

SECOND-ORDER NORMAL FORMS FOR N-DIMENSIONAL SYSTEMS WITH A NILPOTENT POINT

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Abstract Normal forms theory is one of the most powerful tools for the study of nonlinear differential equations, in particular, for stability and bifurcation analysis. Many works paid attention to normal forms associated with nilpotent Jacobian where the critical eigenvalues have algebraic multiplicity k ($k > 1$) and geometric multiplicity one, and in particular, the case $k > 2$ is more complicated for determining unfolding. Despite a lot of theoretical results on nilpotent normal forms have been obtained, computation developing can not satisfy practical applications. To our knowledge, no results have been reported on the computation of explicit formulas of the nilpotent normal forms for $k > 3$ with perturbation parameters. The main difficulty is how to determine the complementary spaces of the Lie transformation. In this paper, we achieve the following results. (1) A simple dimension formula for the complementary space of the Lie transform; (2) a simple direct method to determine a basis of the complementary spaces; (3) a simple direct method to determine the projection of any vector to the complementary spaces. Using this method, the second-order normal forms for any n -dimensional nilpotent systems can be given easily. As an illustrative application, the normal forms for the vector field with triple-zero or four-fold zero singularity and functional differential equation with a triple-zero singularity are presented, and explicit formulas for the normal form coefficients with three or four unfolding parameters are obtained.

Keywords Normal forms, nilpotent, Lie Bracket Operation, bifurcation.

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1. Introduction

Studying dynamical systems with multiple zero critical singularity is not only theoretically significant but also important in real applications. When the Jacobian of a dynamical system evaluated at a critical point contains one or two zero eigenvalues, the so-called simple zero or double zero bifurcation may occur. A nilpotent singularity corresponding to a double-zero eigenvalue with geometric multiplicity one is known as codimension-2 Bogdanov-Takens (B-T) singularity, which

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can yield homoclinic orbits to saddle equilibria near the critical point. Since Bogdanov [20] and Takens [2] obtained the normal forms of B-T bifurcation and gave a very detailed bifurcation analysis, many works have been done in this area (e.g. see [1, 4, 9, 13, 14, 19, 21, 24] and references therein). The triple-zero eigenvalue with geometric multiplicity one called codimension-3 singularity has also been considered, see Ref. [3, 7, 15, 17, 22, 23].

There are few studies of codimension-4 or higher codimension problems with non-semisimple nilpotent singularities, perhaps due to the relative rarity of higher codimension singularities in ordinary differential equation (ODE) models. However, in delay differential equations (DDEs) higher codimension singularities seem to occur more frequently.

The method of normal forms provides a powerful tool in finding a simple form which keeps the fundamental dynamics of the original system unchanged [6]. For a practical system, not only the possible qualitative dynamical behavior of the system is of concern, but also the quantitative relationship between the normal forms and the original system needs to be established. For a general singular vector field with non-semisimple nilpotent singularities, the computation of the normal forms is very complicated. In particular, finding the explicit formulas of normal forms in terms of the original systems coefficients with nilpotent singularities is very difficult. Therefore, the crucial part in computing a normal form is the computational efficiency in finding the normal forms coefficients. In this study, we consider the following vector field.

$$\dot{x} = Jx + F_2(x) + F_3(x) + \cdots + F_{r-1}(x) + O(|x|^r), \quad x \in \mathbb{R}^n, \quad (1.1)$$

where J is the canonical Jordan nilpotent form, and $F_i(x)$ represents the i th-degree homogeneous polynomial in the Taylor expansion of $F(x)$. Introducing the coordinate transformation,

$$x = y + h_2(y), \quad (1.2)$$

where $h_2(y)$ denotes the 2nd-degree homogeneous polynomial in y , and substituting (1.2) into (1.1) yields

$$\dot{y} = Jy + Jh_2(y) - Dh_2(y)Jy + F_2(y) + F_3(y) + \cdots + F_{r-1}(y) + O(|y|^r). \quad (1.3)$$

The basic ideal of normal forms method is to choose a specific form for $h_2(y)$ so as to simplify the 2nd-degree terms as much as possible.

Let H_n^2 be the linear space of 2nd-degree homogeneous polynomials. Further, we introduce the following linear map of H_n^2 into H_n^2 :

$$h_2(y) \mapsto Dh_2(y)Jy - Jh_2(y).$$

Due to its presence in Lie algebra theory, this map is often denoted as

$$L_{N_n}(h_2(y)) = -(Dh_2(y)Jy - Jh_2(y)),$$

and is called Lie bracket operation. Assume that H_n^2 can be (non-uniquely) decomposed as

$$H_n^2 = L_{N_n}(H_n^2) \oplus W,$$

where W is a complementary spaces of H_n^2 .

The purpose of normal forms method is to choose $h_2(y)$ so that only the terms in W are retained. We denote these terms by $F_2^r(y)$. Thus, (1.3) can be simplified to

$$\dot{y} = Jy + F_2^r(y) + \tilde{F}_3(y) + \dots + \tilde{F}_{r-1}(y) + O(|y|^r), \tag{1.4}$$

and so the second-order terms have been simplified.

To determine the nature of the second-order terms that cannot be eliminated (i.e., $F_2^r(y)$), we must investigate the space complementary to $L_{N_n}(H_n^2)$. Solving this problem involves the following three main tasks:

- (1) determining the dimension of the complementary space of the Lie transform;
- (2) determining the basis of the complementary space of the Lie transform; and
- (3) determining the projection of any vector in H_n^2 to the complementary space.

Using this method in this paper, the second-order normal forms for any n -dimensional nilpotent systems can be given easily. We will present detailed steps to show how to fulfill these tasks. Our goal is to analyze the codimension- $n(n > 2)$ singularity corresponding to n -zero eigenvalues with geometric multiplicity one.

In Section 2, we define Lie bracket operation (Choquet-Bruhat et al. Ref. [5]). Using the linear transformation L_{N_n} we determine the dimensions of the complementary space of $L_{N_n}(H_n^2)$ and obtain the basis of the complementary space of $L_{N_n}(H_n^2)$, which is the key step in calculating the non-semisimple normal form and the explicit coefficients of the normal form. More precisely, we present the following results in this section.

- (1) A simple dimension formula for the complementary spaces of the Lie transform;
- (2) a simple direct method to determine the basis of the complementary spaces; and
- (3) a simple direct method to determine the projection of any vector in H_n^2 to the complementary spaces.

In Section 3, we obtain results for the complementary space of $L_{N_n}(H_{n+p}^2)$ with p parameters.

In Sections 4, 5 and 6, as an illustrative application, the normal forms for the vector field and functional differential equation with triple-zero and four-fold zero singularity are considered using the results of section 3. We derive the explicit normal form for the triple-zero and four-fold zero singularity, which are of primary importance in applications. On the one hand, we can determine the terms that are inessential in determining the dynamical and bifurcation behaviors of the system. On the other hand, as we can compute the normal form coefficients, we can identify the parameter values for which nonlinear degeneracies take place. Near these critical parameter values, more complicated bifurcation phenomena can occur.

2. Complementary Space of $L_{N_n}(H^n)$

Let m, n be positive integers, \mathbb{R} the real number field. We denote H_n^2 the following space of 2nd degree homogeneous polynomials with n variables:

$$H_n^2 = \{P(x) | P_m(X) = \sum_{1 \leq i \leq j \leq n} a_{i,j,m} x_i x_j, a_{i,j,m} \in \mathbb{R}, 1 \leq m \leq n\},$$

where $X = (x_1, x_2, \dots, x_n)^T$; $P(x) = (P_1(x), P_2(x), \dots, P_n(x))^T$.

Obviously, H_n^2 is a real inner product space. Let

$$e_1 = (1, 0, \dots, 0)^T, e_2 = (0, 1, 0, \dots, 0)^T, \dots, e_n = (0, \dots, 0, 1)^T.$$

Then, $f_{m,1} = x_1x_1e_m, f_{m,2} = x_1x_2e_m, \dots, f_{m,\frac{n(n+1)}{2}} = x_nx_ne_m, m = 1, 2, \dots, n,$ consisting of a standard orthogonal basis of H_n^2 , called a natural basis.

Further, let

$$U_m = \{P(X)e_m \mid P(X) = \sum_{1 \leq i \leq j \leq n} a_{i,j}x_ix_j, a_{i,j} \in \mathbb{R}\}, \quad m = 1, 2, \dots, n;$$

$$V_h = \{P(X)e_1 \mid P(X) = \sum_{1 \leq i \leq j \leq n, i+j=h} a_{i,j}x_ix_j, a_{i,j} \in \mathbb{R}\}, \quad h = 2, \dots, 2n;$$

$$V_1 = V_{2n+1} = \{0\}.$$

For example, when $n = 3, h = 4$, we have

$$U_1 = \{(a_{11}x_1x_1 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{22}x_2x_2 + a_{23}x_2x_3 + a_{33}x_3x_3)e_1 \mid a_{ij} \in \mathbb{R}\},$$

$$V_4 = \{(a_{13}x_1x_3 + a_{22}x_2x_2)e_1 \mid a_{13}, a_{22} \in \mathbb{R}\}.$$

Then the following result is obvious :

Lemma 2.1. U_m is a subspace of H_n^2, V_h is a subspace of U_1 , and

- (1) $H_n^2 = \bigoplus_{m=1}^n U_m; U_1 = \bigoplus_{h=2}^{2n} V_h;$
- (2) $\dim(H_n^2) = \frac{n^2(n+1)}{2}, \dim(U_m) = \frac{n(n+1)}{2}, m = 1, 2, \dots, n.$

Let

$$N_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}. \tag{2.1}$$

Define the linear transformation L_{N_n} on H_n^2 by N_n as follows:

$$L_{N_n}(P(X)) = D_X P(X)N_n X - N_n P(X),$$

where

$$D_X P(X) = \begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_1}{\partial x_2} & \dots & \frac{\partial P_1}{\partial x_n} \\ \frac{\partial P_2}{\partial x_1} & \frac{\partial P_2}{\partial x_2} & \dots & \frac{\partial P_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial P_n}{\partial x_1} & \frac{\partial P_n}{\partial x_2} & \dots & \frac{\partial P_n}{\partial x_n} \end{pmatrix}.$$

In this section, we investigate the dimensions of the complementary space of $L_{N_n}(H_n^2)$. It is easy to see that the following lemma is true.

