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VARIATIONAL METHODS TO THE FOURTH-ORDER LINEAR AND NONLINEAR DIFFERENTIAL EQUATIONS WITH NON-INSTANTANEOUS IMPULSES

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Abstract In this paper, the existence of solutions for the fourth-order linear and nonlinear differential equations with non-instantaneous impulses is studied by applying variational methods. The interesting point lies in that the variational structures corresponding to the fourth-order linear and nonlinear differential equations with non-instantaneous impulses are established for the first time.

Keywords Variational methods, non-instantaneous impulses, fourth-order, linear and nonlinear differential equations.

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1. Introduction

In recent years, impulsive differential equations have emerged as a valuable branch of mathematics which is widely used in physical, chemistry, biology and engineering. Many authors studied the existence of solution for the problems with instantaneous impulses in research papers, we refer the readers to [3, 14, 15, 18–20].

However, in many realistic applications, we need to describe the change in the process which is transient but continues influence for a limited time interval. Non-instantaneous impulsive, a new class of impulse was introduced by Hernádez and ORegan in [8]. Many different motivations for the study of this type of problems have been proposed. For example, some dynamic changes of blood in patients during drug therapy, which cause impulsive jump starting abruptly at any fixed point and keep continues process for a finite time interval.

By and large, the study of the existence and multiplicity of solutions for non-instantaneous impulsive differential equations has attracted more and more attention, see, for example [1, 2, 7, 8, 10, 11, 16, 21, 22, 24]. They are extensively studied by using various tools [7, 16, 22], such as monotone iterative technique, the theory of analytic semigroup, fixed point theory. In [8], Hernádez and O'Regan first introduced the non-instantaneous impulses and studied the mild and classical solutions

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for the non-instantaneous impulsive differential equation which was of the form

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)),\ t \in (s_i, t_{i+1}],\ i = 0, 1, 2, ..., m, \\ x(t) = g_i(t, x(t)),\ t \in (t_i, s_i],\ i = 1, 2, ..., m, \\ x(0) = x_0 \in X. \end{cases} \tag{1.1}$$

In [10], Bai and Nieto first revealed the variational structure of the following linear equation with non-instantaneous impulses

$$\begin{cases} -u''(t) = \sigma_i(t),\ t \in (s_i, t_{i+1}],\ i = 0, 1, 2, ..., N, \\ u'(t) = \alpha_i,\ t \in (t_i, s_i],\ i = 1, 2, ..., N, \\ u'(s_i^-) = u'(s_i^+),\ i = 1, 2, ..., N, \\ u(0) = u(T) = 0,\ u'(0) = \alpha_0. \end{cases} \tag{1.2}$$

They got the existence and uniqueness of weak solutions as critical points by the variational methods. It is the first time that the critical point theory has been applied to consider this kind of problems. In [21], by applying variational methods, Tian and Zhang studied the existence of solutions for the following second-order differential equations with instantaneous and non-instantaneous impulses

$$\begin{cases} -u''(t) = f_i(t, u(t)),\ t \in (s_i, t_{i+1}],\ i = 0, 1, 2, ..., N, \\ \Delta u'(t) = I_i(u(t_i)),\ i = 1, 2, ..., N, \\ u'(t) = u'(t_i^+),\ t \in (t_i, s_i],\ i = 1, 2, ..., N, \\ u'(s_i^+) = u'(s_i^-), i = 1, 2, ..., N, \\ u(0) = u(T) = 0. \end{cases} \tag{1.3}$$

In recent years, a great deal of papers have studied the fourth-order differential equations [4, 12, 18, 23].

Motivated by the above work, we investigate the existence of solutions for the fourth-order linear differential equation with non-instantaneous impulsive as follows

$$\begin{cases} u^{iv}(t) + Au''(t) + Bu(t) = f_i(t),\ t \in (s_i, t_{i+1}],\ i = 0, 1, 2, ..., N, \\ u'''(t) + Au'(t) = u'''(t_i^+) + Au'(t_i^+),\ t \in (t_i, s_i],\ \ i = 1, 2, ..., N, \\ u'''(t_i^+) = u'''(t_i^-),\ u'''(s_i^+) = u'''(s_i^-),\ i = 1, 2, ..., N, \\ u(0) = u(T) = 0,\ u'(0) = u'(T) = 0, \end{cases} \tag{1.4}$$

where $A < 0$, $B \in \mathbb{R}$, $0 = s_0 < t_1 < s_1 < t_2 < \cdots < s_N < t_{N+1} = T$, $f_i \in C((s_i, t_{i+1}], \mathbb{R})$. Furthermore, we also consider the existence result of the corresponding nonlinear problems as follows

$$\begin{cases} u^{iv}(t) + Au''(t) + Bu(t) = f_i(t, u(t)),\ t \in (s_i, t_{i+1}],\ i = 0, 1, 2, ..., N, \\ u'''(t) + Au'(t) = u'''(t_i^+) + Au'(t_i^+),\ t \in (t_i, s_i],\quad i = 1, 2, ..., N, \\ u'''(t_i^+) = u'''(t_i^-),\ u'''(s_i^+) = u'''(s_i^-),\ i = 1, 2, ..., N, \\ u(0) = u(T) = 0,\ u'(0) = u'(T) = 0, \end{cases} \tag{1.5}$$

where $A < 0$, $B \in \mathbb{R}$, $0 = s_0 < t_1 < s_1 < t_2 < \cdots < s_N < t_{N+1} = T$, $f_i \in C((s_i, t_{i+1}] \times \mathbb{R}, \mathbb{R})$.

