GLOBAL DYNAMICS OF A FILIPPOV PLANT DISEASE MODEL WITH AN ECONOMIC THRESHOLD OF INFECTED-SUSCEPTIBLE RATIO*

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Abstract This paper presents a Filippov plant disease model incorporating an economic threshold of infected-susceptible ratio, above which control strategies of replanting or removing are needed to be carried out. Based on the Filippov approach, we study the sliding mode dynamics and further the global dynamics. It is shown that there is a unique equilibrium, which is a disease-free equilibrium, an endemic equilibrium or a pseudo-equilibrium. Moreover, the equilibrium is proved to be globally asymptotically stable. Our results indicate that the control goal can be achieved by taking appropriate replanting and removing rate.

Keywords Filippov systems, plant disease model, economic threshold, stability.

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1. Introduction

Recently plant diseases have become a threat to crop yield as well as quality, and can further lead to health problem and unstable social effects [11, 21, 25]. Therefore a wide array of measures have been developed to control plant diseases. The more effective strategy is the integrated disease management, which combines several control measures to minimize losses and maximize returns. Among these measures, the cultural control measure including replanting and/or removing diseased plants is widely accepted due to the little environmental influence, see [1, 6, 23, 24, 26, 34].

One important approach to understand disease transmission mechanisms is mathematical modeling, see [1–5, 9, 10, 12–20, 22, 24, 26–34]. Fishman et al. [9] considered the Citrus tristeza virus temporal spread model in a closed plant population with periodic complete removal of infected plants. However, eradicating the infected completely is generally not possible, nor biologically or economically. It is more reasonable and feasible to bring down the number of infected under an economic threshold, under which economic damage is acceptable. Thus by incorporating an economic threshold, some plant disease models were established, see [1, 24, 26, 34].

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Note that the plants population might not be closed, and the number of the infected and the susceptible plants might be always changed due to removing, recruiting as well as replanting. It is more realistic and flexible to consider an economic threshold as the ratio of the infected to the susceptible plants. Based on this motivation, we extend the model in [26] with an economic threshold of infected-susceptible ratio as follows

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = A - \beta SI - \eta_1 S + pS\Phi(S, I), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta SI - \eta_2 I - vI\Phi(S, I) \end{cases} \tag{1.1}$$

with

$$\Phi(S, I) = \begin{cases} 0, & \frac{I}{S} < k, \\ 1, & \frac{I}{S} > k, \end{cases} \tag{1.2}$$

where $S(t)$ and $I(t)$ represent the number of susceptible plants and infected plants at the time t respectively; the constant A is the recruitment rate of susceptible plants; β is the infectious rate; η_1 and η_2 are the mortality rates of susceptible and infected plants respectively; p and v denote the replanting rate of susceptible plants and the removing rate of infected plants respectively and $k \geq 0$ denotes the economic threshold of infected-susceptible ratio. In this model, when the ratio of the infected to the susceptible is less than k, no control measures need to be implemented. However, once the ratio exceeds the threshold k, one should remove the infected and/or replant the susceptible to control the disease.

Throughout this paper, we assume $p < kv$. When the infected-susceptible ratio exceeds the economic threshold k, the approach for removing the infected, as the leading role to control the plant disease, are mainly implemented to effectively bring down the ratio of the infected to the susceptible under the ratio threshold k. Furthermore, we assume $p < \eta_1$ indicating that the replanting rate is less than the death rate of susceptible plants. Then the replanting rate p meets $p < \min\{kv, \eta_1\}$, which not only contributes to minimizing the loss and maximizing the production but also prevents the excessive increase in infected plants from replanting appropriate number of susceptible plants.

The goal of this paper is to study the global dynamics of the model (1.1) with (1.2). We find that the plant disease is able to be controlled by choosing appropriate removing and replanting rate. The rest of the paper is organized as follows. In section 2, we provide preliminaries for planar Filippov systems and dynamics analysis of the subsystems. Sliding mode dynamics are exhibited in Section 3. Section 4 is devoted to the analysis of the global dynamics of the model (1.1). Finally, we discuss biological implications in Section 5.

2. Preliminaries

In this section, we give some preliminaries for planar Filippov systems and discuss the dynamics of two subsystems.

Since the system (1.1) is piecewise continuous, we consider its solutions in Filippov sense. Thus some essential definitions on Filippov systems are given as follows based on references [7, 27, 28].

Let $R_+^2 = \{X = (S, I)^T | S \geq 0, I \geq 0\}$,

$$F_1(X) = (A - \beta SI - \eta_1 S, \ \beta SI - \eta_2 I)^T$$

and

$$F_2(X) = (A - \beta SI - \eta_1 S + pS, \ \beta SI - \eta_2 I - vI)^T.$$

Then the system (1.1) with (1.2) can be written as the following generic planar Filippov system:

$$\dot{X} = \begin{cases} F_1(X), \ X \in G_1, \\ F_2(X), \ X \in G_2, \end{cases} \tag{2.1}$$

where $G_1 = \{X \in R_+^2 | H(X) < 0\}$ and $G_2 = \{X \in R_+^2 | H(X) > 0\}$ with $H(X) = I - kS$ as a smooth scale function.

Suppose $H_X(X)$ directs to G_2, where $H_X(X)$ represents the gradient of $H(X)$ and $\langle \cdot, \ \cdot \rangle$ is the standard scalar product. The separating boundary $\Sigma = \{X \in R_+^2 | H(X) = 0\}$ can be partitioned by the following regions:

(1) $\Sigma_c \subset \Sigma$ is the crossing region if $\langle H_X(X), F_1(X) \rangle \langle H_X(X), F_2(X) \rangle > 0$;

(2) $\Sigma_s \subset \Sigma$ is the sliding region if $\langle H_X(X), F_1(X) \rangle > 0$ and $\langle H_X(X), F_2(X) \rangle < 0$;

(3) $\Sigma_e \subset \Sigma$ is the escaping region if $\langle H_X(X), F_1(X) \rangle < 0$ and $\langle H_X(X), F_2(X) \rangle > 0$.

Now we give definitions of some types of singular points for the system (2.1).