Lemma 2.2. U_1 is an L_{N_n} -invariant subspace and

- (1) $L_{N_n}(V_h) \subseteq V_{h+1}, h = 2, 3, \dots, 2n;$
- (2) $L_{N_n}(U_1) = \bigoplus_{h=2}^{2n} L_{N_n}(V_h).$

Now, we prove the following result.

Lemma 2.3. Suppose $n \geq 2$.

- (1) If $2 \leq h \leq n$, then L_{N_n} is an injective linear mapping from V_h to V_{h+1} .
- (2) If $n + 1 \leq h \leq 2n$, then L_{N_n} is a surjective linear mapping from V_h to V_{h+1} .

Proof. (1) For $2 \leq h \leq n$, if there exists $f(x)$ such that $f(X)e_1 \in V_h$, satisfying $L_{N_n}(f(X)e_1) = 0$, then $f(X) = 0$.

Case 1. $h = 2$. In this case, $f(X) = ax_1x_1, L_{N_n}(f(X)e_1) = 2ax_1x_2e_1 = 0$. So $a = 0, f(X) = 0$.

Case 2. $3 \leq h \leq n$, and h is an odd number. In this case, we can write

$$f(X) = a_1x_1x_{h-1} + a_2x_2x_{h-2} + \dots + a_{\frac{h-3}{2}}x_{\frac{h-3}{2}}x_{\frac{h+3}{2}} + a_{\frac{h-1}{2}}x_{\frac{h-1}{2}}x_{\frac{h+1}{2}}.$$

Since

$$L_{N_n}(f(X)e_1) = [a_1x_1x_h + (a_1 + a_2)x_2x_{h-1} + (a_2 + a_3)x_3x_{h-2} + \dots + (a_{\frac{h-3}{2}} + a_{\frac{h-1}{2}})x_{\frac{h-1}{2}}x_{\frac{h+3}{2}} + a_{\frac{h-1}{2}}x_{\frac{h+1}{2}}x_{\frac{h+1}{2}}]e_1 = 0.$$

we have

$$a_1 = 0, a_1 + a_2 = 0, a_2 + a_3 = 0, \dots, a_{\frac{h-3}{2}} + a_{\frac{h-1}{2}} = 0, a_{\frac{h-1}{2}} = 0.$$

Thus, $a_1 = a_2 = \dots = a_{\frac{h-1}{2}} = 0$, and $f(X) = 0$ follows.

Case 3. $3 \leq h \leq n$, and h is an even number. In this case, we can write

$$f(X) = a_1x_1x_{h-1} + a_2x_2x_{h-2} + \dots + a_{\frac{h-2}{2}}x_{\frac{h-2}{2}}x_{\frac{h+2}{2}} + a_{\frac{h}{2}}x_{\frac{h}{2}}x_{\frac{h}{2}}.$$

Since

$$L_{N_n}(f(X)e_1) = [a_1x_1x_h + (a_1 + a_2)x_2x_{h-1} + (a_2 + a_3)x_3x_{h-2} + \dots + (a_{\frac{h-4}{2}} + a_{\frac{h-2}{2}})x_{\frac{h-2}{2}}x_{\frac{h+4}{2}} + (a_{\frac{h-2}{2}} + 2a_{\frac{h}{2}})x_{\frac{h}{2}}x_{\frac{h+2}{2}}]e_1 = 0,$$

we obtain

$$a_1 = 0, a_1 + a_2 = 0, a_2 + a_3 = 0, \dots, a_{\frac{h-2}{2}} + 2a_{\frac{h}{2}} = 0,$$

which yields $a_1 = a_2 = \dots = a_{\frac{h}{2}} = 0$, and so $f(X) = 0$.

(2) For $n + 1 \leq h \leq 2n$, we can prove that for any $g(X)e_1 \in V_{h+1}$, there exists $f(X)e_1 \in V_h$ such that $L_{N_n}(f(X)e_1) = g(X)e_1$.

Case 4. $n + 1 \leq h \leq 2n - 1$, and h is an odd number. In this case, $2 \leq h - n + 1 \leq n$, we write

$$g(X) = b_{h-n+1}x_{h-n+1}x_n + b_{h-n+2}x_{h-n+2}x_{n-1} + \dots + b_{\frac{h-1}{2}}x_{\frac{h-1}{2}}x_{\frac{h+3}{2}} + b_{\frac{h+1}{2}}x_{\frac{h+1}{2}}x_{\frac{h+1}{2}}.$$

Choose $a_{h-n}, a_{h-n+1}, \dots, a_{\frac{h-1}{2}}$ as follows:

$$\begin{cases} a_{\frac{h-1}{2}} = b_{\frac{h+1}{2}}, \\ a_{\frac{h-3}{2}} = b_{\frac{h-1}{2}} - a_{\frac{h-1}{2}}, \\ a_{\frac{h-5}{2}} = b_{\frac{h-3}{2}} - a_{\frac{h-3}{2}}, \\ \dots \\ a_{h-n+1} = b_{h-n+2} - a_{h-n+2}, \\ a_{h-n} = b_{h-n+1} - a_{h-n+1}. \end{cases}$$

Then taking

$$f(X) = a_{h-n}x_{h-n}x_n + a_{h-n+1}x_{h-n+1}x_{n-1} + \dots + a_{\frac{h-1}{2}}x_{\frac{h-1}{2}}x_{\frac{h+1}{2}},$$

yields $f(X)e_1 \in V_h$ and so $L_{N_n}(f(X)e_1) = g(X)e_1$.

Case 5. $n + 1 \leq h \leq 2n - 1$, and h is an even number. In this case, $2 \leq h - n + 1 \leq n$, we write

$$g(X) = b_{h-n+1}x_{h-n+1}x_n + b_{h-n+2}x_{h-n+2}x_{n-1} + \dots + b_{\frac{h-2}{2}}x_{\frac{h-2}{2}}x_{\frac{h+4}{2}} + b_{\frac{h}{2}}x_{\frac{h}{2}}x_{\frac{h+2}{2}}.$$

We choose $a_{h-n+1}, a_{h-n+2}, \dots, a_{\frac{h}{2}}$ as

$$\begin{cases} a_{h-n+1} = b_{h-n+1}, \\ a_{h-n+2} = b_{h-n+2} - a_{h-n+1}, \\ a_{h-n+3} = b_{h-n+3} - a_{h-n+2}, \\ \dots, \\ a_{\frac{h-2}{2}} = b_{\frac{h-2}{2}} - a_{\frac{h-4}{2}}, \\ a_{\frac{h}{2}} = \frac{1}{2}(b_{\frac{h}{2}} - a_{\frac{h-2}{2}}), \end{cases}$$

and take

$$f(X) = a_{h-n+1}x_{h-n+1}x_{n-1} + a_{h-n+2}x_{h-n+2}x_{n-2} + \dots + a_{\frac{h}{2}}x_{\frac{h}{2}}x_{\frac{h}{2}}.$$

Then, $f(X)e_1 \in V_h$ and so $L_{N_n}(f(X)e_1) = g(X)e_1$.

Case 6. $h = 2n$. In this case, $g(X)e_1 \in V_{2n+1} = \{0\}$, so $g(X) = 0$. Choose $f(X) = x_nx_n$, then $f(X)e_1 \in V_{2n}$ and so $L_{N_n}(f(X)e_1) = 0 = g(X)e_1$. \square

The following theorem provides a general formula for determining the dimension of the complementary space $(L_{N_n}(H_n^2))^c$.

