Journal of Applied Analysis and Computation
Volume 10, Number 6, December 2020, 2506–2520
Website:http://www.jaac-online.com
DOI:10.11948/20190391

RESEARCH ON THE COMPOSITION CENTER OF A CLASS OF RIGID DIFFERENTIAL SYSTEMS

Yuexin Yan[1] and Zhengxin Zhou[2,†]

Abstract In this paper, we answer the question: under what conditions a class of rigid differential systems have a composition center. We give the sufficient and necessary conditions for these systems to have a center at origin point. At the same time, we give the formula of focal values and the highest order of fine focus.

Keywords Uniformly isochronous center, rigid system, composition conjecture, composition center, center conditions.

MSC(2010) 34C07, 34C05, 34C25, 37G15.

1. Introduction

Consider the system

$$\begin{cases} x' = -y + x(P_1(x,y) + P_2(x,y) + ... + P_n(x,y)) = -y + xP, \\ y' = x + y(P_1(x,y) + P_2(x,y) + ... + P_n(x,y)) = x + yP, \end{cases} \tag{1.1}$$

where $P_k(x,y) = \sum_{i+j=k} p_{ij}x^iy^j$, p_{ij} are real numbers. The system in polar coordinates becomes

$$r' = P_1r^2 + P_2r^3 + ... + P_nr^{n+1}, \, \theta' = 1.$$

This system is called a **rigid system** [3] because the derivative of the angular variable is constant. It is clear that the origin is the only critical point and if it is a center then it is a **uniformly isochronous center** [12]. In [9, 16], the authors have proved that a planar polynomial differential system of degree n+1 has a center at the origin of coordinates, then this center is uniform isochronous if and only if by doing a linear change of variables and a scaling of time it can be written as (1.1). The interest in the uniform isochronous centers has attracted people's attention since the 17th century. So far, there are many people who have strong interest in this problem and have achieved fruitful results [2,8,9,12,16]. In [1,2] the authors have used techniques based on normal forms and commutation and have proved that the rigid system (1.1), in the cases: $P = P_1 + P_n$ or $P = P_2 + P_{2m}$ or $P = P_1 + P_2 + P_3 + P_4$, it has a center if and only if it is reversible. In [20],

†The corresponding author.
Email address: zxzhou@yzu.edu.cn (Z. Zhou), yxyan@yzu.edu.cn (Y. Yan)
[1]Guangling college of Yangzhou university, 225002, Yangzhou, China
[2]School of mathematical sciences, Yangzhou university, 225002, Yangzhou, China

the author calculated by computer to get the center condition for this system with $P = P_k + P_{2k}$ $(k = 2, 3, 4, 5)$. In [14, 15], the numbers of limit cycles of (1.1) have been discussed. In [18, 19], some new methods have been used to studied the center problem of this system.

In this paper, we consider the following rigid system

$$\begin{cases} x' = -y + x(P_1(x,y) + P_3(x,y) + P_7(x,y)), \\ y' = x + y(P_1(x,y) + P_3(x,y) + P_7(x,y)). \end{cases} \tag{1.2}$$

By [5, 6], this system has a center at $(0, 0)$ if and only if all solutions $r(\theta)$ of periodic differential equation

$$\frac{dr}{d\theta} = r(P_1(\cos\theta, \sin\theta)r + P_3(\cos\theta, \sin\theta)r^3 + P_7(\cos\theta, \sin\theta)r^7) \tag{1.3}$$

near the solution $r = 0$ are periodic, $r(0) = r(2\pi)$. In such case it is said that equation (1.3) has a center at $r = 0$.

As we known, the derivation of conditions for a center is a difficult and long-standing problem in the theory of nonlinear differential equations, however due to complexity of the problem necessary and sufficient conditions are known only for a very few families of polynomial systems [4, 13, 17]. In [5, 6] the authors introduce a simple condition called **Composition Conditions**, which ensures that the Abel equation

$$\frac{dr}{d\theta} = A(\theta)r^2 + B(\theta)r^3 \tag{1.4}$$

has a center. Roughly speaking the composition condition says that the primitives of the functions A and B depend functionally on a new 2π-periodic function. When an Abel equation has a center because A and B satisfy the composition condition we will say that the equation has a **Composition Center** [6].

The **Composition Conjecture** is that the composition condition is not only the sufficient but also necessary condition for a center. This conjecture first appeared in [7] with classes of coefficients which are polynomial functions in t, or trigonometric polynomials. A counterexample was presented in [5] to demonstrate that the conjecture is not true. To find the restrictive conditions under which the composition conjecture is true, this is an open problem which has attracted during the last years a wide interest. In [5] the author has proved that for a family of cubic system the composition conjecture is valid. [21, 22] the author used the different method from [1, 2] to prove that for system (1.1) with $P = P_1 + P_n$ and $P = P_2 + P_{2m}$, the composition conjecture is true. The authors in paper [6, 10, 11] give the sufficient and necessary conditions for the $r = 0$ of the Able equation (1.4) to be a composition center.

In this paper, we find out all the restrictive conditions under which the origin point of (1.3) is a composition center. At the same time, we give the sufficient and necessary conditions for equation (1.2) to have a center at origin point by using a different method from [1, 2, 20]. These center conditions are more succinct and beautiful than those calculated by computer.

2. Several Lemmas

Alwash and Lloyd [5–7] proved the following statement.

Lemma 2.1 ([5,6]). *If there exists a differentiable function u of period 2π such that*

$$\tilde{A}_1(\theta) = u'(\theta)\hat{A}_1(u(\theta)),\ \tilde{A}_2(\theta) = u'(\theta)\hat{A}_2(u(\theta))$$

for some continuous functions $\hat{A}_1$ and $\hat{A}_2$, then the Abel differential equation

$$\frac{dr}{d\theta} = \tilde{A}_1(\theta)r^2 + \tilde{A}_2(\theta)r^3$$

has a center at $r \equiv 0$.

The condition in Lemma2.1 is called the **Composition Condition**. This is a sufficient but not a necessary condition for $r = 0$ to be a center [7,10].

The following statement presents a generalization of Lemma 2.1.

Lemma 2.2 ([23]). *If there exists a differentiable function u of period 2π such that*

$$\tilde{A}_i(\theta) = u'\hat{A}_i(u),\ (i = 1, 2, ..., n) \tag{2.1}$$

for some continuous functions $\hat{A}_i\ (i = 1, 2, ..., n)$, then the differential equation

$$\frac{dr}{d\theta} = r\sum_{i=1}^{n} \tilde{A}_i(\theta)r^i$$

has a center at $r = 0$.

Denote :

$$P_k = \sum_{i+j=k} p_{ij}\cos^i\theta\sin^j\theta,\ \bar{P}_k = \int_0^\theta P_k d\theta,\ C_k^i = \frac{k!}{i!(k-i)!}.$$

Lemma 2.3. *If $p_{10}^2 + p_{01}^2 \neq 0$ and*

$$\int_0^{2\pi} \bar{P}_1^{2i+1} P_3 d\theta = 0,\ (i = 0, 1),\ \int_0^{2\pi} \bar{P}_1^{2j+1} P_7 d\theta = 0,\ (j = 0, 1, 2, 3),$$

then

$$P_3 = P_1(\lambda_1 + 2\lambda_2\bar{P}_1 + 3\lambda_3\bar{P}_1^2),\ P_7 = P_1(\mu_1 + 2\mu_2\bar{P}_1 + ... + 7\mu_7\bar{P}_1^6),$$

where $\mu_j\ (j = 1, 2, ..., 7)$ are real numbers and

$$\begin{aligned}\lambda_1 =& \frac{1}{(p_{10}^2 + p_{01}^2)^3}(p_{30}p_{10}(p_{10}^4 + 3p_{01}^4) + p_{21}p_{01}(p_{10}^2 - p_{01}^2)^2 \\ &+ 2p_{12}p_{01}^2p_{10}(p_{10}^2 - p_{01}^2) + 4p_{03}p_{01}^3p_{10}^2), \\ \lambda_2 =& -3\lambda_3 p_{01},\ \lambda_3 = -\frac{1}{3(p_{10}^2 + p_{01}^2)^3}((p_{30} - p_{12})(p_{10}^3 - 3p_{10}p_{01}^2) \\ &+ (p_{03} - p_{21})(p_{01}^3 - 3p_{01}p_{10}^2)).\end{aligned}$$