To the best of our knowledge, the structure of the fourth-order differential equation with non-instantaneous impulses has not been yet developed. In order to fill this gap, we consider the linear and nonlinear fourth-order differential equation with non-instantaneous impulse simultaneously. With the non-instantaneous impulsive effect and fourth-order taken into consideration, some difficulties need to be overcome, such as how to find the corresponding variational functional J should be overcome. The variational structure of the fourth-order differential equation with the non-instantaneous impulse are established for the first time. Furthermore, we overcome the difficulties that the weak solution is the classical solution under the non-instantaneous impulsive effect.

This paper is organized as follows. In Section 2, we demonstrate a few preliminaries, definitions, lemmas and our main tools which are to be employed in exhibiting our essential outcomes of the rest of the article. In Section 3, we discuss the existence of solutions for the linear problem (1.4). In Section 4, we give the main results that the nonlinear problem (1.5) has infinitely many pairs of distinct classical solutions. Moreover, a concrete example of application is given.

2. Preliminaries

Let us consider the space

$$X = H_0^2([0, T]) = \{u \in W^{2,2}[0, T] \mid u(0) = u(T) = 0,\ u'(0) = u'(T) = 0\}$$

be equipped with the inner product

$$(u, v) = \int_0^T u''(t)v''(t)dt,\ \forall u, v \in X,$$

which induces the usual norm

$$\|u\| = \left(\int_0^T |u''|^2 dt\right)^{\frac{1}{2}}.$$

It is clear that $(X, \|\cdot\|)$ is a Hilbert space. Then the following norm

$$\|u\|_X = \left(\int_0^T |u''|^2 - A|u'|^2 dt\right)^{\frac{1}{2}}$$

is equivalent to the usual one. In fact, for any $u \in X$, let us recall Poincáre type inequality $\int_0^T |u'|^2 dt \leq \frac{T^2}{\pi^2} \int_0^T |u''|^2 dt$, then one has $\int_0^T |u''|^2 dt \leq \int_0^T |u''|^2 - A|u'|^2 dt \leq (1 - A\frac{T^2}{\pi^2}) \int_0^T |u''|^2 dt$. Let $\delta_1 = (1 - A\frac{T^2}{\pi^2})^{\frac{1}{2}}$, we obtain

$$\|u\| \leq \|u\|_X \leq \delta_1 \|u\|. \tag{2.1}$$

Thus, $(X, \|\cdot\|_X)$ is also a Hilbert space.

Lemma 2.1. *If $u \in X$, then $\|u\|_\infty \leq M_1 \|u\|_X$,*
where $M_1 = \max \left\{ T^{\frac{1}{2}}, \frac{T^{\frac{2}{3}}}{\pi} \right\}$, $\|u\|_\infty = \max \left\{ \max_{t\in[0,T]} |u(t)|, \max_{t\in[0,T]} |u'(t)| \right\}$.

Proof. For any $u \in X$, by Hölder's inequality and Poincáre type inequality,

$$\begin{aligned} |u(t)| &= \left| \int_0^t u'(s) ds \right| \leq T^{\frac{1}{2}} \left(\int_0^T |u'(s)|^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{T^{\frac{2}{3}}}{\pi} \left(\int_0^T |u''(s)|^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{T^{\frac{2}{3}}}{\pi} \|u\|_X \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} |u'(t)| &= \left| u'(0) + \int_0^t u''(s) ds \right| \leq \int_0^T |u''(s)| ds \\ &\leq T^{\frac{1}{2}} \left(\int_0^T |u''(s)|^2 ds \right)^{\frac{1}{2}} \\ &\leq T^{\frac{1}{2}} \|u\|_X. \end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we have $\|u\|_\infty \leq M_1 \|u\|_X$, where $M_1 = \max \left\{ \mathrm{T}^{\frac{1}{2}}, \frac{\mathrm{T}^{\frac{2}{3}}}{\pi} \right\}$. □

For the linear problem (1.4), we define functional $J : X \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} \int_0^T |u''(t)|^2 - A|u'(t)|^2 dt + \frac{1}{2} \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} Bu^2(t) dt - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} f_i(t) u(t) dt. \tag{2.4}$$

Obviously, J is a Gâteaux differentiable functional and its Gâteaux derivation at the point $u \in X$ is

$$\begin{aligned} \langle J'(u), v \rangle = &\int_0^T u''(t) v''(t) - Au'(t) v'(t) dt + \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} Bu(t) v(t) dt \\ &- \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} f_i(t) v(t) dt \end{aligned} \tag{2.5}$$

for all $v \in X$.

For the nonlinear problem (1.5), we define the following functional on X

$$I(u) = \frac{1}{2} \int_0^T |u''(t)|^2 - A|u'(t)|^2 dt + \frac{1}{2} \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} Bu^2(t) dt - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} F_i(t, u(t)) dt, \tag{2.6}$$

where $F_i(t,u)=\int_0^u f_i(t,s)ds$.

Obviously, I is a Gâteaux differentiable functional and its Gâteaux derivation at the point $u\in X$ is

$$\langle I'(u),v\rangle=\int_0^T u''(t)v''(t)-Au'(t)v'(t)dt+\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}Bu(t)v(t)dt$$
$$-\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}f_i(t,u(t))v(t)dt. \tag{2.7}$$

for all $v\in X$.

Definition 2.1. A function $u\in X\cap C^4((0,t_1]\cup(s_1,t_2]\cup\cdots\cup(s_N,T])$ is said to be a classical solution of problem (1.4) (*resp.* (1.5) if u simultaneously satisfies equation , non-instantaneous impulsive condition and boundary conditions in (1.4) (*resp.* (1.5)).

