prove the following lemma:

Lemma 3.4. *Assume that the assumption ($\mathcal{R}_3$) holds, then the operator $\mathcal{P} : \mathbb{E} \to \mathbb{E}$ defined in (3.2), satisfies $\mathcal{P}\mathcal{B}_\ell \subseteq \mathcal{B}_\ell$ where:*

$$\mathcal{B}_\ell = \{u \in \mathbb{E} | \ \|u\|_{\mathbb{E}} \leq \ell\}, \tag{3.8}$$

with radius:

$$\ell \geq |1 - \Phi\mathcal{L} - \Psi|^{-1} \Phi M,$$

where Φ and Ψ are defined in (3.4) and $M = \max_{t\in[0,1]} |f(t,0,0)|$.

4. Existence Results

In the present section, the existence of solutions for the boundary value problem (1.9)-(1.10) is investigated by means of applying the Schauder fixed point theorem. Also, we provide an example to illustrate the applicability of our results in this section.

Theorem 4.1. *Assume that the assumptions ($\mathcal{R}_1$) and ($\mathcal{R}_2$) hold. Then, the boundary value problem (1.9)-(1.10) has at least one solution in* $[0,1]$.

Proof. In addition to results of Lemmas 3.1 and 3.2, it suffices to prove that the operator $\mathcal{P}_\alpha : S_r \to S_r$ if $\alpha < 1$ and the operator $\mathcal{P} : S_\ell \to S_\ell$ if $\alpha = 1$ are completely continuous. Then with applying the Schauder fixed point theorem, we show the existence of solutions for the boundary value problem (1.9)-(1.10). In order to show that, the assumptions ($\mathcal{R}_1$) and $u(t) \in \mathbb{E}$ yield the continuity of the operators $\mathcal{P}_\alpha$ and $\mathcal{P}$. Now, let $0 \leq t_1 < t_2 \leq 1$ and

$$N = \max_{t\in[0,1]} |f(s, u(s),{}^c D^\alpha u(s))|.$$

From the definition of E_α in Lemma 3.4, we can get:

$$|E_\alpha(t_2) - E_\alpha(t_1)| \leq \frac{2(t_2^{\alpha+1} - t_1^{\alpha+1})\Gamma(\alpha+2) + |\lambda|(1+\alpha)(t_2^\alpha - t_1^\alpha - t_2^{\alpha+1} + t_1^{\alpha+1})}{|\Delta|\Gamma(\alpha+2)}$$

$$\leq \frac{2(t_2^{\alpha+1}-t_1^{\alpha+1})\Gamma(\alpha+2)+|\lambda|(1+\alpha)(t_2^{\alpha}-t_1^{\alpha})}{|\Delta|\Gamma(\alpha+2)}.$$

Similarly, for F_a, E and F, we can deduce that:

$$\begin{aligned}
|F_\alpha(t_2)-F_\alpha(t_1)| &\leq \frac{|\lambda|(1+\alpha)(t_2^{\alpha}-t_1^{\alpha})}{|\Delta|},\\
|E(t_2)-E(t_1)| &\leq \frac{|\lambda|(t_2-t_1)+|e^{-\lambda t_2}-e^{-\lambda t_1}|}{\lambda(1-e^{-\lambda})},\\
|F(t_2)-F(t_1)| &\leq \frac{|\lambda|(t_2-t_1)+|e^{-\lambda t_2}-e^{-\lambda t_1}|}{\lambda(1-e^{-\lambda})}.
\end{aligned}$$

Thus, for $\alpha<1$, we get:

$$\begin{aligned}
&|\mathcal{P}_\alpha u(t_2)-\mathcal{P}_\alpha u(t_1)|\\
\leq&\frac{N}{\Gamma(\alpha+\beta)}\int_0^{t_1}\left[(t_2-s)^{\alpha+\beta-1}-(t_1-s)^{\alpha+\beta-1}\right]ds\\
&+\frac{N}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}ds+\frac{|\lambda|r}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right]ds\\
&+\frac{|\lambda|r}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds+|E_\alpha(t_2)-E_\alpha(t_1)|C_1+|F_\alpha(t_2)-F_\alpha(t_1)|C_2\\
\leq&\frac{N(t_2^{\alpha+\beta}-t_1^{\alpha+\beta})}{\Gamma(\alpha+\beta+1)}+\frac{2|\lambda|r(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)}\\
&+\frac{2(t_2^{\alpha+1}-t_1^{\alpha+1})\Gamma(\alpha+2)+|\lambda|(1+\alpha)(t_2^{\alpha}-t_1^{\alpha})}{|\Delta|\Gamma(\alpha+2)}C_1+\frac{|\lambda|(1+\alpha)(t_2^{\alpha}-t_1^{\alpha})}{|\Delta|}C_2,
\end{aligned}$$

