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EXISTENCE AND MULTIPLICITY OF WEAK SOLUTIONS FOR A CLASS OF FRACTIONAL STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS WITH IMPULSIVE CONDITIONS*

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Abstract In this paper, we consider the existence and multiplicity of weak solutions for a class of fractional differential equations with non-homogeneous Sturm-Liouville conditions and impulsive conditions by using the critical point theory. In addition, at the end of this paper, we also give the existence results of infinite weak solutions of fractional differential equations under homogeneous Sturm-Liouville boundary value conditions. Finally, several examples are given to illustrate our main results.

Keywords Fractional differential equations, Sturm-Liouville boundary conditions, impulsive conditions, critical point theory.

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1. Introduction

Fractional calculus can be applied to many fields, such as fluid flow, chemical physics and signal processing, probability and statistics, control, electrochemistry and so on (see [1, 5, 15, 17, 23, 29, 34] and references therein). In the past long time, the research on boundary value problems of fractional differential equations has been in full swing (see [7, 11, 24, 32] and references therein).

The main focus of this paper is to study the existence and multiplicity of solutions to the boundary value problems of fractional differential equations by using the critical point theory. Some common methods have been used to discuss fractional differential problem in the literature, whether using fixed point theory (see [4, 6] and references therein), topological degree theory (see [8, 30] and references therein), upper and lower solution method (see [27, 28] and references therein), or variational method, critical point theory (see [14, 19] and references therein).

In this paper, we consider the existence and multiplicity of weak solutions for a class of fractional Sturm-Liouville boundary value problems with impulsive condi-

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tions

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}\ {}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}\ {}_tD_T^{-\beta}(u'(t))\right) + \lambda f(t, u(t)) = 0,\ t \neq t_i,\ a.e.\ t \in [0, T], \\ a\left(\dfrac{1}{2}\ {}_0D_t^{-\beta}(u'(0)) + \dfrac{1}{2}\ {}_tD_T^{-\beta}(u'(0))\right) - bu(0) = A, \\ c\left(\dfrac{1}{2}\ {}_0D_t^{-\beta}(u'(T)) + \dfrac{1}{2}\ {}_tD_T^{-\beta}(u'(T))\right) + du(T) = B, \\ \Delta\left(\ {}_0D_t^{-\beta}(u'(t_i)) + \ {}_tD_T^{-\beta}(u'(t_i))\right) = I_i(u(t_i)), \quad i = 1, 2, \cdots, m, \end{cases} \tag{1.1}$$

where $0 \leq \beta < 1$, $\lambda > 0$, $a, b, c, d > 0$, A and B are constants. ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ denote the left and right Riemann-Liouville fractional integrals of order β, respectively. $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous, $I_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, \cdots, m)$ are continuous, $0 = t_0 < t_1 < \cdots < t_{m+1} = T$. Besides,

$$\begin{aligned} \Delta\left(\ {}_0D_t^{-\beta}(u'(t_i)) + \ {}_tD_T^{-\beta}(u'(t_i))\right) =& \left(\ {}_0D_t^{-\beta}(u'(t_i^+)) + \ {}_tD_T^{-\beta}(u'(t_i^+))\right) \\ &- \left(\ {}_0D_t^{-\beta}(u'(t_i^-)) + \ {}_tD_T^{-\beta}(u'(t_i^-))\right), \end{aligned}$$

where

$$\begin{aligned} {}_0D_t^{-\beta}(u'(t_i^+)) + \ {}_tD_T^{-\beta}(u'(t_i^+)) &= \lim_{t \to t_i^+}\left(\ {}_0D_t^{-\beta}(u'(t)) + \ {}_tD_T^{-\beta}(u'(t))\right), \\ {}_0D_t^{-\beta}(u'(t_i^-)) + \ {}_tD_T^{-\beta}(u'(t_i^-)) &= \lim_{t \to t_i^-}\left(\ {}_0D_t^{-\beta}(u'(t)) + \ {}_tD_T^{-\beta}(u'(t))\right). \end{aligned}$$

Our motivation for studying boundary value problem (for convenience, abbreviated as BVP) (1.1) is largely derived from the fact that it can be used to simulate physical phenomena such as anomalous diffusion. In other words, the traditional second-order convection-diffusion equations cannot accurately simulate diffusion. Therefore, we use the extension of classical advection - dispersion equation, namely fractional advection - dispersion equation, to simulate anomalous diffusion under certain conditions, or to describe nonsymmetric or symmetric transition and solute transportation and so on, for example, in [10, 18, 19, 25, 31] and their references.

In [3], the existence and multiplicity of solutions for the following integer order nonlinear boundary value problems with impulsive conditions are obtained by using the critical point theory

$$\begin{cases} (\rho(t)\Phi_p(u'(t)))' - s(t)\Phi_p(u(t)) + \lambda f(t, u(t)) = 0,\ t \neq t_j,\ a.e.\ t \in [a, b], \\ \alpha_1 u'(a) - \alpha_2 u(a) = A, \\ \beta_1 u'(b) + \beta_2 u(b) = B, \\ -\Delta(\rho(t_j)\Phi_p(u'(t_j))) = I_j(u(t_j)), \quad i = 1, 2, \cdots, l, \end{cases} \tag{1.2}$$

where α_1, α_2, β_1, β_2 are positive constants, A and B are constants. $\lambda \in (0, +\infty)$ is a parameter. $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous.

In [25], the authors applied the critical point theory of non-differentiable functions to establish the existence of infinite solutions for the following boundary value problems

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t))+\dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\right)=\lambda f(u(t)),\ a.e.\ t\in[0,T],\\ au(0)-b\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}u'(0)+\dfrac{1}{2}\,{}_tD_T^{-\beta}u'(0)\right)=0,\\ cu(T)+d\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}u'(T)+\dfrac{1}{2}\,{}_tD_T^{-\beta}u'(T)\right)=0, \end{cases} \tag{1.3}$$

where $0\le\beta<1$, $a,c>0$, $b,d\ge0$, λ is a positive parameter. $f:\mathbb{R}\to\mathbb{R}$ is an almost everywhere continuous function.

As is known to all, the most prominent feature of impulsive differential equation is that it can fully consider the effect of instantaneous catastrophe on state, and can reflect the changing law of things more profoundly and accurately. With the development of science and technology, many scholars pay more and more attention to the theoretical significance and practical application of impulsive differential equation (see [12,13,22,26] and references therein).

Inspired by the above research, we study the BVP (1.1) in this paper. Compared with the BVP (1.2), the BVP (1.1) studies fractional order, which is undoubtedly the progress and innovation of our research. Further attention is paid to the BVP (1.3), the BVP (1.1) will study non-homogeneous Sturm-Liouville and impulsive boundary value conditions. Thus, the study of the BVP (1.1) is necessary and meaningful.

2. Preliminaries

For convenience, we will recall the necessary definitions and lemmas of fractional calculus.

Definition 2.1 ([2,20])**.** Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional integrals of order $0<\gamma\le1$ for the function u denoted by ${}_aD_t^{-\gamma}u(t)$ and ${}_tD_b^{-\gamma}u(t)$, respectively, are defined by

$${}_aD_t^{-\gamma}u(t)=\frac{1}{\Gamma(\gamma)}\int_a^t(t-s)^{\gamma-1}u(s)\mathrm{d}s,\ t\in[a,b],\ \gamma>0$$

and

$${}_tD_b^{-\gamma}u(t)=\frac{1}{\Gamma(\gamma)}\int_t^b(s-t)^{\gamma-1}u(s)\mathrm{d}s,\ t\in[a,b],\ \gamma>0,$$

provided the right-hand sides are pointwise defined on $[a,b]$, where $\Gamma>0$ is the gamma function.