Theorem 2.4. *Suppose $n \geq 2$. The dimension of any complementary space $L_{N_n}(H_n^2)^c$ is given by*

$$\dim(L_{N_n}(H_n^2))^c = \begin{cases} \frac{1}{8}(3n^2 + 2n), & n \equiv 0 \pmod{4}; \\ \frac{1}{8}(3n^2 + 2n + 3), & n \equiv 1 \pmod{4}; \\ \frac{1}{8}(3n^2 + 2n), & n \equiv 2 \pmod{4}; \\ \frac{1}{8}(3n^2 + 2n - 1), & n \equiv 3 \pmod{4}. \end{cases}$$

Proof. Denote A the matrix of $L_{N_n}|_{U_1}$ on the standard basis $x_i x_j e_1, 1 \leq i \leq j \leq n$. Then,

$$M = \begin{pmatrix} A & -E_s & 0 & 0 & \dots & 0 & 0 \\ 0 & A & -E_s & 0 & \dots & 0 & 0 \\ 0 & 0 & A & -E_s & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & A & -E_s \\ 0 & 0 & 0 & 0 & \dots & 0 & A \end{pmatrix}$$

is the matrix of L_{N_n} on the standard basis of H_n^2 ,

$$x_i x_j e_m, \quad 1 \leq i \leq j \leq n, \quad 1 \leq m \leq n,$$

where E_s is the $s \times s$ identity matrix, $s = \frac{n(n+1)}{2}$. Simplifying the matrix M by elementary column transformation yields

$$M \rightarrow \begin{pmatrix} E_s & 0 & 0 & \dots & 0 & 0 & 0 \\ -A & E_s & 0 & \dots & 0 & 0 & 0 \\ 0 & -A & E_s & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -A & E_s & 0 \\ 0 & 0 & 0 & \dots & 0 & -A & A^n \end{pmatrix}.$$

Thus,

$$\dim(L_{N_n}(H_n^2))^c = \dim U_1 - \dim L_{N_n}^n(U_1) = \frac{n(n+1)}{2} - \sum_{h=2}^{2n} \dim L_{N_n}^n(V_h).$$

By Lemma 2.3, we have

$$\dim L_{N_n}^n(V_h) = \min\{\dim V_h, \dim V_{h+n}\}.$$

Since

$$\dim(V_h) = 0, \quad h = 2n + 1, 2n + 2, \dots,$$

we obtain

$$\sum_{h=2}^{2n} \dim L_{N_n}^n(V_h) = \sum_{h=2}^n \min\{\dim(V_h), \dim(V_{h+n})\}.$$

If $n \equiv 0 \pmod{4}$, denote $n = 4k + 4$. Then

$$\sum_{h=2}^n \min\{\dim(V_h), \dim(V_{h+n})\} = 4(1 + 2 + 3 + \dots + k) + 3(k + 1) = \frac{n^2 + 2n}{8},$$

and so

$$\dim(L_{N_n}(H_n^2))^c = \frac{1}{8}(3n^2 + 2n).$$

If $n \equiv 1 \pmod{4}$, denote $n = 4k + 1$. Then, we obtain

$$\sum_{h=2}^n \min\{\dim(V_h), \dim(V_{h+n})\} = 4(1 + 2 + 3 + \dots + k) = \frac{(n-1)(n+3)}{8},$$

yielding

$$\dim(L_{N_n}(H_n^2))^c = \frac{1}{8}(3n^2 + 2n + 3).$$

If $n \equiv 2 \pmod{4}$, denote $n = 4k + 2$. Then, we have

$$\sum_{h=2}^n \min\{\dim(V_h), \dim(V_{h+n})\} = 4(1 + 2 + 3 + \dots + k) + (k + 1) = \frac{n^2 + 2n}{8},$$

which leads to

$$\dim(L_{N_n}(H_n^2))^c = \frac{1}{8}(3n^2 + 2n).$$

If $n \equiv 3 \pmod{4}$, denote $n = 4k + 3$. Then,

$$\sum_{h=2}^n \min\{\dim(V_h), \dim(V_{h+n})\} = 4(1 + 2 + 3 + \dots + k) + 2(k + 1) = \frac{n^2 + 2n + 1}{8},$$

and thus

$$\dim(L_{N_n}(H_n^2))^c = \frac{3n^2 + 2n - 1}{8}.$$

□

The following example illustrates Theorem 2.4.

Example 2.5. The dimensions of the complementary space $(L_{N_n}(H_n^2))^c$ for $n = 2, 3, \dots, 15$, is given by

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\dim(L_{N_n}(H_n^2))^c$	2	4	7	11	15	20	26	33	40	48	57	67	77	88

Corollary 2.6. Let $n \geq 2$, A be the matrix of $L_{N_n}|_{U_1}$ on the standard basis $f_{1,1}, \dots, f_{1,s}$. Then,

$$\text{rank}(A^n) + \dim(L_{N_n}(H_n^2))^c = \frac{n(n+1)}{2}.$$

Corollary 2.7. Let $n \geq 2, s = \frac{n(n+1)}{2}, t = \dim(L_{N_n}(H_n^2))^c, A$ be the matrix of $L_{N_n}|_{U_1}$ on the standard basis $f_{1,1}, \dots, f_{1,s}$. Then, there exists matrix $B \in \mathbb{R}^{s \times t}$ such that $\text{rank} B = t$ and $\text{rank}(A^n : B) = s$. Moreover, for any $\beta \in \mathbb{R}^s$, there exists $Y_0 \in \mathbb{R}^s, Z_0 \in \mathbb{R}^t$, such that $A^n Y_0 + B Z_0 = \beta$.

In the following, a method to construct a complementary space to $L_{N_n}(H_n^2)$, and a very simple algorithm to calculate the projection from H_n^2 to the complementary space are presented. We have the following theorem.

Theorem 2.8. Let $n \geq 2, s = \frac{n(n+1)}{2}, t = \dim(L_{N_n}(H_n^2))^c, A$ be the matrix of $L_{N_n}|_{U_1}$ on the standard basis $f_{1,1}, \dots, f_{1,s}, B \in \mathbb{R}^{s \times t}$ satisfying $\text{rank} B = t$ and $\text{rank}(A^n : B) = s$. Then,

$$H_n^2 = L_{N_n}(H_n^2) \oplus W,$$

where

$$W = \text{span}\{g_1, g_2, \dots, g_t\}, \quad (g_1, g_2, \dots, g_t) = (f_{n,1}, f_{n,2}, \dots, f_{n,s})B.$$

Moreover,

$$(g_1(X), g_2(X), \dots, g_t(X))Z_0$$

is the projection of $f(X) = \sum_{i=1}^n \sum_{j=1}^s a_{ij} f_{ij}(X) \in H_n^2$ to W (along $L_{N_n}(H_n^2)$), where $(Y_0, Z_0), Y_0 \in \mathbb{R}^n, Z_0 \in \mathbb{R}^t$, is a solution of the equation:

$$A^n Y_0 + B Z_0 = A^{n-1} \begin{pmatrix} a_{11} \\ a_{12} \\ \dots \\ a_{1,s} \end{pmatrix} + A^{n-2} \begin{pmatrix} a_{21} \\ a_{22} \\ \dots \\ a_{2,s} \end{pmatrix} + \dots + A \begin{pmatrix} a_{n-1,1} \\ a_{n-1,2} \\ \dots \\ a_{n-1,s} \end{pmatrix} + \begin{pmatrix} a_{n1} \\ a_{n2} \\ \dots \\ a_{n,s} \end{pmatrix}.$$

Proof. By Theorem 2.4, we obtain that

$$M = \begin{pmatrix} A & -E_s & 0 & 0 & \dots & 0 & 0 \\ 0 & A & -E_s & 0 & \dots & 0 & 0 \\ 0 & 0 & A & -E_s & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & A & -E_s \\ 0 & 0 & 0 & 0 & \dots & 0 & A \end{pmatrix}$$

is the matrix of L_{N_n} on the standard basis

$$f_{1,1}, \dots, f_{1,s}, \dots, f_{n,1}, \dots, f_{n,s}.$$

Take

$$\begin{aligned} & (\tilde{f}_{1,1}, \dots, \tilde{f}_{1,s}, \dots, \tilde{f}_{n,1}, \dots, \tilde{f}_{n,s}) \\ & = (f_{1,1}, \dots, f_{1,s}, \dots, f_{n,1}, \dots, f_{n,s}) \begin{pmatrix} A & -E_s & 0 & 0 & \dots & 0 & 0 \\ 0 & A & -E_s & 0 & \dots & 0 & 0 \\ 0 & 0 & A & -E_s & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & A & -E_s \\ 0 & 0 & 0 & 0 & \dots & 0 & A \end{pmatrix}. \end{aligned}$$

Then, $L_{N_n}(H_n^2) = \text{span}(\tilde{f}_{1,1}, \dots, \tilde{f}_{1,s}, \dots, \tilde{f}_{n,1}, \dots, \tilde{f}_{n,s})$.

It follows from the proof of Theorem 2.4, that

$$\text{rank}H_n^2 - \text{rank}(L_{N_n}) = s - \text{rank}(A^n),$$

and thus

$$\begin{aligned} & \text{rank} \begin{pmatrix} A & -E_s & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & A & -E_s & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & A & -E_s & \dots & 0 & 0 & 0 \\ \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & A & -E_s & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & A & B \end{pmatrix} \\ &= \text{rank} \begin{pmatrix} E_s & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -A & E_s & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -A & E_s & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots \\ 0 & 0 & 0 & \dots & -A & E_s & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -A & A^n & B \end{pmatrix} = ns. \end{aligned}$$

So

$$H_n^2 = L_{N_n}(H_n^2) \oplus W.$$

Then, for any

$$f(X) = \sum_{i=1}^n \sum_{j=1}^s a_{ij} f_{ij}(X) \in H_n^2,$$

denote

$$\alpha_1 = \begin{pmatrix} a_{11} \\ a_{12} \\ \dots \\ a_{1,s} \end{pmatrix}, \alpha_2 = \begin{pmatrix} a_{21} \\ a_{22} \\ \dots \\ a_{2,s} \end{pmatrix}, \dots, \alpha_n = \begin{pmatrix} a_{n1} \\ a_{n2} \\ \dots \\ a_{n,s} \end{pmatrix}.$$

