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# FURTHER DISCUSSION ON KATO'S CHAOS IN SET-VALUED DISCRETE SYSTEMS*

Risong Li[1,4], Tianxiu Lu[2,†], Guanrong Chen[3] and Xiaofang Yang[2]

**Abstract** For a compact metric space $Y$ and a continuous map $g : Y \to Y$, the collective accessibility and collectively Kato chaotic of the dynamical system $(Y, g)$ were defined. The relations between topologically weakly mixing and collective accessibility, or strong accessibility, or strongly Kato chaos were studied. Some common properties of $g$ and $\overline{g}$ were given. Where $\overline{g} : \kappa(Y) \to \kappa(Y)$ is defined as $\overline{g}(B) = g(B)$ for any $B \in \kappa(Y)$, and $\kappa(Y)$ is the collection of all nonempty compact subsets of $Y$. Moreover, it is proved that $g$ is collectively accessible (or strongly accessible) if and only if $\overline{g}$ in $w^e$-topology is collectively accessible (or strongly accessible).

**Keywords** Kato's chaos, collective accessibility, strongly accessible.

**MSC(2010)** 37B40, 37D45, 54H20.

## 1. Introduction

Since Li and Yorke [21] gave the first definition of chaos in 1975, the description of chaos was highly explored. For example, Devaney chaos [8], Auslander-Yorke chaos [1], dense chaos [25], distributional chaos [24], distributional chaos in a sequence [29], $\mathcal{F}$-chaos [27], shadowing properties [35], sensitivity and transitivity [17] and others (see [5,6,9,19,20,23,31–34,37], for example). Topological dynamical systems have been intensively discussed because they model many phenomena from various disciplines of science.

In [14], H. Kato introduced a kind of chaos, named Kato's chaos or everywhere chaos. And an equivalent characterization of Kato's chaos for a continuous map on

†The corresponding author. Email address: lubeeltx@163.com (T. Lu)
[1]School of Mathematics and Computer Science, Guangdong Ocean University, Zhanjiang, 524025, China
[2]School of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, 643000, China
[3]Department of Electrical Engineering, City University of Hong Kong, Hong Kong SAR, 999077, China
[4]Artificial Intelligence Key Laboratory of Sichuan Province, Bridge Nondestruction Detecting and Engineering Computing Key Laboratory of Sichuan Province, Key Laboratory of Higher Education of Sichuan Province for Enterprise Informationalization and Internet of Things, Zigong, 643000, China
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a compact metric space was given by a way similar to Li-Yorke's chaos. Also, it is claimed that any topologically mixing map is Kato chaotic. In [10], some relations between Kato's chaoticity of a dynamical system $(Y, g)$ and that of the set-valued discrete system induced by $(Y, g)$ were studied. It is showed that, if the set-valued discrete system is Kato chaotic in the Vietoris topology, then so is the system $(Y, g)$. And the system $(Y, g)$ is Kato chaotic if and only if so is the set-valued discrete system in $w^e$-topology. Moreover, for a continuous map which has a fixed point on a complete metric space without isolated point, if it is Ruelle-Takens chaotic, then it is Kato chaotic. But the converse does not hold in general. Consequently, one can see that Kato's chaos is strictly weaker than Ruelle-Takens chaos.

In [18], we explored Kato's chaos, sensitivity and accessibility of a given Cournot map and presented a sufficient condition and a necessary condition for a Cournot map to be Kato chaotic. Also, the accessibility of the product map $g_1 \times g_2$ was posed. Where $g_1$ and $g_2$ are accessible continuous maps on the metric spaces $Y_1$ and $Y_2$, respectively. In [33], X. Wu and J. Wang gave some characteristics of accessibility or Kato's chaos, and constructed a dynamical system $(Y, g)$ which is accessible but its product system $(Y \times Y, g \times g)$ is not accessible.

The current work will further discuss Kato's chaos for continuous maps on compact metric spaces. The relations between topologically weakly mixing and collectively accessible, or strongly accessible, or strongly Kato chaotic were studied. Some necessary and sufficient conditions of $g$ is collectively accessible (or strongly accessible) were obtained. The results in this paper improve the corresponding ones in [10] or others.

## 2. Preliminaries

A pair $(u, v) \in Y \times Y$ is called a Li-Yorke pair of a system $(Y, g)$ (or the map $g : Y \to Y$) on metric space $(Y, d)$ if

$$\limsup_{k\to\infty} d(g^k(u), g^k(v)) > 0$$

and

$$\liminf_{k\to\infty} d(g^k(u), g^k(v)) = 0.$$

A subset $A \subset Y$ having at least two points is a LY-scrambled set for the system $(Y, g)$ (or the map $g : Y \to Y$) if any $(u, v) \in A \times A : u \neq v$ is a Li-Yorke pair of the system $(Y, g)$ (or the map $g : Y \to Y$). A system $(Y, g)$ (or the map $g : Y \to Y$) is Li-Yorke chaotic if it has an uncountable LY-scrambled set.

Assume that $t \geq 2$ is an integer, and that $\zeta$ is the product metric on the product space $Y^{(t)} = \underbrace{Y \times Y \times \cdots \times Y}_{t}$ defined by

$$\zeta((u_1, u_2, \cdots, u_t), (v_1, v_2, \cdots, v_t)) = \max_{i=1}^{t}\{d(u_i, v_i)\}$$

for any $(u_1, u_2, \cdots, u_t), (v_1, v_2, \cdots, v_t) \in Y^{(t)}$.

Let $(Y, d)$ be a metric space. A dynamic system $(Y, g)$ (or the map $g : Y \to Y$) is transitive if for any nonempty open subsets $A_1, A_2 \subset Y$, $g^k(A_1) \cap A_2 \neq \emptyset$ for some integer $k > 0$. A dynamic system $(Y, g)$ (or the map $g : Y \to Y$) is topologically mixing if for any nonempty open subsets $A_1, A_2 \subset Y$, $g^p(A_1) \cap A_2 \neq \emptyset$ for some

integer $k > 0$ and any integer $p > k$. A dynamic system $(Y, g)$ (or the map $g : Y \to Y$) is sensitive if there is a $N > 0$ such that for any given $\varepsilon > 0$ and any given $u \in Y$, there exists a point $v \in Y : d(u, v) < \varepsilon$ such that $d(g^k(u), g^k(v)) > N$ for some integer $k > 0$, where $N$ is called a sensitivity constant of $g$. A dynamic system $(Y, g)$ (or the map $g : Y \to Y$) is accessible if for any $\varepsilon > 0$ and any two nonempty open subsets $U_1, U_2 \subset Y$, there are two points $u \in U_1$ and $v \in U_2$ such that $d(g^k(u), g^k(v)) < \varepsilon$ for some integer $k > 0$. A dynamic system $(Y, g)$ (or the map $g : Y \to Y$) is chaotic in the sense of Ruelle and Takens [10] if it is transitive and sensitive. A dynamic system $(Y, g)$ (or the map $g : Y \to Y$) is Kato chaotic if it is sensitive and accessible.