Definition 2.2 ([2,20])**.** Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order $0<\gamma\le1$ for the function u denoted by ${}_aD_t^{\gamma}u(t)$ and ${}_tD_b^{\gamma}u(t)$, respectively, are defined by

$${}_aD_t^{\gamma}u(t)=\frac{\mathrm{d}}{\mathrm{d}t}\,{}_aD_t^{\gamma-1}u(t)=\frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^t(t-s)^{-\gamma}u(s)\mathrm{d}s\right)$$

and

$$ {}_tD_b^{\gamma}u(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\ {}_tD_b^{\gamma-1}u(t) = \frac{-1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_t^b (s-t)^{-\gamma}u(s)\mathrm{d}s\right), $$

where $t \in [a,b]$.

Definition 2.3 ([2, 20]). Let $u(t) \in AC([0,T],\mathbb{R})$, then the left and right Caputo fractional derivatives of order $0 < \gamma \leq 1$ for the function u denoted by ${}_a^cD_t^{\gamma}u(t)$ and ${}_t^cD_b^{\gamma}u(t)$, respectively, are defined by

$$ {}_a^cD_t^{\gamma}u(t) = {}_aD_t^{\gamma-1}u'(t) = \frac{1}{\Gamma(1-\gamma)}\int_a^t (t-s)^{-\gamma}u'(s)\mathrm{d}s $$

and

$$ {}_t^cD_b^{\gamma}u(t) = -\ {}_tD_b^{\gamma-1}u'(t) = \frac{-1}{\Gamma(1-\gamma)}\int_t^b (s-t)^{-\gamma}u'(s)\mathrm{d}s, $$

where $t \in [a,b]$.

Proposition 2.1 ([2]). *Let u is continuous for a.e. $t \in [a,b]$, the left and right Riemann-Liouville fractional integral operators have the following properties*

$$ \begin{aligned} {}_aD_t^{-\gamma_1}({}_aD_t^{-\gamma_2}u(t)) &= {}_aD_t^{-\gamma_1-\gamma_2}u(t), \\ {}_tD_b^{-\gamma_1}({}_tD_b^{-\gamma_2}u(t)) &= {}_tD_b^{-\gamma_1-\gamma_2}u(t), \quad \gamma_1,\gamma_2 > 0. \end{aligned} $$

Proposition 2.2 ([2]). *If $u \in L^{p'}([a,b],\mathbb{R}^N)$, $v \in L^{q'}([a,b],\mathbb{R}^N)$ and $p' \geq 1, q' \geq 1, \frac{1}{p'} + \frac{1}{q'} \leq 1+\gamma$ or $p' \neq 1, q' \neq 1, \frac{1}{p'} + \frac{1}{q'} = 1+\gamma$. Then*

$$ \int_a^b [{}_aD_t^{-\gamma}u(t)]v(t)\mathrm{d}t = \int_a^b [{}_tD_b^{-\gamma}v(t)]u(t)\mathrm{d}t,\ \gamma > 0. $$

Proposition 2.3 ([2]). *If $0 < \gamma \leq 1$ and $u \in AC([a,b],\mathbb{R}^N)$, then*

$$ {}_aD_t^{-\gamma}({}_a^cD_t^{\gamma}u(t)) = u(t) - u(a) $$

and

$$ {}_tD_b^{-\gamma}({}_t^cD_b^{\gamma}u(t)) = u(t) - u(b). $$

Let $L^p([0,T],\mathbb{R})(1 \leq p < \infty)$ and $C([0,T],\mathbb{R})$ be the $p-$Lebesgue space and continuous function space, respectively, with the norms

$$ \|u\|_{L^p} = (\int_0^T |u(t)|^p\mathrm{d}t)^{\frac{1}{p}}, \quad u \in L^p([0,T],\mathbb{R})\ (1 \leq p < \infty) $$

and

$$ \|u\|_{\infty} = \max_{t\in[0,T]} |u(t)|, \quad u \in C([0,T],\mathbb{R}). $$

Definition 2.4. Let $\frac{1}{2} < \alpha \leq 1$ and $1 \leq p < +\infty$. The fractional derivative space $E^{\alpha,p}$ is defined as the closure of $C^{\infty}([0,T],\mathbb{R})$, that is, $E^{\alpha,p} = \overline{C^{\infty}([0,T],\mathbb{R})}$ with the norm

$$ \|u\|_{\alpha,p} = (\int_0^T |{}_0^cD_t^{\alpha}u(t)|^p\mathrm{d}t + \int_0^T |u(t)|^p\mathrm{d}t)^{\frac{1}{p}}. $$

It is obvious that $E^{\alpha,p}$ is the space of functions $u(t) \in L^p([0,T],\mathbb{R})$ with an α order Caputo fractional derivative ${}_0^cD_t^\alpha u(t) \in L^p([0,T],\mathbb{R})$. When $p=2$, we denote $E^{\alpha,2}$ as X.

Lemma 2.1 ([9]). *The space X is a reflexive and separable Banach space.*

Lemma 2.2 ([9,25]). *Let $0<\alpha\le 1$ and $1\le p<+\infty$. For any $u\in L^p([0,T],\mathbb{R})$, we have*

(1)

$$\|{}_0D_\xi^{-\alpha}u\|_{L^p([0,t])} \le \frac{t^\alpha}{\Gamma(\alpha+1)}\|u\|_{L^p([0,t])} \quad \textit{for } \xi\in[0,t],\ t\in[0,T]. \tag{2.1}$$

(2)

$$\|{}_\xi D_T^{-\alpha}u\|_{L^p([t,T])} \le \frac{(T-t)^\alpha}{\Gamma(\alpha+1)}\|u\|_{L^p([t,T])} \quad \textit{for } \xi\in[t,T],\ t\in[0,T]. \tag{2.2}$$

In this paper, we treat the BVP (1.1) in the Hilbert space X with the inner product and the corresponding norm defined by

$$< u,v >= \int_0^T ({}_0^cD_t^\alpha u(t){}_0^cD_t^\alpha v(t))\mathrm{d}t + \int_0^T u(t)v(t)\mathrm{d}t,\ \forall u,v\in X$$

and

$$\|u\|_{\alpha,2} = (\int_0^T |{}_0^cD_t^\alpha u(t)|^2\mathrm{d}t + \int_0^T |u(t)|^2\mathrm{d}t)^{\frac{1}{2}},\ \forall u\in X.$$

Lemma 2.3 ([25]). *If $\frac{1}{2}<\alpha\le 1$, then for any $u\in X$, we have*

$$\begin{aligned} -\cos\pi\alpha\int_0^T |{}_0^cD_t^\alpha u(t)|^2\mathrm{d}t &\le -\int_0^T ({}_0^cD_t^\alpha u(t){}_t^cD_T^\alpha u(t))\mathrm{d}t \\ &\le -\frac{1}{\cos\pi\alpha}\int_0^T |{}_0^cD_t^\alpha u(t)|^2\mathrm{d}t. \end{aligned} \tag{2.3}$$

Lemma 2.4. *Let $\frac{1}{2}<\alpha\le 1$ and $u\in X$, the norm $\|u\|_{\alpha,2}$ is equivalent to*

$$\|u\| = \Big(-\int_0^T ({}_0^cD_t^\alpha u(t){}_t^cD_T^\alpha u(t))\mathrm{d}t + \frac{b}{a}(u(0))^2 + \frac{d}{c}(u(T))^2\Big)^{\frac{1}{2}}. \tag{2.4}$$