and

$$\begin{aligned}
&|^cD^\alpha(\mathcal{P}_\alpha u(t_2))-^cD^\alpha(\mathcal{P}_\alpha u(t_1))|\\
\leq&\frac{N}{\Gamma(\beta)}\int_0^{t_1}\left[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right]ds\\
&+\frac{N}{\Gamma(\beta)}\int_{t_1}^{t_2}(t_2-s)^{\beta-1}ds+|\lambda||u(t_2)-u(t_1)|\\
&+|^cD^\alpha E_\alpha(t_2)-^cD^\alpha E_\alpha(t_1)|C_1+|^cD^\alpha F_\alpha(t_2)-^cD^\alpha F_\alpha(t_1)|C_2\\
\leq&\frac{N(t_2^{\beta}-t_1^{\beta})}{\Gamma(\beta+1)}+|\lambda|\left(\frac{N(t_2^{\alpha+\beta}-t_1^{\alpha+\beta})}{\Gamma(\alpha+\beta+1)}+\frac{2|\lambda|r(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)}\right)\\
&+|\lambda|\left(\frac{2(t_2^{\alpha+1}-t_1^{\alpha+1})\Gamma(\alpha+2)+|\lambda|(1+\alpha)(t_2^{\alpha}-t_1^{\alpha})}{|\Delta|\Gamma(\alpha+2)}C_1+\frac{|\lambda|(1+\alpha)(t_2^{\alpha}-t_1^{\alpha})}{|\Delta|}C_2\right)\\
&+\frac{|2\Gamma(\alpha+2)+\lambda(1+\alpha)|(t_2-t_1)}{|\Delta|}C_1+\frac{2|\lambda|\Gamma(\alpha+2)(t_2-t_1)}{|\Delta|}C_2,
\end{aligned}$$

where

$$\begin{aligned}
C_1&=\frac{|\mu|}{\Gamma(\gamma)}\int_0^{\eta}(\eta-s)^{\gamma-1}|u(s)|ds-\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}|f(s,u(s),^cD^\alpha u(s))|ds,\\
C_2&=\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}|f(s,u(s),^cD^\alpha u(s))|ds+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|u(s)|ds.
\end{aligned}$$

It is obvious that, they are independent of u and if $t_2 - t_1 \to 0$, then $\mathcal{P}_\alpha u(t_2) - \mathcal{P}_\alpha u(t_1) \to 0$ and ${}^cD^\alpha(\mathcal{P}_\alpha u(t_2)) - {}^cD^\alpha(\mathcal{P}_\alpha u(t_1)) \to 0$. These mean that the operator $\mathcal{P}_\alpha : \mathbb{E} \to \mathbb{E}$ is equicontinuous.

When $\alpha = 1$, we have:

$$\begin{aligned}
|\mathcal{P}u(t_2) - \mathcal{P}u(t_1)| \leq & \frac{N}{\Gamma(\beta+1)} \int_0^{t_1} s^\beta |e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)}| ds \\
& + \frac{N}{\Gamma(\beta+1)} \int_{t_1}^{t_2} s^\beta e^{-\lambda(t_2-s)} ds \\
& + |E(t_2) - E(t_1)| C_3 + |F(t_2) - F(t_1)| C_4 \\
\leq & \frac{N|e^{-\lambda t_2} - e^{-\lambda t_1}| t_1^\beta}{\Gamma(\beta+1)} \int_0^{t_1} e^{\lambda s} ds + \frac{N e^{-\lambda t_2} t_2^\beta}{\Gamma(\beta+1)} \int_{t_1}^{t_2} e^{\lambda s} ds \\
& + \frac{|\lambda|(t_2 - t_1) + |e^{-\lambda t_2} - e^{-\lambda t_1}|}{\lambda(1 - e^{-\lambda})} (C_3 + C_4) \\
= & \frac{N|e^{-\lambda t_2} - e^{-\lambda t_1}| t_1^\beta}{\Gamma(\beta+1)} \frac{e^{\lambda t_1} - 1}{\lambda} + \frac{N e^{-\lambda t_2} t_2^\beta}{\Gamma(\beta+1)} \frac{1 - e^{\lambda(t_2 - t_1)}}{\lambda} \\
& + \frac{|\lambda|(t_2 - t_1) + |e^{-\lambda t_2} - e^{-\lambda t_1}|}{\lambda(1 - e^{-\lambda})} (C_3 + C_4),
\end{aligned}$$