Let $\kappa(Y)$ be the collection of all nonempty compact subsets of $Y$. The Hausdorff metric $d_H$ on the space $\kappa(Y)$ is defined as

$$d_H(E, F) = \max\{\varrho(E, F), \varrho(F, E)\}$$

for any $E, F \in \kappa(Y)$, where $\varrho(E, F) = \inf\{\lambda > 0 \mid d(y, E) < \lambda, y \in F\}$. For any compact metric space $(Y, d)$, the topology on $\kappa(Y)$ which is induced by $d_H$ is the same as the Vietoris generated by a basis consisting of all sets of the form,

$$\{W_1, W_2, \cdots, W_m\} = \left\{ B \in \kappa(Y) \mid B \subset \bigcup_{1 \le i \le m} W_i, B \cap W_i \neq \emptyset, 1 \le i \le m \right\},$$

where $W_i$ is a nonempty open subset of $Y$ for any $i \in \{1, 2, \cdots, m\}$. It is known that this topology is admissible in the sense that the map $id : Y \to \kappa(Y)$ which is defined by $id(y) = \{y\}$ for any $y \in Y$ is continuous. And $\kappa(Y)$ is compact if and only if $Y$ is compact. Let $\mathcal{F}(Y)$ be the set of all finite subsets of $Y$. Under this topology, $\mathcal{F}(Y)$ is dense in $\kappa(Y)$ (see [2, 22, 26]). For any continuous self-map $g : Y \to Y$, a continuous map $\overline{g} : \kappa(Y) \to \kappa(Y)$ is defined as $\overline{g}(B) = g(B)$ for any $B \in \kappa(Y)$. If a point $y \in Y$ is identified as a subset $\{y\}$ of $Y$, the system $(Y, g)$ is a subsystem of its induced system $(\kappa(Y), \overline{g})$ (see [3, 10–13, 16, 26, 36]).

Now, some strong forms of accessible, sensitive, or Kato's chaos were given.

**Definition 2.1.** Let $(Y, g)$ be a dynamical system on a metric space $(Y, d)$. The system $(Y, g)$ (or the map $g$) is said to be collectively accessible, if for any $\varepsilon > 0$ and any nonempty open subsets $S_1^{(1)}, S_2^{(1)}, \cdots, S_s^{(1)}, S_1^{(2)}, S_2^{(2)}, \cdots, S_t^{(2)} \subset Y$, there exist $y_i^{(1)} \in S_i^{(1)}$ for any $i \in \{1, 2, \cdots, s\}$ and $y_j^{(2)} \in S_j^{(2)}$ for any $j \in \{1, 2, \cdots, t\}$ such that one of the following holds:

(i) there is an $i_0 \in \{1, 2, \cdots, s\}$ such that $d(g^m(y_{i_0}^{(1)}), g^m(y_j^{(2)})) < \varepsilon$ for any $j \in \{1, 2, \cdots, t\}$ and some integer $m > 0$.

(ii) there is a $j_0 \in \{1, 2, \cdots, t\}$ such that $d(g^m(y_i^{(1)}), g^m(y_{j_0}^{(2)})) < \varepsilon$ for any $i \in \{1, 2, \cdots, s\}$ and some integer $m > 0$.

Clearly, collective accessibility implies accessibility.

**Definition 2.2.** ( [23]) Let $(Y, g)$ be a dynamical system on a metric space $(Y, d)$ and $\lambda > 0$ a constant. The system $(Y, g)$ (or the map $g$) is said to be collectively sensitive with the collective sensitivity constant $\lambda$ if for any finitely many distinct points $a_1, a_2, \cdots, a_m \in Y$ and any $\varepsilon > 0$, there exist $m$ distinct points $b_1, b_2, \cdots, b_m$ in $Y$ such that the following two conditions are satisfied:

(i) $d(a_j, b_j) < \varepsilon$ for all $1 \le j \le m$;

(ii) there exist $i_0$ and $j_0$ satisfying $1 \le i_0, j_0 \le m$ such that $d(g^k(a_j), g^k(b_{j_0})) > \lambda (1 \le j \le m)$ or $d(g^k(a_{i_0}), g^k(b_j)) > \lambda (1 \le j \le m)$ for some integer $k > 0$.

Obviously, collective sensitivity implies sensitivity.

**Definition 2.3.** Let $(Y, g)$ be a dynamical system on a metric space $(Y, d)$. The system $(Y, g)$ (or the map $g$) is said to be collectively Kato chaotic if it is collectively accessible and collectively sensitive.

It is clear that collective Kato's chaos implies Kato's chaos.

**Definition 2.4.** ( [30]) Let $(Y, g)$ be a dynamical system on a metric space $(Y, d)$. The system $(Y, g)$ (or the map $g$) is said to be strongly accessible, if for any $\varepsilon > 0$ and any nonempty open subsets $S_1, S_2, \cdots, S_t \subset Y$, there exist $y_i \in S_i$ for any $i \in \{1, 2, \cdots, t\}$ such that $d(g^m(y_i), g^m(y_j)) < \varepsilon$ for any $j \in \{1, 2, \cdots, t\}$ and some integer $m > 0$.