Definition 2.1. A point X^* is called a real equilibrium of the system (2.1) if $F_1(X^*) = 0$, $X^* \in G_1$ or $F_2(X^*) = 0$, $X^* \in G_2$. A point X^* is called a virtual equilibrium of the system (2.1) if $F_1(X^*) = 0$, $X^* \in G_2$ or $F_2(X^*) = 0$, $X^* \in G_1$. A point X^* is called a boundary equilibrium of the system (2.1) if $F_1(X^*) = 0$, $X^* \in \Sigma$ or $F_2(X^*) = 0$, $X^* \in \Sigma$.

Definition 2.2. A point X^* is called a tangent point of the system (2.1) if $X^* \in \Sigma$ and $\langle H_X(X^*), F_1(X^*) \rangle \langle H_X(X^*), F_2(X^*) \rangle = 0$.

Definition 2.3. A point X^* is called a pseudo-equilibrium if it is an equilibrium of the sliding mode of the system (2.1), i.e. $\lambda F_1(X^*) + (1 - \lambda) F_2(X^*) = 0$ with $0 < \lambda < 1$, where

$$\lambda = \frac{\langle H_X(X^*), F_2(X^*) \rangle}{\langle H_X(X), F_2(X^*) - F_1(X^*) \rangle}.$$

The following propositions imply that the solutions of the model (2.1) with any initial values in R_+^2 are positive and bounded.

Proposition 2.1. *Supposing that* $(S(t), I(t))$ *is a solution of the system (1.1) with* $S(0) = S_0 \geq 0$ *and* $I(0) = I_0 \geq 0$ *on* $[0, T)$, *where* $T \in (0, +\infty]$, *then* $S(t) \geq 0$ *and* $I(t) \geq 0$ *for* $t \in [0, T)$.

Proof.   According to the first equation of the system (1.1)

$$\left. \frac{\mathrm{d}S}{\mathrm{d}t} \right|_{S=0} = (A - \beta SI - \eta_1 S - pS\Phi(S, I)) \, |_{S=0} = A > 0,$$

we have $S(t) \geq 0$ for $t \in [0, T)$ as long as $S_0 \geq 0$. Consider the second equation of the system (1.1)

$$\frac{\mathrm{d}I}{\mathrm{d}t}\bigg|_{I=0} = (\beta S - \eta_2 - v\Phi(S, I))I\,|_{I=0} = 0.$$

If $I_0 = 0$, then $I(t) = 0$ holds for all $t \in [0, T)$. If $I_0 > 0$, we claim $I(t) > 0$ holds for all $t \in [0, T)$. Otherwise, there exists $t_1 = inf\{t : I(t) = 0\}$ with $t_1 > 0$ such that $I(t_1) = 0$ and $I(t) > 0$ for $t \in [0, t_1)$. Note that

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta SI - \eta_2 I - vI\Phi(S, I) \geq -(\eta_2 + v)I.$$

Thus for $t \in [0, t_1)$, we have

$$0 = I(t_1) \geq I_0 e^{-(\eta_2+v)t_1} > 0,$$

which is a contradiction. Thus $I(t) \geq 0$ for all $t \in [0, T)$ when $I(0) \geq 0$. □

Proposition 2.2. *The set $\Omega = \{(S, I) \in R_+^2 \mid S + I \leq \frac{A}{\mu}\}$ is a positively invariant and attracting region for the system (1.1) with any given initial conditions in R_+^2, where $\mu = min\{\eta_1 - p, \eta_2\} > 0$.*

Proof. It follows from (1.1) that

$$\frac{\mathrm{d}(S + I)}{\mathrm{d}t} \leq A - \eta_1 S - \eta_2 I + pS \leq A - \mu(S + I), \tag{2.2}$$

where $\mu = min\{\eta_1 - p, \eta_2\} > 0$. Thus $\frac{\mathrm{d}(S+I)}{\mathrm{d}t} \leq 0$ if $S + I = \frac{A}{\mu}$, which means Ω is positively invariant. Notice that from (2.2) we can obtain

$$S(t) + I(t) \leq \frac{A}{\mu} + \left(S(0) + I(0) - \frac{A}{\mu}\right) e^{-\mu t},$$

which means that $\lim_{t \to +\infty} [S(t) + I(t)] \leq \frac{A}{\mu}$ if $S(0) + I(0) > \frac{A}{\mu}$. Hence the set Ω is attracting. □

The dynamics of the subsystems will play an important role in the analysis of the global dynamical behavior of the system (2.1). Next, we examine the global stability for the subsystems

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = A - \beta SI - \eta_1 S, \\ \frac{\mathrm{d}I}{\mathrm{d}t} = \beta SI - \eta_2 I, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = A - \beta SI - \eta_1 S + pS, \\ \frac{\mathrm{d}I}{\mathrm{d}t} = \beta SI - \eta_2 I - vI \end{cases} \tag{2.4}$$

respectively. For the subsystem (2.3), the basic production number is $R_1 = \frac{A\beta}{\eta_1 \eta_2}$ and there are two possible equilibria, a disease-free equilibrium E_0^1 and an endemic equilibrium E_1, where

$$E_0^1 = (\frac{A}{\eta_1}, 0), \qquad E_1 = (S_1, I_1) = (\frac{\eta_2}{\beta}, \frac{A\beta - \eta_1 \eta_2}{\eta_2 \beta}).$$

Proposition 2.3. *For the subsystem (2.3), the disease-free equilibrium E_0^1 is globally asymptotically stable if $R_1 < 1$, while the endemic equilibrium E_1 is globally asymptotically stable if $R_1 > 1$.*

Proof. According to [26], for $R_1 > 1$, rewrite the subsystem (2.3) as

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = -\eta_1(S - S_1) - \beta I(S - S_1) - \beta S_1(I - I_1), \\ \frac{\mathrm{d}I}{\mathrm{d}t} = \beta I(S - S_1) \end{cases} \tag{2.5}$$

and consider the Lyapunov function

$$V_1(S, I) = \frac{1}{2}(S - S_1)^2 + S_1(I - I_1 - I_1 \ln \frac{I}{I_1}).$$

The time derivative of V_1 along the solutions of the system (2.5) is

$$\frac{\mathrm{d}}{\mathrm{d}t} V_1(S(t), I(t)) = -(\eta_1 + \beta I)(S - S_1)^2 \leq 0.$$

Then by utilizing LaSalle's invariance set principle, we conclude that the endemic equilibrium E_1 is globally asymptotically stable.