Proof. First, we will show that there exists a constant $K>0$ such that $\|u\|_{\alpha,2}\le K\|u\|$. According to Property 2.3, we deduce that $u(t) = {}_0D_t^{-\alpha}({}_0^cD_t^\alpha u(t)) + u(0)$. Then, based on (2.1) and (2.3), we have

$$\begin{aligned} \|u\|_{\alpha,2}^2 &= \int_0^T |u(t)|^2\mathrm{d}t + \int_0^T |{}_0^cD_t^\alpha u(t)|^2\mathrm{d}t \\ &= \int_0^T |{}_0D_t^{-\alpha}({}_0^cD_t^\alpha u(t)) + u(0)|^2\mathrm{d}t + \int_0^T |{}_0^cD_t^\alpha u(t)|^2\mathrm{d}t \\ &\le 2\int_0^T (|{}_0D_t^{-\alpha}({}_0^cD_t^\alpha u(t))|^2 + |u(0)|^2)\mathrm{d}t + \int_0^T |{}_0^cD_t^\alpha u(t)|^2\mathrm{d}t \\ &\le 2T(u(0))^2 + 2\|{}_0D_t^{-\alpha}({}_0^cD_t^\alpha u)\|_{L^2}^2 + \|{}_0^cD_t^\alpha u\|_{L^2}^2 \end{aligned}$$

$$
\begin{aligned}
&\leq 2T(u(0))^2 + \frac{2T^{2\alpha}}{(\Gamma(\alpha+1))^2}\|{}_0^cD_t^\alpha u\|_{L^2}^2 + \|{}_0^cD_t^\alpha u\|_{L^2}^2 \\
&\leq \frac{2Ta}{b}\frac{b}{a}(u(0))^2 + \frac{1}{\cos\pi\alpha}\left(\frac{2T^{2\alpha}}{(\Gamma(\alpha+1))^2}+1\right)\int_0^T ({}_0^cD_t^\alpha u(t){}_t^cD_T^\alpha u(t))\mathrm{d}t \\
&\leq \max\{\frac{2Ta}{b}, -\frac{1}{\cos\pi\alpha}\left(\frac{2T^{2\alpha}}{(\Gamma(\alpha+1))^2}+1\right)\}\|u\|^2.
\end{aligned}
$$

Obviously, we can find a constant $K>0$ so that

$$\|u\|_{\alpha,2} \leq K\|u\|,$$

where

$$K = \left(\max\{\frac{2Ta}{b}, -\frac{1}{\cos\pi\alpha}(\frac{2T^{2\alpha}}{(\Gamma(\alpha+1))^2}+1)\}\right)^{\frac{1}{2}}. \tag{2.5}$$

On the other hand, we must find a constant $H>0$ satisfying $\|u\| \leq H\|u\|_{\alpha,2}$. Based on Property 2.3 again, we get that $u(0) = u(t) - {}_0D_t^{-\alpha}({}_0^cD_t^\alpha u(t))$. Then from (2.1) and Hölder's inequality, we get

$$
\begin{aligned}
(u(0))^2 &= \frac{1}{T}\int_0^T (u(0))^2\mathrm{d}t \\
&= \frac{1}{T}\int_0^T \left(u(t) - {}_0D_t^{-\alpha}({}_0^cD_t^\alpha u(t))\right)^2\mathrm{d}t \\
&\leq \frac{2}{T}\int_0^T \left(|u(t)|^2 + |{}_0D_t^{-\alpha}({}_0^cD_t^\alpha u(t))|^2\right)\mathrm{d}t \\
&\leq \frac{2}{T}\|u\|_{L^2}^2 + \frac{2}{T}\|{}_0D_t^{-\alpha}({}_0^cD_t^\alpha u)\|_{L^2}^2 \\
&\leq \frac{2}{T}\|u\|_{L^2}^2 + \frac{2T^{2\alpha-1}}{(\Gamma(\alpha+1))^2}\|{}_0^cD_t^\alpha u\|_{L^2}^2 \\
&\leq 4\max\{\frac{1}{T}, \frac{T^{2\alpha-1}}{(\Gamma(\alpha+1))^2}\}\|u\|_{\alpha,2}^2 \\
&= H_1\|u\|_{\alpha,2}^2.
\end{aligned}
$$

It follows from Property 2.3 that $u(T) = u(t) - {}_tD_T^{-\alpha}({}_t^cD_T^\alpha u(t))$. Then according to (2.2) and Hölder's inequality, we can obtain

$$
\begin{aligned}
(u(T))^2 &= \frac{1}{T}\int_0^T (u(T))^2\mathrm{d}t \\
&= \frac{1}{T}\int_0^T \left(u(t) - {}_tD_T^{-\alpha}({}_t^cD_T^\alpha u(t))\right)^2\mathrm{d}t \\
&\leq \frac{2}{T}\int_0^T \left(|u(t)|^2 + |{}_tD_T^{-\alpha}({}_t^cD_T^\alpha u(t))|^2\right)\mathrm{d}t \\
&\leq \frac{2}{T}\|u\|_{L^2}^2 + \frac{2}{T}\|{}_tD_T^{-\alpha}({}_t^cD_T^\alpha u)\|_{L^2}^2 \\
&\leq \frac{2}{T}\|u\|_{L^2}^2 + \frac{2T^{2\alpha-1}}{(\Gamma(\alpha+1))^2}\|{}_t^cD_T^\alpha u\|_{L^2}^2.
\end{aligned}
$$

After calculation, we can get

$$\|{}_t^cD_T^\alpha u\|_{L^2}^2 \leq \frac{1}{(\cos\pi\alpha)^2}\|{}_0^cD_t^\alpha u\|_{L^2}^2.$$

By integrating the above two formulas, one has

$$\begin{aligned}(u(T))^2 &\leq \frac{2}{T}\|u\|_{L^2}^2 + \frac{2T^{2\alpha-1}}{(\Gamma(\alpha+1))^2(\cos\pi\alpha)^2}\|{}_0^cD_t^\alpha u\|_{L^2}^2\\ &\leq 2\max\{\frac{2}{T}, \frac{2T^{2\alpha-1}}{(\Gamma(\alpha+1))^2(\cos\pi\alpha)^2}\}\|u\|_{\alpha,2}^2\\ &= H_2\|u\|_{\alpha,2}^2.\end{aligned}$$

So

$$\|u\|^2 \leq (-\frac{1}{\cos\pi\alpha} + \frac{b}{a}H_1 + \frac{d}{c}H_2)\|u\|_{\alpha,2}^2.$$

In other words, we can find a constant $H > 0$ so that

$$\|u\| \leq H\|u\|_{\alpha,2},$$

where

$$H = \left(-\frac{1}{\cos\pi\alpha} + \frac{b}{a}H_1 + \frac{d}{c}H_2\right)^{\frac{1}{2}}$$

and

$$\begin{aligned}H_1 &= 4\max\{\frac{1}{T}, \frac{T^{2\alpha-1}}{(\Gamma(\alpha+1))^2}\},\\ H_2 &= 2\max\{\frac{2}{T}, \frac{2T^{2\alpha-1}}{(\Gamma(\alpha+1))^2(\cos\pi\alpha)^2}\}.\end{aligned}$$

□

Lemma 2.5. *For $u \in X$, there is $\Lambda > 0$ such that*

$$\|u\|_\infty \leq \Lambda\|u\|, \tag{2.6}$$

where

$$\Lambda = \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}\sqrt{|\cos\pi\alpha|}} + \sqrt{2}K\max\{T^{-\frac{1}{2}}, \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha+1)}\},$$

and K is defined (2.5).

Proof. Similar to the proof of *Lemma* 5.4 in [25], we can take

$$\Lambda = \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}\sqrt{|\cos\pi\alpha|}} + \sqrt{2}K\max\{T^{-\frac{1}{2}}, \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha+1)}\},$$

and then we can get $\|u\|_\infty \leq \Lambda\|u\|$. □

Lemma 2.6 ([25]). *Let $u, v \in L^1([0,T],\mathbb{R}^N)$. Suppose*

$$\int_0^T (u(t),\phi'(t))\mathrm{d}t = -\int_0^T (v(t),\phi(t))\mathrm{d}t$$

for every $\phi \in C_0^\infty[0,T]$, then

$$u(t) = \int_0^T v(x)\mathrm{d}x + C$$

for a.e. $t \in [0,T]$ and $C \in \mathbb{R}^N$.