and, similarity

$$\begin{aligned}
& \left| \frac{d}{dt}(\mathcal{P}_\alpha u(t))_{t=t_2} - \frac{d}{dt}(\mathcal{P}_\alpha u(t))_{t=t_1} \right| \\
\leq & \frac{N|e^{-\lambda t_2} - e^{-\lambda t_1}| t_1^\beta}{\Gamma(\beta+1)} \frac{e^{\lambda t_1} - 1}{\lambda} + \frac{N e^{-\lambda t_2} t_2^\beta}{\Gamma(\beta+1)} \frac{1 - e^{\lambda(t_2 - t_1)}}{\lambda} \\
& + \frac{N(t_2^\beta - t_1^\beta)}{\Gamma(\beta+1)} + \frac{|e^{-\lambda t_2} - e^{-\lambda t_1}|}{|1 - e^{-\lambda}|} (C_3 + C_4),
\end{aligned}$$

where

$$\begin{aligned}
C_3 &= \frac{1}{\Gamma(\beta)} \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s (s-v)^{\beta-1} |f(v, u(v), u'(v))| dv \right) ds, \\
C_4 &= \frac{|\mu|}{\Gamma(\gamma)} \int_0^\eta (\eta - s)^{\gamma-1} |u(s)| ds + \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} |f(s, u(s), u'(s))| ds.
\end{aligned}$$

It is obvious that, they are independent of u and if $t_2 - t_1 \to 0$, then $\mathcal{P}u(t_2) - \mathcal{P}u(t_1) \to 0$ and $D(\mathcal{P}u(t_2)) - D(\mathcal{P}u(t_1)) \to 0$. These mean that the operator $\mathcal{P} : \mathbb{E} \to \mathbb{E}$ is equicontinuous.

Therefore, all assumptions of the Schauder fixed point theorem hold which leads to the boundary value problem (1.9)-(1.10) has at least one solution in $[0,1]$. $\square$

Example 4.1. Consider the following boundary value problem for fractional Langevin equations:

$$\begin{cases} {}^cD^{\frac{3}{2}}({}^cD^\alpha + 1/4)u(t) = f(t, u(t), {}^cD^\alpha u(t)), & 0 < t < 1, \\ u(0) + u(1) = 0, \quad u'(0) = 0, \quad {}^cD^\alpha u(1) = \dfrac{5}{\Gamma(5/2)} \displaystyle\int_0^\eta (\eta - s)^{3/2} u(s) ds. & \end{cases} \tag{4.1}$$

Here we take $\beta = 3/2,\ \mu = 5,\ \gamma = 5/2,\ \lambda = 1/4, \eta = 1/2$, and

$$f(t,x,y) = \frac{t^2\sin(\pi t)}{1+\sqrt{t}} + \pi\int_0^t x^{\delta_1}\sin^2(\pi s)ds + y^{\delta_2}\cos^2(\pi t).$$

It is easy to show that:

$$|f(t,x,y)| \le \frac{t^2}{1+\sqrt{t}} + \pi|x|^{\delta_1}\int_0^1 \sin^2(\pi s)ds + |y|^{\delta_2} \le t^{\frac{3}{2}} + \frac{\pi}{2}|x|^{\delta_1} + |y|^{\delta_2},$$

which means that $c = \frac{\pi}{2},\ k = 1$ and $\omega(t) = t^{\frac{3}{2}} \in L_1[0,1]$ equipped with the norm:

$$\begin{aligned}\|\omega\|_{\mathbb{E}} &= \max_{t\in[0,1]}\{t^{\frac{3}{2}}\} + \max_{t\in[0,1]}\{{}^cD^\alpha t^{\frac{3}{2}}\}\\ &= 1 + \max_{t\in[0,1]}\left\{\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2}-\alpha)}t^{\frac{3}{2}-\alpha}\right\} = 1 + \frac{\sqrt{3\pi}}{4\Gamma(\frac{5}{2}-\alpha)}.\end{aligned}$$

Thus, the function $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ satisfies the assumptions $(\mathcal{R}_1)$ and $(\mathcal{R}_2)$ for all $0 < \delta_1, \delta_2 < 1$ and $0 < \alpha \le 1$. Therefore, according to Theorem 4.1, the boundary value problem (4.1) has at least one solution on $[0,1]$.