Definition 2.2. A function $u\in X$ is said to be a weak solution of problem (1.4) (*resp.* (1.5)) if u satisfies $\langle J'(u),v\rangle=0$ (*resp.* $\langle I'(u),v\rangle=0$) for all $v\in X$.

To show Lemma 2.4, we need the following Fundamental Lemmas.

Lemma 2.2 (Fundamental Lemma 1)**.** *Let $u,v\in L^1[0,T]$, if for every $f\in H_0^2([0,T])$, we have $\int_0^T u(t)f''(t)dt=\int_0^T v(t)f(t)dt$, then there exist two constants C_1, $C_2\in\mathbb{R}$ such that $u(t)=\int_0^t\int_0^s v(\theta)d\theta ds+C_1t+C_2$ for a.e $t\in[0,T]$.*

Proof. Let $w\in C[0,T]$ by $w(t)=\int_0^t\int_0^s v(\theta)d\theta ds$, one has

$$\int_0^T w(t)f''(t)dt=\int_0^T\left(\int_0^t\int_0^s v(\theta)d\theta ds\right)f''(t)dt.$$

By the Fubini theorem, one has

$$\begin{aligned}\int_0^T w(t)f''(t)dt&=\int_0^T\int_s^T\int_0^s v(\theta)d\theta f''(t)dtds=\int_0^T\int_0^s v(\theta)d\theta(f'(T)-f'(s))ds\\&=-\int_0^T\int_0^s v(\theta)d\theta f'(s)ds=-\int_0^T\int_\theta^T v(\theta)f'(s)dsd\theta\\&=-\int_0^T v(\theta)(f(T)-f(\theta))d\theta=\int_0^T v(\theta)f(\theta)d\theta.\end{aligned} \tag{2.8}$$

In virtue of $\int_0^T u(t)f''(t)dt=\int_0^T v(t)f(t)dt$, one has

$$\int_0^T(u(t)-w(t))\,f''(t)dt=0 \tag{2.9}$$

for every $f\in H_0^2([0,T])$. In particular, we choose

$$f(t)=\int_0^t\int_0^s(u(\theta)-w(\theta)-C_1\theta-C_2)\,d\theta ds,$$

where

$$C_1=\frac{6}{T^2}\int_0^T(u(s)-w(s))\,ds-\frac{12}{T^3}\int_0^T\int_0^s(u(\theta)-w(\theta))\,d\theta ds,$$

$$C_2 = \frac{6}{T^2}\int_0^T\int_0^s (u(\theta)-w(\theta))\,d\theta ds - \frac{2}{T}\int_0^T (u(s)-w(s))\,ds.$$

By computation, we have $f'(0) = f'(T) = f(0) = f(T) = 0$. By integration by parts, we have

$$\int_0^T (C_1 t + C_2) f''(t)dt = 0. \tag{2.10}$$

From (2.9) and (2.10), one has $\int_0^T (u(t)-w(t)-C_1t-C_2)f''(t)dt = 0$, i.e., $\int_0^T |u(t)-w(t)-C_1t-C_2|^2 dt = 0$. So we have

$$u(t) = \int_0^t\int_0^s v(\theta)d\theta ds + C_1 t + C_2.$$

for a.e. $t \in [0,T]$. □

Lemma 2.3 (Fundamental Lemma 2)**.** *Let $u \in H^2([0,T])$, if for every $v \in H_0^2([0,T])$ and a nonzero constant $C \in \mathbb{R}$, we have $\int_0^T u''v''dt = C\int_0^T u'v'dt$, then there exist two constants C_1, $C_2 \in \mathbb{R}$ such that $u''(t) = -C\int_0^t u'(s)ds + C_1t + C_2$ for a.e $t \in [0,T]$.*

Proof. Let $w \in C[0,T]$ by $w(t) = -C\int_0^t u'(s)ds$ and $v \in H_0^2([0,T])$ one has

$$\int_0^T w(t)v''(t)dt = w(t)v'(t)|_0^T - \int_0^T w'(t)v'(t)dt = C\int_0^T u'(t)v'(t)dt.$$

In virtue of $\int_0^T u''v''dt = C\int_0^T u'v'dt$, one has

$$\int_0^T (u''(t)-w(t))\,v''(t)dt = 0 \tag{2.11}$$

for every $v \in H_0^2([0,T])$. In particular, we choose

$$v(t) = \int_0^t\int_0^s (u''(\theta)-w(\theta)-C_1\theta-C_2)\,d\theta ds,$$

where

$$C_1 = \frac{6}{T^2}\int_0^T (u''(s)-w(s))\,ds - \frac{12}{T^3}\int_0^T\int_0^s (u''(\theta)-w(\theta))\,d\theta ds,$$

$$C_2 = \frac{6}{T^2}\int_0^T\int_0^s (u''(\theta)-w(\theta))\,d\theta ds - \frac{2}{T}\int_0^T (u''(s)-w(s))\,ds.$$

By computation, we have $v'(0) = v'(T) = v(0) = v(T) = 0$. By integration by parts, we have

$$\int_0^T (C_1 t + C_2) v''(t)dt = 0. \tag{2.12}$$

From (2.11) and (2.12), one has $\int_0^T (u''(t)-w(t)-C_1t-C_2)v''(t)dt = 0$, i.e., $\int_0^T |u''(t)-w(t)-C_1t-C_2|^2dt = 0$. So we have

$$u''(t) = -C\int_0^t u'(s)ds + C_1t + C_2.$$

for a.e. $t \in [0,T]$. □

Lemma 2.4. *If $u \in X$ is a weak solution of problem* (1.4)*, then u is a classical solution of problem* (1.4)*.*

Proof. Clearly, $u \in X$ which means that the boundary conditions are satisfied. If u is a weak solution of problem (1.4), then $\langle J'(u), v\rangle = 0$ for all $v \in X$, i.e.,

$$\int_0^T u''(t)v''(t) - Au'(t)v'(t)dt + \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} Bu(t)v(t)dt - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} f_i(t)v(t)dt = 0. \tag{2.13}$$

We will divide three steps to complete the proof.