Proof. Denote

$$a = \frac{p_{10}}{\sqrt{p_{10}^2 + p_{01}^2}}, b = \frac{p_{01}}{\sqrt{p_{10}^2 + p_{01}^2}},$$

$$U = a\cos\theta + b\sin\theta,\ V = a\sin\theta - b\cos\theta,$$

then

$$\begin{aligned}
&P_1 = \sqrt{p_{10}^2 + p_{01}^2}U,\ \bar{P}_1 = \sqrt{p_{10}^2 + p_{01}^2}V + p_{01},\\
&a^2 + b^2 = 1,\ U^2 + V^2 = 1,\ \int_0^{2\pi} U^2 d\theta = \int_0^{2\pi} V^2 d\theta = \pi,\\
&\int_0^{2\pi} U^{2k} d\theta = \int_0^{2\pi} V^{2k} d\theta = \frac{(2k-1)!!}{(2k)!!}2\pi.
\end{aligned}$$

Obviously, P_3 and P_7 can be rewritten in the following forms

$$\begin{aligned}
P_3 &= m_1U + m_3U^3 + n_1V + n_3V^3,\\
P_7 &= s_1U + s_3U^3 + s_5U^5 + s_7U^7 + t_1V + t_3V^3 + t_5V^5 + t_7V^7,
\end{aligned}$$

where $n_i(i = 1, 3), s_j, t_j(j = 1, 3, 5, 7)$ are real numbers and

$$\begin{aligned}
m_1 &= 3p_{30}ab^2 + p_{21}(b^3 - 2a^2b) + p_{12}(a^3 - 2ab^2) + 3p_{03}a^2b,\\
m_3 &= (p_{30} - p_{12})(a^3 - 3ab^2) + (p_{03} - p_{21})(b^3 - 3a^2b).
\end{aligned}$$

By $\int_0^{2\pi} \bar{P}_1^{2i+1} P_3 d\theta = 0$, $(i = 0, 1)$, we get

$$\int_0^{2\pi} (n_1V^2 + n_3V^4) d\theta = 0,\ \int_0^{2\pi} (n_1V^4 + n_3V^6) d\theta = 0,$$

i.e.,

$$n_1 + \frac{3}{4}n_3 = 0,\ \frac{3}{4}n_1 + \frac{5}{8}n_3 = 0,$$

solving these equations we get $n_1 = n_3 = 0$, so,

$$P_3 = m_1U + m_3U^3 = P_1(\lambda_1 + 2\lambda_2\bar{P}_1 + 3\lambda_3\bar{P}_1^2),$$

where $\lambda_1 = \frac{m_1+m_3}{\sqrt{p_{10}^2+p_{01}^2}} + 3p_{01}^2\lambda_3$, $\lambda_2 = -3\lambda_3p_{01}$, $\lambda_3 = -\frac{m_3}{3(p_{10}^2+p_{01}^2)^{\frac{3}{2}}}$.

By

$$\int_0^{2\pi} \bar{P}_1^{2j+1} P_7 d\theta = 0,\ (j = 0, 1, 2, 3),$$

we get

$$\begin{aligned}
&\int_0^{2\pi} (t_1V^2 + t_3V^4 + t_5V^6 + t_7V^8) d\theta = 0,\\
&\int_0^{2\pi} (t_1V^4 + t_3V^6 + t_5V^8 + t_7V^{10}) d\theta = 0,\\
&\int_0^{2\pi} (t_1V^6 + t_3V^8 + t_5V^{10} + t_7V^{12}) d\theta = 0,\\
&\int_0^{2\pi} (t_1V^8 + t_3V^{10} + t_5V^{12} + t_7V^{14}) d\theta = 0,
\end{aligned}$$

i.e.,

$$\begin{pmatrix} 1 & \frac{3}{4} & \frac{5}{8} & \frac{35}{64} \\ \frac{3}{4} & \frac{5}{8} & \frac{35}{64} & \frac{63}{128} \\ \frac{5}{8} & \frac{35}{64} & \frac{63}{128} & \frac{231}{512} \\ \frac{35}{64} & \frac{63}{128} & \frac{231}{512} & \frac{429}{1024} \end{pmatrix} \begin{pmatrix} t_1 \\ t_3 \\ t_5 \\ t_7 \end{pmatrix} = 0,$$

since the value of the determinant of the coefficient matrix of the above equations is not equal to zero, so $t_1 = t_3 = t_5 = t_7 = 0$ and

$$P_7 = s_1 U + s_3 U^3 + s_5 U^5 + s_7 U^7 = P_1(\mu_1 + 2\mu_2 \bar{P}_1 + ... + 7\mu_7 \bar{P}_1^6).$$

□

3. Main results

Consider equation

$$\frac{dr}{d\theta} = r(P_1 r + P_3 r^3 + P_7 r^7), \tag{3.1}$$

where $P_k = \sum_{i+j=k} p_{ij} \cos^i\theta \sin^j\theta$, p_{ij} $(i, j = 0, 1, 2, ..., k,\ k = 1, 3, 7)$ are real numbers and $p_{10}^2 + p_{01}^2 \neq 0$.

Theorem 3.1. *Suppose that one of the following conditions:*

1. $(14 + \lambda_3)(35 + 23\lambda_3)(330 + 703\lambda_3 + 65\lambda_3^2) \neq 0$;
2. $14 + \lambda_3 = 0$, $p_{01} \neq 0$;
3. $14 + \lambda_3 = 0$, $p_{01} = 0, b_3 = 0$;
4. $35 + 23\lambda_3 = 0$, $p_{01} \neq 0$;
5. $35 + 23\lambda_3 = 0$, $p_{01} = 0$, $b_5 = 0$;
6. $330 + 703\lambda_3 + 65\lambda_3^2 = 0$, $p_{01} \neq 0$;
7. $330 + 703\lambda_3 + 65\lambda_3^2 = 0$, $p_{01} = 0$, $b_7 = 0$,

is satisfied. Then $r = 0$ is a center of (3.1) *if and only if*

$$\int_0^{2\pi} \bar{P}_1^{2i+1} P_3 d\theta = 0,\ (i = 0, 1),\ \int_0^{2\pi} \bar{P}_1^{2j+1} P_7 d\theta = 0,\ (j = 0, 1, 2, 3). \tag{3.2}$$

I.e.,

$$p_{10}p_{21} - p_{01}p_{12} + 3p_{10}p_{03} - 3p_{01}p_{30} = 0; \tag{3.3}$$

$$p_{30}p_{01}^3 - p_{21}p_{01}^2 p_{10} + p_{12}p_{01}p_{10}^2 - p_{03}p_{10}^3 = 0; \tag{3.4}$$

$$p_{10}(5p_{61} + 3p_{43} + 5p_{25} + 35p_{07}) - p_{01}(5p_{16} + 3p_{34} + 5p_{52} + 35p_{70}) = 0; \tag{3.5}$$

$$\begin{aligned} &p_{10}^3(3p_{61}+3p_{43}+7p_{25}+63p_{07}) - 3p_{10}^2 p_{01}(3p_{52}+3p_{34}+7p_{70}+7p_{16}) \\ &+ 3p_{10}p_{01}^2(3p_{25}+3p_{43} + 7p_{07}+7p_{61}) - p_{01}^3(3p_{16}+3p_{34}+7p_{52}+63p_{70}) = 0; \end{aligned} \tag{3.6}$$

$$\begin{aligned} &p_{10}^5(5p_{61}+7p_{43}+21p_{25}+231p_{07}) - 5p_{10}^4 p_{01}(5p_{52}+7p_{70}+7p_{34}+21p_{16}) \\ &+ 10p_{10}^3 p_{01}^2(5p_{43}+7p_{61}+7p_{25}+21p_{07}) - 10p_{10}^2 p_{01}^3(5p_{34}+7p_{16} + 7p_{52}+21p_{70}) \\ &+ 5p_{10}p_{01}^4(5p_{25}+7p_{07}+7p_{43}+21p_{61}) - p_{01}^5(5p_{16}+7p_{34}+21p_{52}+231p_{70}) = 0; \end{aligned} \tag{3.7}$$

$$\begin{aligned} &p_{70}p_{01}^7 - p_{61}p_{01}^6 p_{10} + p_{52}p_{01}^5 p_{10}^2 - p_{43}p_{01}^4 p_{10}^3 + p_{34}p_{01}^3 p_{10}^4 - p_{25}p_{01}^2 p_{10}^5 \\ &+ p_{16}p_{01}p_{10}^6 - p_{07}p_{10}^7 = 0. \end{aligned} \tag{3.8}$$

Moreover, this center is a composition center and uniformly isochronous center. Where λ_3 is the same as it is in Lemma2.3, $b_i = \frac{1}{\pi}\int_0^{2\pi} P_7 \sin i\theta d\theta$, $(i = 3, 5, 7)$.