**Definition 2.5.** ( [30]) Let $(Y, g)$ be a dynamical system on a metric space $(Y, d)$ and $N \ge 2$ be an integer. Then $\lambda > 0$ is a $N$-sensitive coefficient of the dynamical system $(Y, g)$ (or the map $g$), if there is an integer $m > 0$ such that for any nonempty open set $V \subset Y$, there exist $N$ points $v_1, v_2, \cdots, v_N \in V$ satisfying

$$\min\{d(g^m(v_i), g^m(v_j)) \mid i, j \in \{1, 2, \cdots, N\}, i \ne j\} \ge \lambda.$$

The supremum of all $N$-sensitive coefficients of the system $(Y, g)$ is denoted by $\lambda_N$ which is called the $N$-critically sensitive coefficient of the system $(Y, g)$. For any $v_1, v_2, \cdots, v_N \in Y$, denote

$$r(v_1, v_2, \cdots, v_N) = \min\{d(v_i, v_j) \mid i, j \in \{1, 2, \cdots, N\}, i \ne j\}$$

and put

$$r_N = \sup_{v_1, v_2, \cdots, v_N \in Y} \{r(v_1, v_2, \cdots, v_N)\}.$$

Obviously, $\lambda_N \le r_N$, and both of them monotonically decrease to 0.

The dynamical system $(Y, g)$ (or the map $g$) is said to be $N$-maximum sensitive if $\lambda_N = r_N$ and is said to be totally maximum sensitive (TMS, for short) if for every integer $N > 0$, $\lambda_N = r_N$.

**Definition 2.6.** ( [30]) Let $(Y, g)$ be a dynamical system on a metric space $(Y, d)$. The system $(Y, g)$ (or the map $g$) is said to be chaotic in the strong sense of Kato, if it is both TMS and strongly accessible.

**Definition 2.7.** Let $(Y, g)$ be a dynamical system on a metric space $(Y, d)$. The system $(Y, g)$ (or the map $g$) is topologically transitive if for any nonempty open sets $U, V \subset Y$, there is an integer $m > 0$ such that $g^m(U) \cap V \ne \phi$. The system $(Y, g)$ (or the map $g$) is topologically weakly mixing if $g \times g$ is topologically transitive.

In this paper, the $w^e$-topology on $\kappa(Y)$ is just the topology which is generated by the sets $e(B) = \{C \in \kappa(Y) \mid C \subset B\}$, where $B \subset Y$ is an open set.

## 3. Kato chaoticity

For giving the proofs of main results, the following lemmas are needed.

**Lemma 3.1** ( [16]). *Let $g : Y \to Y$ be a continuous map on a compact metric space $(Y, d)$. Then $g$ is topologically weakly mixing if and only if $g^{(t)}$ is transitive for any integer $t \geq 2$, where $g^{(t)} = \underbrace{g \times g \times \cdots \times g}_{t}$.*

**Lemma 3.2** ( [4, 15, 16]). *Let $g$ be a continuous self-map on the compact interval $[0, 1]$. If $g$ is topologically transitive, then one of the following statements is hold.*

*(i) The map $g$ is topologically mixing.*

*(ii) There exists a fixed point $a$ of $g$ in $(0, 1)$ such that $g^2|_{[0,a]}$ and $g^2|_{[a,1]}$ are all topologically mixing.*

**Lemma 3.3** ( [28]). *Let $g$ be a continuous self-map on the compact interval $[0, 1]$. Then $g$ is topologically transitive if and only if $g$ is Devaney chaotic.*

**Lemma 3.4** ( [7]). *Let $g_i$ be continuous self-maps on a metric space $Y_i$ for any $i \in \{1, 2\}$. Then $g_1 \times g_2$ is sensitive if and only if so is at least one of $g_1$ and $g_2$.*

**Lemma 3.5.** *Let $g$ be a topologically mixing map on a metric space $(Y, d)$. Then, for any $\varepsilon > 0$ and any nonempty open sets $A, B \subset Y$, there exists an integer $l > 0$ such that, for any $t \geq l$, $d(g^t(a_t), g^t(b_t)) < \varepsilon$ for some $a_t \in A$ and $b_t \in B$.*

**Proof.** Choose $c \in Y$. Since $g$ is topologically mixing, then $g \times g$ is topologically mixing. Therefore, for any $\varepsilon > 0$ and any nonempty open sets $A, B \subset Y$, there exists an integer $l > 0$ such that, for any $t \geq l$,

$$d(g^t(a_t), g^t(c)) < \frac{1}{2}\varepsilon \quad \text{and} \quad d(g^t(c), g^t(b_t)) < \frac{1}{2}\varepsilon$$

for some $a_t \in A$ and $b_t \in B$. Consequently,

$$d(g^t(a_t), g^t(b_t)) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

□

**Lemma 3.6.** *Let $g_i (i \in \{1, 2\})$ be continuous self-maps on the compact interval $[0, 1]$. If $g_i$ is topologically transitive for any $i \in \{1, 2\}$, then, for any $\varepsilon > 0$ and any nonempty open sets $A_i, B_i \subset [0, 1](i \in \{1, 2\})$, there exists an integer $l > 0$ such that, for any $t \geq l$, $d(g_1^{2t}(a_2), g_1^{2t}(a_1)) < \varepsilon$ and $d(g_2^{2t}(b_2), g_2^{2t}(b_1)) < \varepsilon$ for some $(a_1, b_1) \in A_1 \times B_1$ and $(a_2, b_2) \in A_2 \times B_2$.*

**Proof.** If $g_i$ is topologically mixing for any $i \in \{1, 2\}$, then, by Lemma 3.5 and its proof, for any $\varepsilon > 0$ and any nonempty open sets $A_i, B_i \subset [0, 1](i \in \{1, 2\})$, there exists an integer $l > 0$ such that, for any $t \geq l$, $d(g_1^{2t}(a_2), g_1^{2t}(a_1)) < \varepsilon$ and $d(g_2^{2t}(b_2), g_2^{2t}(b_1)) < \varepsilon$ for some $(a_1, b_1) \in A_1 \times B_1$ and $(a_2, b_2) \in A_2 \times B_2$.

If $g_1$ is topologically mixing but $g_2$ is not topologically mixing, then, by hypothesis and Lemma 3.2, there is a fixed point $c \in (0, 1)$ of $g_2$ satisfying that $g_2^2|_{[0,c]}$ and $g_2^2|_{[c,1]}$ are all topologically mixing. By Lemma 3.5 and the above argument, for any $\varepsilon > 0$ and any nonempty open sets $A_i, B_i \subset [0, 1](i \in \{1, 2\})$, there is an integer $l > 0$ such that, for any $t \geq l$, $d(g_1^{2t}(a_2), g_1^{2t}(a_1)) < \varepsilon$ and $d(g_2^{2t}(b_2), g_2^{2t}(b_1)) < \varepsilon$ for some $(a_1, b_1) \in A_1 \times B_1$ and $(a_2, b_2) \in A_2 \times B_2$.