Similarly, for the case where $R_1 < 1$, taking a Lyapunov function

$$V_{10}(t) = \frac{1}{2}(S - \frac{A}{\eta_1})^2 + \frac{A}{\eta_1} I$$

and employing LaSalle's invariance set principle, we claim that the disease-free equilibrium E_0^1 is globally asymptotically stable. □

For the subsystem (2.4), the basic production number is $R_2 = \frac{A\beta}{(\eta_1 - p)(\eta_2 + v)}$ and the possible equilibria are E_0^2 and E_2, where

$$E_0^2 = (\frac{A}{\eta_1 - p}, 0), \qquad E_2 = (S_2, I_2) = (\frac{\eta_2 + v}{\beta}, \frac{A\beta - (\eta_1 - p)(\eta_2 + v)}{\beta(\eta_2 + v)}).$$

Then we have the following proposition, whose proof is very similar to that of Proposition 2.3 and is omitted.

Proposition 2.4. *For the subsystem (2.4), the disease-free equilibrium $E_0^2 = (\frac{A}{\eta_1 - p}, 0)$ is globally asymptotically stable if $R_2 < 1$, whereas the endemic equilibrium $E_2 = (S_2, I_2)$ is globally asymptotically stable if $R_2 > 1$.*

At last in this section, it is remarked that for the system (2.1) the disease-free equilibrium E_0^1 is always real and the disease-free equilibrium E_0^2 is always virtual. Furthermore when $R_1 > 1$, the endemic equilibrium E_1 is real (virtual) if and only if $A\beta - \eta_1\eta_2 - k\eta_2^2 < 0 \ (> 0)$, while it is boundary if and only if $A\beta - \eta_1\eta_2 - k\eta_2^2 = 0$. When $R_2 > 1$, the endemic equilibrium E_2 is virtual (real) if and only if $A\beta - k(\eta_2 + v)^2 - (\eta_1 - p)(\eta_2 + v) < 0 \ (> 0)$, while it is boundary if and only if $A\beta - k(\eta_2 + v)^2 - (\eta_1 - p)(\eta_2 + v) = 0$.

3. Sliding mode dynamics

In this section, we will study the sliding mode dynamics for the system (2.1). Choose vector $\mathbf{n} = (-k, 1)$ as a normal vector on discontinuous boundary Σ. To determine

the tangent points and further the sliding region on Σ, we need the following functions which are defined by

$$g_1(S) = \langle \mathbf{n}, F_1(S, kS) \rangle = k[\beta(k+1)S^2 + (\eta_1 - \eta_2)S - A] \qquad (3.1)$$

and

$$g_2(S) = \langle \mathbf{n}, F_2(S, kS) \rangle = k[\beta(k+1)S^2 + (\eta_1 - \eta_2 - p - v)S - A].$$

Solving $g_1(S) = 0$ and $g_2(S) = 0$ yields the horizontal ordinates S_T^1 and S_T^2 of tangent points respectively, where

$$S_T^1 = \frac{-(\eta_1 - \eta_2) + \sqrt{(\eta_1 - \eta_2)^2 + 4\beta A(k+1)}}{2\beta(k+1)}, \qquad (3.2)$$

$$S_T^2 = \frac{-(\eta_1 - \eta_2 - p - v) + \sqrt{(\eta_1 - \eta_2 - p - v)^2 + 4\beta A(k+1)}}{2\beta(k+1)}. \qquad (3.3)$$

It can be checked that $0 < S_T^1 < S_T^2$ if $p + v > 0$ and $S_T^1 = S_T^2$ if $p + v = 0$ by simple calculation. Moreover, we can get the sliding region on Σ as

$$\Sigma_s = \{(S, I) \in R_+^2 \mid S_T^1 < S < S_T^2, I = kS\},$$

and the sliding mode equation on Σ_s as

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{1}{(p+v)} f(S), \quad I = kS \qquad (3.4)$$

according to Filippov convex method [8], where

$$f(S) = \beta(p - kv)S^2 - (v\eta_1 + p\eta_2)S + vA. \qquad (3.5)$$

Under the assumption $p < kv$, there is a unique zero S^* of the function $f(S)$, where

$$S^* = \frac{(v\eta_1 + p\eta_2) - \sqrt{(v\eta_1 + p\eta_2)^2 - 4vA\beta(p - kv)}}{2\beta(p - kv)}.$$

Moreover, $E^* = (S^*, kS^*)^{\mathrm{T}}$ is the only possible pseudo-equilibrium for the system (2.1) and this pseudo-equilibrium exists if $S_T^1 < S^* < S_T^2$. The following lemma can help us to verify whether $E^* = (S^*, kS^*)^{\mathrm{T}}$ is a pseudo-equilibrium.

Lemma 3.1. *If $p < kv$, then the following assertions hold:*

(i) $\mathrm{sign}(S^* - S_T^1) = \mathrm{sign}(A\beta - k\eta_2^2 - \eta_1\eta_2);$

(ii) $\mathrm{sign}(S^* - S_T^2) = \mathrm{sign}(A\beta - k(\eta_2 + v)^2 - (\eta_1 - p)(\eta_2 + v))$, *where* $\mathrm{sign}(\cdot)$ *is the sign function*

$$\mathrm{sign}(\rho) = \begin{cases} 1, & \rho > 0, \\ 0, & \rho = 0, \\ -1, & \rho < 0. \end{cases}$$

Proof. We only give the proof of the assertion (i), since the assertion (ii) can be similarly obtained. Without loss of generality, we prove that $S_T^1 > S^*$ is equivalent to $A\beta - k\eta_2^2 - \eta_1\eta_2 < 0$. By $p < kv$, $f(S^*) = 0$ and the definition (3.5) of $f(S)$, we derive that $S_T^1 > S^*$ if and only if $f(S_T^1) < 0$. Notice that $g_1(S_T^1) = 0$, i.e.,

$$\beta(k+1)(S_T^1)^2 + (\eta_1 - \eta_2)S_T^1 - A = 0.$$

Thus we can obtain

$$f(S_T^1) = -\frac{(p+v)(\eta_1 + k\eta_2)}{k+1}\left(S_T^1 - \frac{A}{\eta_1 + k\eta_2}\right),$$

which implies that $f(S_T^1) < 0$ if and only if $S_T^1 > \dfrac{A}{\eta_1 + k\eta_2}$. On the other hand, the definition (3.1) of $g_1(S)$ means that $S_T^1 > \dfrac{A}{\eta_1 + k\eta_2}$ is equivalent to

$$g_1\left(\frac{A}{\eta_1 + k\eta_2}\right) = \frac{Ak(1+k)}{(\eta_1 + k\eta_2)^2}(A\beta - k\eta_2^2 - \eta_1\eta_2) < 0.$$