We take

$$\begin{cases} Y_1 = AY_0 - \alpha_1, \\ Y_2 = AY_1 - \alpha_2, \\ \dots \\ Y_{n-1} = AY_{n-2} - \alpha_{n-1}, \end{cases}$$

and so obtain

$$AY_{n-1} = A^n Y_0 - (A^{n-1}\alpha_1 + A^{n-2}\alpha_2 + \dots + \alpha_n) + \alpha_n = \alpha_n - BZ_0.$$

Hence,

$$\begin{aligned}
 & (\tilde{f}_{1,1}, \dots, \tilde{f}_{2,1}, \dots, \tilde{f}_{n,s}) \begin{pmatrix} Y_0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_{n-1} \end{pmatrix} \\
 &= (f_{1,1}, \dots, f_{1,s}, \dots, f_{n,1}, \dots, f_{n,s}) \begin{pmatrix} A & -E_s & \dots & 0 & 0 \\ 0 & A & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A & -E_s \\ 0 & 0 & \dots & 0 & A \end{pmatrix} \begin{pmatrix} Y_0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_{n-1} \end{pmatrix} \\
 &= (f_{1,1}, \dots, f_{1,s}, \dots, f_{n,1}, \dots, f_{n,s}) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n - BZ_0 \end{pmatrix} \\
 &= f(X) - (g_1, g_2, \dots, g_t)Z_0.
 \end{aligned}$$

Therefore,

$$f(X) = (\tilde{f}_{1,1}, \dots, \tilde{f}_{1,s}, \dots, \tilde{f}_{n,1}, \dots, \tilde{f}_{n,s}) \begin{pmatrix} Y_0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_{n-1} \end{pmatrix} + (g_1, g_2, \dots, g_t)Z_0.$$

□

Remark 2.9. Let $n \geq 2, s = \frac{n(n+1)}{2}, t = \dim(L_{N_n}(H_n^2))^c, A$ be the matrix of $L_{N_n}|_{U_1}$ on the standard basis $f_{1,1}, \dots, f_{1,s}$. Take $B = (\xi_1, \xi_2, \dots, \xi_t)$ as an orthonormal basic system solution for the homogeneous linear equation, $A^n x = 0$. Then B satisfies

$$B \in \mathbb{R}^{s \times t}, \quad \text{rank} B = t, \quad \text{rank}(A^n : B) = s, \quad B^T A^n = 0, \quad B^T B = E_t.$$

In this case, $A^n Y_0 + BZ_0 = \beta$ implies $Z_0 = B^T \beta$ for any $\beta \in \mathbb{R}^s$.

Example 2.10. For $n = 2$, we have $s = 3, t = 2$ and

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Choose $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$(f_{21}, f_{22}, f_{23})B = (x_1^2 e_2, x_1 x_2 e_2, x_2^2 e_2)B = (x_1^2 e_2, x_1 x_2 e_2),$$

and hence

$$H_2^2 = L_{N_2}(H_2^2) \oplus W, \quad W = \text{span} \left\{ \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2 \end{pmatrix} \right\}.$$

For any $f(X) = \sum_{i=1}^2 \sum_{j=1}^3 a_{ij} f_{ij}(X) \in H_2^2$, solving the equation:

$$A^2 Y_0 + B Z_0 = A \alpha_1 + \alpha_2, \quad \alpha_1 = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix}, \alpha_2 = \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix},$$

we have

$$Z_0 = B^T (A \alpha_1 + \alpha_2) = \begin{pmatrix} a_{21} \\ 2a_{11} + a_{22} \end{pmatrix}.$$

So

$$\text{Proj}_W f(X) = \left(\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2 \end{pmatrix} \right) Z_0 = \begin{pmatrix} 0 \\ a_{21} x_1^2 + (2a_{11} + a_{22}) x_1 x_2 \end{pmatrix}.$$

3. Complementary Space of $L_{N_n}(H_{n+p}^2)$ with Parameters

Now we want to extend the normal form techniques to systems with parameters. The goal is to transform the system into normal form near the fixed point in both phase space and parameter space.

Let $n \geq 2, p \geq 1, \mu_1, \mu_2, \dots, \mu_p$ be independent parameters, and

$$H_{n+p}^2 = \{P(X, \mu) | P_m(X, \mu) = \sum_{1 \leq i \leq j \leq n} a_{i,j} x_i x_j + \sum_{1 \leq i \leq j \leq p} b_{i,j} \mu_i \mu_j\}$$

$$+ \sum_{1 \leq i \leq n, 1 \leq j \leq p} c_{i,j} x_i \mu_j, a_{i,j}, b_{i,j}, c_{i,j} \in \mathbb{R}, 1 \leq m \leq n\},$$

where

$$X = (x_1, x_2, \dots, x_n)^T, \quad \mu = (\mu_1, \mu_2, \dots, \mu_p)^T, \\ P(X, \mu) = (P_1(X, \mu), P_2(X, \mu) \dots, P_n(X, \mu))^T.$$

Obviously, H_{n+p}^2 is a real inner product space with

$$\dim H_{n+p}^2 = \frac{1}{2}(n+p)(n+p+1).$$

Moreover, the following

$$f_{m,1} = x_1 x_1 e_m, f_{m,2} = x_1 x_2 e_m, \dots, f_{m,s-1} = x_{n-1} x_n e_m, f_{m,s} = x_n x_n e_m, \\ f_{m,s+1} = x_1 \mu_1 e_m, f_{m,s+2} = x_1 \mu_2 e_m, \dots, f_{m,t} = x_n \mu_p e_m, \\ f_{m,t+1} = \mu_1 \mu_1 e_m, f_{m,t+2} = \mu_1 \mu_2 e_m, \dots, f_{m,q} = \mu_p \mu_p e_m, \\ m = 1, 2, \dots, n, s = \frac{n(n+1)}{2}, t = \frac{n(n+1)}{2} + np, q = \frac{(n+p)(n+p+1)}{2},$$

constitutes a standard orthogonal basis of H_{n+p}^2 , called a natural basis.

Define the linear transformation L_{N_n} on H_{n+p}^2 by N_n , which is defined in (2.1), as follows:

$$L_{N_n}(P(X, \mu)) = D_X P(X, \mu) N_n X - N_n P(X, \mu),$$

where

$$D_X P(X, \mu) = \begin{pmatrix} \frac{\partial P_1(X, \mu)}{\partial x_1} & \frac{\partial P_1(X, \mu)}{\partial x_2} & \dots & \frac{\partial P_1(X, \mu)}{\partial x_n} \\ \frac{\partial P_2(X, \mu)}{\partial x_1} & \frac{\partial P_2(X, \mu)}{\partial x_2} & \dots & \frac{\partial P_2(X, \mu)}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial P_n(X, \mu)}{\partial x_1} & \frac{\partial P_n(X, \mu)}{\partial x_2} & \dots & \frac{\partial P_n(X, \mu)}{\partial x_n} \end{pmatrix}.$$

In this section, we study the complementary spaces $L_{N_n}(H_{n+p}^2)^c$.

Denote $U_m = \text{span}\{f_{m1}, f_{m2}, \dots, f_{mq}\}, m = 1, 2, \dots, n$.

Theorem 3.1. *Suppose $n \geq 2, p \geq 1$. Then, the dimension of the complementary space $L_{N_n}(H_{n+p}^2)^c$ is*

$$\dim(L_{N_n}(H_{n+p}^2))^c = \begin{cases} \frac{3n^2+2n}{8} + \frac{p(p+1)}{2} + np, & n \equiv 0 \pmod{4}; \\ \frac{3n^2+2n+3}{8} + \frac{p(p+1)}{2} + np, & n \equiv 1 \pmod{4}; \\ \frac{3n^2+2n}{8} + \frac{p(p+1)}{2} + np, & n \equiv 2 \pmod{4}; \\ \frac{3n^2+2n-1}{8} + \frac{p(p+1)}{2} + np, & n \equiv 3 \pmod{4}. \end{cases}$$

Proof. Let A be the matrix of $L_{N_n}|_{U_1}$ on the standard basis of U_1 . Then,

$$M = \begin{pmatrix} A & -E_q & 0 & 0 & \dots & 0 & 0 \\ 0 & A & -E_q & 0 & \dots & 0 & 0 \\ 0 & 0 & A & -E_q & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & A & -E_q \\ 0 & 0 & 0 & 0 & \dots & 0 & A \end{pmatrix}$$

is the matrix of L_{N_n} on the standard basis of H_{n+p}^2 , where

$$A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0_{p \times p} & & & & \\ E_p & 0_{p \times p} & & & \\ & E_p & 0_{p \times p} & & \\ & & \ddots & \ddots & \\ & & & E_p & 0_{p \times p} \end{pmatrix}_{np \times np},$$

where A_1 is the matrix of $L_{N_n}|_W$ on the standard basis of $W = \text{span}\{f_{11}, f_{12}, \dots, f_{1s}\}$, and A_2 is the matrix of $L_{N_n}|_V$ on the standard basis of $V = \text{span}\{f_{1,s+1}, f_{1,s+2}, \dots, f_{1,t}\}$. Simplifying the matrix M by elementary column transformation, we have

$$M \rightarrow \begin{pmatrix} -E_q & 0 & 0 & \dots & 0 & 0 & 0 \\ A & -E_q & 0 & \dots & 0 & 0 & 0 \\ 0 & A & -E_q & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A & -E_q & 0 \\ 0 & 0 & 0 & \dots & 0 & A & A^n \end{pmatrix}.$$

Because

$$A_2^n = \begin{pmatrix} 0_{p \times p} & & & & \\ E_p & 0_{p \times p} & & & \\ & E_p & 0_{p \times p} & & \\ & & \ddots & \ddots & \\ & & & E_p & 0_{p \times p} \end{pmatrix}_{np \times np}^n = 0,$$

so we obtain

$$A^n = \begin{pmatrix} A_1^n & 0 & 0 \\ 0 & A_2^n & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1^n & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, with a similar proof to that for Theorem 2.4, we can prove that

$$\begin{aligned} \dim(L_{N_n}(H_{n+p}^2))^c &= \frac{(n+p)(n+p+1)}{2} - \text{rank} A_1^n \\ &= \begin{cases} \frac{3n^2+2n}{8} + \frac{p(p+1)}{2} + np, & n \equiv 0(\text{mod}4); \\ \frac{3n^2+2n+3}{8} + \frac{p(p+1)}{2} + np, & n \equiv 1(\text{mod}4); \\ \frac{3n^2+2n}{8} + \frac{p(p+1)}{2} + np, & n \equiv 2(\text{mod}4); \\ \frac{3n^2+2n-1}{8} + \frac{p(p+1)}{2} + np, & n \equiv 3(\text{mod}4). \end{cases} \end{aligned}$$

□

Example 3.2. We list below the dimensions of the complementary spaces $(L_{N_n}(H_{n+p}^2))^c$ for $n = 2, 3, 4, 5, 6; p = 1, 2, 3, 4, 5, 6$.