First of all, we show that u satisfies the equation in (1.4). Without loss of generality, we assume that $v \in H_0^2([s_i, t_{i+1}])$, $v(t) \equiv 0$ for $t \in [0, s_i) \cup (t_{i+1}, T]$. Then, substituting $v(t)$ into (2.13), we get

$$\int_{s_i}^{t_{i+1}} u''(t)v''(t) - Au'(t)v'(t)dt = \int_{s_i}^{t_{i+1}} (f_i(t) - Bu(t))v(t)dt. \tag{2.14}$$

By integration by parts, one has

$$\begin{aligned} \int_{s_i}^{t_{i+1}} Au'(t)v'(t)dt &= \int_{s_i}^{t_{i+1}} A\left(u'(s_i) + \int_{s_i}^{t} u''(s)ds\right) v'(t)dt \\ &= A\int_{s_i}^{t_{i+1}} \left(\int_{s_i}^{t} u''(s)ds\right) v'(t)dt \\ &= A\left(v(t)\int_{s_i}^{t} u''(s)ds\Big|_{s_i}^{t_{i+1}} - \int_{s_i}^{t_{i+1}} u''(t)v(t)dt\right) \\ &= -A\int_{s_i}^{t_{i+1}} u''(t)v(t)dt. \end{aligned} \tag{2.15}$$

Substituting (2.15) into (2.14), we get

$$\int_{s_i}^{t_{i+1}} u''(t)v''(t)dt = \int_{s_i}^{t_{i+1}} \left(f_i(t) - Bu(t) - Au''(t)\right) v(t)dt$$

for $i = 0, 1, 2, ..., N$. By Lemma 2.2, we have $u''(t) = \int_{s_i}^{t}\int_{s_i}^{s}[f_i(\theta) - Bu(\theta) - Au''(\theta)]d\theta ds + C_1 t + C_2$. Since $f \in C(s_i, t_{i+1}]$, $u \in C(s_i, t_{i+1}]$, one has $u'' \in C(s_i, t_{i+1}]$ and $u^{iv} \in C(s_i, t_{i+1}]$. Thus, we have

$$u^{iv}(t) = f_i(t) - Au''(t) - Bu(t) \text{ for } t \in (s_i, t_{i+1}] \text{ and } i = 0, 1, 2, ..., N, \tag{2.16}$$

i.e., u satisfies the equation in (1.4).

Secondly, we will show the non-instantaneous impulsive conditions holds. Substituting (2.16) into (2.13), one has

$$\int_0^T u''(t)v''(t) - Au'(t)v'(t)dt = \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} (u^{iv}(t) + Au''(t))v(t)dt. \tag{2.17}$$

By integration by parts, we have

$$\begin{aligned}\int_{s_i}^{t_{i+1}} u^{iv}(t)v(t)dt =& u'''(t)v(t)|_{s_i^+}^{t_{i+1}^-} - \int_{s_i}^{t_{i+1}} u'''(t)v'(t)dt \\ =& u'''(t_{i+1}^-)v(t_{i+1}) - u'''(s_i^+)v(s_i) \\ & - u''(t_{i+1}^-)v'(t_{i+1}) + u''(s_i^+)v'(s_i) + \int_{s_i}^{t_{i+1}} u''(t)v''(t)dt\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}A\int_{s_i}^{t_{i+1}} u''(t)v(t)dt &= A\left(u'(t)v(t)|_{s_i^+}^{t_{i+1}^-} - \int_{s_i}^{t_{i+1}} u'(t)v'(t)dt\right) \\ &= A\left(u'(t_{i+1}^-)v(t_{i+1}) - u'(s_i^+)v(s_i) - \int_{s_i}^{t_{i+1}} u'(t)v'(t)dt\right).\end{aligned} \tag{2.19}$$

Substituting (2.18) and (2.19) into (2.17), one has

$$\begin{aligned}0 =& \sum_{i=1}^{N}\int_{t_i}^{s_i} u''(t)v''(t) - Au'(t)v'(t)dt \\ & - \sum_{i=0}^{N}[u'''(t_{i+1}^-)v(t_{i+1}) - u'''(s_i^+)v(s_i) - u''(t_{i+1}^-)v'(t_{i+1}) + u''(s_i^+)v'(s_i)] \\ & - \sum_{i=0}^{N} A[u'(t_{i+1}^-)v(t_{i+1}) - u'(s_i^+)v(s_i)].\end{aligned} \tag{2.20}$$

Without loss of generality, we assume that $v \in H_0^2([t_i, s_i])$, $v(t) \equiv 0$ for $t \in [0, t_i) \cup (s_i, T]$. Substituting $v(t)$ into (2.20), we get

$$\int_{t_i}^{s_i} u''(t)v''(t)dt = \int_{t_i}^{s_i} Au'(t)v'(t)dt.$$

By Lemma 2.3, one has $u''(t) = -A\int_{t_i}^{t} u'(s)ds + C_1 t + C_2$. Since $u' \in C(t_i, s_i]$, one has $u'''(t) + Au'(t) = constant$, $t \in (t_i, s_i]$, $i = 1, 2, ..., N$, which means the non-instantaneous impulsive equation

$$\begin{aligned}u'''(t) + Au'(t) &= u'''(t_i^+) + Au'(t_i^+) \\ &= u'''(s_i^-) + Au'(s_i^-) \text{ for } t \in (t_i, s_i] \text{ and } i = 1, 2, ..., N\end{aligned} \tag{2.21}$$

in (1.4) holds.