Consequently, $S_T^1 > S^*$ if and only if $A\beta - k\eta_2^2 - \eta_1\eta_2 < 0$. The proof is completed. $\qquad\square$

4. Analysis of global dynamics

In this section, we concentrate on the global dynamics for the system (2.1). We start by giving two lemmas to preclude closed orbits. Denote the right-hand sides of the system (2.1) F_i by $f^i(X)$, where $f^i(X) = (f_1^i(X), f_2^i(X))$, $i = 1, 2$.

Lemma 4.1. *There is no closed orbit that contains a part of the closure of the sliding mode $\overline{\Sigma}_s$ for the system (2.1).*

Proof. We give the proof by the way of contradiction. Without loss of generality, suppose that E_1 is real and E_2 is virtual and there exists a closed orbit C that contains a part of $\overline{\Sigma}_s$. Then the closed orbit C must start from the tangent point T_1 and reach $\overline{\Sigma}_s$ after some time, as shown in Fig. 1. This implies that any orbit outside C cannot converge to E_1, which leads to a contradiction with the global asymptotical stability of E_1 in region G_1 for the system (2.3). Analogously, the orbit starting from the tangent T_2 will not reach $\overline{\Sigma}_s$ either. $\qquad\square$
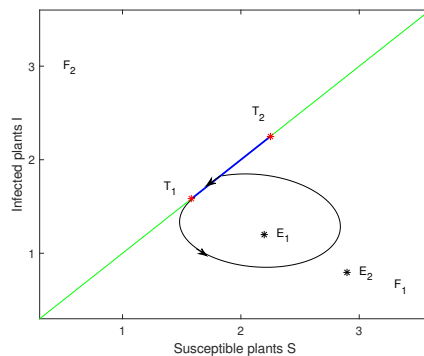

Figure 1. The possible closed orbit containing a part of $\overline{\Sigma}_s$ if E_1 is real and E_2 is virtual.

Lemma 4.2. *There is no closed orbit surrounding $\overline{\Sigma}_s$ for the system (2.1).*

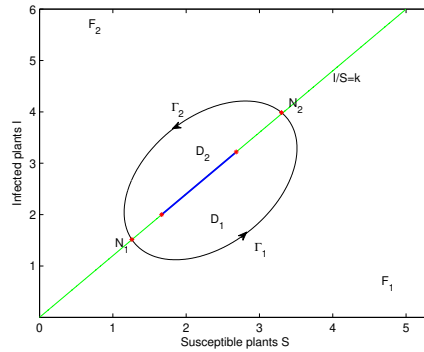**Proof.** Suppose there exists a closed orbit Γ that surrounds $\overline{\Sigma}_s$, as shown in Fig.2.

Figure 2. The possible closed orbit surrounding $\overline{\Sigma}_s$.

Denote the part below Σ as Γ_1 and the part above Σ as Γ_2. The closed orbit Γ intersects with Σ at the points N_1 and N_2. Let D_1 be the bounded region delimited by Γ_1 and N_1N_2 and D_2 be the region bounded by Γ_2 and N_1N_2. Let $B(S, I) = \frac{1}{SI}$, then

$$\sum_{i=1}^{2} \iint_{D_i} [\frac{\partial(Bf_1^i)}{\partial S} + \frac{\partial(Bf_2^i)}{\partial I}] \mathrm{d}S\mathrm{d}I = \sum_{i=1}^{2} \iint_{D_i} (-\frac{A}{S^2 I}) \mathrm{d}S\mathrm{d}I < 0, \qquad (4.1)$$

where i=1,2. Applying Green's theorem, we have

$$\iint_{D_1} [\frac{\partial(Bf_1^1)}{\partial S} + \frac{\partial(Bf_2^1)}{\partial I}] \mathrm{d}S\mathrm{d}I$$

$$= \oint_{\Gamma_1 \cup \overrightarrow{N_2 N_1}} Bf_1^1 \mathrm{d}I - \oint_{\Gamma_1 \cup \overrightarrow{N_2 N_1}} Bf_2^1 \mathrm{d}S$$

$$= \int_{\Gamma_1} Bf_1^1 \mathrm{d}I + \int_{\overrightarrow{N_2 N_1}} Bf_1^1 \mathrm{d}I - (\int_{\Gamma_1} Bf_2^1 \mathrm{d}S + \int_{\overrightarrow{N_2 N_1}} Bf_2^1 \mathrm{d}S)$$

$$= \int_{\Gamma_1} B(f_1^1 f_2^1 - f_2^1 f_1^1) \mathrm{d}t + \int_{\overrightarrow{N_2 N_1}} Bf_1^1 \mathrm{d}I - \int_{\overrightarrow{N_2 N_1}} Bf_2^1 \mathrm{d}S$$

$$= \int_{\overrightarrow{N_2 N_1}} (Bf_1^1 \mathrm{d}I - Bf_2^1 \mathrm{d}S)$$

$$= \int_{\overrightarrow{N_2 N_1}} [\frac{A}{S^2} + \frac{\eta_2 - \eta_1}{S} - \beta(k+1)] \mathrm{d}S$$

$$= -A(\frac{1}{N_1} - \frac{1}{N_2}) - \beta(k+1)(N_1 - N_2) + (\eta_2 - \eta_1) \ln|\frac{N_1}{N_2}|,$$

where $\mathrm{d}S = f_1^1 \mathrm{d}t$ and $\mathrm{d}I = f_2^1 \mathrm{d}t$ along Γ_i $(i = 1, 2)$. Similarly,

$$\iint_{D_2} [\frac{\partial(Bf_1^2)}{\partial S} + \frac{\partial(Bf_2^2)}{\partial I}] \mathrm{d}S\mathrm{d}I$$

$$= \int_{\overrightarrow{N_1 N_2}} (Bf_1^2 \mathrm{d}I - Bf_2^2 \mathrm{d}S)$$

$$= -A(\frac{1}{N_2} - \frac{1}{N_1}) - \beta(k+1)(N_2 - N_1) + (\eta_2 + v - \eta_1 + p) \ln|\frac{N_2}{N_1}|.$$