$\dim(L_{N_n}(H_{n+p}^2))^c$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$p = 1$	5	8	12	17	22
$p = 2$	9	13	18	24	30
$p = 3$	14	19	25	32	39
$p = 4$	20	26	33	41	49
$p = 5$	27	34	42	51	60
$p = 6$	35	43	52	62	72

Theorem 3.3. For any $n \geq 2, p \geq 1, q = \frac{(n+p)(n+p+1)}{2}$, let A be the matrix of $L_{N_n}|_{U_1}$ on the standard basis $f_{1,1}, \dots, f_{1,q}, t = q - \text{rank} A^n$, matrix $B \in \mathbb{R}^{q \times t}$ satisfying $\text{rank} B = t$ and $\text{rank}(A^n \cdot B) = q$. Further, if

$$(g_1(X), g_2(X), \dots, g_t(X)) = (f_{n,1}, f_{n,2}, \dots, f_{n,q})B,$$

and $W = \text{span}\{g_1(X), g_2(X), \dots, g_t(X)\} \subseteq U_n$, then

$$H_{n+p}^2 = L_{N_n}(H_{n+p}^2) \oplus W.$$

Moreover,

$$(g_1(X), g_2(X), \dots, g_t(X))Z_0, \quad Z_0 = (z_1, z_2, \dots, z_n)^T \in \mathbb{R}^t$$

is the projection of $f(X) = \sum_{i=1}^n \sum_{j=1}^q a_{ij} f_{ij}(X) \in H_{n+p}^2$ to W (along $L_{N_n}(H_{n+p}^2)$) if and only if there exists $Y_0 \in \mathbb{R}^q$ such that

$$A^n Y_0 + B Z_0 = A^{n-1} \alpha_1 + A^{n-2} \alpha_2 + \dots + A \alpha_{n-1} + \alpha_n,$$

where

$$\alpha_i = (a_{i1}, a_{i2}, \dots, a_{iq})^T, \quad i = 1, 2, \dots, n.$$

Proof. The proof is similar to that for Theorem 2.8, and thus omitted. □

Example 3.4. For $n = p = 3$, we have

$$A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{21 \times 21}, A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 \\ E_3 & 0 & 0 \\ 0 & E_3 & 0 \end{pmatrix}_{9 \times 9}.$$

Since

$$A_1^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{pmatrix}, A_1^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{pmatrix}, A_2^2 = \begin{pmatrix} 0 & 0 \\ E_3 & 0 \end{pmatrix},$$

$$A_2^3 = 0, A^2 = \begin{pmatrix} A_1^2 & 0 & 0 \\ 0 & A_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A^3 = \begin{pmatrix} 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 19} \\ 6 & 0 & 0_{1 \times 19} \\ 0 & 3 & 0_{1 \times 19} \\ 0_{15 \times 1} & 0_{15 \times 1} & 0_{15 \times 19} \end{pmatrix},$$

we can choose

$$B = \begin{pmatrix} E_4 & 0_{4 \times 15} \\ 0_{2 \times 4} & 0_{2 \times 15} \\ 0_{15 \times 4} & E_{15} \end{pmatrix}.$$

Then,

$$H_{3+3}^2 = L_{N_3}(H_{3+3}^2) \oplus W, \\ W = \text{span}\{g_1(X), g_2(X), \dots, g_{19}(X)\},$$

where

$$(g_1(X), g_2(X), \dots, g_{19}(X)) = (x_1^2 e_3, x_1 x_2 e_3, \dots, \mu_3^2 e_3) B \\ = (x_1^2 e_3, x_1 x_2 e_3, x_1 x_3 e_3, x_2^2 e_3, x_1 \mu_1 e_3, x_1 \mu_2 e_3, x_1 \mu_3 e_3, x_2 \mu_1 e_3, x_2 \mu_2 e_3, x_2 \mu_3 e_3, \\ x_3 \mu_1 e_3, x_3 \mu_2 e_3, x_3 \mu_3 e_3, \mu_1^2 e_3, \mu_1 \mu_2 e_3, \mu_1 \mu_3 e_3, \mu_2^2 e_3, \mu_2 \mu_3 e_3, \mu_3^2 e_3).$$

For any $f(X) = \sum_{i=1}^3 \sum_{j=1}^{21} a_{ij} f_{ij}(X) \in H_{3+3}^2$, we solve the equation:

$$A^3 Y_0 + B Z_0 = A^2 \alpha_1 + A \alpha_2 + \alpha_3,$$

where

$$\begin{aligned} \alpha_1 &= (a_{1,1}, a_{1,2}, \dots, a_{1,21})^T, \alpha_2 = (a_{2,1}, a_{2,2}, \dots, a_{2,21})^T, \dots, \alpha_3 \\ &= (a_{3,1}, a_{3,2}, \dots, a_{3,21})^T. \end{aligned}$$

Since $B^T B = E_{19}, B^T A^3 = 0$, we have $Z_0 = B^T (A^2 \alpha_1 + A \alpha_2 + \alpha_3)$. So

$$\begin{aligned} & \text{Proj}_W f(X) \\ &= a_{3,1} \begin{pmatrix} 0 \\ 0 \\ x_1^2 \end{pmatrix} + (a_{3,2} + 2a_{2,1}) \begin{pmatrix} 0 \\ 0 \\ x_1 x_2 \end{pmatrix} + (a_{3,3} + a_{2,2} + 2a_{1,1}) \begin{pmatrix} 0 \\ 0 \\ x_1 x_3 \end{pmatrix} \\ &+ (a_{3,4} + a_{2,2} + 2a_{1,1}) \begin{pmatrix} 0 \\ 0 \\ x_2^2 \end{pmatrix} + a_{3,7} \begin{pmatrix} 0 \\ 0 \\ x_1 \mu_1 \end{pmatrix} + a_{3,8} \begin{pmatrix} 0 \\ 0 \\ x_1 \mu_2 \end{pmatrix} + a_{3,9} \begin{pmatrix} 0 \\ 0 \\ x_1 \mu_3 \end{pmatrix} \\ &(a_{3,10} + a_{2,7}) \begin{pmatrix} 0 \\ 0 \\ x_2 \mu_1 \end{pmatrix} + (a_{3,11} + a_{2,8}) \begin{pmatrix} 0 \\ 0 \\ x_2 \mu_2 \end{pmatrix} + (a_{3,12} + a_{2,9}) \begin{pmatrix} 0 \\ 0 \\ x_2 \mu_3 \end{pmatrix} \\ &+ (a_{3,13} + a_{2,10} + a_{1,7}) \begin{pmatrix} 0 \\ 0 \\ x_3 \mu_1 \end{pmatrix} + (a_{3,14} + a_{2,11} + a_{1,8}) \begin{pmatrix} 0 \\ 0 \\ x_3 \mu_2 \end{pmatrix} \\ &+ (a_{3,15} + a_{2,12} + a_{1,9}) \begin{pmatrix} 0 \\ 0 \\ x_3 \mu_3 \end{pmatrix} + a_{3,16} \begin{pmatrix} 0 \\ 0 \\ \mu_1^2 \end{pmatrix} + a_{3,17} \begin{pmatrix} 0 \\ 0 \\ \mu_1 \mu_2 \end{pmatrix} \\ &+ a_{3,18} \begin{pmatrix} 0 \\ 0 \\ \mu_1 \mu_3 \end{pmatrix} + a_{3,19} \begin{pmatrix} 0 \\ 0 \\ \mu_2 \mu_2 \end{pmatrix} + a_{3,20} \begin{pmatrix} 0 \\ 0 \\ \mu_2 \mu_3 \end{pmatrix} + a_{3,21} \begin{pmatrix} 0 \\ 0 \\ \mu_3 \mu_3 \end{pmatrix}, \end{aligned}$$

that is

$$\begin{aligned} & \text{Proj}_W f(X) \\ &= [a_{3,1} x_1^2 + (a_{3,2} + 2a_{2,1}) x_1 x_2 + (a_{3,3} + a_{2,2} + 2a_{1,1}) x_1 x_3 + (a_{3,4} + a_{2,2} + 2a_{1,1}) x_2^2 \\ &+ a_{3,7} x_1 \mu_1 + a_{3,8} x_1 \mu_2 + a_{3,9} x_1 \mu_3 + (a_{3,10} + a_{2,7}) x_2 \mu_1 + (a_{3,11} + a_{2,8}) x_2 \mu_2 \\ &+ (a_{3,12} + a_{2,9}) x_2 \mu_3 + (a_{3,13} + a_{2,10} + a_{1,7}) x_3 \mu_1 + (a_{3,14} + a_{2,11} + a_{1,8}) x_3 \mu_2 \end{aligned}$$

$$A_1^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_2^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_4 & 0 & 0 & 0 \\ 0 & E_4 & 0 & 0 \end{pmatrix},$$

$$A_2^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_4 & 0 & 0 & 0 \end{pmatrix}, \quad A_2^4 = 0_{16 \times 16}, \quad A^4 = \begin{pmatrix} A_1^4 & 0 \\ 0 & 0 \end{pmatrix},$$

we can choose

$$B = \begin{pmatrix} E_6 & 0_{6 \times 1} & 0_{6 \times 26} \\ 0_{1 \times 6} & 3 & 0_{1 \times 26} \\ 0_{1 \times 6} & -4 & 0_{1 \times 26} \\ 0_{2 \times 6} & 0 & 0_{2 \times 29} \\ 0_{26 \times 6} & 0 & E_{26} \end{pmatrix}_{36 \times 33}.$$