Proof. **Necessity:** Let $r(\theta, c)$ be the solution of (3.1) such that $r(0, c) = c\,(0 < c \ll 1)$. We write

$$r(\theta, c) = c\sum_{n=0}^{\infty} a_n(\theta)c^n,$$

where $a_0(0) = 1$ and $a_n(0) = 0$ for $n \geq 1$. The origin of (3.1) is a center if and only if $r(\theta + 2\pi, c) = r(\theta, c)$, i.e., $a_0(2\pi) = 1$, $a_n(2\pi) = 0\,(n = 1, 2, 3, ...)$ [10,12].

Substituting $r(\theta, c)$ into (3.1) we obtain

$$\sum_{n=0}^{\infty} a'_n(\theta)c^n = cP_1(\sum_{n=0}^{\infty} a_n(\theta)c^n)^2 + c^3P_3(\sum_{n=0}^{\infty} a_n(\theta)c^n)^4 + c^7P_7(\sum_{n=0}^{\infty} a_n(\theta)c^n)^8. \quad (3.9)$$

Equating the corresponding coefficients of c^n of (3.9), we get

$$\begin{aligned}
&a'_0(\theta) = 0,\ a_0(0) = 1,\\
&a'_1(\theta) = P_1a_0^2,\ a_1(0) = 0,\\
&a'_2(\theta) = 2a_0a_1P_1,\ a_2(0) = 0,\\
&a'_3(\theta) = (2a_0a_2 + a_1^2)P_1 + a_0^4P_3, a_3(0) = 0,\\
&a'_4(\theta) = (2a_0a_3 + 2a_1a_2)P_1 + 4a_0^3a_1P_3,\ a_4(0) = 0,
\end{aligned}$$

solving these equations we obtain

$$a_0 = 1,\ a_1 = \bar{P}_1,\ a_2 = \bar{P}_1^2,\ a_3 = \bar{P}_1^3 + \gamma_0,\ a_4 = \bar{P}_1^4 + \gamma_1,$$

where

$$\gamma_0 = \bar{P}_3,\ \gamma_1 = 2\bar{P}_1\bar{P}_3 + 2\overline{\bar{P}_1P_3}. \quad (3.10)$$

By this we see that $a_i(2\pi) = 0$, $(i = 0, 1, 2, 3)$ and from $a_4(2\pi) = 0$ implies that

$$\int_0^{2\pi} \bar{P}_1P_3d\theta = 0. \quad (3.11)$$

Denote:

$$\phi = f + \gamma c^3 + \alpha c^6 + \beta c^7 + \delta c^{10},$$

where

$$f = \sum_{i=0}^{\infty} \bar{P}_1^ic^i,\ \gamma = \sum_{i=0}^{\infty} \gamma_ic^i,\ \alpha = \sum_{i=0}^{\infty} \alpha_ic^i,\ \beta = \sum_{i=0}^{\infty} \beta_ic^i,\ \delta = \sum_{i=0}^{\infty} \delta_ic^i.$$

Thus

$$\begin{aligned}
\phi^2 =& f^2 + 2f\gamma c^3 + (\gamma^2 + 2f\alpha)c^6 + 2f\beta c^7 + 2\gamma\alpha c^9 + 2(f\delta + \gamma\beta)c^{10}\\
&+ \alpha^2c^{12} + 2(\gamma\delta + \alpha\beta)c^{13} + \beta^2c^{14} + ...,
\end{aligned} \quad (3.12)$$

$$\begin{aligned}
\phi^4 =& f^4 + 4f^3\gamma c^3 + (6f^2\gamma^2 + 4f^3\alpha)c^6 + 4f^3\beta c^7 + (4f\gamma^3 + 12f^2\gamma\alpha)c^9\\
&+ (4f^3\delta + 12f^2\gamma\beta)C^{10} + (\gamma^4 + 12f\gamma^2\alpha + 6f^2\alpha^2)c^{12} + ...,
\end{aligned} \quad (3.13)$$

$$\phi^8 = f^8 + 8f^7\gamma c^3 + (28f^6\gamma^2 + 8f^7\alpha)c^6 + 8\beta f^7c^7 + ..., \quad (3.14)$$

$$f^m = \sum_{k=0}^{\infty} C_{m+k-1}^{m-1} \bar{P}_1^k c^k. \tag{3.15}$$

Substituting $r = \phi c = (f + \gamma c^3 + \alpha c^6 + \beta c^7 + \delta c^{10})c$ into (3.9) and using (3.12)–(3.15) we get

$$a_5' = P_1(5\bar{P}_1^5 + 2\sum_{i+j=1} \bar{P}_1^i \gamma_j) + P_3 C_5^3 \bar{P}_1,$$

solving this equation we obtain

$$a_5 = \bar{P}_1^5 + \gamma_2, \tag{3.16}$$

where

$$\gamma_2 = 3\bar{P}_1^2\bar{P}_3 + 4\bar{P}_1\overline{\bar{P}_1 P_3} + 3\overline{\bar{P}_1^2 P_3}. \tag{3.17}$$

By (3.11) and (3.16) and (3.17) we have $a_5(2\pi) = 0$.

Applying (3.9) and (3.12)–(3.15) we obtain

$$a_6' = P_1(6\bar{P}_1^5 + 2\sum_{i+j=2} \bar{P}_1^i \gamma_j) + P_3(C_6^3\bar{P}_1^3 + C_4^1\gamma_0),$$

solving this equation we get

$$a_6 = \bar{P}_1^6 + \gamma_3 + \alpha_0, \tag{3.18}$$

where

$$\gamma_3 = 4\bar{P}_1^3\bar{P}_3 + 6\bar{P}_1^2\overline{\bar{P}_1 P_3} + 6\bar{P}_1\overline{\bar{P}_1^2 P_3} + 4\overline{\bar{P}_1^3 P_3},\ \alpha_0 = 2\bar{P}_3^2. \tag{3.19}$$

By (3.11) and (3.18) and (3.19) we see that if $a_6(2\pi) = 0$, then

$$\int_0^{2\pi} \bar{P}_1^3 P_3 d\theta = 0. \tag{3.20}$$

Using (3.11) and (3.20) and Lemma2.3 we get

$$P_3 = P_1(\lambda_1 + 2\lambda_2\bar{P}_1 + 3\lambda_3\bar{P}_1^2), \tag{3.21}$$

where $\lambda_i\,(i = 1, 2, 3)$ are the same as they are in Lemma2.3. Therefore,

$$\bar{P}_3 = \lambda_3\bar{P}_1^3 + \lambda_2\bar{P}_1^2 + \lambda_1\bar{P}_1. \tag{3.22}$$

By (3.21) and (3.22) we see that $\gamma_k(k = 0, 1, 2, 3)$ are the polynomials with respect to $\bar{P}_1$ of degree $k + 3$, α_0 is a polynomial on $\bar{P}_1$ of degree 6 , so they are 2π periodic functions.