Since $N_2 > N_1$, then

$$
\sum_{i=1}^{2} \iint_{D_i} [\frac{\partial(Bf_1^i)}{\partial S} + \frac{\partial(Bf_2^i)}{\partial I}] \mathrm{d}S \mathrm{d}I
$$

$$
= \iint_{D_1} [\frac{\partial(Bf_1^1)}{\partial S} + \frac{\partial(Bf_2^1)}{\partial I}] \mathrm{d}S \mathrm{d}I + \iint_{D_2} [\frac{\partial(Bf_1^2)}{\partial S} + \frac{\partial(Bf_2^2)}{\partial I}] \mathrm{d}S \mathrm{d}I
$$

$$
= \int_{\overrightarrow{N_2 N_1}} (Bf_1^1 \mathrm{d}I - Bf_2^1 \mathrm{d}S) + \int_{\overrightarrow{N_1 N_2}} (Bf_1^2 \mathrm{d}I - Bf_2^2 \mathrm{d}S)
$$

$$
= (p + v) \ln |\frac{N_2}{N_1}| > 0,
$$

which contradicts with (4.1). Hence there is no closed orbit surrounding $\overline{\Sigma}_s$. □

Lemma 4.3 (See [8], §13). *If a half trajectory T^+ is bounded for the system (2.1), then its limit set $\Omega(T)$ contains either an equilibrium or a closed trajectory.*

In the following, we consider global dynamics of all possible equilibria including the disease-free equilibrium E_0^1, the endemic equilibrium E_1 or E_2 and the pseudo-equilibrium E^*.

Theorem 4.1. *Suppose that $p < \min\{kv, \eta_1\}$ and $R_1 < 1$. Then the disease-free equilibrium E_0^1 is globally asymptotically stable for the system (2.1).*

Proof. By the assertion (i) of Lemma 3.1, $R_1 = \frac{A\beta}{\eta_1 \eta_2} < 1$ yields $A\beta - k\eta_2^2 - \eta_1\eta_2 < 0$ and further gives $S^* < S_T^1$. Notice that

$$
\begin{aligned}
&A\beta - k(\eta_2 + v)^2 - (\eta_1 - p)(\eta_2 + v) \\
=&A\beta - k\eta_2^2 - \eta_1\eta_2 + (p - kv)(v + \eta_2) - (\eta_1 + k\eta_2)v,
\end{aligned}
\tag{4.2}
$$

we obtain that $A\beta - k(\eta_2 + v)^2 - (\eta_1 - p)(\eta_2 + v) < 0$ if $R_1 < 1$ and $p < kv$.

Since $A\beta - k\eta_2^2 - \eta_1\eta_2 < 0$ and $A\beta - k(\eta_2 + v)^2 - (\eta_1 - p)(\eta_2 + v) < 0$, then $S^* < S_T^1 < S_T^2$ by Lemma 3.1, which reveals that there is no pseudo-equilibrium on the sliding mode Σ_s. Also, the orbit on the sliding mode Σ_s goes down along the Σ_s as $f(S) < 0$ for $S \in (S_T^1, S_T^2)$ from the equation (3.5). Besides, Proposition 2.2 explains that solutions of the system (2.1) are bounded and no closed orbit exists for the system (2.1) on the basis of Lemma 4.1 and Lemma 4.2. In this case the ω-limit set of the system (2.1) is the unique real equilibrium E_0^1 by Lemma 4.3 so that any solution of system (2.1) eventually stabilizes at the disease-free equilibrium E_0^1, as shown in Fig. 3. □

For $R_1 > 1$, by the equation (4.2) and $p < kv$, we have $A\beta - k(\eta_2 + v)^2 - (\eta_1 - p)(\eta_2 + v) < A\beta - k\eta_2^2 - \eta_1\eta_2$, which contradicts with the case $A\beta - k\eta_2^2 - \eta_1\eta_2 < 0$ and $A\beta - k(\eta_2 + v)^2 - (\eta_1 - p)(\eta_2 + v) > 0$. Then we have the following theorem.

Theorem 4.2. *Suppose that $p < \min\{kv, \eta_1\}$ and $R_1 > 1$. Then for the system (2.1) the following assertions hold:*

(i) *the real equilibrium E_1 is globally asymptotically stable if $A\beta - k\eta_2^2 - \eta_1\eta_2 < 0$ and $A\beta - k(\eta_2 + v)^2 - (\eta_1 - p)(\eta_2 + v) < 0$;*

(ii) *the real equilibrium E_2 is globally asymptotically stable if $A\beta - k\eta_2^2 - \eta_1\eta_2 > 0$ and $A\beta - k(\eta_2 + v)^2 - (\eta_1 - p)(\eta_2 + v) > 0$;*
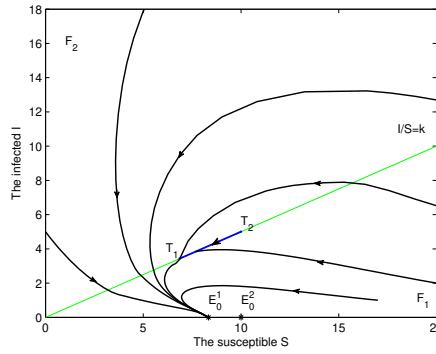
Figure 3. Global asymptotic stability of the disease-free equilibrium E_0^1, where the parameter are chosen as follows: $\eta_1 = 0.6, \eta_2 = 0.9, A = 5, \beta = 0.1, p = 0.1, v = 0.6, k = 0.5$.