Lemma 2.7 ([9]). *Assume that $\frac{1}{2} < \alpha \leq 1$ and the sequence $\{u_k\}$ converges weakly to u in X, denote by $u_k \rightharpoonup u$ in X, then $u_k \to u$ in $C([0,T],\mathbb{R})$, that is, $\|u_k - u\|_\infty \to 0$ as $k \to +\infty$.*

According to Property 2.1, then using Definition 2.3, the BVP (1.1) can be transformed into the following boundary value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \dfrac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\right) + \lambda f(t,u(t)) = 0, \\ t \neq t_i,\ a.e.\ t \in [0,T], \\ a\left(\dfrac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(0)) - \dfrac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(0))\right) - bu(0) = A, \\ c\left(\dfrac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(T)) - \dfrac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(T))\right) + du(T) = B, \\ \Delta\left(\ {}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t_i)) -\ {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t_i))\right) = I_i(u(t_i)), \quad i = 1,2,\cdots,m, \end{cases} \tag{2.7}$$

where $\alpha = 1 - \frac{\beta}{2} \in (\frac{1}{2}, 1]$. Therefore, we look for a weak solution u of the BVP (1.1) which corresponds to the weak solution u of the BVP (2.7).

Take $v \in X$ and multiply the two sides of the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\right) + \lambda f(t,u(t)) = 0$$

by v and integrate from 0 to T, we have

$$\int_0^T \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\right) + \lambda f(t,u(t))\right] v(t)\mathrm{d}t = 0.$$

Moreover, we get

$$\begin{aligned} &\int_0^T \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\right)\right] v(t)\mathrm{d}t \\ =& \sum_{i=0}^m \int_{t_i}^{t_{i+1}} \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\right)\right] v(t)\mathrm{d}t \\ =& \sum_{i=0}^m \frac{1}{2}\left[\ {}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) -\ {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\right] v(t)\Big|_{t_i^+}^{t_{i+1}^-} \end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=0}^{m}\frac{1}{2}\int_{t_i}^{t_{i+1}}\Big[\,{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t))-\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big]v'(t)\mathrm{d}t\\
=&-\frac{1}{2}\sum_{i=1}^{m}\Delta({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_i))-\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_i)))v(t_i)\\
&-\frac{1}{2}({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(0))-\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(0)))v(0)\\
&+\frac{1}{2}({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(T))-\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(T)))v(T)\\
&-\frac{1}{2}\int_0^T({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t))-\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t)))v'(t)\mathrm{d}t\\
=&-\frac{1}{2}\sum_{i=1}^{m}I_i(u(t_i))v(t_i)-\frac{A+bu(0)}{a}v(0)+\frac{B-du(T)}{c}v(T)\\
&+\frac{1}{2}\int_0^T\Big(({}_0^cD_t^{\alpha}u(t))({}_t^cD_T^{\alpha}v(t))+({}_t^cD_T^{\alpha}u(t))({}_0^cD_t^{\alpha}v(t))\Big)\mathrm{d}t.
\end{aligned}
$$

Above all, we have

$$
\begin{aligned}
&-\frac{1}{2}\int_0^T\Big(({}_0^cD_t^{\alpha}u(t))({}_t^cD_T^{\alpha}v(t))+({}_t^cD_T^{\alpha}u(t))({}_0^cD_t^{\alpha}v(t))\Big)\mathrm{d}t+\frac{bu(0)}{a}v(0)+\frac{du(T)}{c}v(T)\\
&+\frac{A}{a}v(0)-\frac{B}{c}v(T)+\frac{1}{2}\sum_{i=1}^{m}I_i(u(t_i))v(t_i)-\lambda\int_0^T f(t,u(t))v(t)\mathrm{d}t=0.
\end{aligned}
$$

We are now introduce the concept of a weak solution for the BVP (2.7).

Definition 2.5. The weak solution of the BVP (2.7) is $u\in X$ satisfying the following equation

$$
\begin{aligned}
&-\frac{1}{2}\int_0^T\Big(({}_0^cD_t^{\alpha}u(t))({}_t^cD_T^{\alpha}v(t))+({}_t^cD_T^{\alpha}u(t))({}_0^cD_t^{\alpha}v(t))\Big)\mathrm{d}t+\frac{bu(0)}{a}v(0)+\frac{du(T)}{c}v(T)\\
&+\frac{A}{a}v(0)-\frac{B}{c}v(T)+\frac{1}{2}\sum_{i=1}^{m}I_i(u(t_i))v(t_i)-\lambda\int_0^T f(t,u(t))v(t)\mathrm{d}t=0,\ \forall v\in X.
\end{aligned}
$$

For $\forall u\in X$, we consider the functional $J:X\to\mathbb{R}$, that is $J\in C^1(X,\mathbb{R})$ as follows

$$
\begin{aligned}
J(u)=&\frac{1}{2}\Big(-\int_0^T({}_0^cD_t^{\alpha}u(t))({}_t^cD_T^{\alpha}u(t))\mathrm{d}t+\frac{b}{a}(u(0))^2+\frac{d}{c}(u(T))^2\Big)+\frac{A}{a}u(0)\\
&-\frac{B}{c}u(T)+\frac{1}{2}\sum_{i=1}^{m}\int_0^{u(t_i)}I_i(t)\mathrm{d}t-\lambda\int_0^T F(t,u(t))\mathrm{d}t\\
=&\frac{1}{2}\|u\|^2+\frac{A}{a}u(0)-\frac{B}{c}u(T)+\frac{1}{2}\sum_{i=1}^{m}\int_0^{u(t_i)}I_i(t)\mathrm{d}t-\lambda\int_0^T F(t,u(t))\mathrm{d}t,
\end{aligned}
\tag{2.8}
$$

where $\|u\|$ is defined by (2.4) and $F(t,u)=\int_0^u f(t,s)\mathrm{d}s$ for all $(t,u)\in[0,T]\times\mathbb{R}$. It is easy to get that the functional J is differentiable on X and

$$
J'(u)v=-\frac{1}{2}\int_0^T\Big(({}_0^cD_t^{\alpha}u(t))({}_t^cD_T^{\alpha}v(t))+({}_t^cD_T^{\alpha}u(t))({}_0^cD_t^{\alpha}v(t))\Big)\mathrm{d}t+\frac{bu(0)}{a}v(0)
$$

$$
\begin{aligned}
&+\frac{du(T)}{c}v(T)+\frac{A}{a}v(0)-\frac{B}{c}v(T)+\frac{1}{2}\sum_{i=1}^{m}I_i(u(t_i))v(t_i)\\
&-\lambda\int_0^T f(t,u(t))v(t)\mathrm{d}t, \qquad \forall v\in X. \qquad (2.9)
\end{aligned}
$$

Lemma 2.8. *If $u \in X$ is a critical point of J in X, then u is a weak solution of the BVP* (2.7).