Applying (3.9) and (3.12)–(3.15) we get

$$a_{7+k} = \bar{P}_1^{3+k} + \gamma_{4+k} + \alpha_{1+k} + \beta_k,\ (k = 0, 1, 2), \tag{3.23}$$

where

$$\gamma_k = \sum_{j=0}^{k} (k + 1 - j)(1 + j)\bar{P}_1^{k-j}\overline{\bar{P}_1^j P_3}, \tag{3.24}$$

$$\beta_k = \sum_{j=0}^{k} (k+1-j) C_{5+j}^{j} \bar{P}_1^{k-j} \overline{\bar{P}_1^{j} P_7}, \tag{3.25}$$

$$\begin{aligned}
\alpha_0 &= 2\bar{P}_3^2, \\
\alpha_1 &= 8\bar{P}_3 \overline{\bar{P}_1 P_3} + 2\overline{\bar{P}_1 P_3 \bar{P}_3} + 5\bar{P}_1 \bar{P}_3^2, \\
\alpha_2 &= 4\bar{P}_1 \overline{\bar{P}_1 P_3 \bar{P}_3} + 12\bar{P}_3 \overline{\bar{P}_1^2 P_3} + 9\bar{P}_1^2 \bar{P}_3^3 + 20\bar{P}_1 \bar{P}_3 \overline{\bar{P}_1 P_3} + 10\overline{\bar{P}_1 P_3}^2 + 6\overline{\bar{P}_1^2 \bar{P}_3 P_3}, \\
\alpha_3 &= 6\bar{P}_1^2 \overline{\bar{P}_1 P_3 \bar{P}_3} + 14\bar{P}_1^3 \bar{P}_3^2 + 24\bar{P}_1 \overline{\bar{P}_1 P_3}^2 + 12\bar{P}_1 \overline{\bar{P}_1^2 P_3 \bar{P}_3} + 16\bar{P}_3 \overline{\bar{P}_1^3 P_3} \\
&\quad + 36\bar{P}_1^2 \bar{P}_3 \overline{\bar{P}_1 P_3} + 30\bar{P}_1 \bar{P}_3 \overline{\bar{P}_1^2 P_3} + 30\overline{\bar{P}_1 P_3}\, \overline{\bar{P}_1^2 P_3} + 6\overline{\bar{P}_1^2 P_3 \overline{\bar{P}_1 P_3}} + 12\overline{\bar{P}_1^3 \bar{P}_3 P_3} + \frac{14}{3}\bar{P}_3^3.
\end{aligned}$$

By these relations and (3.11) and (3.20)-(3.22) we see that γ_{4+k} and α_{1+k} are 2π-periodic functions, thus from $a_{7+k}(2\pi) = 0$, $(k = 0, 1, 2)$ imply that

$$\int_0^{2\pi} \bar{P}_1 P_7 d\theta = 0. \tag{3.26}$$

Applying (3.9) and (3.12)– (3.15) we obtain

$$a_{10+k} = \bar{P}_1^{10+k} + \gamma_{7+k} + \alpha_{4+k} + \beta_{3+k} + \delta_k, \ (k = 0, 1, 2, ..., 5), \tag{3.27}$$

where γ_k, β_k are expressed by (3.24) and (3.25), respectively, α_k is the polynomial on $\bar{P}_1$ of degree $6+k$, δ_k $(k = 0, 1, 2, ..., 5)$ are the solutions of the following equations:

$$\begin{aligned}
\delta_0' &= 4P_3\beta_0 + 8P_7\gamma_0, \\
\delta_1' &= 2P_1(\delta_0 + \gamma_0\beta_0) + 4P_3 \sum_{i+j=1} C_{2+i}^{2} \bar{P}_1^{i} \beta_j + 8P_7 \sum_{i+j=1} C_{6+i}^{i} \bar{P}_i \gamma_j, \\
\delta_2' &= 2P_1 \sum_{i+j=1} (\bar{P}_1^{i}\delta_j + \gamma_i\beta_j) + 4P_3 \sum_{i+j=2} C_{2+i}^{2} \bar{P}_1^{i} \beta_j + 8P_7 \sum_{i+j=2} C_{6+i}^{i} \bar{P}_1^{i} \gamma_j, \\
\delta_3' &= 2P_1 \sum_{i+j=2} (\bar{P}_1^{i}\delta_j + \gamma_i\beta_j) + 4P_3 (\sum_{i+j=3} C_{2+i}^{2} \bar{P}_1^{i} \beta_j + \delta_0 + 3\gamma_0\beta_0) \\
&\quad + P_7 (8 \sum_{i+j=3} C_{6+i}^{i} \bar{P}_1^{i} \gamma_j + 28\gamma_0^2 + 8\alpha_0), \\
\delta_4' &= 2P_1 (\sum_{i+j=3} (\bar{P}_1^{i}\delta_j + \gamma_i\beta_j) + \gamma_0\delta_0 + \alpha_0\beta_0) + 4P_3 (\sum_{i+j=4} C_{2+i}^{2} \bar{P}_1^{i} \beta_j + \sum_{i+j=1} C_{2+i}^{2} \bar{P}_1^{i} \delta_j \\
&\quad + 3 \sum_{i+j+l=1} (i+1) \bar{P}_1^{i} \gamma_j \beta_l) + P_7 (8 \sum_{i+j=4} C_{6+i}^{i} \bar{P}_1^{i} \gamma_j + 28 \sum_{i+j=1} C_{5+i}^{i} \bar{P}_1^{i} \sum_{i_1+i_2=j} \gamma_{i_1}\gamma_{i_2} \\
&\quad + 8 \sum_{i+j=1} C_{6+i}^{i} \bar{P}_1^{i} \alpha_j + 8\beta_0). \\
\delta_5' &= 2P_1 (\sum_{i+j=4} (\bar{P}_1^{i}\delta_j + \gamma_i\beta_j) + \sum_{i+j=1} (\gamma_i\delta_j + \alpha_i\beta_j) + \frac{1}{2}\beta_0^2) + P_3 (4 \sum_{i+j=5} C_{2+i}^{i} \bar{P}_1^{i} \beta_j \\
&\quad + 4 \sum_{i+j=2} C_{2+i}^{i} \bar{P}_1^{i} \delta_j + 12 \sum_{i+j+l=2} (i+1) \bar{P}_1^{i} \gamma_j \beta_l) + P_7 (8 \sum_{i+j=5} C_{6+i}^{i} \bar{P}_1^{i} \gamma_j \\
&\quad + 28 \sum_{i+j+l=2} C_{5+i}^{i} \bar{P}_1^{i} \gamma_j \gamma_l + 8 \sum_{i+j=2} C_{6+i}^{i} \bar{P}_1^{i} \alpha_j + 8 \sum_{i+j=1} C_{6+i}^{i} \bar{P}_1^{i} \beta_j).
\end{aligned}$$

Solving these equations we get

$$\delta_0 = 4\bar P_3\bar P_7 + 4\overline{\bar P_3 P_7}; \tag{3.28}$$

$$\delta_1 = 8\bar P_1\overline{\bar P_3 P_7}+24\bar P_3\overline{\bar P_1 P_7}+10\bar P_7\overline{\bar P_1 P_3}+10\bar P_1\bar P_3\bar P_7+30\overline{\bar P_1\bar P_3 P_7} + 6\overline{\overline{\bar P_1 P_3}P_7}; \tag{3.29}$$

$$\begin{aligned}\delta_2 =&12\bar P_1^2\overline{\bar P_3P_7}+84\bar P_3\overline{\bar P_1^2 P_7}+60\bar P_1\bar P_3\overline{\bar P_1 P_7}+60\overline{\bar P_1 P_3}\,\overline{\bar P_1 P_7}+18\bar P_1^2\bar P_3\bar P_7\\ &+ 24\bar P_1\bar P_7\overline{\bar P_1P_3} + 60\bar P_1\overline{\bar P_1\bar P_3 P_7} + 12\bar P_1\overline{\overline{\bar P_1 P_3}P_7} + 18\bar P_7\overline{\bar P_1^2 P_3}\\ &+ 48\overline{\bar P_1 P_7\overline{\bar P_1P_3}} + 6\overline{P_7\overline{\bar P_1^2P_3}} + 126\overline{\bar P_1^2\bar P_3 P_7};\end{aligned} \tag{3.30}$$