Then,

$$H_{4+4}^2 = L_{N_4}(H_{4+4}^2) \oplus W, \\ W = \text{span}\{g_1(X), g_2(X), \dots, g_{33}(X)\},$$

where

$$(g_1(X), g_2(X), \dots, g_{33}(X)) = (f_{4,1}, \dots, f_{4,36})B \\ = (f_{4,1}, \dots, f_{4,6}, 3f_{4,7}, -4f_{4,8}, f_{4,11}, \dots, f_{4,36}).$$

For any $f(X) = \sum_{i=1}^4 \sum_{j=1}^{36} a_{ij} f_{ij}(X) \in H_{4+4}^2$, let

$$\alpha_1 = (a_{1,1}, \dots, a_{1,36})^T; \\ \alpha_2 = (a_{2,1}, \dots, a_{2,36})^T; \\ \alpha_3 = (a_{3,1}, \dots, a_{3,36})^T; \\ \alpha_4 = (a_{4,1}, \dots, a_{4,36})^T.$$

we solve the equation:

$$A^4 Y_0 + B Z_0 = A^3 \alpha_1 + A^2 \alpha_2 + A \alpha_3 + \alpha_4.$$

Since $B^T B = \begin{pmatrix} E_6 & 0_{6 \times 1} & 0_{6 \times 26} \\ 0_{1 \times 6} & 25 & 1 \times 26 \\ 0_{26 \times 6} & 0_{26 \times 1} & E_{26} \end{pmatrix}$, $B^T A^4 = 0$, we have $Z_0 = (B^T B)^{-1} B^T (A^3 \alpha_1 + A^2 \alpha_2 + A \alpha_3 + \alpha_4)$. So

$$\begin{aligned} & \text{Proj}_W f(X) \\ & = (g_1(x), g_2(x), \dots, g_{33}(x)) Z_0 = a_{4,1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1^2 \end{pmatrix} + (a_{4,2} + 2a_{3,1}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1 x_2 \end{pmatrix} \\ & \quad + (a_{4,3} + a_{3,2} + 2a_{2,1}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1 x_3 \end{pmatrix} + (a_{4,4} + a_{3,3} + a_{2,2} + 2a_{1,1}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1 x_4 \end{pmatrix} \\ & \quad + (a_{4,5} + a_{3,2} + 2a_{2,1}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_2^2 \end{pmatrix} + (6a_{1,1} + 3a_{2,2} + a_{3,3} + 2a_{3,5} + a_{4,6}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_2 x_3 \end{pmatrix} \\ & \quad + \frac{1}{25} (2a_{2,3} - 2a_{2,5} + 3a_{3,4} - a_{3,6} + 3a_{4,7} - 4a_{4,8}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3x_2 x_4 - 4x_3^2 \end{pmatrix} \\ & \quad + \sum_{j=1}^4 a_{4,10+j} \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1 \mu_j \end{pmatrix} + \sum_{j=1}^4 (a_{3,10+j} + a_{4,14+j}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_2 \mu_j \end{pmatrix} \\ & \quad + \sum_{j=1}^4 (a_{2,10+j} + a_{3,14+j} + a_{4,18+j}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_3 \mu_j \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^4 (a_{1,10+j} + a_{2,14+j} + a_{3,18+j} + a_{4,22+j}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_4 \mu_j \end{pmatrix} \\
 & + \sum_{j=1}^4 a_{4,26+j} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mu_1 \mu_j \end{pmatrix} + \sum_{j=1}^4 a_{4,26+j} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mu_1 \mu_j \end{pmatrix} \\
 & + \sum_{j=1}^3 a_{4,30+j} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mu_2 \mu_{j+1} \end{pmatrix} + \sum_{j=1}^2 a_{4,33+j} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mu_3 \mu_{j+2} \end{pmatrix} + a_{4,36} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mu_4^2 \end{pmatrix}.
 \end{aligned}$$

4. Normal forms for 3-dimensional Vector Field with a Nilpotent Point with Three Parameters

Consider the vector field

$$\dot{x} = Jx + F_2(x, \mu) + F_3(x, \mu) + \dots + F_r(x, \mu) + O(|x|^r), \quad x \in \mathbb{R}^3, \quad \mu \in I \subset \mathbb{R}^3, \quad (4.1)$$

where $J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is the canonical Jordan nilpotent form, and $F_i(x, \mu)$ represents

the i th-degree homogeneous polynomial in the Taylor expansion of $F(x, \mu)$.

According to Theorem 3.3 and Example 3.4, we have the following normal form of system (4.1):

$$\begin{aligned}
 \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + [a_{3,1}x_1^2 + (a_{3,2} + 2a_{2,1})x_1x_2 \\
 & + (a_{3,3} + a_{2,2} + 2a_{1,1})x_1x_3 + (a_{3,4} + a_{2,2} + 2a_{1,1})x_2^2 + a_{3,7}x_1\mu_1 + a_{3,8}x_1\mu_2 \\
 & + a_{3,9}x_1\mu_3 + (a_{3,10} + a_{2,7})x_2\mu_1 + (a_{3,11} + a_{2,8})x_2\mu_2 + (a_{3,12} + a_{2,9})x_2\mu_3 \\
 & + (a_{3,13} + a_{2,10} + a_{1,7})x_3\mu_1 + (a_{3,14} + a_{2,11} + a_{1,8})x_3\mu_2 \\
 & + (a_{3,15} + a_{2,12} + a_{1,9})x_3\mu_3 + a_{3,16}\mu_1^2 + a_{3,17}\mu_1\mu_2 + a_{3,18}\mu_1\mu_3 + a_{3,19}\mu_2\mu_2 \\
 & + a_{3,20}\mu_2\mu_3 + a_{3,21}\mu_3\mu_3](0, 0, 1)^T.
 \end{aligned}$$

Ignoring the higher order terms $\mu_i\mu_j$ ($i, j = 1, 2, 3$), we obtain

Theorem 4.1. *If the Jacobian of vector field (4.1) evaluated at a critical point involves a triple-zero eigenvalue with geometric multiplicity one, ignoring the higher order terms $\mu_i\mu_j$ ($i, j = 1, 2, 3$), we obtain the reduced normal form with unfolding on the center manifold near $(x, \mu) = (0, 0)$ as follows:*

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \eta_1 x_1^2 + \eta_2 x_2^2 + \eta_3 x_1 x_2 + \eta_4 x_1 x_3, \end{cases} \quad (4.2)$$

where

$$\begin{aligned} \lambda_1 &= a_{3,7}\mu_1 + a_{3,8}\mu_2 + a_{3,9}\mu_3, \\ \lambda_2 &= (a_{3,10} + a_{2,7})\mu_1 + (a_{3,11} + a_{2,8})\mu_2 + (a_{3,12} + a_{2,9})\mu_3, \\ \lambda_3 &= (a_{3,13} + a_{2,10} + a_{1,7})\mu_1 + (a_{3,14} + a_{2,11} + a_{1,8})\mu_2 + (a_{3,15} + a_{2,12} + a_{1,9})\mu_3, \\ \eta_1 &= a_{3,1}, \\ \eta_2 &= (a_{3,4} + a_{2,2} + 2a_{1,1}), \\ \eta_3 &= (a_{3,2} + 2a_{2,1}), \\ \eta_4 &= (a_{3,3} + a_{2,2} + 2a_{1,1}). \end{aligned}$$

Generically, we expect the fixed points to move as the parameters are varied. This does not happen in (4.2); the origin always remains a fixed point. This situation is easy to remedy. Notice from the form of (4.2) that any fixed point must have $x_1 = x_2 = 0$. Suppose that system (4.2) satisfies $\eta_1 \neq 0$. We make the coordinate transformation [18]

$$\begin{cases} x = x_1 + \frac{\lambda_1}{2\eta_1}, \\ y = x_2, \\ z = x_3, \end{cases}$$

then Eq.(4.2) becomes

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = \kappa_1 + \kappa_2 y + \kappa_3 z + \eta_1 x^2 + \eta_2 y^2 + \eta_3 xy + \eta_4 xz, \end{cases} \quad (4.3)$$

where $\kappa_1 = -\frac{\lambda_1^2}{4\eta_1}$, $\kappa_2 = \lambda_2 - \frac{\lambda_1\eta_3}{2\eta_1}$, $\kappa_3 = \lambda_3 - \frac{\lambda_1\eta_4}{2\eta_1}$. Using the method developed in Ref [8, 10, 12], we can obtain the following truncated hypernormal form up to second order:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = \kappa_1 + \kappa_2 y + \kappa_3 z - \frac{1}{2}x^2 + \gamma_1 xy + \gamma_2 xz, \end{cases} \quad (4.4)$$

and $\gamma_1 = \frac{\eta_3}{-2\eta_1}$, $\gamma_2 = \frac{\eta_4}{-2\eta_1}$.