Similarly, one can assume that $g_1$ and $g_2$ are all not topologically mixing. Then, for any $\varepsilon > 0$ and any nonempty open sets $A_i, B_i \subset [0,1] (i \in \{1,2\})$, there is an integer $l > 0$ such that, for any $t \geq l$, $d(g_1^{2t}(a_2), g_1^{2t}(a_1)) < \varepsilon$ and $d(g_2^{2t}(b_2), g_2^{2t}(b_1)) < \varepsilon$ for some $(a_1, b_1) \in A_1 \times B_1$ and $(a_2, b_2) \in A_2 \times B_2$. □

In [14], it is pointed out that a topologically mixing map $g$ on a compact metric space $Y$ is Kato chaotic. In [33], it is proved that any topologically transitive continuous interval map is accessible. While, the following Theorem 3.1 and Theorem 3.2 will improve and extend these results.

**Theorem 3.1.** *Let $g$ be a continuous self-map on a compact metric space $(Y, d)$. If $g$ is topologically weakly mixing, then $g^{(t)}$ is Kato chaotic for any integer $t > 0$.*

**Proof.** Firstly, the following shows that if a map is topologically weakly mixing, then it is Kato chaotic.

Let $\varepsilon > 0$ and choose $b \in Y$. Write

$$V = B(b, \frac{1}{2}\varepsilon) = \{a \in Y \mid d(a, b) < \frac{1}{2}\varepsilon\}.$$

Since $g$ is topologically weakly mixing, by the definition, for any nonempty open sets $U_1, U_2 \subset Y$, there exists an integer $l \geq 1$ satisfying $(g \times g)^l (U_1 \times U_2) \cap (V \times V) \neq \emptyset$. That is, there exist $a \in U_1$, $b \in U_2$ such that $g^l(a), g^l(b) \in V$. So, $d(g^l(a), g^l(b)) \leq d(g^l(a), b) + d(b, g^l(b)) < \varepsilon$. This implies that $g$ is accessible.

On the other hand, $g$ is topologically weakly mixing, so it is sensitive. Thus, by the definition, $g$ is Kato chaotic.

Moreover, by Lemma 2.1, $g$ is topologically weakly mixing if and only if so is $g^{(t)}$ for any integer $t \geq 2$. Thus, $g^{(t)}$ is Kato chaotic for any integer $t \geq 0$. □

**Theorem 3.2.** *Let $t$ be a given positive integer and $g_i (i \in \{1, 2, \cdots, t\})$ be continuous self-maps on the compact interval $[0,1]$. If $g_i$ is topologically transitive for any $i \in \{1, 2, \cdots, t\}$, then $g_1 \times g_2 \times \cdots \times g_t$ is Kato chaotic.*

**Proof.** By hypothesis and Lemma 3.3, $g_i$ is sensitive for any $i \in \{1, 2, \cdots, t\}$. By Lemma 3.4, $g_1 \times g_2 \times \cdots \times g_t$ is sensitive.

By Lemma 3.2 and Lemma 3.4, for any $\varepsilon > 0$ and any nonempty open sets $A_i, B_i \subset [0,1] (i \in \{1, 2, \cdots, t\})$, there is an integer $l > 0$ such that, for any $m \geq l$, there are $(a_{i,m}, b_{i,m}) \in A_i \times B_i$ for any $i \in \{1, 2, \cdots, t\}$ satisfying

$$d(g^{2m}(a_{i,m}), g^{2m}(a_{j,m})) < \varepsilon \quad \textit{and} \quad d(g^{2m}(b_{i,m}), g^{2m}(b_{j,m})) < \varepsilon$$

for any $i, j \in \{1, 2, \cdots, t\} : i \neq j$. So, $g_1 \times g_2 \times \cdots \times g_t$ is accessible. Consequently, $g_1 \times g_2 \times \cdots \times g_t$ is Kato chaotic. □

**Corollary 3.1.** Let $g$ be a continuous map on the compact interval $[0,1]$. If $g$ is topologically transitive, then $g^{(t)}$ is Kato chaotic for any integer $t > 0$.

**Proof.** By Theorem 3.2, Corollary 3.1 is true. □

**Remark 3.1.** Is there a continuous interval map Kato chaotic but not topologically transitive?

## 4. Strong forms of Kato chaoticity

**Theorem 4.1.** *Any topologically weakly mixing map $g$ on a compact metric space $(Y, d)$ is collectively accessible.*

**Proof.** Let $b \in Y$ be a given point. For any $\varepsilon > 0$, denote

$$S = B(b, \frac{1}{2}\varepsilon) = \{a \in Y \mid d(a, b) < \frac{1}{2}\varepsilon\} \subset Y.$$

By Lemma 3.1, $g$ is topologically weakly mixing if and only if so is $g^{(n)} = \underbrace{g \times g \times \cdots \times g}_{n}$ for any integer $n \geq 2$. Then, for any nonempty open subsets $S_1^{(1)}, S_2^{(1)}, \cdots, S_s^{(1)}, S_1^{(2)}, S_2^{(2)}, \cdots, S_t^{(2)} \subset Y$ and the above subset $S \subset Y$, there exist $y_i^{(1)} \in S_i^{(1)}$ for any $i \in \{1, 2, \cdots, s\}$, $y_j^{(2)} \in S_j^{(2)}$ for any $j \in \{1, 2, \cdots, t\}$ and an integer $m > 0$ such that $g^m(y_i^{(1)}) \in S$ for any $i \in \{1, 2, \cdots, s\}$ and $g^m(y_j^{(2)}) \in S$ for any $j \in \{1, 2, \cdots, t\}$. This implies that $d(g^m(y_i^{(1)}), g^m(y_j^{(2)})) < \varepsilon$ for any $i \in \{1, 2, \cdots, s\}$ and any $j \in \{1, 2, \cdots, t\}$. Consequently, $g$ is collectively accessible. □

**Corollary 4.1.** *Any topologically weakly mixing map $g$ on a compact metric space $(Y, d)$ is collectively Kato chaotic.*

**Proof.** By Theorem 4.1 in [36], if $g$ is a topologically weakly mixing map, then it is collectively sensitive. By Theorem 4.1 and hypothesis, $g$ is collectively accessible. Thus, $g$ is collectively Kato chaotic. □

**Theorem 4.2.** *Any topologically weakly mixing map $g$ on a compact metric space $(Y, d)$ is strongly accessible.*

**Proof.** Suppose that $g$ is a topologically weakly mixing map on a compact metric space $(Y, d)$. Then, for any integer $N > 0$, $g^{(N)} = \underbrace{g \times g \times \cdots \times g}_{N}$ is topologically transitive.