$$\begin{aligned}\delta_3 =&28\bar P_1^3\bar P_3\bar P_7 + 16\bar P_1^3\overline{\bar P_3P_7} + 108\bar P_1^2\bar P_3\overline{\bar P_1P_7} + 42\bar P_1^2\bar P_7\overline{\bar P_1P_3} + 90\bar P_1^2\overline{\bar P_1\bar P_3P_7}\\ &+ 18\bar P_1^2\overline{\overline{\bar P_1P_3}P_7} + 210\bar P_1\bar P_3\overline{\bar P_1^2P_7} + 144\bar P_1\overline{\bar P_1P_3}\,\overline{\bar P_1P_7} + 252\bar P_1\overline{\bar P_1^2\bar P_3P_7}\\ &+ 42\bar P_1\bar P_7\overline{\bar P_1^2P_3} + 96\bar P_1\overline{\bar P_1P_7\overline{\bar P_1P_3}} + 12\bar P_1\overline{P_7\overline{\bar P_1^2P_3}} + 224\bar P_3\overline{\bar P_1^3P_7}\\ &+ 14\bar P_3^2\bar P_7 + 16\bar P_3\overline{\bar P_3P_7} + 28\bar P_7\overline{\bar P_1^3P_3} + 108\overline{\bar P_1^2P_3}\,\overline{\bar P_1P_7} + 210\overline{\bar P_1P_3}\,\overline{\bar P_1^2P_7}\\ &+ 4\overline{\overline{\bar P_1^3P_3}P_7} + 392\overline{\bar P_1^3\bar P_3P_7} + 210\overline{\bar P_1^2P_7\overline{\bar P_1P_3}} + 54\overline{\bar P_1P_7\overline{\bar P_1^2P_3}} + 14\overline{\bar P_3^2P_7};\end{aligned} \tag{3.31}$$

$$\begin{aligned}\delta_4 =&40\bar P_1^4\bar P_3\bar P_7 + 20\bar P_1^4\overline{\bar P_3P_7} + 168\bar P_1^3\bar P_3\overline{\bar P_1P_7} + 64\bar P_1^3\bar P_7\overline{\bar P_1P_3} + 120\bar P_1^3\overline{\bar P_1\bar P_3P_7}\\ &+ 24\bar P_1^3\overline{\overline{\bar P_1P_3}P_7} + 378\bar P_1^2\bar P_3\overline{\bar P_1^2P_7} + 252\bar P_1^2\overline{\bar P_1P_3}\,\overline{\bar P_1P_7} + 378\bar P_1^2\overline{\bar P_1^2\bar P_3P_7}\\ &+ 72\bar P_1^2\bar P_7\overline{\bar P_1^2P_3} + 144\bar P_1^2\overline{\bar P_1P_7\overline{\bar P_1P_3}} + 18\bar P_1^2\overline{\overline{\bar P_1^2P_3}P_7} + 560\bar P_1\bar P_3\overline{\bar P_1^3P_7}\\ &+ 40\bar P_1\bar P_3^2\bar P_7 + 40\bar P_1\bar P_3\overline{\bar P_3P_7} + 64\bar P_1\bar P_7\overline{\bar P_1^3P_3} + 252\bar P_1\overline{\bar P_1^2P_3}\,\overline{\bar P_1P_7}\\ &+ 504\bar P_1\overline{\bar P_1P_3}\,\overline{\bar P_1^2P_7} + 8\bar P_1\overline{P_7\overline{\bar P_1^3P_3}} + 784\bar P_1\overline{\bar P_1^3\bar P_3P_7} + 420\bar P_1\overline{\bar P_1^2P_7\overline{\bar P_1P_3}}\\ &+ 108\bar P_1\overline{\bar P_1P_7\overline{\bar P_1^2P_3}} + 28\bar P_1\overline{\bar P_3^2P_7} + 84\bar P_3^2\overline{\bar P_1P_7} + 504\bar P_3\overline{\bar P_1^4P_7} + 64\bar P_3\bar P_7\overline{\bar P_1P_3}\\ &+ 120\bar P_3\overline{\bar P_1\bar P_3P_7} + 24\bar P_3\overline{\overline{\bar P_1P_3}P_7} + 16\bar P_7\overline{\bar P_1P_3\bar P_3} + 40\bar P_7\overline{\bar P_1^4P_3} + 4\bar P_7^2\\ &+ 168\overline{\bar P_1^3P_3}\,\overline{\bar P_1P_7} + 378\overline{\bar P_1^2P_3}\,\overline{\bar P_1^2P_7} + 560\overline{\bar P_1P_3}\,\overline{\bar P_1^3P_7} + 40\overline{\bar P_1P_3}\,\overline{\bar P_3P_7}\\ &+ 48\overline{\bar P_1P_7\overline{\bar P_1^3P_3}} + 252\overline{\bar P_1^2P_7\overline{\bar P_1^2P_3}} + 672\overline{\bar P_1^3P_7\overline{\bar P_1P_3}} + 48\overline{\bar P_3P_7\overline{\bar P_1P_3}}\\ &+ 1008\overline{\bar P_1^4\bar P_3P_7} + 120\overline{\bar P_1\bar P_3^2P_7};\end{aligned} \tag{3.32}$$

$$\begin{aligned}\delta_5 =& 54\bar P_1^5\bar P_3\bar P_7 + 24\bar P_1^5\overline{\bar P_3P_7} + 240\bar P_1^4\bar P_3\overline{\bar P_1P_7} + 90\bar P_1^4\bar P_7\overline{\bar P_1P_3} + 150\bar P_1^4\overline{\bar P_1\bar P_3P_7}\\ &+ 30\bar P_1^4\overline{\overline{\bar P_1P_3}P_7} + 588\bar P_1^3\bar P_3\overline{\bar P_1^2P_7} + 384\bar P_1^3\overline{\bar P_1P_3}\,\overline{\bar P_1P_7} + 504\bar P_1^3\overline{\bar P_1^2\bar P_3P_7}\\ &+ 108\bar P_1^3\bar P_7\overline{\bar P_1^2P_3} + 192\bar P_1^3\overline{\bar P_1P_7\overline{\bar P_1P_3}} + 24\bar P_1^3\overline{\overline{\bar P_1^2P_3}P_7} + 1008\bar P_1^2\bar P_3\overline{\bar P_1^3P_7}\\ &+ 81\bar P_1^2\bar P_3^2\bar P_7 + 72\bar P_1^2\bar P_3\overline{\bar P_3P_7} + 108\bar P_1^2\bar P_7\overline{\bar P_1^3P_3} + 432\bar P_1^2\overline{\bar P_1^2P_3}\,\overline{\bar P_1P_7}\\ &882\bar P_1^2\overline{\bar P_1P_3}\,\overline{\bar P_1^2P_7} + 12\bar P_1^2\overline{P_7\overline{\bar P_1^3P_3}} + 1176\bar P_1^2\overline{\bar P_1^3\bar P_3P_7} + 630\bar P_1^2\overline{\bar P_1^2P_7\overline{\bar P_1P_3}}\\ &+ 162\bar P_1^2\overline{\bar P_1P_7\overline{\bar P_1^2P_3}}+42\bar P_1^2\overline{\bar P_3^2P_7}+240\bar P_1\bar P_3^2\overline{\bar P_1P_7}+1260\bar P_1\bar P_3\overline{\bar P_1^4P_7}\\ &+ 180\bar P_1\bar P_3\bar P_7\overline{\bar P_1P_3} + 300\bar P_1\bar P_3\overline{\bar P_1\bar P_3P_7} + 60\bar P_1\bar P_3\overline{\overline{\bar P_1P_3}P_7} + 36\bar P_1\bar P_7\overline{\bar P_1P_3\bar P_3}\\ &+ 90\bar P_1\bar P_7\overline{\bar P_1^4P_3} + 9\bar P_1\bar P_7^2 + 384\bar P_1\overline{\bar P_1^3P_3}\,\overline{\bar P_1P_7} + 882\bar P_1\overline{\bar P_1^2P_3}\,\overline{\bar P_1^2P_7}\\ &+ 1344\bar P_1\overline{\bar P_1P_3}\,\overline{\bar P_1^3P_7} + 96\bar P_1\overline{\bar P_1P_3}\,\overline{\bar P_3P_7} + 96\bar P_1\overline{\bar P_1P_7\overline{\bar P_1^3P_3}} + 504\bar P_1\overline{\bar P_1^2P_7\overline{\bar P_1^2P_3}}\end{aligned}$$