5. Normal form of a DDE system associated with triple-zero singularity

In this section, we present the normal form of a functional differential equation associated with a trip-zero singularity.

Let us consider an abstract retarded functional differential equation with parameters in the phase space $C = C([-\tau, 0]; R^n)$, described by

$$\dot{u}(t) = L(\mu)u_t + F(u_t, \mu), \tag{5.1}$$

where $u_t \in C$, is defined by $u_t(\theta) = u(t + \theta)$, $-\tau \leq \theta \leq 0$, the parameter $\mu \in R^p$ is a parameter vector in a neighborhood V of zero. $L(\mu) : V \rightarrow L(C, R^n)$ is C^{k-1} and $F : C \times R^p \rightarrow R^n$ is C^k ($k \geq 2$) with $F(0, \mu) = 0, DF(0, \mu) = 0$. Define $L = L(0)$ and

$$G(u_t, \mu) = F(u_t, \mu) + (L(\mu) - L(0))u_t,$$

under which system (5.1) can be rewritten as

$$\dot{u}(t) = Lu_t + G(u_t, \mu). \tag{5.2}$$

Then, the linear homogeneous retard functional differential equation (5.2) can be written as

$$\dot{u}(t) = Lu_t, \tag{5.3}$$

where L is a bounded linear operator and satisfies

$$L\varphi = \int_{-\tau}^0 d\eta(\theta)\varphi(\theta), \quad \forall \varphi \in C. \tag{5.4}$$

Here, $\eta(\theta)(\theta \in [-\tau, 0])$ is an $n \times n$ matrix function of bounded variation. Let A_0 be the infinitesimal generator such that

$$A_0\varphi = \dot{\varphi}, D(A_0) = \left\{ \varphi \in C^1([-\tau, 0], R^n) : \dot{\varphi}(0) = \int_{-\tau}^0 d\eta(\theta)\varphi(\theta) \right\},$$

and its adjoint is given by

$$A_0^*\psi = \dot{\psi}, D(A_0^*) = \left\{ \psi \in C^1([0, \tau], R^{n*}) : \dot{\psi}(0) = - \int_{-\tau}^0 d\eta(\theta)\psi(-\theta) \right\}.$$

Define the bilinear form between C and $C' = C([0, \tau], R^{n*})$ by

$$(\psi, \varphi) = \psi(0)\varphi(0) - \int_{-\tau}^0 \int_0^\theta \psi(\xi - \theta)d\eta(\theta)\varphi(\xi)d\xi, \forall \psi \in C^1, \forall \varphi \in C.$$

In the following, we will consider the case for which L has a triple-zero eigenvalue and all other eigenvalues have negative real parts.

Let Λ be the set of eigenvalues with zero real part and P be the generalized eigenspace associated with Λ which has a triple-zero eigenvalue and P^* the space adjoint with P . Then, C can be decomposed as

$$C = P \oplus Q, \quad \text{where } Q = \left\{ \varphi \in C : \langle \varphi, \psi \rangle = 0, \forall \psi \in P^* \right\},$$

with $\dim p = 3$. Choose the bases Φ and Ψ for P and P^* respectively such that

$$\langle \Psi, \Phi \rangle = I, \quad \dot{\Phi} = \Phi B, \quad \dot{\Psi} = -B\Psi,$$

where I is the $m \times m$ identity matrix and $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Following the ideas in [9],

we consider the enlarged phase space BC ,

$$BC = \left\{ \varphi : [-\tau, 0] \rightarrow \mathbb{R}^n : \varphi \text{ continuous on } [-\tau, 0), \exists \lim_{\theta \rightarrow 0^-} \varphi(\theta) \in \mathbb{R}^n \right\}.$$

Then, the elements of BC can be expressed as $\psi = \varphi + x_0\alpha$, $\varphi \in C$, $\alpha \in \mathbb{R}^n$ and

$$x_0(\theta) = \begin{cases} 0, & -\tau \leq \theta < 0, \\ I, & \theta = 0, \end{cases}$$

where I is the identity operator on C . The space BC has the norm $\|\varphi + u_0\alpha\| = \|\varphi\|_c + \|\alpha\|_{\mathbb{R}^n}$. Then, the continuous projection $\pi : BC \rightarrow P$, defined by

$$\pi(\varphi + u_0\alpha) = \Phi[(\Psi, \varphi) + \Psi(0)\alpha],$$

allows us to decompose the enlarged phase space $BC = P \oplus \text{Ker}\pi$. Let $u = \Phi x + y$. Then, system (5.1) can be decomposed as

$$\begin{cases} \dot{x} = Bx + \Psi(0)G(\Phi x + y, \mu), \\ \frac{dy}{dt} = A_{Q^1}y + (I - \pi)u_0G(\Phi x + y, \mu) \end{cases} \quad z \in \mathbb{R}^3, y \in Q^1, \quad (5.5)$$

for $y \in Q^1 = Q \cap C^1 \subset \text{Ker}\pi$, where A_{Q^1} is the restriction of A_0 as an operator from Q^1 to Banach space $\text{Ker}\pi$. Employing Taylor's theorem, system (5.5) becomes

$$\begin{cases} \dot{x} = Bx + \sum_{j \geq 2} \frac{1}{j} f_j^1(z, y, \mu), \\ \frac{dy}{dt} = A_{Q^1}y + \sum_{j \geq 2} \frac{1}{j} f_j^2(x, y, \mu), \end{cases} \quad (5.6)$$

where $f_j^i(x, y, \mu)$ ($i = 1, 2$) denotes the homogeneous polynomials of degree j in variables (x, y, μ) . For $J = B$, as defined above, the non-resonance conditions are naturally satisfied. According to normal form theory for DDEs in [9], system (5.6) can be transformed into the following normal form on the center manifold,

$$\dot{x} = Bx + \frac{1}{2}g_2^1(x, 0, \mu) + h.o.t. \quad (5.7)$$

For a normed space Z , denote $V_j^6(Z)$ the linear space of homogeneous polynomials of $(x, \mu) = (x_1, x_2, x_3, \mu_1, \mu_2, \mu_3)$ with degree j and with coefficients in Z and define M_j the operator in $V_j^6(\mathbb{R}^3 \times \text{Ker}\pi)$ with the range in the same space by

$$M_j(p, h) = (M_j^1 p, M_j^2 h),$$

where

$$M_j^1 p = M_j^1 \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \frac{\alpha p_1}{\alpha x_1} x_2 + \frac{\alpha p_1}{\alpha x_2} x_3 - p_2 \\ \frac{\alpha p_2}{\alpha x_1} x_2 + \frac{\alpha p_2}{\alpha x_2} x_3 - p_3 \\ \frac{\alpha p_3}{\alpha x_1} x_2 + \frac{\alpha p_3}{\alpha x_2} x_3 \end{pmatrix},$$

$$M_j^2 h = M_j^2 h(z, \mu) = D_x h(x, \mu) Bx - A_{Q^1} h(x, \mu).$$

Using M_j^1 , we have the following decompositions,

$$V_j^6(R^3) = \text{Im}(M_j^1) \oplus (\text{Im}(M_j^1))^c, V_j^6(R^3) = \text{Ker}(M_j^1) \oplus (\text{Ker}(M_j^1))^c.$$

Then, $g_2^1(z, 0, \mu)$ can be expressed as

$$g_2^1(x, 0, \mu) = \text{Proj}_{(\text{Im}(M_2^1))^c} f_2^1(x, 0, \mu).$$

From the results obtained in section 3, we know that a basis of $(\text{Im}(M_2^1))^c$ can be taken as the set composed by the elements

$$\begin{pmatrix} 0 \\ 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_1 x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_1 x_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_1 \mu_1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_1 \mu_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_1 \mu_3 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 0 \\ x_2 \mu_1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_2 \mu_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_2 \mu_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_3 \mu_1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_3 \mu_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_3 \mu_3 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 0 \\ \mu_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \mu_1 \mu_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \mu_1 \mu_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \mu_2 \mu_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \mu_2 \mu_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \mu_3 \mu_3 \end{pmatrix}.$$