Finally, we will show $u'''(t_i^+) = u'''(t_i^-)$, $u'''(s_i^+) = u'''(s_i^-)$ in (1.4) hold. In virtue of (2.20), by integration by parts, we obtain

$$\begin{aligned}0 =& \sum_{i=1}^{N}\left(u''(t)v'(t)|_{t_i^+}^{s_i^-} - \int_{t_i}^{s_i} u'''(t)v'(t)dt\right) - \sum_{i=1}^{N}\int_{t_i}^{s_i} Au'(t)v'(t)dt \\ & - \sum_{i=0}^{N}[u'''(t_{i+1}^-)v(t_{i+1}) - u'''(s_i^+)v(s_i) - u''(t_{i+1}^-)v'(t_{i+1}) + u''(s_i^+)v'(s_i)]\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=0}^{N} A[u'(t_{i+1}^-)v(t_{i+1}) - u'(s_i^+)v(s_i)] \\
&= -\sum_{i=1}^{N} \int_{t_i}^{s_i} (u'''(t) + Au'(t))v'(t)dt + \sum_{i=1}^{N} \left[u''(s_i^-)v'(s_i) - u''(t_i^+)v'(t_i)\right] \\
&-\sum_{i=0}^{N} [u'''(t_{i+1}^-)v(t_{i+1}) - u'''(s_i^+)v(s_i) - u''(t_{i+1}^-)v'(t_{i+1}) + u''(s_i^+)v'(s_i)] \\
&-\sum_{i=0}^{N} A[u'(t_{i+1}^-)v(t_{i+1}) - u'(s_i^+)v(s_i)]. \qquad (2.22)
\end{aligned}
$$

Substituting (2.21) into (2.22), we have

$$
\begin{aligned}
0 &= -\sum_{i=1}^{N} \int_{t_i}^{s_i} (u'''(t_i^+) + Au'(t_i^+))v'(t)dt + \sum_{i=1}^{N} \left[u''(s_i^-)v'(s_i) - u''(t_i^+)v'(t_i)\right] \\
&-\sum_{i=0}^{N} [u'''(t_{i+1}^-)v(t_{i+1}) - u'''(s_i^+)v(s_i) - u''(t_{i+1}^-)v'(t_{i+1}) + u''(s_i^+)v'(s_i)] \\
&-\sum_{i=0}^{N} A[u'(t_{i+1}^-)v(t_{i+1}) - u'(s_i^+)v(s_i)] \\
&= -\sum_{i=1}^{N} (u'''(t_i^+) + Au'(t_i^+))(v(s_i) - v(t_i)) + \sum_{i=1}^{N} \left[u''(s_i^-)v'(s_i) - u''(t_i^+)v'(t_i)\right] \\
&-\sum_{i=1}^{N} [u'''(t_i^-)v(t_i) - u'''(s_i^+)v(s_i) - u''(t_i^-)v'(t_i) + u''(s_i^+)v'(s_i)] \\
&-\sum_{i=1}^{N} A[u'(t_i^-)v(t_i) - u'(s_i^+)v(s_i)] \\
&= \sum_{i=1}^{N} \left[u'''(s_i^+) + Au'(s_i^+) - (u'''(t_i^+) + Au'(t_i^+))\right] v(s_i) \\
&+ \sum_{i=1}^{N} \left[u'''(t_i^+) + Au'(t_i^+) - (u'''(t_i^-) + Au'(t_i^-))\right] v(t_i). \qquad (2.23)
\end{aligned}
$$

Let us choose

$$v_1(t) = (t - s_0)^2(t - t_1)\dots(t - s_{i-1})(t - s_i)(t - t_{i+1})\dots(t - T)^2.$$

Obviously, $v_1(t) \in X$, $v_1(s_i) = 0$ for $i = 1, 2, ..., N$, $v_1(t_j) = 0$ for $j \neq i$ and $v_1(t_i) \neq 0$. Substituting $v_1(t)$ into (2.23), we obtain $u'''(t_i^+) + Au'(t_i^+) - \big(u'''(t_i^-) + Au'(t_i^-)\big) = 0$, i.e.,

$$u'''(t_i^+) = u'''(t_i^-), \ i = 1, 2, ..., N.$$

Similarly, we choose

$$v_2(t) = (t - s_0)^2(t - t_1)\dots(t - t_i)(t - t_{i+1})\dots(t - T)^2.$$

Obviously, $v_2(t) \in X$, $v_2(s_i) \neq 0$, $v_2(s_j) = 0$ for $j \neq i$ and $v_2(t_i) = 0$ for $i = 1, 2, ..., N$. Substituting $v_2(t)$ into (2.23), we obtain $u'''(s_i^+) + Au'(s_i^+) - \big(u'''(t_i^+) + Au'(t_i^+)\big) = 0$. From (2.21), we have $u'''(t_i^+) + Au'(t_i^+) = u'''(s_i^-) + Au'(s_i^-)$, combining two equations and $u \in X$, one has

$$u'''(s_i^+) = u'''(s_i^-), i = 1, 2, ..., N.$$

From the above, according to the definition 2.1, it follows that u is a classical solution of the problem (1.4). The proof is complete. □

Similarly we can prove the following Lemma.

Lemma 2.5. *If $u \in X$ is a weak solution of problem* (1.5)*, then u is a classical solution of problem* (1.5)*.*

In this paper, the following theorems will be used in our main results.