Proof. Suppose $u \in X$ is a critical point of J in X, that is,

$$
\begin{aligned}
&-\frac{1}{2}\int_0^T\Big(({}_0^cD_t^{\alpha}u(t))({}_t^cD_T^{\alpha}v(t))+({}_t^cD_T^{\alpha}u(t))({}_0^cD_t^{\alpha}v(t))\Big)\mathrm{d}t+\frac{bu(0)}{a}v(0)+\frac{du(T)}{c}v(T)\\
&+\frac{A}{a}v(0)-\frac{B}{c}v(T)+\frac{1}{2}\sum_{i=1}^{m}I_i(u(t_i))v(t_i)-\lambda\int_0^T f(t,u(t))v(t)\mathrm{d}t=0,\ \forall v\in X.
\end{aligned}
\tag{2.10}
$$

Without loss of generality, for any $i \in \{0, 1, 2, \cdots, m\}$, take $v \in C_0^{\infty}(t_i, t_{i+1})$, then (2.10) can be sorted into

$$
\begin{aligned}
&-\frac{1}{2}\int_0^T\Big(({}_0^cD_t^{\alpha}u(t))({}_t^cD_T^{\alpha}v(t))+({}_t^cD_T^{\alpha}u(t))({}_0^cD_t^{\alpha}v(t))\Big)\mathrm{d}t\\
&-\lambda\int_0^T f(t,u(t))v(t)\mathrm{d}t=0,
\end{aligned}
$$

that is,

$$
\frac{1}{2}\int_0^T\Big(\ {}_0D_t^{-\beta}(u'(t))+\ {}_tD_T^{-\beta}(u'(t))\Big)v'(t)\mathrm{d}t=\lambda\int_0^T f(t,u(t))v(t)\mathrm{d}t,\ \forall v\in C_0^{\infty}.
$$

According to Lemma 2.6, we get

$$
\frac{1}{2}\Big(\ {}_0D_t^{-\beta}(u'(t))+\ {}_tD_T^{-\beta}(u'(t))\Big)=-\lambda\int_0^t f(x,u(x))\mathrm{d}x+C,\ C\in\mathbb{R},
$$

then

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\ {}_0D_t^{-\beta}(u'(t))+\ {}_tD_T^{-\beta}(u'(t))\Big)+\lambda f(t,u(t))=0. \tag{2.11}
$$

Multiply both sides of (2.11) by v, then integrate from 0 to T, and we have

$$
\int_0^T\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\ {}_0D_t^{-\beta}(u'(t))+\ {}_tD_T^{-\beta}(u'(t))\Big)v(t)\mathrm{d}t+\lambda\int_0^T f(t,u(t))v(t)\mathrm{d}t=0. \tag{2.12}
$$

In addition,

$$
\begin{aligned}
&\int_0^T\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\ {}_0D_t^{-\beta}(u'(t))+\ {}_tD_T^{-\beta}(u'(t))\Big)v(t)\mathrm{d}t\\
=&\int_0^T\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\ {}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t))-\ {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big)v(t)\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{m}\int_{t_i}^{t_{i+1}} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big)v(t)\mathrm{d}t \\
&= \frac{1}{2}\sum_{i=0}^{m}\Big[\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_{i+1}^-)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_{i+1}^-))\Big)v(t_{i+1}^-) \\
&\quad - \Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_{i+1}^+)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_{i+1}^+))\Big)v(t_{i+1}^+) \\
&\quad - \int_{t_i}^{t_{i+1}}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big)v'(t)\mathrm{d}t\Big] \\
&= \frac{1}{2}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(T)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(T))\Big)v(T) \\
&\quad - \frac{1}{2}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(0)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(0))\Big)v(0) \\
&\quad - \frac{1}{2}\sum_{i=1}^{m}\Delta\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_i)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_i))\Big)v(t_i) \\
&\quad - \frac{1}{2}\int_0^T\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big)v'(t)\mathrm{d}t. \qquad (2.13)
\end{aligned}
$$

It follows from (2.10), (2.12) and (2.13) that

$$
\begin{aligned}
&\Big(\frac{bu(0)}{a} + \frac{A}{a} - \frac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(0)) + \frac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(0))\Big)v(0) \\
&+\Big(- \frac{du(T)}{c} + \frac{B}{c} - \frac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(T)) + \frac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(T))\Big)v(T) \\
&+\Big(\frac{1}{2}\sum_{i=1}^{m}I_i(u(t_i))v(t_i) - \frac{1}{2}\sum_{i=1}^{m}\Delta({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_i)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_i)))\Big) = 0.
\end{aligned}
$$

Therefore, one has

$$
\begin{aligned}
&a\Big(\frac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(0)) - \frac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(0))\Big) - bu(0) = A, \\
&c\Big(\frac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(T)) - \frac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(T))\Big) + du(T) = B, \\
&\Delta\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_i)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_i))\Big) = I_i(u(t_i)), \quad i = 1, 2, \cdots, m.
\end{aligned}
$$

□

Lemma 2.9 ([16]). *If φ is sequentially weakly lower semi-continuous on a reflexive Banach space E and has a bounded minimizing sequence, then φ has a minimum on E.*

Lemma 2.10 ([33]). *For the functional $\varphi : \Theta \subseteq E \to [-\infty, +\infty]$ with $\Theta \neq \emptyset$, $\min_{u\in\Theta}\varphi(u) = \epsilon$ has a solution when the following hold*

(1) E is a real reflexive Banach space.

(2) Θ is bounded and weak sequentially closed, i.e., by definition, for each sequence $\{u_n\}$ in Θ such that $u_n \rightharpoonup u$ as $n \to +\infty$, we always have $u \in \Theta$.

(3) φ *is sequentially weakly lower semi-continuous on* Θ.

Lemma 2.11 ([21]). *Let* E *be a real Banach space and* $\varphi \in C^1(E, \mathbb{R})$ *satisfy the Palais-Smale condition ((PS) -condition for short). Suppose that* φ *satisfies the following conditions*

(1) $\varphi(0) = 0$.

(2) There exist $\rho, \sigma > 0$ *such that* $\varphi(u_0) \geq \sigma$ *for every* $u_0 \in E$ *with* $\|u_0\| = \rho$.

(3) There exists $u_1 \in E$ *with* $\|u_1\| \geq \rho$ *such that* $\varphi(u_1) < \sigma$.

Then, φ *possesses a critical value* $\epsilon \geq \sigma$. *Moreover,* ϵ *can be characterized as*

$$\epsilon = \inf_{g \in \Delta} \max_{s \in [0,1]} \varphi(g(s)),$$

where

$$\Delta = \{g \in C([0,1], E) : g(0) = u_0, g(1) = u_1\}.$$

Lemma 2.12 ([21]). *Let* E *be an infinite-dimensional Banach space and let* $\varphi \in C^1(E, \mathbb{R})$ *be even, satisfy the* (PS)*-condition, and have* $\varphi(0) = 0$. *Suppose that* $E = V \bigoplus W$, *where* V *is finite dimensional, and* φ *satisfies the following*

(1) there exist $\varrho > 0$ *and* $\rho > 0$ *such that* $\varphi(u) \geq \varrho$ *for all* $u \in E$ *with* $\|u\| = \rho$, *and*

(2) for any finite-dimensional subspace $Z \subset E$ *there is* $R = R(Z)$ *such that* $\varphi(u) \leq 0$ *for all* $u \in Z$ *with* $\|u\| \geq R$.

Then φ *possesses an unbounded sequence of critical values.*

3. Main result

In this section, the existence and multiplicity of weak solutions for the BVP (1.1) are considered.

In the rest of the article, we make the following assumptions

(H_1) There exist some nonnegative constants l_i, $\bar{l}_i$, k_i, $\bar{k}_i$ and $0 \leq \sigma_i < 1$, $0 \leq \bar{\sigma}_i < 1$, for $i = 1, 2, \cdots, m$, such that

$$-l_i - k_i|y|^{\sigma_i} \leq I_i(y) \leq \bar{l}_i + \bar{k}_i|y|^{\bar{\sigma}_i}$$

for $y \in \mathbb{R}$.

(H_2) There exist $0 \leq \theta < 2$ and $\tau(t) \in C([0,T])$ with $ess \inf \tau > 0$ such that

$$\limsup_{|u| \to +\infty} \frac{F(t,u)}{|u|^\theta} < \tau(t)$$

uniformly for a.e. $t \in [0,T]$.