$$
\begin{aligned}
&+1344\bar{P}_1\overline{\bar{P}_1^3P_7\overline{\bar{P}_1P_3}}+96\bar{P}_1\overline{\bar{P}_3P_7\overline{\bar{P}_1P_3}}+2016\bar{P}_1\overline{\bar{P}_1^4\bar{P}_3P_7}+240\bar{P}_1\overline{\bar{P}_1\bar{P}_3^2P_7}\\
&+54\overline{\bar{P}_1^5P_3}\bar{P}_7+240\overline{\bar{P}_1^4P_3}\,\overline{\bar{P}_1P_7}+588\overline{\bar{P}_1^3P_3}\,\overline{\bar{P}_1^2P_7}+1008\overline{\bar{P}_1^2P_3}\,\overline{\bar{P}_1^3P_7}\\
&+1260\overline{\bar{P}_1P_3}\,\overline{\bar{P}_1^4P_7}+1008\overline{\bar{P}_1^5P_7}\bar{P}_3+72\overline{\bar{P}_1^2P_3}\,\overline{\bar{P}_3P_7}+294\bar{P}_3^2\overline{\bar{P}_1^2P_7}\\
&+480\overline{\bar{P}_1\bar{P}_3P_3}\,\overline{\bar{P}_1P_7}+384\overline{\overline{\bar{P}_1P_3}P_3}\,\overline{\bar{P}_1P_7}+162\overline{\bar{P}_1^2P_3\bar{P}_3}\,\bar{P}_7+90\overline{\bar{P}_1P_3}^2\bar{P}_7\\
&+300\overline{\bar{P}_1P_3}\,\overline{\bar{P}_1P_7\bar{P}_3}+60\overline{\bar{P}_1P_3}\,\overline{\overline{\bar{P}_1P_3}P_7}+108\overline{\overline{\bar{P}_1^2P_3}P_3}\,\bar{P}_7+192\bar{P}_3\,\overline{\bar{P}_1P_7\overline{\bar{P}_1P_3}}\\
&+24\bar{P}_3\,\overline{P_7\overline{\bar{P}_1^2P_3}}+504\overline{\bar{P}_1^2P_7\bar{P}_3}\,\bar{P}_3+48\bar{P}_7\,\overline{\bar{P}_1P_7}+6\overline{\bar{P}_1\bar{P}_7P_7}-6\overline{\overline{\bar{P}_1^5P_3}P_7}\\
&+30\overline{\overline{\bar{P}_1^4P_3}P_7\bar{P}_1}+252\overline{\overline{\bar{P}_1^3P_3}P_7\bar{P}_1^2}+840\overline{\overline{\bar{P}_1^2P_3}P_7\bar{P}_1^3}+1764\overline{\overline{\bar{P}_1P_3}P_7\bar{P}_1^4}\\
&+2268\overline{\bar{P}_1^5\bar{P}_3P_7}+168\overline{\overline{\bar{P}_1^2P_3}P_7\bar{P}_3}+567\overline{\bar{P}_3^2\bar{P}_1^2P_7}-372\overline{\overline{\bar{P}_1P_3\bar{P}_3}P_7\bar{P}_1}\\
&+42\overline{\overline{\bar{P}_1P_3}^2P_7}+804\overline{\overline{\bar{P}_1P_3}P_7\bar{P}_1\,\bar{P}_3}-114\overline{\overline{\overline{\bar{P}_1^2P_3}P_3}P_7}\\
=&\ \frac{\lambda_3}{280}(1184850+310617\lambda_3)\overline{\bar{P}_1^8P_7}+\frac{\lambda_2}{35}(139058+75104\lambda_3)\overline{\bar{P}_1^7P_7}+\ldots.. \qquad (3.33)
\end{aligned}
$$

By (3.11) and (3.20) and Lemma2.3 we see that γ_i and α_j are the polynomials functions with respect to $\bar{P}_1$ and they are 2π-periodic. Thus, by (3.27) we see that if $a_{10+k}(2\pi)=0$, then $\beta_{3+k}(2\pi)+\delta_k(2\pi)=0$, $(k=0,1,...,5)$. By $a_{10}(2\pi)=0$ and (3.28) we get $\beta_3(2\pi)+\delta_0(2\pi)=0$, i.e.,

$$
56\int_0^{2\pi}\bar{P}_1^3P_7d\theta+4\int_0^{2\pi}\bar{P}_3P_7d\theta=4(14+\lambda_3)\int_0^{2\pi}\bar{P}_1^3P_7d\theta=0. \qquad (3.34)
$$

Case 1. If $(14+\lambda_3)(35+23\lambda_3)(330+703\lambda_3+65\lambda_3^2)\neq 0$. By (3.34) we have

$$
\int_0^{2\pi}\bar{P}_1^3P_7d\theta=0. \qquad (3.35)
$$

Using (3.26) and (3.35) and (3.29) we see that $a_{11}(2\pi)=0$. Using (3.30) we see that if $a_{12}(2\pi)=0$, then

$$
\begin{aligned}
\beta_5(2\pi)+\delta_2(2\pi)&=\int_0^{2\pi}(252\bar{P}_1^5P_7+48\bar{P}_1P_7\overline{\bar{P}_1P_3}+6P_7\overline{\bar{P}_1^2P_3}+126\bar{P}_1^2\bar{P}_3P_7)d\theta\\
&=\frac{36}{5}(35+23\lambda_3)\int_0^{2\pi}\bar{P}_1^5P_7d\theta=0, \qquad (3.36)
\end{aligned}
$$

by the hypothesis , it implies

$$
\int_0^{2\pi}\bar{P}_1^5P_7d\theta=0. \qquad (3.37)
$$

By (3.26) and (3.35) (3.37) we see that $a_{13}(2\pi)=0$. Applying (3.32) we see that if $a_{14}(2\pi)=0$, then

$$
\begin{aligned}
&\beta_7(2\pi)+\delta_4(2\pi)\\
=&\int_0^{2\pi}(792\bar{P}_1^7P_7+48\bar{P}_1P_7\overline{\bar{P}_1^3P_3}+252\bar{P}_1^2P_7\overline{\bar{P}_1^2P_3})d\theta\\
&+\int_0^{2\pi}(672\bar{P}_1^3P_7\overline{\bar{P}_1P_3}+48\bar{P}_3P_7\overline{\bar{P}_1P_3}+1008\bar{P}_1^4\bar{P}_3P_7+120\bar{P}_1\bar{P}_3^2P_7)d\theta
\end{aligned}
$$

$$=\frac{12}{5}(330+703\lambda_3+65\lambda_3^2)\int_0^{2\pi}\bar{P}_1^7P_7d\theta=0, \tag{3.38}$$

by the hypothesis, it implies

$$\int_0^{2\pi}\bar{P}_1^7P_7d\theta=0. \tag{3.39}$$

In summary, by (3.11) and (3.20) and (3.26), (3.35),(3.37) and (3.39) imply that the condition (3.2) is the necessary for the origin to be a center of equation (3.1).

Case 2. If $14+\lambda_3=0, p_{01}\neq 0$, then $\lambda_2=-3\lambda_3p_{01}\neq 0$ and (3.34) is an identity, by this we see that $a_{10}(\theta)$ is 2π-periodic and the coefficient function $2(4\overline{\bar{P}_3P_7}+56\overline{\bar{P}_1^3P_7})$ of $\bar{P}_1$ of $\beta_4(\theta)+\delta_1(\theta)$ is 2π-periodic. Thus, by $a_{11}(2\pi)=0$ implies that

$$\begin{aligned}\beta_4(2\pi)+\delta_1(2\pi)&=\int_0^{2\pi}(30\bar{P}_1\bar{P}_3P_7+6\overline{\bar{P}_1P_3}P_7)d\theta\\&=\frac{69}{2}\lambda_3\int_0^{2\pi}\bar{P}_1^4P_7d\theta+34\lambda_2\int_0^{2\pi}\bar{P}_1^3P_7d\theta+33\lambda_1\int_0^{2\pi}\bar{P}_1^2P_7d\theta\\&=34\lambda_2\int_0^{2\pi}\bar{P}_1^3P_7d\theta=0,\end{aligned}$$

so, $\int_0^{2\pi}\bar{P}_1^3P_7d\theta=0$, i.e., the identity (3.35) is valid. Similar to case 1, using (3.35) we can get that the identities (3.37) and (3.39) are valid. Therefore, the condition (3.2) is the necessary for the origin to be a center of equation (3.1).