Theorem 2.1 (Theorem 2.1, [15])**.** *Let H be a reflexive Hilbert space. Let a: $H \times H \to \mathbb{R}$ be a bounded bilinear form. If a is coercive, i.e., there exists $\delta > 0$ such that $a(u, u) \geq \delta \|u\|^2$ for every $u \in H$, then for any $f \in H'$ (the dual of H) there exists a unique $u \in H$ such that*

$$a(u, v) = \langle f, v \rangle\,, \quad for\ every \ \ u \in H.$$

Moreover, if a is also symmetric, then the function φ: $H \to \mathbb{R}$ defined by

$$\varphi(v) = \frac{1}{2}a(v, v) - \langle f, v \rangle$$

attains its minimum at u.

Theorem 2.2 (Theorem 5.23, [6])**.** *Let X be a Banach space, and $I \in C^1(X, \mathbb{R})$ be an even function satisfying the (PS) condition. Assume $\alpha < \beta$ and either $I(0) < \alpha$ or $I(0) > \beta$. If further,*

(1) there is an m-dimensional linear subspace E and $\rho > 0$ such that $\sup_{t \in E \cap \partial B_\rho(0)} I(t) \leq \beta$, where $\partial B_\rho(0) = \{t \in X : \|t\| = \rho\}$,

(2) there is a j-dimensional linear subspace F such that $\inf_{t \in F^\perp} I(t) > \alpha$, where $F_\perp$ is a complementary space of F,

(3) $m > j$,
then I has at least $m - j$ pairs of distinct critical points.

3. Linear non-instantaneous impulsive problem (1.4)

We define $a : H_0^2([0, T]) \times H_0^2([0, T]) \to \mathbb{R}$ by

$$a(u, v) = \int_0^T u''(t)v''(t) - Au'(t)v'(t)dt + \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} Bu(t)v(t)dt$$

and $l : H_0^2([0, T]) \to \mathbb{R}$ by

$$l(v) = \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} f_i(t)v(t)dt.$$

We see that finding the weak solutions of (1.4) is equivalent to finding $u \in H_0^2(0,T)$ such that

$$a(u,v) = l(v), \text{ for every } v \in H_0^2(0,T).$$

Theorem 3.1. *Non-instantaneous impulsive linear problem* (1.4) *has a unique weak solution* $u \in H_0^2([0,T])$ *for any* $f_i \in C((s_i, t_{i+1}], \mathbb{R})$, *when* $B > -\frac{1}{\max\left\{T^2, \frac{T^{\frac{7}{3}}}{\pi^2}\right\}}$. *Moreover, u is a classical solution and u minimizes the functional J(u).*

Proof. We will apply Theorem 2.6 to show the existence of solutions of linear problems (1.4). It is evident that a is bilinear, symmetric, and l is linear. Firstly, we will show a is bounded, by Hölder's inequality, (2.1) and Lemma 2.1, we have

$$\begin{aligned}
|a(u,v)| &= \left| \int_0^T u''(t)v''(t) - Au'(t)v'(t)dt + \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} Bu(t)v(t)dt \right| \\
&\leq \int_0^T |u''(t)v''(t)|\, dt - A \int_0^T |u'(t)v'(t)|\, dt + \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} |Bu(t)v(t)|\, dt \\
&\leq \|u\|\|v\| - A\|u\|_\infty \|v\|_\infty T + |B|\|u\|_\infty \|v\|_\infty T \\
&\leq \|u\|_X \|v\|_X - AM_1^2 \|u\|_X \|v\|_X T + |B| M_1^2 \|u\|_X \|v\|_X T \\
&\leq \delta_0 \|u\|_X \|v\|_X,
\end{aligned} \tag{3.1}$$

where $\delta_0 = 1 - AM_1^2 T + |B|\, M_1^2 T$. We have $|a(u,v)| \leq \delta_0 \|u\|_X \|v\|_X$, i.e., $a(u,v)$ is bounded.

Now, we will show a is coercive. By the definition of a, one has

$$\begin{aligned}
a(u,u) &= \int_0^T |u''(t)|^2 - A\, |u'(t)|^2\, dt + \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} Bu^2(t)dt \\
&= \|u\|_X^2 + \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} Bu^2(t)dt.
\end{aligned} \tag{3.2}$$

For $B \geq 0$, we can easily get $a(u,u) \geq \|u\|_X^2$ for every $u \in H_0^2([0,T])$. For $B < 0$, by Lemma 2.1, one has

$$\sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} u^2(t)dt \leq \sum_{i=0}^{N} \|u\|_\infty^2 (t_{i+1} - s_i) \quad \leq M_1^2 \|u\|_X^2 \sum_{i=0}^{N} (t_{i+1} - s_i) \leq M_2 \|u\|_X^2, \tag{3.3}$$

where $M_2 = M_1^2 T$. By (3.2) and (3.3), one has $a(u,u) \geq (1 + BM_2)\|u\|_X^2$, for every $u \in H_0^2([0,T])$. By Lemma 2.1 and the assumption in Theorem 3.1, one has $1 + BM_2 > 0$. Thus, $a(u,v)$ is coercive.