5. Uniqueness Results

In the present section, the uniqueness of solution for the boundary value problem (1.9)-(1.10) is investigated by means of applying the Banach fixed point theorem. Also, we provide an example to illustrate the applicability of our results in this section.

Theorem 5.1. *Assume that the assumptions $(\mathcal{R}_1)$ and $(\mathcal{R}_3)$ hold. Then, the boundary value problem (1.9)-(1.10) has a unique solution in $[0,1]$ if $\mathcal{L}Q + R < 1$ for $\alpha < 1$ and if $\mathcal{L}\Phi + \Psi < 1$ for $\alpha = 1$, where Q, R and Φ, Ψ are defined in (3.3) and (3.4), respectively.*

Proof. In addition to results of Lemmas 3.3 and 3.4, it suffices to prove that the operator $\mathcal{P}_\alpha : \mathcal{B}_r \to \mathcal{B}_r$ if $\alpha < 1$ and the operator $\mathcal{P} : \mathcal{B}_\ell \to \mathcal{B}_\ell$ if $\alpha = 1$ satisfy the contraction condition. Consider $u, v \in \mathcal{B}_r$ and $t \in [0,1]$, by the assumption $(\mathcal{R}_3)$ for $\alpha < 1$, we have:

$$\begin{aligned}&|f(s,u(s),{}^cD^\alpha u(s)) - f(s,v(s),{}^cD^\alpha v(s))|\\ &\le \mathcal{L}(|u(t)-v(t)| + |{}^cD^\alpha u(s) - {}^cD^\alpha v(s)|)\\ &\le \mathcal{L}(\max_{t\in[0,1]}|u(t)-v(t)| + \max_{t\in[0,1]}|{}^cD^\alpha u(t) - {}^cD^\alpha v(t)|)\\ &\le \mathcal{L}\|u-v\|_{\mathbb{E}},\end{aligned}$$

which leads to:

$$\begin{aligned}&|\mathcal{P}_\alpha u(t) - \mathcal{P}_\alpha v(t)|\\ \le&\frac{\mathcal{L}\|u-v\|_{\mathbb{E}}}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}ds + \frac{|\lambda|\|u-v\|_{\mathbb{E}}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\\ &+ |E_\alpha(t)|\left(\frac{|\mu|\|u-v\|_{\mathbb{E}}}{\Gamma(\gamma)}\int_0^\eta (\eta-s)^{\gamma-1}ds + \frac{\mathcal{L}\|u-v\|_{\mathbb{E}}}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}ds\right)\end{aligned}$$

$$
\begin{aligned}
&+|F_\alpha(t)|\left(\frac{\mathcal{L}\|u-v\|_{\mathbb{E}}}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}ds+\frac{|\lambda|\|u-v\|_{\mathbb{E}}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}ds\right)\\
=&(\mathcal{L}Q_1+R_1)\|u-v\|_{\mathbb{E}},
\end{aligned}
$$

and also

$$|{}^cD^\alpha\mathcal{P}_\alpha u(t)-{}^cD^\alpha\mathcal{P}_\alpha v(t)|\le(\mathcal{L}Q_2+R_2)\|u-v\|_{\mathbb{E}}.$$

Thus,

$$\|\mathcal{P}_\alpha u-\mathcal{P}_\alpha v\|_{\mathbb{E}}\le(\mathcal{L}Q+R)\|u-v\|_{\mathbb{E}}.$$