Case 3. If $14+\lambda_3=0$, $p_{01}=0$, $b_3=0$, then $p_{10}\neq 0$, $P_1=p_{10}\cos\theta, \bar{P}_1=p_{10}\sin\theta$. By (3.26) we get $\int_0^{2\pi}\sin\theta P_7d\theta=0$ and

$$\int_0^{2\pi}\bar{P}_1^3P_7d\theta=\frac{p_{10}^3}{4}\int_0^{2\pi}(3\sin\theta-\sin 3\theta)P_7d\theta=0.$$

i.e., the identity (3.35) is valid. Similar to case 2, we know that the condition (3.2) is the necessary for the origin to be a center of equation (3.1).

Case 4. If $35+23\lambda_3=0, p_{01}\neq 0$, then $\lambda_2=-3\lambda_3p_{01}\neq 0$ and the identity (3.35) is valid, (3.36) is an identity, and $a_{11}(\theta)$ and $a_{12}(\theta)$ are 2π-periodic functions, by this we see that the coefficient function $2(252\overline{\bar{P}_1^5P_7}+48\bar{P}_1\overline{\bar{P}_7\overline{\bar{P}_1P_3}}+6\overline{P_7\overline{\bar{P}_1^2P_3}}+126\overline{\bar{P}_1^2\bar{P}_3P_7})$ of $\bar{P}_1$ in the formula $\beta_6(\theta)+\delta_3(\theta)$ is 2π-periodic and from $a_{13}(2\pi)=0$ follows that

$$\begin{aligned}&\beta_6(2\pi)+\delta_3(2\pi)\\&=\int_0^{2\pi}(4\overline{\bar{P}_1^3P_3}P_7+392\bar{P}_1^3\bar{P}_3P_7+210\bar{P}_1^2P_7\overline{\bar{P}_1P_3}+54\bar{P}_1P_7\overline{\bar{P}_1^2P_3}+14\bar{P}_3^2P_7)d\theta\\&=\lambda_2(28\lambda_3+\frac{2803}{5})\int_0^{2\pi}\bar{P}_1^5P_7d\theta=0,\end{aligned}$$

as $\lambda_3=-\frac{35}{23}$, $\lambda_2=-3\lambda_3p_{01}\neq 0$, from above follows that

$$\int_0^{2\pi}\bar{P}_1^5P_7d\theta=0,$$

i.e., the identity (3.37) is valid. Similar to case 1, we can get (3.39). Thus the condition (3.2) is the necessary for the origin to be a center of equation (3.1).

Case 5. If $35 + 23\lambda_3 = 0, p_{01} = 0, b_5 = 0$, then $p_{10} \neq 0$, $P_1 = p_{10}\cos\theta, \bar{P}_1 = p_{10}\sin\theta$. By (3.26) and (3.35) we get $\int_0^{2\pi} \sin\theta P_7 d\theta = 0$, $\int_0^{2\pi} \sin 3\theta P_7 d\theta = 0$, thus

$$\int_0^{2\pi} \bar{P}_1^5 P_7 d\theta = \frac{p_{10}^5}{16}\int_0^{2\pi} (10\sin\theta - 5\sin 3\theta + \sin 5\theta) P_7 d\theta = 0,$$

i.e., the identity (3.37) is valid. Similar to case 4, we get (3.39). Thus the condition (3.2) is the necessary for the origin to be a center of equation (3.1).

Case 6. If $330 + 703\lambda_3 + 65\lambda_3^2 = 0, p_{01} \neq 0$, then $\lambda_2 = -3\lambda_3 p_{01} \neq 0$, $(14 + \lambda_3)(35+23\lambda_3) \neq 0$ and $139058 + 75104\lambda_3 \neq 0$, (3.35) and (3.37) are valid and (3.38) is an identity, $a_{14}(\theta)$ is 2π-periodic function, by this we see that the coefficient function $2(792\overline{\bar{P}_1^7 P_7} + 48\overline{\bar{P}_1 P_7 \overline{\bar{P}_1^3 P_3}} + 252\overline{\bar{P}_1^2 P_7 \overline{\bar{P}_1^2 P_3}} + 672\overline{\bar{P}_1^3 P_7 \overline{\bar{P}_1 P_3}} + 48\overline{\bar{P}_3 P_7 \overline{\bar{P}_1 P_3}} + 1008\overline{\bar{P}_1^4 \bar{P}_3 P_7} + 120\overline{\bar{P}_1 \bar{P}_3^2 P_7})$ of $\bar{P}_1$ in the formula $\beta_8(\theta) + \delta_5(\theta)$ is 2π-periodic, by this from $a_{15}(2\pi) = 0$ follows that

$$\begin{aligned}
&\beta_8(2\pi) + \delta_5(2\pi) \\
=&\int_0^{2\pi} (6\bar{P}_1 \bar{P}_7 P_7 - 6\overline{\bar{P}_1^5 P_3} P_7 + 30\overline{\bar{P}_1^4 P_3} P_7 \bar{P}_1 + 252\overline{\bar{P}_1^3 P_3} P_7 \bar{P}_1^2 \\
&+ 840\overline{\bar{P}_1^2 P_3} P_7 \bar{P}_1^3 + 1764\overline{\bar{P}_1 P_3} P_7 \bar{P}_1^4 + 2268\bar{P}_1^5 \bar{P}_3 P_7 + 168\overline{\bar{P}_1^2 P_3} P_7 \bar{P}_3 + 567\bar{P}_3^2 \bar{P}_1^2 P_7 \\
&- 372\overline{\bar{P}_1 P_3 \bar{P}_3} P_7 \bar{P}_1 + 42\overline{\bar{P}_1 P_3}^2 P_7 + 804\overline{\bar{P}_1 P_3} P_7 \bar{P}_1\, \bar{P}_3 - 114\overline{\overline{\bar{P}_1^2 P_3} P_3} P_7) d\theta \\
=&\lambda_2(139058 + 75104\lambda_3)\int_0^{2\pi} \bar{P}_1^7 P_7 d\theta = 0,
\end{aligned}$$

thus the identity (3.39) is valid . Therefore, the condition (3.2) is the necessary for the origin to be a center of equation (3.1).

Case 7. If $330 + 703\lambda_3 + 65\lambda_3^2 = 0, p_{01} = 0, b_7 = 0$, then $p_{10} \neq 0$, $P_1 = p_{10}\cos\theta, \bar{P}_1 = p_{10}\sin\theta$. By (3.26) and (3.35) and (3.37) we get $\int_0^{2\pi} \sin\theta P_7 d\theta = 0$, $\int_0^{2\pi} \sin 3\theta P_7 d\theta = 0$, $\int_0^{2\pi} \sin 5\theta P_7 d\theta = 0$, thus

$$\int_0^{2\pi} \bar{P}_1^7 P_7 d\theta = \frac{p_{10}^7}{64}\int_0^{2\pi} (35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta) P_7 d\theta = 0,$$

i.e., the identity (3.39) is valid. Thus the condition (3.2) is the necessary for the origin to be a center of equation (3.1).

Sufficiency: Now, we show that the condition (3.2) is also sufficient for $r = 0$ to be a center.

By Lemma2.3, using (3.2) we have

$$P_3 = P_1(\lambda_1 + 2\lambda_2 \bar{P}_1 + 3\lambda_3 \bar{P}_1^2), P_7 = P_1 \sum_{k=1}^{7} k\mu_k \bar{P}_1^{k-1},$$

where λ_i, μ_j are real numbers. As P_1, $\bar{P}_1$ are 2π-periodic functions , by Lemma2.2, $r = 0$ is a center and composition center of (3.1).

In summary, the present theorem has been proved. □

Obviously, by this theorem implies the following result.