Let $\varepsilon > 0$. Pick $y_0 \in Y$. Write

$$V = \{y \in Y \mid d(y_0, y) < \frac{1}{2}\varepsilon\}.$$

By the transitivity of $g^{(N)}$, for any $\varepsilon > 0$, any integer $N > 1$ and any nonempty open sets $U_1, U_2, \cdots, U_N \subset Y$, there exists an integer $m \geq 0$ satisfying $g^m(U_j) \cap V \neq \emptyset$ for any $j \in \{1, 2, \cdots, N\}$. So, for any $j \in \{1, 2, \cdots, N\}$, there exist $x_j \in U_j$ satisfying $g^m(x_j) \in V$. This implies that $d(g^m(x_i), g^m(x_j)) < \varepsilon$ for any $i, j \in \{1, 2, \cdots, N\}$. Thus, $g$ is strongly accessible. □

**Theorem 4.3.** *Any topologically weakly mixing map $g$ on a compact metric space $(Y, d)$ is chaotic in the strong sense of Kato.*

**Proof.** Assume that $g$ is a topologically weakly mixing map on a compact metric space $(Y, d)$. Then, for any integer $N > 0$, $g^{(N)}$ is topologically transitive.

The following will prove that the set

$$Tran(g^{(N)}) = \{(y_1, y_2, \cdots, y_N) \in Y^{(N)} \mid \overline{orb((y_1, y_2, \cdots, y_N), g^{(N)})} = Y^{(N)}\}$$

which consists of all transitive points of the system $(Y^{(N)}, g^{(N)})$, is a dense $G_\delta$ set.

Since $Y$ is a compact metric space, it has a countable topological basis $\mathfrak{B} = \{B_j\}_{j=1}^{\infty}$. This means that $\mathfrak{B}^{(N)}$ is a topological basis of $Y^{(N)}$. Obviously, one has

that

$$Tran(g^{(N)}) = \bigcap_{i_1,\cdots,i_N\in\{1,2,\cdots\}} \bigcup_{m\in\{0,1,\cdots\}} \underbrace{(g^m \times g^m \times \cdots \times g^m)}_{N}(B_{i_1}\times B_{i_2}\times\cdots\times B_{i_N}).$$

By the transitivity of $(Y^{(N)}, g^{(N)})$, for any nonempty open sets

$$U_1 \times U_2 \times \cdots \times U_N, V_1 \times V_2 \times \cdots \times V_N \subset Y^{(N)},$$

there exists an integer $m \geq 0$ satisfying

$$\underbrace{(g^m \times g^m \times \cdots \times g^m)}_{N}(U_1 \times U_2 \times \cdots \times U_N) \cap (V_1 \times V_2 \times \cdots \times V_N) \neq \emptyset.$$

So, $Tran(g^{(N)})$ is a dense $G_\delta$ set.

Now, it can be proved that $g$ is totally maximum sensitive (TMS, for short). By the definition of $r_N(g)$, for any $\lambda_0 \in (0, r_N(g))$, there exist points $y_1, y_2, \cdots, y_N \in Y$ satisfying

$$\min\{d(y_i, y_j) \mid i, j \in \{1, 2, \cdots, N\}, i \neq j\} \geq r_N(g) - \frac{1}{2}\lambda_0.$$

From the above argument, one can see that for any nonempty open set $(U_1 \times U_2 \times \cdots \times U_N) \subset Y^{(N)}$, there exists

$$(y_1, y_2, \cdots, y_N) \in (U_1 \times U_2 \times \cdots \times U_N) \cap Tran(g^{(N)})$$

and an integer $m \geq 0$ satisfying

$$\max\{d(x_i, g^m(y_i)) \mid i, j \in \{1, 2, \cdots, N\}, i \neq j, x_i \in U_i\} < \frac{1}{4}\lambda_0.$$

Therefore, for any $i, j \in \{1, 2, \cdots, N\}, i \neq j$, one has that

$$d(g^m(y_i), g^m(y_j)) \geq d(x_i, x_j) - d(x_j, g^m(y_j)) - d(x_i, g^m(y_i)) \geq r_N(g) - \lambda_0.$$

Hence $r_N(g) - \lambda_0$ is a $N$-sensitive coefficient of $(Y, g)$. By the arbitrariness of $N$ and $\lambda_0$, $g$ is TMS. □

For a continuous self-map $g$ on a compact metric space $(Y, d)$, [10] proved that if $\overline{g}$ is accessible, then $g$ is accessible. Now, it will be shown that there are similar conclusions about collectively accessible and collectively Kato chaotic. However, [10] did not tell us whether Kato's chaoticity of $g$ on a compact metric space $(Y, d)$ implies Kato's chaoticity of $\overline{g}$. The following will give some results to answer this question.

**Theorem 4.4.** *For a continuous self-map $g$ on a compact metric space $(Y, d)$. If $\overline{g}$ is collectively accessible, so is $g$.*

**Proof.** Let $S_1^{(1)}, S_2^{(1)}, \cdots, S_s^{(1)}, S_1^{(2)}, S_2^{(2)}, \cdots, S_t^{(2)} \subset Y$ be nonempty open sets and $\varepsilon > 0$. By Lemma 3.1 in [10], both $e(S_i^{(1)})$ and $e(S_j^{(2)})$ are nonempty open subsets of $\kappa(Y)$ for any $i \in \{1, 2, \cdots, s\}$ and any $j \in \{1, 2, \cdots, t\}$. Since $\overline{g}$ is collectively accessible, then there exist $K_i^{(1)} \in e(S_i^{(1)})$ for any $i \in \{1, 2, \cdots, s\}$ and $K_j^{(2)} \in e(S_j^{(2)})$ for any $j \in \{1, 2, \cdots, t\}$ such that one of the following holds:

(1) there is an $i_0 \in \{1, 2, \cdots, s\}$ such that $d_H(\overline{g}^m(K_{i_0}^{(1)}), \overline{g}^m(K_j^{(2)})) < \varepsilon$ for any $j \in \{1, 2, \cdots, t\}$ and some integer $m > 0$.