(H_3) There exists $\eta(t) \in C([0,T])$ with $ess \inf \eta > 0$ such that

$$\limsup_{|u| \to +\infty} \frac{F(t,u)}{|u|^2} < \eta(t)$$

uniformly for a.e. $t \in [0,T]$.
(H_4) There exists $\mu > 2$ such that

$$0 < \mu F(t,y) \leq y f(t,y)$$

for any $t \in [0,T]$ and $y \in \mathbb{R}\backslash\{0\}$.
(H_5) $f(t,y)$ is odd in y, i.e., $f(t,-y) = -f(t,y)$ for all $t \in [0,T]$ and $y \in \mathbb{R}$. $I_i(i = 1, 2, \cdots, m)$ are odd and increasing.

Theorem 3.1. *Assume that (H_1) and one of the following conditions hold, then the BVP (2.7) admits at least one weak solution, and thus the BVP (1.1) has at least one weak solution.*

(1) (H_2) holds and $\lambda \in (0, +\infty)$.
(2) (H_3) holds and $\lambda \in (0, \frac{1}{2\Lambda^2 \int_0^T \eta(t)\mathrm{d}t})$.

Proof. Let u_k converges weakly to u in X, then u_k converges uniformly to u in $[0,T]$. Based on the continuity of f and $I_i(i = 1, 2, \cdots, m)$, it is logical to get

$$\begin{aligned}&\lim_{k\to+\infty}\left(\frac{1}{2}\sum_{i=1}^{m}\int_0^{u_k(t_i)} I_i(t)\mathrm{d}t - \lambda\int_0^T F(t,u_k(t))\mathrm{d}t + \frac{A}{a}u_k(0) - \frac{B}{c}u_k(T)\right)\\ =&\frac{1}{2}\sum_{i=1}^{m}\int_0^{u(t_i)} I_i(t)\mathrm{d}t - \lambda\int_0^T F(t,u(t))\mathrm{d}t + \frac{A}{a}u(0) - \frac{B}{c}u(T).\end{aligned}$$

Notice that

$$\liminf_{k\to+\infty}\frac{1}{2}\|u_k\|^2 \geq \frac{1}{2}\|u\|^2,$$

so we conclude that $J(u)$ is sequentially weakly lower semi-continuous.

If (H_2) holds, one can get that there exists $L > 0$ such that

$$F(t,u) \leq \tau(t)|u|^\theta + L$$

for a.e. $t \in [0,T]$ and $u \in \mathbb{R}$, therefore, combined with (2.6), we have

$$\begin{aligned}\int_0^T F(t,u(t))\mathrm{d}t &\leq \int_0^T \tau(t)|u(t)|^\theta\mathrm{d}t + \int_0^T L\mathrm{d}t\\ &\leq \Lambda^\theta\|u\|^\theta\int_0^T \tau(t)\mathrm{d}t + LT.\end{aligned}$$

According to (H_1) and (2.6), we can obtain

$$\begin{aligned}\sum_{i=1}^{m}\int_0^{u(t_i)} I_i(t)\mathrm{d}t &\geq -\sum_{i=1}^{m}(l_i\|u\|_\infty + \frac{k_i}{\sigma_i+1}\|u\|_\infty^{\sigma_i+1})\\ &\geq -\sum_{i=1}^{m}(l_i\Lambda\|u\| + \frac{k_i}{\sigma_i+1}\Lambda^{\sigma_i+1}\|u\|^{\sigma_i+1}).\end{aligned}$$

Combining the above two formulas and (2.8), we infer that

$$J(u) \geq \frac{1}{2}\|u\|^2 - \frac{|A|}{a}\Lambda\|u\| - \frac{|B|}{c}\Lambda\|u\| - \frac{1}{2}\sum_{i=1}^{m}(l_i\Lambda\|u\| + \frac{k_i}{\sigma_i+1}\Lambda^{\sigma_i+1}\|u\|^{\sigma_i+1})$$

$$-\lambda\Lambda^{\theta}\|u\|^{\theta}\int_0^T\tau(t)\mathrm{d}t-\lambda LT.$$

On account of $0\le\theta<2$ and $0\le\sigma_i<1$, we get that $\lim_{\|u\|\to+\infty}J(u)=+\infty$. This means that $J(u)$ has a bounded minimizing sequence. So the BVP (2.7) exists at least one weak solution, that is, the BVP (1.1) admits at least one weak solution.

If (H_3) holds. (H_3) implies that there exists $Q>0$ such that

$$F(t,u)\le\eta(t)|u|^2+Q$$

for a.e. $t\in[0,T]$ and $u\in\mathbb{R}$.

Similar to the above discussion, we can get

$$J(u)\ge\frac{1}{2}\|u\|^2-\frac{|A|\Lambda}{a}\|u\|-\frac{|B|\Lambda}{c}\|u\|-\frac{1}{2}\sum_{i=1}^m(l_i\Lambda\|u\|+\frac{k_i}{\sigma_i+1}\Lambda^{\sigma_i+1}\|u\|^{\sigma_i+1})$$
$$-\lambda\Lambda^2\|u\|^2\int_0^T\eta(t)\mathrm{d}t-\lambda QT.$$

Notice that $\lambda\in(0,\frac{1}{2\Lambda^2\int_0^T\eta(t)\mathrm{d}t})$, we get that $\lim_{\|u\|\to+\infty}J(u)=+\infty$. □

Lemma 3.1. *Assume that (H_1) and (H_4) hold, then J satisfies the (PS)-condition.*

Proof. Assume that $\{u_k\}_{k\in\mathbb{N}}\subset X$ is a sequence such that $\{J(u_k)\}_{k\in\mathbb{N}}$ is bounded and $\lim_{k\to+\infty}J'(u_k)=0$. It follows from $\mu>2$, (H_1) and (H_4) that

$$\begin{aligned}\mu J(u_k)-J'(u_k)u_k=&(\frac{\mu}{2}-1)\|u_k\|^2+(\mu-1)\frac{A}{a}u_k(0)-(\mu-1)\frac{B}{c}u_k(T)\\&+\frac{\mu}{2}\sum_{i=1}^m\int_0^{u_k(t_i)}I_i(t)\mathrm{d}t-\frac{1}{2}\sum_{i=1}^mI_i(u_k(t_i))u_k(t_i)\\&+\lambda\Big(\int_0^Tf(t,u_k(t))u_k(t)\mathrm{d}t-\mu\int_0^TF(t,u_k(t))\mathrm{d}t\Big)\\\ge&(\frac{\mu}{2}-1)\|u_k\|^2-(\mu-1)\frac{|A|\Lambda}{a}\|u_k\|-(\mu-1)\frac{|B|\Lambda}{c}\|u_k\|\\&-\frac{\mu}{2}\sum_{i=1}^m(l_i\Lambda\|u_k\|+\frac{k_i}{\sigma_i+1}\Lambda^{\sigma_i+1}\|u_k\|^{\sigma_i+1})\\&-\frac{1}{2}\sum_{i=1}^m(\bar{l}_i\Lambda\|u_k\|+\bar{k}_i\Lambda^{\bar{\sigma}_i+1}\|u_k\|^{\bar{\sigma}_i+1}),\end{aligned}$$

which indicates that $\{u_k\}$ is bounded in X.