Corollary 3.1. *Suppose that $\lambda_3 \geq 0$. Then $r = 0$ is a center of* (3.1)*, if and only if*

$$\int_0^{2\pi} \bar{P}_1^{2i+1} P_3 d\theta = 0, (i = 0, 1), \int_0^{2\pi} \bar{P}_1^{2j+1} P_7 d\theta = 0, \ (j = 0, 1, 2, 3).$$

Moreover, this center is a composition center. (3.3) –(3.8) *are all the focus values of system* (1.2).

Remark 3.1. By Theorem3.1, the focal values of system (1.2) is a constant multiple of six definite integrals (3.2) and the highest order of the fine focus is seven.

Theorem 3.2. *For equation* (3.1)*, if one of the following conditions:*

1. $14 + \lambda_3 = 0$, $p_{01} = 0, b_3 \neq 0$;
2. $35 + 23\lambda_3 = 0$, $p_{01} = 0$, $b_5 \neq 0$;
3. $330 + 703\lambda_3 + 65\lambda_3^2 = 0$, $p_{01} = 0$, $b_7 \neq 0$,

is satisfied, then the origin point of (3.1) *can't be a composition center, where* λ_3, $b_i (i = 3, 5, 7)$ *are the same as they are in Theorem 3.1.*

Proof. Now we only prove that if the first condition of the present theorem is satisfied, then the origin point of (3.1) can't be a composition center. Conversely, assuming that $14 + \lambda_3 = 0$, $p_{01} = 0, b_3 \neq 0$ and $r = 0$ is a composition center, then $p_{10} \neq 0$ and $P_1 = p_{10} \cos\theta$, $\bar{P}_1 = p_{10} \sin\theta$,

$$\int_0^{2\pi} \bar{P}_1 P_3 d\theta = 0, \int_0^{2\pi} \bar{P}_1^3 P_3 d\theta = 0,$$

by Lemma 2.3

$$P_3 = P_1 (\frac{p_{30}}{p_{10}} - \frac{p_{30} - p_{12}}{p_{10}^3} \bar{P}_1^2),$$

i.e., P_1 and P_3 satisfy the composition conditions (2.1) with $u = \bar{P}_1$ (if taking $u = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}$, then it is not a 2π-periodic function and does not meet the requirements of the Lemma2.2). As $r = 0$ is a composition center of (3.1),

$$P_7 = P_1 \psi(\bar{P}_1) = p_{10} \cos\theta \psi(p_{10} \sin\theta)$$

and

$$b_3 = \frac{1}{\pi} \int_0^{2\pi} P_7 \sin 3\theta d\theta = \frac{1}{\pi} \int_0^{2\pi} P_{10} \cos\theta \psi(p_{10} \sin\theta)(3 \sin\theta - 4 \sin^3\theta) d\theta = 0,$$

it is contradict with $b_3 \neq 0$. Therefore, if the first condition of the present theorem is satisfied , then $r = 0$ can't be a composition center of (3.1). Similarly, in the other cases, the present conclusions are valid. $\square$

Example 3.1. The system

$$\begin{cases} x' = -y + x(x+y)(d_1 + d_1(x^2 - 4xy + y^2) + d_2 x^6 + (-d_2 + d_3) x^5 y \\ \qquad + (d_2 - d_3 + d_4) x^4 y^2 + (-d_2 + d_3 - d_4 + d_5) x^3 y^3 \\ \qquad + (d_2 - d_3 + d_4) x^2 y^4 + (-d_2 + d_3) x y^5 + d_2 y^6), \\ y' = x + y(x+y)(d_1 + d_1(x^2 - 4xy + y^2) + d_2 x^6 + (-d_2 + d_3) x^5 y \\ \qquad + (d_2 - d_3 + d_4) x^4 y^2 + (-d_2 + d_3 - d_4 + d_5) x^3 y^3 \\ \qquad + (d_2 - d_3 + d_4) x^2 y^4 + (-d_2 + d_3) x y^5 + d_2 y^6) \end{cases}$$

has a composition center at $(0,0)$, where, $d_1 \neq 0$, $d_i\,(i=1,2...,5)$ are arbitrary numbers. In this example $\lambda_3 = \frac{2}{3d_1^2} > 0$ and the conditions of Theorem3.1 are satisfied.

References

[1] A. Algaba and M. Reyes, *Computing center conditions for vector fields with constant angular speed*, J. Comput. Appl. Math., 2003, 154(1), 143–159.

[2] A. Algaba, M. Reyes and A. Bravo, *Uniformly isochronous quintic planar vector fields*, Proceedings of the Equadiff'99, Berlin, 1999.

[3] M. A. M. Alwash, *Periodic solutions of Abel differential equations*, J. Math. Anal. Appl., 2007, 329, 1161–1169.

[4] M. A. M. Alwash,*Composition conditions for two-dimensional polynomial systems*, Differ. Equ. Appl., 2013, 5(1), 1–12.

[5] M. A. M. Alwash, *On the center conditions of certain cubic systems*, Proc. Amer. Math. Soc., 1998, 126, 3335–3336.

[6] M. A. M. Alwash, *The composition conjecture for Abel equation*, Expos. Math., 2009, 27, 241–250.

[7] M. A. M. Alwash and N. G. Lloyd, *Non-autonomous equations related to polynomial two-dimensional systems*, Proc. Roy. Soc. Edinburgh, 1987, 105, 129–152.

[8] V. V. Amel'kin and A. E. Rudenok, *Centers and isochronous centers of Liénard systems*, Differ. Uravn., 2019, 55(3), 294–303.

[9] J. C. Artés, J. Itikawab and J. Llibre, *Uniform isochronous cubic and quartic centers*, J. Comput. Appl. Math., 2017, 313, 448–453.

[10] A. Cima, A. Gasull and F. Manosas, *Centers for trigonometric Abel equations*, Qual. Theory Dyn. Syst. , 2012, 11, 19–37.

[11] A. Cima, A. Gasull and F. Mañosas, *A simple solution of some composition conjectures for Abel equations*, J. Math. Anal. Appl., 2013, 398, 477–486.

[12] R. Conti, *Uniformly isochronous centers of polynomial systems in R^2*, in: Lecture Notes in Pure and Appl. Math., 1994, 152, 21–31.

[13] J. Devlin, N. G. Lloyd and J. M. Pearson, *Cubic systems and Abel equations*, J. Differ. Equ., 1998, 147, 435–454;

[14] A. Gasull, R. Prohens and J. Torregrosa, *Limit cycles of rigid cubic systems*, J. Math. Anal. Appl., 2005, 304, 391–404.

[15] A. Gasull and J. Torregrosa, *Exact number of limit cycles for a family of rigid systems*, Proc. Amer. Math. Soc., 2005, 133, 751–758.

[16] J. Llibre, R. Ramírez, V. Ramírez and N. Sadovskaia, *Centers and uniform isochronous centers of planar polynomial differential systems*, J. Dyn. Diff. Equ., 2018, 30, 1295–1310.

[17] N. G. Lloyd and J. M. Pearson, *Computing centre conditions for certain cubic systems*, J. Comput. Appl. Math., 1992, 40, 323–336.

[18] V. I. Mironenko and V. V. Mironenko, *Perturbations of linear differential systems that do not change the Poincaré map*, Probl. Fiz. Mat. Tekh., 2017, 30(1), 54–58.

[19] A. P. Sadovskii, *Center-focus problem for analytic systems with nonzero linear part*, Differ. Uravn., 2017, 53(10), 1413–1417.

[20] J. Xu and Z. Lu, *Sufficient and necessary center conditions for the Poincaré systems $P(2, n)(n \leq 5)$* , J. Appl. Math., 2011, 762762, 18.

[21] Z. Zhou, *The Poincaré center-focus problem for a class of higher order polynomial differential systems*, Math. Meth. Appl. Sci., 2019, DOI:10.1002/mma.5475, 1–15.

[22] Z. Zhou, *On the center-focus problem for a family of high-order polynomial differential systems*, J. Dyn. Differ. Equ., 2020, 32, 393–418.

[23] Z. Zhou and V. G. Romanovski, *The center problem and the composition condition for a family of quartic differential systems*, Electron. J. Qual. Theory. Differ. Equ., 2018, 15, 1–17.