(iii) *the pseudo-equilibrium E^* is globally asymptotically stable if $A\beta - k\eta_2^2 - \eta_1\eta_2 > 0$ and $A\beta - k(\eta_2 + v)^2 - (\eta_1 - p)(\eta_2 + v) < 0$.*

Proof. We first give the proof of the assertion (i). If $A\beta - k\eta_2^2 - \eta_1\eta_2 < 0$ and $A\beta - k(\eta_2 + v)^2 - (\eta_1 - p)(\eta_2 + v) < 0$, then by Lemma 3.1 we have $S^* < S_T^1 < S_T^2$ which explains that there is no pseudo-equilibrium. The orbit on Σ_s goes downward along the Σ_s as $f(S) < 0$ for $S \in (S_T^1, S_T^2)$ by the equation (3.5). Besides, $A\beta - k(\eta_2 + v)^2 - (\eta_1 - p)(\eta_2 + v) < 0$ indicates that the endemic equilibrium E_2 is virtual or the disease-free equilibrium E_0^2 exists, while the endemic equilibrium E_1 is a real equilibrium if $A\beta - \eta_1\eta_2 - k\eta_2^2 < 0$ and $R_1 > 1$. Therefore, by Lemma 4.3, the ω-limit set of the system (2.1) is the unique real equilibrium E_1 so that in this case E_1 is globally asymptotically stable for the system (2.1), as shown in Fig. 4. Analogously, we can obtain the assertion (ii), as shown in Fig. 5.
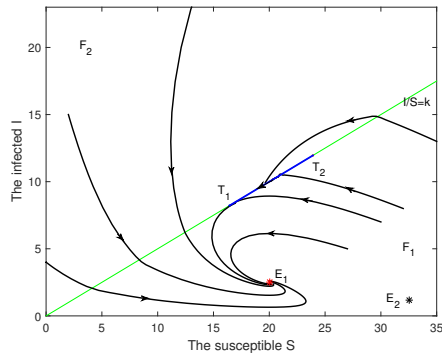

Figure 4. Global asymptotic stability of the endemic equilibrium E_1, where the parameters are chosen as follows: $\eta_1 = 0.3, \eta_2 = 0.8, A = 8, \beta = 0.04, p = 0.1, v = 0.5, k = 0.5$.
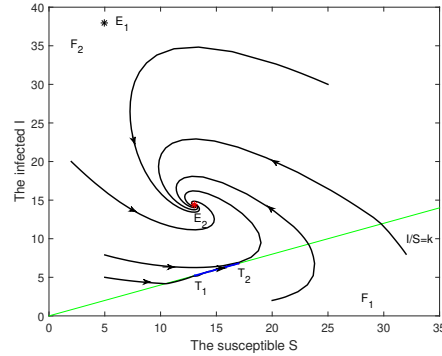
Figure 5. Global asymptotic stability of the endemic equilibrium E_2, where the parameters are fixed as follows: $\eta_1 = 0.2, \eta_2 = 0.5, A = 20, \beta = 0.1, p = 0.1, v = 0.8, k = 0.4$.

Now we turn to the proof of the assertion (iii). The conditions $A\beta - \eta_1\eta_2 - k\eta_2^2 > 0$ and $A\beta - (\eta_1 - p)(\eta_2 + v) - k(\eta_2 + v)^2 < 0$ imply that $S_T^1 < S^* < S_T^2$, which indicates that there exists a pseudo-equilibrium E^*. The pseudo-equilibrium E^* is stable on

the sliding mode Σ_s since $f(S) > 0$ for $S \in (S_T^1, S^*)$ and $f(S) < 0$ for $S \in (S^*, S_T^2)$. In addition, $A\beta - \eta_1\eta_2 - k\eta_2^2 > 0$ and $A\beta - k(\eta_2 + v)^2 - (\eta_1 - p)(\eta_2 + v) < 0$ explain that solutions of both subsystems cannot converge to their own equilibria. Accordingly, in this case the pseudo-equilibrium E^* is globally asymptotically stable for the system (2.1) (see Fig. 6). □
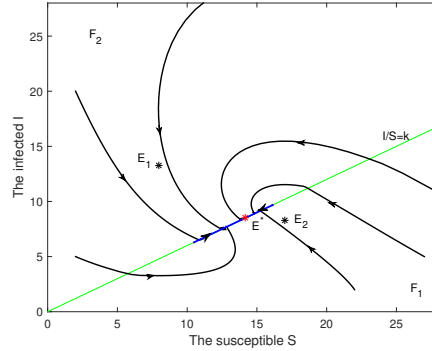

Figure 6. Global asymptotic stability of the pseudo-equilibrium E^*, where the parameters are fixed as follows: $\eta_1 = 0.55, \eta_2 = 0.8, A = 15, \beta = 0.1, p = 0.5, v = 0.9, k = 0.6$.

5. Biological implications

In this paper, we put forward a Filippov plant disease model incorporating a threshold policy and control strategies for cutting off the infected and replanting the susceptible. Our goal is to establish conditions for controlling the number of infected plants below a given tolerable threshold so that the farmers can evaluate and choose appropriate rouging and replanting control strategies to scientifically minimize the economic losses and maximize the production.

Making use of the Filippov approach for sliding mode and global dynamics, we investigate the global dynamics of the model (2.1) and acquire the global stability of all possible equilibria including disease free equilibrium E_0^1, pseudo-equilibrium E^*, endemic equilibrium E_1 or E_2 as the value of parameters varies, which is summarized in Theorem 4.1 and Theorem 4.2.

These theorems also reveal that the choice of replanting rate p and the rouging rate v are of great importance in controlling the plant disease and reduce the losses. Next by varying the control parameters p and v under the assumption $p < \min\{kv, \eta_1\}$, we distinguish three cases to illustrate how to achieve our control goal: to maintain the infected-susceptible ratio below the ratio threshold k.

Case 1: $R_1 < 1$

In this case, the infected plants will eventually goes extinct whether the control measures are implemented or not by Proposition 2.3 and Theorem 4.1. Compared with no control measures ($p = 0, v = 0$), taking appropriate control measures ($p \geq 0, v > 0, p < \min\{kv, \eta_1\}$) helps control the plants disease faster. As shown in Fig.7, without any control measures, the infected-susceptible plants ratio (i.e. $\frac{I(t)}{S(t)}$) stay below the threshold k after time t_1. By contrast, with the same initial values and other parameters, it takes time t_2 for $\frac{I(t)}{S(t)}$ maintaining below k when control

measures are carried out ($p = 0.4$, $v = 0.4$). Due to $t_2 < t_1$, we conclude that taking control measures contributes to achieving control goal faster, which reduces the losses.