Because $\{u_k\}$ is bounded in X, there exists a subsequence of $\{u_k\}$, which converges weakly to a subsequence of u in X. For convenience, we still record this subsequence as $\{u_k\}$. Then $\{u_k\}$ converges uniformly to u on $[0,T]$. Thus

$$\begin{aligned}&u_k(0)\to u(0),\\&u_k(T)\to u(T),\\&\sum_{i=1}^m(I_i(u_k(t_i))-I_i(u(t_i)))(u_k(t_i)-u(t_i))\to0,\end{aligned}$$

$$\int_0^T (f(t,u_k(t)) - f(t,u(t)))(u_k(t) - u(t))\mathrm{d}t \to 0,$$

as $k \to +\infty$. Since $\lim_{k\to+\infty} J'(u_k) = 0$ and $\{u_k\}$ converges weakly to some u, we get

$$< J'(u_k) - J'(u), u_k - u > \to 0,$$

as $k \to +\infty$. On the other hand, notice that

$$\begin{aligned}
&< J'(u_k) - J'(u), u_k - u > \\
=& -\int_0^T \Big({}_0^cD_t^\alpha(u_k(t) - u(t)),\ {}_t^cD_T^\alpha(u_k(t) - u(t))\Big)\mathrm{d}t + \frac{b(u_k(0) - u(0))^2}{a} \\
&+ \frac{d(u_k(T) - u(T))^2}{c} + \frac{1}{2}\sum_{i=1}^m \Big(I_i(u_k(t_i)) - I_i(u(t_i))\Big)(u_k(t_i) - u(t_i)) \\
&- \lambda\int_0^T (f(t,u_k(t)) - f(t,u(t)))(u_k(t) - u(t))\mathrm{d}t \\
=& \|u_k - u\|^2 + \frac{1}{2}\sum_{i=1}^m \Big(I_i(u_k(t_i)) - I_i(u(t_i))\Big)(u_k(t_i) - u(t_i)) \\
&- \lambda\int_0^T (f(t,u_k(t)) - f(t,u(t)))(u_k(t) - u(t))\mathrm{d}t.
\end{aligned}$$

According to the previous discussion, $\|u_k - u\| \to 0$ as $k \to +\infty$ is proved. That is, $u_k \to u$ in X. In summary, J satisfies the (PS)-condition. □

Remark 3.1. From (H_4), we know that there exists $P > 0$ such that

$$F(t,u) \le F^0|u|^\mu,\ (t,u) \in [0,T] \times [-1,1] \tag{3.1}$$

and

$$F(t,u) \ge F_0|u|^\mu - P,\ (t,u) \in [0,T] \times \mathbb{R}, \tag{3.2}$$

where

$$F^0 = \max_{t\in[0,T],|u|=1} F(t,u) > 0$$

and

$$F_0 = \min_{t\in[0,T],|u|=1} F(t,u) > 0.$$

Theorem 3.2. *If $M > 0$, and (H_1) and (H_4) hold, then the BVP (2.7) admits at least two different weak solutions for each $\lambda \in (0, \frac{M}{F^0T})$, where*

$$M = \frac{1}{2\Lambda^2} - \frac{|A|}{a} - \frac{|B|}{c} - \frac{1}{2}\sum_{i=1}^m (l_i + \frac{k_i}{\sigma_i + 1}).$$

That is, the BVP (1.1) exists at least two different weak solutions.

Proof. Let B_r denotes the open ball in X with radius r and centered at 0 and let ∂B_r and $\bar{B}_r$ represent the boundary and closure of B_r, respectively. It is easy to get $\bar{B}_{\frac{1}{\Lambda}}$ is a bounded weak closed set.

Next, we have known that $J(u)$ is sequentially weakly lower semi-continuous in X, based on Lemma 2.10, we get $J(u)$ has a local minimum point u_0 in $\bar{B}_{\frac{1}{\Lambda}}$, in other word, $J(u_0) \le J(0) = 0$.

It follows from (2.6) that $\|u\| \le \frac{1}{\Lambda}$ implies that $\|u\|_\infty \le 1$. According to (3.1), one has

$$\int_0^T F(t,u(t))\mathrm{d}t \le F^0 \int_0^T \|u\|_\infty^\mu \mathrm{d}t \le F^0 T \Lambda^\mu \|u\|^\mu, \quad \|u\| \le \frac{1}{\Lambda}. \tag{3.3}$$

For any $u \in \partial B_r (r \le \frac{1}{\Lambda})$, combined with (2.8), we have

$$J(u) \ge \frac{1}{2}r^2 - \frac{\Lambda|A|r}{a} - \frac{\Lambda|B|r}{c} - \frac{1}{2}\sum_{i=1}^m (l_i \Lambda r + \frac{k_i}{\sigma_i+1}\Lambda^{\sigma_i+1} r^{\sigma_i+1}) - \lambda T F^0 \Lambda^\mu r^\mu.$$

And then for all $u \in \partial B_{\frac{1}{\Lambda}}$, one has

$$J(u) \ge \frac{1}{2\Lambda^2} - \frac{|A|}{a} - \frac{|B|}{c} - \frac{1}{2}\sum_{i=1}^m (l_i + \frac{k_i}{\sigma_i+1}) - \lambda T F^0 = M_\lambda.$$

In view of $\lambda \in (0, \frac{M}{F^0 T})$, we obtain that $J(u) = M_\lambda > J(0) \ge J(u_0)$ for any $u \in \partial B_{\frac{1}{\Lambda}}$.

Therefore $\inf_{u\in\partial B_{\frac{1}{\Lambda}}} J(u) > J(u_0)$, and $J(u)$ has a local minimum $u_0 \in \partial B_{\frac{1}{\Lambda}}$.

Let $\omega > 0$. Since $\omega \in X$, in view of (H_1) and (3.2), we get that

$$J(\omega) \le (\frac{b}{2a}+\frac{d}{2c})\omega^2 + \frac{|A|}{a}\omega + \frac{|B|}{c}\omega + \frac{1}{2}\sum_{i=1}^m (\bar{l}_i \omega + \frac{\bar{k}_i}{\bar{\sigma}_i+1}\omega^{\bar{\sigma}_i+1}) - \lambda F_0 T \omega^\mu + \lambda P T.$$

Notice that $\mu > 2$ and $1 \le \bar{\sigma}_i + 1 < 2$, one has $J(\omega) \to -\infty$ as $\omega \to +\infty$. Thus, there exists $u_1 > 0$ with $\|u_1\| > \frac{1}{\Lambda}$ such that $\inf_{u\in\partial B_{\frac{1}{\Lambda}}} J(u) > J(u_1)$. Based on Lemma 2.11 and Lemma 3.1, it is easy to know that there is $u_2 \in X$ such that $J'(u_2) = 0$ and $J(u_2) > \max\{J(u_0), J(u_1)\}$.

Thus, u_0 and u_2 are two different weak solutions of the BVP (2.7), moreover, they are also two different weak solutions of the BVP (1.1). □

Next, let's consider the case of $A = B = 0$, then the BVP (1.1) becomes the following form

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \dfrac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\right) + \lambda f(t,u(t)) = 0, \\ t \ne t_i,\ a.e.\ t \in [0,T], \\ a\left(\dfrac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(0)) - \dfrac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(0))\right) - bu(0) = 0, \\ c\left(\dfrac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(T)) - \dfrac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(T))\right) + du(T) = 0, \\ \Delta\left(\,{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t_i)) - \,{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t_i))\right) = I_i(u(t_i)), \quad i = 1,2,\cdots,m. \end{cases} \tag{3.4}$$

Theorem 3.3. *Assume that* (H_1), (H_4) *and* (H_5) *hold, then the BVP* (3.4) *admits infinitely many weak solutions.*

Proof. According to the definition of J, it is obvious that $J(0) = 0$. In addition, according to (H_5), J is even. Notice the fact that all norms are equivalent in finite dimensional space, so for any finite dimensional space $\tilde{X}$ in X, and for each $u \in \tilde{X}$, we know that there exists a constant $G' > 0$, so that

$$\|u\|_{L^q} = (\int_0^T |u(t)|^q \mathrm{d}t)^{\frac{1}{q}} \geq G'\|u\|, \ q \geq 1.$$