(2) there is a $j_0 \in \{1, 2, \cdots, t\}$ such that $d_H(\overline{g}^m(K_i^{(1)}), \overline{g}^m(K_{j_0}^{(2)})) < \varepsilon$ for any $i \in \{1, 2, \cdots, s\}$ and some integer $m > 0$.

If (1) holds, by definition of $d_H$, for any $x \in K_{i_0}^{(1)}$,

$$d(g^m(x), g^m(K_j^{(2)})) = \inf\{d(g^m(x), g^m(y)) \mid y \in K_j^{(2)}\} < \varepsilon$$

for any $j \in \{1, 2, \cdots, t\}$. Choose $x_{i_0} \in K_{i_0}^{(1)} \subset S_{i_0}^{(1)}$. Then there exists a point $y_j \in K_j^{(2)} \subset S_j^{(2)}$ satisfying $d(g^m(x_{i_0}), g^m(y_j)) < \varepsilon$ for any $j \in \{1, 2, \cdots, t\}$.

Similarly, if (2) holds, for any $y \in K_{j_0}^{(2)}$,

$$d(g^m(y), g^m(K_i^{(1)})) = \inf\{d(g^m(y), g^m(x)) \mid x \in K_i^{(1)}\} < \varepsilon$$

for any $i \in \{1, 2, \cdots, s\}$. One can choose $y_{j_0} \in K_{j_0}^{(2)} \subset S_{j_0}^{(2)}$. Then there exists a point $x_i \in K_i^{(1)} \subset S_i^{(1)}$ satisfying $p(g^m(y_{j_0}), g^m(x_i)) < \varepsilon$ for any $i \in \{1, 2, \cdots, s\}$. Thus, $g$ is collectively accessible. □

**Corollary 4.2.** *For a continuous self-map $g$ on a compact metric space $(Y, d)$. If $\overline{g}$ is collectively Kato chaotic, so is $g$.*

**Proof.** By Theorem 4.4, If $\overline{g}$ is collectively accessible, so is $g$. By hypothesis and Corollary 2.4 in [36], $g$ is collectively sensitive. So, $g$ is collectively Kato chaotic. □

**Theorem 4.5.** *For a continuous self-map $g$ on a compact metric space $(Y, d)$. If $\overline{g}$ is strongly accessible, so is $g$.*

**Proof.** For any integer $t > 0$, let $S_1, S_2, \cdots, S_t \subset Y$ be nonempty open sets and $\varepsilon > 0$. By Lemma 3.1 in [10], $e(S_j)$ is a nonempty open subset of $\kappa(Y)$ for any $j \in \{1, 2, \cdots, t\}$.

Since $\overline{g}$ is strongly accessible, then there exist $K_j \in e(S_j)$ for any $j \in \{1, 2, \cdots, t\}$ such that $d_H(\overline{g}^m(K_i), \overline{g}^m(K_j)) < \varepsilon(i, j \in \{1, 2, \cdots, t\})$ for some integer $m > 0$. By definition of $d_H$, for any $i \in \{1, 2, \cdots, t\}$ and any $x \in K_i$,

$$d(g^m(x), g^m(K_j)) = \inf\{d(g^m(x), g^m(y)) \mid y \in K_j\} < \varepsilon$$

for any $j \in \{1, 2, \cdots, t\}$. Choose $x_i \in K_i \subset S_i$. Then there exists a point $y_j \in K_j \subset S_j$ such that $p(g^m(x_i), g^m(y_j)) < \varepsilon$ for any $j \in \{1, 2, \cdots, t\}$. Thus, $g$ is strongly accessible. □

**Theorem 4.6.** *For a continuous self-map $g$ on a compact metric space $(Y, d)$. If $\overline{g}$ is chaotic in the strong sense of Kato, so is $g$.*

**Proof.** Firstly, if $\overline{g}$ is TMS, so is $g$. Let $N > 0$ be a given integer, $\overline{g}$ is TMS, and $\lambda > 0$ is a $N$-sensitive coefficient of the dynamical system $(\kappa(Y), \overline{g})$. Then, for any nonempty open set $V \subset Y$ , there exist $N$ points $\overline{v}_1, \overline{v}_2, \cdots, \overline{v}_N \in e(V)$ satisfying

$$\min\{d_H(\overline{g}^m(\overline{v}_i), \overline{g}^m(\overline{v}_j)) \mid i, j \in \{1, 2, \cdots, N\}, i \neq j\} \geq \lambda.$$

So, by

$$d_H(\overline{g}^m(\overline{v}_i), \overline{g}^m(\overline{v}_j)) = \max\{\sup_{a \in \overline{v}_i} d(g^m(a), g^m(\overline{v}_j)), \sup_{b \in \overline{v}_j} d(g^m(a), g^m(\overline{v}_i))\}$$

for any $i, j \in \{1, 2, \cdots, N\}$, $i \neq j$, one has

$$\sup_{a \in \overline{v}_i} d(g^m(a), g^m(\overline{v}_j)) > \lambda$$

or

$$\sup_{a \in \overline{v}_j} d(g^m(a), g^m(\overline{v}_i))\} > \lambda$$

for any $i, j \in \{1, 2, \cdots, N\}$, $i \neq j$. Without loss of generality, one can assume that

$$\sup_{a \in \overline{v}_i} d(g^m(a), g^m(\overline{v}_j)) > \lambda$$

for any $i, j \in \{1, 2, \cdots, N\}$, $i \neq j$. Then, for any $i, j \in \{1, 2, \cdots, N\}$, $i \neq j$, there exists a point $a_i \in \overline{v}_i$ such that

$$d(g^m(a), g^m(\overline{v}_j)) = \inf_{b \in \overline{v}_j} \{d(g^m(a), g^m(b))\} > \lambda.$$

Hence, for any $i, j \in \{1, 2, \cdots, N\}$, $i \neq j$, one can choose a point $b_j \in \overline{v}_j$ satisfying $d(g^m(a), g^m(b_j)) > \lambda$. By the definition and the above argument, $g$ is $N$-sensitive with the $N$-sensitive coefficient $\lambda$. Therefore, the $N$-critically sensitive coefficient $\lambda_N(g)$ of the system $(Y, g)$ is not less than the $N$-critically sensitive coefficient $\lambda_N(\overline{g})$ of the system $(\kappa(Y), \overline{g})$. That is, $\lambda_N(\overline{g}) \leq \lambda_N(g)$ for any integer $N > 0$. Set

$$r_N(Y) = \sup_{y_1, y_2, \cdots, y_N \in Y} r(y_1, y_2, \cdots, y_N)$$

and

$$r_N(\kappa(Y)) = \sup_{\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_N \in \kappa(Y)} r(\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_N).$$

Obviously, $\lambda_N \leq r_N(Y) \leq r_N(\kappa(Y))$ for any integer $N > 0$. Since $\overline{g}$ is TMS, then $g$ is TMS.