For $\alpha=1$, we have:

$$
\begin{aligned}
&|\mathcal{P}u(t)-\mathcal{P}v(t)|\\
\le&\frac{\mathcal{L}\|u-v\|_{\mathbb{E}}}{\Gamma(\beta+1)}\int_0^t s^\beta e^{-\lambda(t-s)}ds+\frac{|E(t)|\mathcal{L}\|u-v\|_{\mathbb{E}}}{\Gamma(\beta+1)}\int_0^t s^\beta e^{-\lambda(t-s)}ds\\
&+|F(t)|\left[\frac{|\mu|\|u-v\|_{\mathbb{E}}}{\Gamma(\gamma)}\int_0^\eta(\eta-s)^{\gamma-1}ds+\frac{\mathcal{L}\|u-v\|_{\mathbb{E}}}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}ds\right]\\
=&\frac{\mathcal{L}(1+E)\|u-v\|_{\mathbb{E}}t^\beta}{\Gamma(\beta+1)}\frac{1-e^{-\lambda t}}{\lambda}+F\left[\frac{|\mu|\eta^\gamma}{\Gamma(\gamma+1)}+\frac{\mathcal{L}}{\Gamma(\beta+1)}\right]\|u-v\|_{\mathbb{E}}\\
\le&(\mathcal{L}\Phi_1+\Psi_1)\|u-v\|_{\mathbb{E}},
\end{aligned}
$$

and also

$$|\mathcal{P}'u(t)-\mathcal{P}'v(t)|\le(\mathcal{L}\Phi_2+\Psi_2)\|u-v\|_{\mathbb{E}}.$$

Thus,

$$\|\mathcal{P}u-\mathcal{P}v\|_{\mathbb{E}}\le(\mathcal{L}\Phi+\Psi)\|u-v\|_{\mathbb{E}}.$$

Since $\mathcal{L}Q+R<1$ and $\mathcal{L}\Phi+\Psi<1$, then the operators $\mathcal{P}_\alpha$ and $\mathcal{P}$ are contraction operators. Therefore, all assumptions of the contraction mapping principle hold which leads to the boundary value problem (1.9)-(1.10) has a unique solution in $[0,1]$. □

Example 5.1.

$$
\begin{cases}
{}^cD^{\frac{3}{2}}({}^cD^\alpha+1/4)u(t)=f(t,u(t),{}^cD^\alpha u(t)), & 0<t<1,\\
u(0)+u(1)=0,\quad u'(0)=0,\quad {}^cD^\alpha u(1)=\dfrac{5}{\Gamma(5/2)}\displaystyle\int_0^\eta(\eta-s)^{3/2}u(s)ds.
\end{cases}
\tag{5.1}
$$

Here we take $\beta=3/2,\ \mu=2/3,\ \gamma=5/2,\ \lambda=1/4,\eta=1/3$, and

$$f(t,x,y)=\mathcal{L}\left(\frac{t^2\sin(\pi t)}{1+\sqrt{t}}+\frac{\pi}{2}\int_0^t x\sin(\pi s)ds+\tan^{-1}y\right).$$

It is easy to show that:

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\le\mathcal{L}\left(|x_1-x_2|\frac{\pi}{2}\int_0^1\sin(\pi s)ds+|y_1-y_2|\right)$$

$$= \mathcal{L}\left(|x_1 - x_2| + |y_1 - y_2|\right),$$

which means that the function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies assumption $(\mathcal{R}_1)$ and the Lipschitz condition in the assumption $(\mathcal{R}_3)$. By carrying out of Mathematica 11 software , we have:

$$Q = 3.17745, \quad R = 0.876652, \quad \Phi = 3.22745, \quad \Psi = 0.0162196,$$

which means that the condition $\mathcal{L}Q + R < 1$ is satisfied if $0 < \mathcal{L} < 0.03882$ when $\alpha = 1/2$ and the condition $\mathcal{L}\Phi + \Psi < 1$ is satisfied if $0 < \mathcal{L} < 0.304816$ when $\alpha = 1$. Therefore, according to Theorem 5.1, the boundary value problem (5.1) has unique solution on $[0,1]$ if $0 < \mathcal{L} < 0.03882$ when $\alpha = 1/2$ and if $0 < \mathcal{L} < 0.304816$ when $\alpha = 1$.

6. Conclusion

As a matter of fact and over this paper, we have determined the incorrect method to use the condition $u'(0) = 0$ in [9] , and it was rectified successfully. Likewise, in the main consequences of a proposed boundary value problem conditional on an anti-periodic and nonlocal integral boundary conditions, we used the technic of Schauder and Banach fixed point theorems to scrutinize the existence and uniqueness upshots respectively. It is worthiness to symbolize each one by an exemplar which already what we did.

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