Case 2: $R_1 > 1$ and $A\beta - k\eta_2^2 - \eta_1\eta_2 < 0$.

Similar to the discussion in Case 1, control goal can be achieved with or without control strategies since any solution of the system (2.1) with any given initial values will eventually stabilize at the endemic equilibrium E_1 by Proposition 2.3 and the assertion (i) of Theorem 4.2. In particular, Fig. 8 suggests that choosing appropriate control parameters p and v can help achieve the control goal faster and minimize the losses.
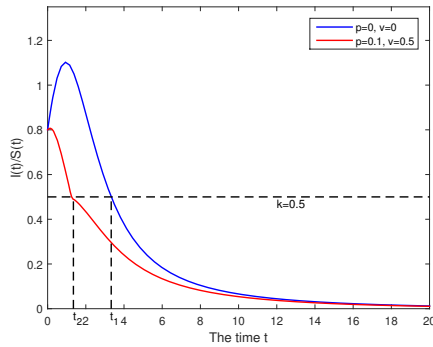

Figure 7. The graph of the infected-susceptible plants ratio $\frac{I(t)}{S(t)}$ with different control parameters p and v when $R_1 < 1$, where the parameters are chosen as follows: $\eta_1 = 0.6, \eta_2 = 0.8, A = 10, \beta = 0.04, k = 0.5$ and the initial value $(S(0), I(0)) = (20, 16)$.
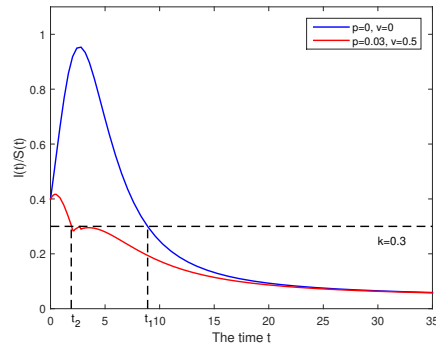
Figure 8. The graph of the infected-susceptible plants ratio $\frac{I(t)}{S(t)}$ with different control parameters p and v when $R_1 > 1$ and $A\beta - k\eta_2^2 - \eta_1\eta_2 < 0$. The parameters are chosen as follows: $\eta_1 = 0.3, \eta_2 = 0.5, A = 4, \beta = 0.04, k = 0.3$ and the initial value $(S(0), I(0)) = (20, 8)$.

Case 3: $R_1 > 1$ and $A\beta - k\eta_2^2 - \eta_1\eta_2 > 0$.

Under the assumption $p < \min\{kv, \eta_1\}$, it is worth noting that whether the control goal can be achieved depends on the control parameters p and v. As shown in Fig. 9, if we choose parameters p and v in the region U_1, the susceptible-infected ratio $\frac{I(t)}{S(t)}$ can be ultimately controlled at the threshold k by the assertion (iii) of Theorem 4.2. Nevertheless, we fail to come to control goal if the parameters p and v are taken in the region U_2 since the ratio $\frac{I(t)}{S(t)}$ always exceed the threshold k by the assertion (ii) of Theorem 4.2.

Notice that the control strategy in this paper is based on the ratio of the infected population over the susceptible population. It should be pointed out that the same analysis would work for the case that the control strategy is based on the ratio of the infected population over the total population. The above analysis indicates that control measures are effective if appropriate removal rate v and replanting rate p are chosen. Note that we only discuss the dynamics of the system (2.1) provided $p < kv$ and more abundant dynamics when $p > kv$ remain for further investigation. In the case $p > kv$, Fig.10 shows that two pseudo-equilibira E_1^*, E_2^* and a real equilibrium E_2 coexist, and there are two heteroclinic orbits between which one connects the two pseudo-equilibira E_1^*, E_2^* and the other connects the pseudo-equilibrium E_2^* and

the real equilibrium E_2.
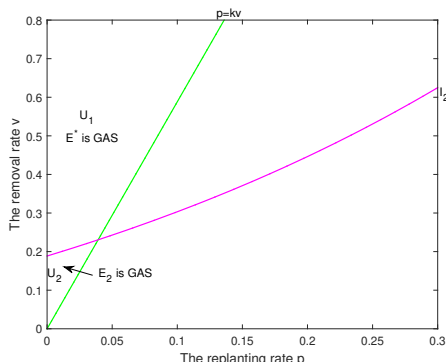

Figure 9. The parameter region in the $p - v$ plane, where the equilibrium E^* or E_2 is globally asymptotically stable (GAS) if $R_1 > 1$ and $A\beta - k\eta_2^2 - \eta_1\eta_2 > 0$. The other parameters are picked up as follows: $\eta_1 = 0.5, \eta_2 = 0.6, A = 10, \beta = 0.05, k = 0.17$. Besides, $U_1 \cup U_2 = \{(p, v)|p < \min\{kv, \eta_1\}, p \geq 0, v \geq 0\}$.
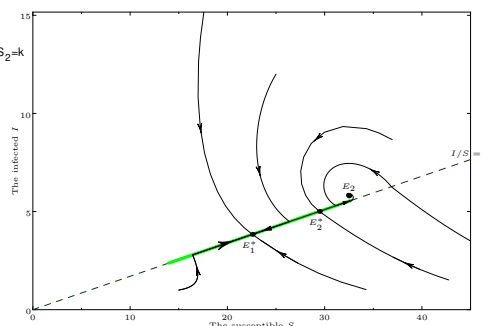
Figure 10. The system (2.1) is bistable if the parameters are chosen as: $\eta_1 = 0.5, \eta_2 = 0.6, A = 10, \beta = 0.05, k = 0.17, p = 0.4821, v = 1.0241$.

Acknowledgments

References

[1] C. Chen and Y. Kang, *Sliding motion and global dynamics of a Filippov fireblight model with economic thresholds*, Nonlinear Anal.-Real World Appl., 2018, 39, 492–519.

[2] T. Chen, L. Huang, P. Yu and W Huang, *Bifurcation of limit cycles at infinity in piecewise polynomial systems*, Nonlinear Anal.-Real World Appl., 2018, 41, 82–106.