Lastly, it remains to check that l is bounded. Since $f_i \in C((s_i, t_{i+1}], \mathbb{R})$, for some $M_0 > 0$, we have $|f_i(t)| < M_0$ for $t \in (s_i, t_{i+1}]$. By Lemma 2.1, one has

$$\begin{aligned}
|l(v)| &= \left| \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} f_i(t)v(t)dt \right| \leq M_0 \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} |v(t)|\, dt \\
&\leq M_0 \|v\|_\infty \sum_{i=0}^{N} (t_{i+1} - s_i)
\end{aligned}$$

$$\leq M_0 M_1 T \|v\|_X. \tag{3.4}$$

Hence, $l \in X'$. By applying Theorem 2.6, the functional J has a unique weak solution. Moreover, u is a classical solution and u minimizes the functional J. □

4. Nonlinear non-instantaneous impulsive problem (1.5)

In this part, we need the following conditions about the nonlinearity

(G_1) $F_i(t,\xi) \geq \varphi_i(t)|\xi|^q$ for$|\xi| \to 0$, $0 \leq q < 2$, $\varphi_i(t) \in C([0,T],\mathbb{R}_+)$, $i = 1,2,...,N$,

(G_2) $0 < \xi f_i(t,\xi) \leq \phi_i(t)|\xi|^\mu$ for $|\xi| > r$, $0 \leq \mu < 2$, $\phi_i(t) \in C([0,T],\mathbb{R}_+)$, $i = 1,2,...,N$,

(G_3) $f_i(t,\xi)$ is odd in ξ, $i = 1,2,...,N$.

Remark 4.1. By (G_2), we can get $F_i(t,\xi) \leq \frac{\phi_i(t)}{\mu}|\xi|^\mu$ for $|\xi| > r$, $0 \leq \mu < 2$, $\phi_i(t) \in C([0,T],\mathbb{R}_+)$, $i = 1,2,...,N$.

Such the above conditions produce the existence of infinitely many pairs of distinct solutions for nonlinear non-instantaneous impulse problem (1.5). Precisely, we obtain the following results

Theorem 4.1. *If (G_1), (G_2) and (G_3) are fulfilled, when $B > -\frac{1}{\max\left\{T^2, \frac{T^{\frac{7}{3}}}{\pi^2}\right\}}$, the problem* (1.5) *has infinitely many pairs of distinct classical solutions.*

Proof. It is obvious that $I \in C^1(X,\mathbb{R})$. By (G_3), I is an even functional with $I(0) = 0$. We complete the proof by the following three steps.
Step 1. We shall prove that the functional I satisfies the (PS)-condition, i.e., every sequence $\{u_m\} \subset X$ for which $\{I(u_m)\}$ is bounded and $I'(u_m) \to 0$ as $n \to \infty$ possesses a convergent subsequence in X. By (2.6) and (2.7), one has $\beta = \frac{1}{\alpha}$, $\alpha > 2$

$$\begin{aligned} M + \beta\|u_m\|_X \geq I(u_m) - \beta\left\langle I'(u_m), u_m\right\rangle \\ = (\frac{1}{2} - \frac{1}{\alpha})\|u_m\|_X^2 + (\frac{1}{2} - \frac{1}{\alpha})\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} Bu_m^2(t)dt \\ - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,u_m(t))dt + \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \frac{1}{\alpha}f_i(t,u_m(t))u_m(t)dt \end{aligned} \tag{4.1}$$

For $B \geq 0$, by (4.1), (G_2), Remark 4.1 and Lemma 2.1, one has

$$\begin{aligned} M + \frac{1}{\alpha}\|u_m\|_X \geq & I(u_m) - \frac{1}{\alpha}\left\langle I'(u_m), u_m\right\rangle \\ \geq & (\frac{1}{2} - \frac{1}{\alpha})\|u_m\|_X^2 - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t,u_m(t))dt \\ & + \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \frac{1}{\alpha}f_i(t,u_m(t))u_m(t)dt \end{aligned}$$

$$
\begin{aligned}
&\geq (\frac{1}{2}-\frac{1}{\alpha})\|u_m\|_X^2-\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\frac{\phi_i(t)}{\mu}\,|u_m|^{\mu}\,dt\\
&\geq (\frac{1}{2}-\frac{1}{\alpha})\|u_m\|_X^2-\|u_m\|_\infty^{\mu}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\frac{\phi_i(t)}{\mu}dt\\
&\geq (\frac{1}{2}-\frac{1}{\alpha})\|u_m\|_X^2-M_1^{\mu}\|u_m\|_X^{\mu}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\frac{\phi_i(t)}{\mu}dt. \qquad (4.2)
\end{aligned}
$$

So $\{u_m\}$ is bounded in X for $B\geq 0$.

For $B<0$, by (3.3), (4.1), (G_2), Remark 4.1 and Lemma 2.1, one has

$$
\begin{aligned}
M+\frac{1}{\alpha}\|u_m\| \geq & I(u_m)-\frac{1}{\alpha}\left\langle I'(u_m),u_m\right\rangle\\
&\geq (\frac{1}{2}-\frac{1}{\alpha})\|u_m\|_X^2+(\frac{1}{2}-\frac{1}{\alpha})\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}Bu_m^2(t)dt\\
&\quad-\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}F_i(t,u_m(t))dt \qquad (4.3)\\
&\geq (\frac{1}{2}-\frac{1}{\alpha})(1+BM_2)\|u_m\|_X^2-M_1^{\mu}\|u_m\|_X^{\mu}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\frac{\phi_i(t)}{\mu}dt.
\end{aligned}
$$

By Lemma 2.1 and the assumption in Theorem 4.1, one has $1+BM_2>0$. So $\{u_m\}$ is bounded in X for $B<0$. Combining (4.2) with (4.3), one has $\{u_m\}$ is bounded in X. Since X is reflexive Banach space, the fact $\{u_m\}$ is bounded in X means that one has weakly convergent subsequence. Without loss of generality, we still denote $\{u_m\}$ is the subsequence of $\{u_m\}$. So, we have $u_m\rightharpoonup u$ in X. Following we will show $\{u_m\}$ strongly converges to u in X. One has

$$
\begin{aligned}
\langle I'(u_m)-I'(u),u_m-u\rangle =&\|u_m-u\|_X^2+\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}B(u_m(t)-u(t))^2dt\\
&-\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\left(f_i(t,u_m(t))-f_i(t,u(t))\right)(u_m(t)-u(t))dt.
\end{aligned} \qquad (4.4)
$$