By Theorem 4.5, $g$ is chaotic in the strong sense of Kato. □

**Theorem 4.7.** *For a topologically transitive map $g$ on the interval $[0, 1]$, $g$ and $\overline{g}$ are collectively accessible.*

**Proof.** Since $g$ is topologically transitive, by Lemma 3.2, there exists a fixed point $a \in (0, 1)$ of $g$ satisfying that $g^2|_{[0,a]}$ and $g^2|_{[a,1]}$ are all topologically mixing. This means that, for any $\varepsilon > 0$ and any nonempty open subsets $S_1^{(1)}, S_2^{(1)}, \cdots, S_s^{(1)}, S_1^{(2)}$, $S_2^{(2)}, \cdots, S_t^{(2)} \subset [0, 1]$, there exists an integer $l > 0$ such that, for any $k \geq l$,

$$d(g^k(y_i(k)), g^k(a)) < \frac{1}{st}\varepsilon$$

for any integer $k \geq l$, $\forall i \in \{1, 2, \cdots, s\}$, and

$$d(gk(a), gk(y_j(k))) < \frac{1}{st}\varepsilon$$

for any integer $k \geq l$, $\forall j \in \{1, 2, \cdots, t\}$. Where $y_i(k) \in S_i^{(1)} (i \in \{1, 2, \cdots, s\})$ and $y_j(k) \in S_j^{(2)} (j \in \{1, 2, \cdots, t\})$. Consequently, $d(g^k(y_i(k)), g^k(y_j(k))) < \varepsilon$ for any integer $k \geq l$, any $i \in \{1, 2, \cdots, s\}$ and any $j \in \{1, 2, \cdots, t\}$. So, $g$ is collectively accessible.

Since $g^2|_{[0,a]}$ and $g^2|_{[a,1]}$ are topologically mixing, then $\overline{g^2|_{[0,a]}}$ and $\overline{g^2|_{[a,1]}}$ are topologically mixing. This means that, for any $\varepsilon > 0$ and any nonempty open subsets $S_1^{(1)}, S_2^{(1)}, \cdots, S_s^{(1)}, S_1^{(2)}, S_2^{(2)}, \cdots, S_t^{(2)} \subset \kappa([0,1])$ there exists an integer $l > 0$ such that, for any $k \geq l$,

$$d_H(\overline{g}^k(y_i(k)), \overline{g}^k(\{a\})) < \frac{1}{st}\varepsilon$$

for any integer $k \geq l$, $\forall i \in \{1, 2, \cdots, s\}$ and

$$d_H(\overline{g}^k(\{a\}), \overline{g}^k(y_j(k))) < \frac{1}{st}\varepsilon$$

for any integer $k \geq l$, $\forall j \in \{1, 2, \cdots, t\}$. Where $y_i(k) \in S_i^{(1)} (i \in \{1, 2, \cdots, s\})$ and $y_j(k) \in S_j^{(2)} (j \in \{1, 2, \cdots, t\})$. Consequently, $d(\overline{g}^k(y_i(k)), \overline{g}^k(y_j(k))) < \varepsilon$ for any integer $k \geq l$, any $i \in \{1, 2, \cdots, s\}$ and any $j \in \{1, 2, \cdots, t\}$. So, $\overline{g}$ is collectively accessible. □

**Corollary 4.3.** *For a topologically transitive map $g$ on the interval $[0,1]$, $g$ and $\overline{g}$ are collectively Kato chaotic.*

**Proof.** Since $g$ is topologically transitive, then $g$ is cofintely sensitive. This implies that $g$ and $\overline{g}$ are collectively sensitive. By the definition and Theorem 4.4, $g$ and $\overline{g}$ are collectively Kato chaotic. □

**Theorem 4.8.** *For a topologically transitive map $g$ on the interval $[0,1]$, $g$ and $\overline{g}$ are strongly accessible.*

**Proof.** Since $g$ is topologically transitive, by Lemma 3.2, there exists a fixed point $a \in (0,1)$ of $g$ satisfying that $g^2|_{[0,a]}$ and $g^2|_{[a,1]}$ are topologically mixing. This means that for any $\varepsilon > 0$ and any nonempty open subsets $S_1, S_2, \cdots, S_t \subset [0,1]$, there exists an integer $l > 0$ such that, for any $k \geq l$, $d(gk(a), gk(y_j(k))) < \frac{1}{t}\varepsilon$ for $y_j(k) \in S_j (j \in \{1, 2, \cdots, t\})$. Consequently, $d(g^k(y_i(k)), g^k(y_j(k))) < \varepsilon$ for any integer $k \geq l$ and any $i, j \in \{1, 2, \cdots, t\}$. So, $g$ is strongly accessible.

Since $g^2|_{[0,a]}$ and $g^2|_{[a,1]}$ are topologically mixing, $\overline{g^2|_{[0,a]}}$ and $\overline{g^2|_{[a,1]}}$ are topologically mixing too. This means that for any $\varepsilon > 0$ and any nonempty open subsets $S_1, S_2, \cdots, S_t \subset \kappa([0,1])$, there exists an integer $l > 0$ such that, for any $k \geq l$,

$$d_H(\overline{g}^k(\{a\}), \overline{g}^k(y_j(k))) < \frac{1}{st}\varepsilon$$

for $y_j(k) \in S_j (j \in \{1, 2, \cdots, t\})$. Consequently, $d(\overline{g}^k(y_i(k)), \overline{g}^k(y_j(k))) < \varepsilon$ for any integer $k \geq l$ and $i, j \in \{1, 2, \cdots, t\}$. So, $\overline{g}$ is strongly accessible. □

For a topologically transitive map $g$ on the interval $[0,1]$, are $g$ and $\overline{g}$ chaotic in the strong sense of Kato? The following Theorem will discuss this question.