[3] Z. Cai, J. Huang and L. Huang, *Periodic orbit analysis for the delayed Filippov system*, Proc. Amer. Math. Soc., 2018, 146 (11), 4667–4682.

[4] Z. Cai, J. Huang and L. Huang, *Generalized Lyapunov-Razumikhin method for retarded differential inclusions: Applications to discontinuous neural networks*, Discrete Contin. Dyn. Syst.-Ser. B, 2017, 22(9), 3591–3614.

[5] N. Chong, *Modeling avian influenza using Filippov systems to determine culling of infected birds and quarantine*, Nonlinear Anal.-Real World Appl., 2015, 24, 196–218.

[6] M. Chan and M. J. Jeger, *An analytical model of plant virus disease dynamics with roguing and replanting*, J. Appl. Ecol., 1994, 31(3), 413–427.

[7] X. Chen and L. Huang, *A Filippov system describing the effect of prey refuge use on a ratio-dependent predator-prey model*, J. Math. Anal. Appl., 2015, 428(2), 817–837.

[8] A. F. Filippov, *Differential Equations with Discontinuous Right-Hand Sides*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1988.

[9] S. Fishman, R. Marcus, H. Talpaz, et al, *Epidemiological and economic models for spread and control of citrus tristeza virus disease*, Phytoparasitica, 1983, 11(1), 39–49.

[10] S. Fishman and R. Marcus, *A model for spread of plant disease with periodic removals*, J. Math. Biol., 1984, 21(2), 149–158.

[11] R. W. Gibson and V. Aritua, *The perspective of sweetpotato chlorotic stunt virus in sweetpotato production in Africa: a review*, 2002.

[12] Z. Guo, L. Huang and X. Zou, *Impact of discontinuous treatments on disease dynamics in an SIR epidemic model*, Math. Biosci. Eng., 2013, 9(1), 97–110.

[13] C. Huang, Z. Yang, T. Yi and X. Zou, *On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities*, J. Differ. Equ., 2014, 256(7), 2101–2114.

[14] H. Hu, X. Yuan, L. Huang and C. Huang, *Global dynamics of an SIRS model with demographics and transfer from infectious to susceptible on heterogeneous networks*, Math. Biosci. Eng, 2019, 16(5), 5729–5749.

[15] C. Huang, Y. Qiao, L. Huang and R. P. Agarwal, *Dynamical behaviors of a food-chain model with stage structure and time delays*, Adv. Differ. Equ., 2018, 2018(1), 186.

[16] M. Han, H. Sun and Z. Balanov, *Upper estimates for the number of periodic solutions to multi-dimensional systems*, J. Differ. Equ., 2019, 266(12), 8281–8293.

[17] M. Han, *On the maximum number of periodic solutions of piecewise smooth periodic equations by average method*, J. Appl. Anal. Comput., 2017, 7(2), 788–794.

[18] H. Hu and X. Zou, *Existence of an extinction wave in the fisher equation with a shifting habitat*, Proc. Amer. Math. Soc., 2017, 145(11), 4763–4771.

[19] F. Jiang and M. Han, *Qualitative analysis of crossing limit cycles in discontinuous Liénard-type differential systems*, Journal of Nonlinear Modeling and Analysis, 2019, 1(4), 527–543.

[20] L. Sheng and M. Han, *Bifurcation of limit cycles from a compound loop with five saddles*, J. Appl. Anal. Comput., 2019, 9(6), 2482–2495.

[21] R. N. Strange and P. R. Scott, *Plant Disease: A threat to global food security*, Annu. Rev. Phytopathol., 2005, 43(1), 83–116.

[22] Y. Tian, M. Han and F. Xu, *Bifurcations of small limit cycles in Liénard systems with cubic restoring terms*, J. Differ. Equ., 2019, 267(3), 1561–1580.

[23] H. R. Thieme and J. A. P. Heesterbeek, *How to estimate the efficacy of periodic control of an infectious plant disease*, Math. Biosci., 1989, 93(1), 0–29.

[24] S. Tang, Y. Xiao and R. A. Cheke, *Dynamical analysis of plant disease models with cultural control strategies and economic thresholds*, Math. Comput. Simul., 2010, 80(5), 894–921.

[25] M. Vurro, B. Bonciani and G. Vannacci, *Emerging infectious diseases of crop plants in developing countries: impact on agriculture and socio-economic consequences*, Food Security, 2010, 2(2), 113–132.

[26] J. Wang, F. Zhang and L. Wang, *Equilibrium, pseudoequilibrium and sliding-mode heteroclinic orbit in a Filippov-type plant disease model*, Nonlinear Anal.-Real World Appl., 2016, 31, 308–324.

[27] A. Wang and Y. Xiao, *A Filippov system describing media effects on the spread of infectious diseases*, Nonlinear Anal.-Hybrid Syst., 2014, 11(1), 84–97.

[28] A. Wang and Y. Xiao, *Sliding bifurcation and global dynamics of a Filippov epidemic model with vaccination*, Int. J. Bifurcation Chaos, 2013, 23(8), 1350144.

[29] J. Wang, X. Chen and L. Huang, *The number and stability of limit cycles for planar piecewise linear systems of node-saddle type*, J. Math. Anal. Appl., 2019, 469(1), 405–427.

[30] J. Wang, C. Huang and L. Huang, *Discontinuity-induced limit cycles in a general planar piecewise linear system of saddle-focus type*, Nonlinear Anal.-Hybrid Syst., 2019, 33, 162–178.

[31] J. Wang, S. He and L. Huang, *Limit cycles induced by threshold nonlinearity in planar piecewise linear systems of node–focus or node–center type*, Int. J. Bifur. Chaos, 2020, DOI:10.1142/S0218127420501606.

[32] R. Yuan and Z. Wang, *A HIV infection model with periodic multidrug therapy*, Journal of Nonlinear Modeling and Analysis, 2019, 1(4), 573–593.

[33] C. Yang, L. Huang and F. Li, *Exponential synchronization control of discontinuous nonautonomous networks and autonomous coupled networks*, Complexity, 2028, 2018, 1–10.

[34] T. Zhao and Y. Xiao, *Non-smooth plant disease models with economic thresholds*, Math. Biosci., 2013, 241(1), 34–48.