Similar to the proof of Proposition 1.2 in [19], $u_m\rightharpoonup u$ implies $\{u_m\}$ uniformly converges to u in $C([0,T])$. So

$$
\begin{cases}
\left\langle I'(u_m)-I'(u),u_m-u\right\rangle\to 0,\\
\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}B(u_m(t)-u(t))^2dt\to 0,\\
\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}(f_i(t,u_m(t))-f_i(t,u(t)))(u_m(t)-u(t))dt\to 0,
\end{cases} \qquad (4.5)
$$

as $m\to 0$. By (4.4) and (4.5), we obtain that $\|u_m-u\|_X^2\to 0$ as $m\to\infty$, which means that $\{u_m\}$ strongly converges to u in X. Therefore, $I(u)$ satisfies (PS)-condition.

Step 2. We shall prove that there exists $\rho>0$ such that $\sup_{t\in E\cap\partial B_\rho(0)}I(t)\leq\beta<0=I(0)$, where $\partial B_\rho(0)=\left\{u\in H_0^2([0,T]):\|u\|_X=\rho\right\}$.

Choosing $E = X$ and $F = \emptyset$, then $F^{\perp} = X$. In virtue of (2.6) and (G_1), we have

$$\begin{aligned}\lim_{s\to 0}\frac{I(su)}{s^q} &= \lim_{s\to 0}\frac{\frac{1}{2}\|su\|_X^2 + \frac{1}{2}B\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}(su)^2dt - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}F_i(t,su)dt}{s^q}\\ &\le \lim_{s\to 0}\left(\frac{1}{2}s^{2-q}\|u\|_X^2 + \frac{1}{2}Bs^{2-q}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}u^2dt - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\varphi_i(t)\,|u|^q\,dt\right)\\ &= -\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\varphi_i(t)\,|u|^q\,dt.\\ &< 0,\end{aligned}$$

for every $u \in H_0^2([0,T]) \setminus \{0\}$ and $s > 0$.

Therefore, there exists $\rho > 0$ small such that $\sup_{t\in E\cap\partial B_\rho(0)} I(t) \le \beta < 0 = I(0)$, where $\partial B_\rho(0) = \left\{u \in H_0^2([0,T]) : \|u\|_X = \rho\right\}$.

Step 3. We shall prove that the functional I is bounded from below on X. For $B \ge 0$, by (G_2) and Remark 4.1, one has

$$\begin{aligned}I(u) &= \frac{1}{2}\|u\|_X^2 + \frac{1}{2}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}Bu^2(t)dt - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}F_i(t,u(t))dt\\ &\ge \frac{1}{2}\|u\|_X^2 - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}F_i(t,u(t))dt\\ &\ge \frac{1}{2}\|u\|_X^2 - M_1\|u\|_X^{\mu}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\frac{\phi_i(t)}{\mu}dt.\end{aligned}$$

For $B < 0$, by (3.3), (G_2) and Remark 4.1, one has

$$\begin{aligned}I(u) &= \frac{1}{2}\|u\|_X^2 + \frac{1}{2}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}Bu^2(t)dt - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}F_i(t,u(t))dt\\ &\ge \frac{1}{2}(1+BM_2)\|u\|_X^2 - M_1^{\mu}\|u\|_X^{\mu}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\frac{\phi_i(t)}{\mu}dt.\end{aligned}$$

Thus, there exists $r > 0$ such that $I(u) > 0$ for $|u| > r$. By $I \in C^1(X,\mathbb{R})$, it follows that I is bounded from below on X.

Moreover, from the all above, it means that there exists $\alpha < \beta$ such that $\inf_{F^{\perp}} I(u) > \alpha$. By applying Theorem 2.7, I has infinitely many pairs of distinct critical points on X. Consequently, the problem (1.5) has infinitely many pairs of distinct classical solutions. □

Example 4.1. Consider the following problem:

$$\begin{cases} u^{iv}(t) - u''(t) + u(t) = tu^{0.6},\ t \in (s_i, t_{i+1}],\ i = 0,1,2,...,N,\\ u'''(t) - u'(t) = u'''(t_i^+) - u'(t_i^+),\ t \in (t_i, s_i],\quad i = 1,2,...,N,\\ u'''(t_i^+) = u'''(t_i^-),\ u'''(s_i^+) = u'''(s_i^-),\ i = 1,2,...,N,\\ u(0) = u(T) = 0,\ u'(0) = u'(T) = 0.\end{cases} \tag{4.6}$$

With regard to the problem (1.5), $f_i(t,u) = tu^{0.6}$, where $F_i(t,u) = \frac{5}{8}tu^{1.6}$. Let $q = 1.8$, $\varphi_i(t) = \frac{5}{16}t$, $\mu = 1.7$, $\phi_i(t) = 2t$. Obviously, $f_i(t,u) = tu^{0.6}$ is odd in u. Besides, $F_i(t,u) = \frac{5}{8}tu^{1.6} \geq \frac{5}{16}t\,|u|^{1.8}$ for $|u| \to 0$. Moreover $0 < tu^{1.6} \leq 2t\,|u|^{1.7}$ for $|u| \to \infty$. Therefore, (G_1) ,(G_2) and (G_3) are satisfied. Applying Theorem 4.1, we obtain the problem (4.6) has infinitely many pairs of distinct classical solutions.

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