Based on (3.2), for any $u \in \tilde{X}$, there is a constant $G > 0$ such that

$$\begin{aligned}\int_0^T F(t, u(t))\mathrm{d}t \geq & F_0\|u\|_{L^\mu}^\mu - PT \\ \geq & F_0 G^\mu \|u\|^\mu - PT,\end{aligned}$$

which yields that

$$\begin{aligned}J(u) = & \frac{1}{2}\|u\|^2 + \frac{1}{2}\sum_{i=1}^m \int_0^{u(t_i)} I_i(t)\mathrm{d}t - \lambda \int_0^T F(t, u(t))\mathrm{d}t \\ \leq & \frac{1}{2}\|u\|^2 + \frac{1}{2}\sum_{i=1}^m (\bar{l}_i \Lambda \|u\| + \frac{\bar{k}_i}{\bar{\sigma}_i + 1}\Lambda^{\bar{\sigma}_i+1}\|u\|^{\bar{\sigma}_i+1}) - \lambda F_0 G^\mu \|u\|^\mu + \lambda PT.\end{aligned}$$

And because $\mu > 2$ and $1 \leq \bar{\sigma}_i + 1 < 2$, we get $J(u) \to -\infty$ as $u \in \tilde{E}$ and $\|u\| \to +\infty$. Then there exists $R = R(\tilde{E})$ such that $J(u) \leq 0$ for all $u \in \tilde{E}$ satisfies $\|u\| \geq R$.

In view of (H_5), we know $\int_0^x I_i(t)\mathrm{d}t$ are even and $\int_0^x I_i(t)\mathrm{d}t \geq 0$ for any $x \geq 0$. Thus

$$\sum_{i=1}^m \int_0^{u(t_i)} I_i(t)\mathrm{d}t \geq 0,$$

combined with (3.3), we get that

$$J(u) \geq \frac{1}{2}\|u\|^2 - \lambda F^0 T \Lambda^\mu \|u\|^\mu, \quad \|u\| \leq \frac{1}{\Lambda}.$$

Since $\mu > 2$, the above inequality implies that we can choose $\delta > 0$ small enough such that $J(u) \geq \varpi > 0$ for $\|u\| = \delta$. Base on Lemma 2.12, the BVP (3.4) has infinitely many weak solutions. □

4. Examples

Example 4.1. Let us consider the following fractional boundary value problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\ {}_0D_t^0(u'(t)) + \frac{1}{2}\ {}_tD_1^0(u'(t))\right) + \lambda f(t, u(t)) = 0, \ t \neq t_1, \ a.e.\ t \in [0,1], \\ 2\left(\frac{1}{2}\ {}_0D_t^0(u'(0)) + \frac{1}{2}\ {}_tD_1^0(u'(0))\right) - u(0) = A, \\ 3\left(\frac{1}{2}\ {}_0D_t^0(u'(1)) + \frac{1}{2}\ {}_tD_1^0(u'(1))\right) + u(1) = B, \\ \Delta\left(\ {}_0D_t^0(u'(t_1)) + \ {}_tD_1^0(u'(t_1))\right) = 1 + |u(t_1)|^{\frac{1}{2}}. \end{cases} \tag{4.1}$$

Then, we easy know that (H_1) holds with $l_1 = 1$, $k_1 = 3$, $\bar{l}_1 = 1$, $\bar{k}_1 = 2$, $\sigma_1 = \frac{1}{2}$ and $\bar{\sigma}_1 = \frac{1}{2}$.

(1) Let $F(t,u) = 2|u| + t$, $\theta = 1$ and $\tau(t) = 3 + t$, then (H_2) holds.

(2) Let $F(t,u) = (1+t)(\frac{1}{6}u^2 - 2|u|)$ and $\eta(t) = \frac{1}{4}(1+t)$, then (H_3) holds.

So, based on Theorem 3.1, for any constants A and B, the BVP (4.1) has at least one weak solution under $\lambda \in (0, +\infty)$ and $\lambda \in (0, \frac{1}{10})$, respectively.

Example 4.2. Consider the following fractional boundary value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}\, {}_0D_t^0(u'(t)) + \dfrac{1}{2}\, {}_tD_1^0(u'(t))\right) + \lambda f(t,u(t)) = 0,\ t \neq t_1,\ a.e.\ t \in [0,1], \\ 2\left(\dfrac{1}{2}\, {}_0D_t^0(u'(0)) + \dfrac{1}{2}\, {}_tD_1^0(u'(0))\right) - u(0) = -\dfrac{1}{60}, \\ 3\left(\dfrac{1}{2}\, {}_0D_t^0(u'(1)) + \dfrac{1}{2}\, {}_tD_1^0(u'(1))\right) + u(1) = \dfrac{1}{40}, \\ \Delta\left({}_0D_t^0(u'(t_1)) + {}_tD_1^0(u'(t_1))\right) = \dfrac{1}{20} + \dfrac{1}{60}|u(t_1)|^{\frac{1}{2}}. \end{cases} \tag{4.2}$$

When $l_1 = \frac{1}{100}$, $\bar{l}_1 = \frac{1}{10}$, $k_1 = \bar{k}_1 = \frac{1}{60}$ and $\sigma_1 = \bar{\sigma}_1 = \frac{1}{2}$, (H_1) holds. Let $F(t,u) = \frac{1}{6}(1+t)u^6$ and $\mu = 3$, we know that $M > 0$ and (H_4) holds. Thus, according to Theorem 3.2, the BVP (4.2) exists at least two different weak solutions for $\lambda \in (0, 0.02)$.

Example 4.3. Consider the following fractional boundary value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}\, {}_0D_t^0(u'(t)) + \dfrac{1}{2}\, {}_tD_1^0(u'(t))\right) + \lambda f(t,u(t)) = 0,\ t \neq t_1,\ a.e.\ t \in [0,1], \\ 2\left(\dfrac{1}{2}\, {}_0D_t^0(u'(0)) + \dfrac{1}{2}\, {}_tD_1^0(u'(0))\right) - u(0) = 0, \\ 3\left(\dfrac{1}{2}\, {}_0D_t^0(u'(1)) + \dfrac{1}{2}\, {}_tD_1^0(u'(1))\right) + u(1) = 0, \\ \Delta\left({}_0D_t^0(u'(t_1)) + {}_tD_1^0(u'(t_1))\right) = \dfrac{1}{2}u(t_1)^{\frac{1}{3}}. \end{cases} \tag{4.3}$$

If $l_1 = \bar{l}_1 = 1$, $k_1 = \bar{k}_1 = \frac{1}{2}$ and $\sigma_1 = \bar{\sigma}_1 = \frac{1}{3}$, (H_1) holds. From $I_1(u) = \frac{1}{2}u^{\frac{1}{3}}$, $F(t,u) = \frac{1}{6}(1+t)u^6$ and $\mu = 3$, we know that (H_4) and (H_5) hold. Therefore, according to Theorem 3.3, the BVP (4.3) admits infinitely many weak solutions for $\lambda \in (0, +\infty)$.

5. Conclusions

In this paper, we study a class of fractional Sturm-Liouville problem with impulsive conditions. The existence of at least one weak solution and at least two different weak solutions of the BVP (1.1) are obtained by using different variational methods. In addition, we also study the BVP (3.4) of the homogeneous Sturm-Liouville boundary condition corresponding to the BVP (1.1), and obtain the existence result of its infinite solutions. The BVP (1.2) in literature [3] is of integer order, and the BVP (1.3) in literature [24] is only a homogeneous Sturm-Liouville boundary condition. So the BVP (1.1) in this paper extends the BVP (1.2) and generalizes the BVP (1.3). Then the research of this paper is necessary and meaningful.

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