Further, since

$$f_2^1(x, y, \mu) = \Psi(0) F_2(\Phi x + y),$$

we let

$$f_2^1(x, 0, \mu) = \sum_{i=1}^n \sum_{j=1}^q a_{ij} f_{ij}(X, \mu).$$

Example 3.4 shows that the projection of $f(X) \in (\text{Im}(M_2^1))^c$ is given by

$$\begin{aligned} & \text{Proj}_{(\text{Im}(M_2^1))^c} f(X, \mu) \\ &= [a_{3,1} x_1^2 + (a_{3,2} + 2a_{2,1}) x_1 x_2 + (a_{4,3} + a_{3,2} + 2a_{2,1}) x_1 x_3 \\ & \quad + (a_{3,4} + a_{2,2} + 2a_{1,1}) x_2^2 + a_{3,7} x_1 \mu_1 + a_{3,8} x_1 \mu_2 + a_{3,9} x_1 \mu_3 + (a_{3,10} + a_{2,7}) x_2 \mu_1 \\ & \quad + (a_{3,11} + a_{2,8}) x_2 \mu_2 + (a_{3,12} + a_{2,9}) x_2 \mu_3 + (a_{3,13} + a_{2,10} + a_{1,7}) x_3 \mu_1 \\ & \quad + (a_{3,14} + a_{2,11} + a_{1,8}) x_3 \mu_2 + (a_{3,15} + a_{2,12} + a_{1,9}) x_3 \mu_3 + a_{3,16} \mu_1^2 + a_{3,17} \mu_1 \mu_2 \\ & \quad + a_{3,18} \mu_1 \mu_3 + a_{3,19} \mu_2 \mu_2 + a_{3,20} \mu_2 \mu_3 + a_{3,21} \mu_3 \mu_3](0, 0, 1)^T. \end{aligned}$$

Ignoring the higher order terms $\mu_i\mu_j$ ($i, j = 1, 2, 3$), we obtain

$$g_2^1(z, 0, \mu) = (\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \eta_1 x_1^2 + \eta_2 x_2^2 + \eta_3 x_1 x_2 + \eta_4 x_1 x_3)(0, 0, 1)^T,$$

where $\lambda_1, \lambda_2, \lambda_3, \eta_1, \eta_2, \eta_3$ and η_4 are given in (4.2).

Summarizing the above results yields the following theorem.

Theorem 5.1. *If Eq.(5.1) has a triple-zero eigenvalue with geometric multiplicity one, then on the center manifold near $(u_t, \mu) = (0, 0)$, the reduced normal form with unfolding has the following form:*

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \eta_1 x_1^2 + \eta_2 x_2^2 + \eta_3 x_1 x_2 + \eta_4 x_1 x_3, \end{cases}$$

where

$$\begin{aligned} \lambda_1 &= a_{3,7}\mu_1 + a_{3,8}\mu_2 + a_{3,9}\mu_3, \\ \lambda_2 &= (a_{3,10} + a_{2,7})\mu_1 + (a_{3,11} + a_{2,8})\mu_2 + (a_{3,12} + a_{2,9})\mu_3, \\ \lambda_3 &= (a_{3,13} + a_{2,10} + a_{1,7})\mu_1 + (a_{3,14} + a_{2,11} + a_{1,8})\mu_2 + (a_{3,15} + a_{2,12} + a_{1,9})\mu_3, \\ \eta_1 &= a_{3,1}, \\ \eta_2 &= (a_{3,4} + a_{2,2} + 2a_{1,1}), \\ \eta_3 &= (a_{3,2} + 2a_{2,1}), \\ \eta_4 &= (a_{3,3} + a_{2,2} + 2a_{1,1}). \end{aligned}$$

This form is consistent with the Ref [11, 16]. If necessary, we can also convert it into (4.3) or (4.4).

6. Normal forms for 4-dimensional Vector Field with a Nilpotent Point with Four Parameters

Consider the vector field

$$\dot{x} = Jx + F_2(x, \mu) + F_3(x, \mu) + \cdots + F_r(x, \mu) + O(|x|^r), \quad x \in \mathbb{R}^4, \quad \mu \in I \subset \mathbb{R}^4, \quad (6.1)$$

where $J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is the canonical Jordan nilpotent form, and $F_i(x, \mu)$ repre-

sents the i th-degree homogeneous polynomial in the Taylor expansion of $F(x, \mu)$.

According to Theorem 3.3 and Example 3.5, we have the following normal form

of system (6.1):

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + [a_{4,1}x_1^2 + (2a_{3,1} + a_{4,2})x_1x_2 \\ &+ (a_{3,3} + a_{2,2} + 2a_{1,1})x_1x_3 + (a_{4,4} + a_{3,3} + a_{2,2} + 2a_{1,1})x_1x_4 \\ &+ (a_{4,5} + a_{3,2} + 2a_{2,1})x_2^2 + (6a_{1,1} + 3a_{2,2} + a_{3,3} + 2a_{3,5} + a_{4,6})x_2x_3 \\ &+ \frac{1}{25}(2a_{2,3} - 2a_{2,5} + 3a_{3,4} - a_{3,6} + 3a_{4,7} - 4a_{4,8})(3x_2x_4 - 4x_3^2) \\ &+ \sum_{j=1}^4 a_{4,10+j}x_1\mu_j + \sum_{j=1}^4 (a_{3,10+j} + a_{4,14+j})x_2\mu_j \\ &+ \sum_{j=1}^4 (a_{2,10+j} + a_{3,14+j} + a_{4,18+j})x_3\mu_j \\ &+ \sum_{j=1}^4 (a_{1,10+j} + a_{2,14+j} + a_{3,18+j} + a_{4,22+j})x_4\mu_j \\ &+ \sum_{j=1}^4 a_{4,26+j}\mu_1\mu_j + \sum_{j=1}^3 a_{4,30+j}\mu_2\mu_{j+1} \\ &+ \sum_{j=1}^2 a_{4,33+j}\mu_3\mu_{j+2} + a_{4,36}\mu_4^2](0, 0, 0, 1)^T. \end{aligned}$$

Ignoring the higher order terms $\mu_i\mu_j(i, j = 1, 2, 3, 4)$, we obtain

Theorem 6.1. *If the Jacobian of vector field (6.1) evaluated at a critical point involves a four-zero eigenvalue with geometric multiplicity one, ignoring the higher order terms $\mu_i\mu_j(i, j = 1, 2, 3, 4)$, we obtain the reduced normal form with unfolding on the center manifold near $(x, \mu) = (0, 0)$ as follows:*

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3 + \lambda_4x_4 + \eta_1x_1^2 + \eta_2x_2^2 + \eta_3x_3^2 \\ \quad + \gamma_1x_1x_2 + \gamma_2x_1x_3 + \gamma_3x_1x_4 + \gamma_4x_2x_3 + \gamma_5x_2x_4, \end{cases} \tag{6.2}$$

where

$$\begin{aligned} \lambda_1 &= \sum_{j=1}^4 a_{4,10+j}, \\ \lambda_2 &= \sum_{j=1}^4 (a_{3,10+j} + a_{4,14+j}), \end{aligned}$$

$$\begin{aligned}
\lambda_3 &= \sum_{j=1}^4 (a_{2,10+j} + a_{3,14+j} + a_{4,18+j}), \\
\lambda_4 &= \sum_{j=1}^4 (a_{1,10+j} + a_{2,14+j} + a_{3,18+j} + a_{4,22+j}), \\
\eta_1 &= a_{4,1}, \\
\eta_2 &= a_{4,5} + a_{3,2} + 2a_{2,1}, \\
\eta_3 &= -\frac{4}{25}(2a_{2,3} - 2a_{2,5} + 3a_{3,4} - a_{3,6} + 3a_{4,7} - 4a_{4,8}), \\
\gamma_1 &= 2a_{3,1} + a_{4,2}, \\
\gamma_2 &= a_{3,3} + a_{2,2} + 2a_{1,1}, \\
\gamma_3 &= a_{4,4} + a_{3,3} + a_{2,2} + 2a_{1,1}, \\
\gamma_4 &= 6a_{1,1} + 3a_{2,2} + a_{3,3} + 2a_{3,5} + a_{4,6}, \\
\gamma_5 &= \frac{3}{25}(2a_{2,3} - 2a_{2,5} + 3a_{3,4} - a_{3,6} + 3a_{4,7} - 4a_{4,8}).
\end{aligned}$$

Generically, we expect the fixed points to move as the parameters are varied. This does not happen in (6.2); the origin always remains a fixed point. This situation is easy to remedy. Notice from the form of (6.2) that any fixed point must have $x_1 = x_2 = 0$. Suppose that system (6.2) satisfies $\eta_1 \neq 0$. We make the coordinate transformation [20]

$$\begin{cases} x = x_1 + \frac{\lambda_1}{2\eta_1}, \\ y = x_2, \\ z = x_3, \\ w = x_4, \end{cases}$$

then Eq. (6.2) becomes

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = w, \\ \dot{w} = \kappa_1 + \kappa_2 y + \kappa_3 z + \kappa_4 w + \eta_1 x^2 + \eta_2 y^2 + \eta_3 z^2 \\ \quad + \gamma_1 xy + \gamma_2 xz + \gamma_3 xw + \gamma_4 yz + \gamma_5 yw, \end{cases} \quad (6.3)$$

where $\kappa_1 = -\frac{\lambda_1^2}{4\eta_1}$, $\kappa_2 = \lambda_2 - \frac{\lambda_1\gamma_2}{2\eta_1}$, $\kappa_3 = \lambda_3 - \frac{\lambda_1\gamma_3}{2\eta_1}$, $\kappa_4 = \lambda_4 - \frac{\lambda_1\gamma_4}{2\eta_1}$.

7. Conclusion

In this paper, we have derived second-order explicit formulas of the normal forms associated with nilpotent critical points. As an application, the explicit formulas have been obtained for normal forms with unfolding associated with a triple-zero and a four-fold zero singularity in vector field and retarded functional differential

equations. The formulas obtained in this paper can be easily implemented using a computer algebra system such as Maple or Mathematica.

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