**Theorem 4.9.** *Let $g : [0,1] \to [0,1]$ be a topologically transitive map. $a \in (0,1)$ is a fixed point of $g$, and $g^2|_{[0,a]}$ and $g^2|_{[a,1]}$ are topologically mixing. If*

$$\lambda_N(g^2|_{[0,a]}) = \lambda_N(g^2|_{[a,1]})$$

*for any integer $N > 0$, then $g$ is chaotic in the strong sense of Kato.*

**Proof.** By Theorem 4.1, it is enough to prove that $g$ is TMS. By [16] and the hypothesis,

$$\lambda_N(g^2|_{[0,a]}) = r_N(g^2|_{[0,a]}) \quad \text{and} \quad \lambda_N(g^2|_{[a,1]}) = r_N(g^2|_{[a,1]})$$

for every integer $N > 0$. Clearly,

$$\lambda_N(g) = \min\{\lambda_N(g^2|_{[0,a]}), \lambda_N(g^2|_{[a,1]})\}$$

and

$$r_N(g) = \max\{r_N(g^2|_{[0,a]}), r_N(g^2|_{[a,1]})\}$$

for every integer $N > 0$. Since

$$\lambda_N(g^2|_{[0,a]}) = \lambda_N(g^2|_{[a,1]}),$$

by Theorem 4.2, $g$ is chaotic in the strong sense of Kato. □

In [10], R. Gu proved that a continuous self-map $g$ on a compact metric space $(Y, d)$ is chaotic in the sense of Kato if and only if $\overline{g}$ is Kato chaotic in $w^e$-topology. Inspired by this result, the following will show that a continuous self-map $g$ on a compact metric space $(Y, d)$ is collectively accessible (or strongly accessible) if and only if so is $\overline{g}$ in $w^e$-topology.

**Theorem 4.10.** *For a continuous self-map $g$ on a compact metric space $(Y, d)$, it is collectively accessible if and only if so is $\overline{g}$ in $w^e$-topology.*

**Proof.** Let $s, t$ be two given integers. Assume that $g$ is a collectively accessible continuous self-map on a compact metric space $(Y, d)$ in $w^e$-topology, and that $\widetilde{U}_j$ and $\widetilde{V}_h$ are nonempty open subsets in the $w^e$-topology of $\kappa(Y)$ for any $j \in \{1, 2, \cdots, s\}$, any $h \in \{1, 2, \cdots, t\}$ and $\varepsilon > 0$. By Lemma 3.1 in [11], for any $k \in \{1, 2, \cdots, s\}$ and any $h \in \{1, 2, \cdots, t\}$, there exist nonempty open subsets $A_{i,k}, B_{j,h} \subset Y$ satisfying $\widetilde{U}_k = \bigcup\limits_i A_{i,k}$ and $\widetilde{V}_h = \bigcup\limits_j B_{j,h}$. Fixed $A_{i_k,k}$ and $B_{j_h,h}$ for any $k \in \{1, 2, \cdots, s\}$ and any $h \in \{1, 2, \cdots, t\}$. Since $g$ is collectively accessible in $w^e$-topology, then there exist $a_{i_k,k} \in A_{i_k,k}$ $(k \in \{1, 2, \cdots, s\})$, $b_{j_h,h} \in B_{j_h,h}$ $(h \in \{1, 2, \cdots, t\})$ and an integer $m \geq 0$ such that one of the following holds:

(1) there is a $k_0 \in \{1, 2, \cdots, s\}$ such that

$$d(g^m(a_{i_{k_0},k_0}), g^m(b_{j_h,h})) < \varepsilon$$

for any $h \in \{1, 2, \cdots, t\}$.

(2) there is an $h_0 \in \{1, 2, \cdots, t\}$ such that

$$d(g^m(a_{i_k,k}), g^m(b_{j_{h_0},h_0})) < \varepsilon$$

for any $k \in \{1, 2, \cdots, s\}$.

Clearly, $\{a_{i_k,k}\} \in e(A_{i_k,k}) \subset \widetilde{U}_k$ and $\{b_{j_h,h}\} \in e(B_{j_h,h}) \subset \widetilde{V}_h$ for any $k \in \{1, 2, \cdots, s\}$ and any $h \in \{1, 2, \cdots, t\}$. Also, one has that

$$d_H(\overline{g}^m(\{a_{i_{k_0},k_0}\}), \overline{g}^m(\{b_{j_h,h}\})) = d(g^m(a_{i_{k_0},k_0}), g^m(b_{j_h,h})) < \varepsilon$$

for any $h \in \{1, 2, \cdots, t\}$ or

$$d_H(\overline{g}^m(\{a_{i_k,k}\}), \overline{g}^m(\{b_{j_{h_0},h_0}\})) = d(g^m(a_{i_k,k}), g^m(b_{j_{h_0},h_0})) < \varepsilon$$

for any $k \in \{1, 2, \cdots, s\}$. Thus, $\overline{g}$ is collectively accessible in $w^e$-topology. □

**Theorem 4.11.** *For a continuous self-map $g$ on a compact metric space $(Y,d)$, it is strongly accessible if and only if so is $\overline{g}$ in $w^e$-topology.*

**Proof.** Let $s$ be a given integer. Assume that $g$ is a strongly accessible map on a compact metric space $(Y,d)$ in $w^e$-topology, $\widetilde{U}_j (j \in \{1,2,\cdots,s\})$ are nonempty open subsets in the $w^e$-topology of $\kappa(Y)$, and $\varepsilon > 0$. By Lemma 3.1 in [11], for any $k \in \{1,2,\cdots,s\}$ there are nonempty open subsets $A_{i,k}, B_{j,h} \subset Y$ satisfying $\widetilde{U}_k = \bigcup\limits_i A_{i,k}$. Fixed $A_{i_k,k}$ for any $k \in \{1,2,\cdots,s\}$. Since $g$ is strongly accessible in $w^e$-topology, there exist $a_{i_k,k} \in A_{i_k,k}$ $(k \in \{1,2,\cdots,s\})$ and an integer $m \geq 0$ such that

$$d(g^m(a_{i_k,k}), g^m(a_{j_h,h})) < \varepsilon$$

for any $k,h \in \{1,2,\cdots,s\}$. Clearly, $\{a_{i_k,k}\} \in e(A_{i_k,k}) \subset \widetilde{U}_k$ for any $k \in \{1,2,\cdots,s\}$. Also, one has that

$$d_H(\overline{g}^m(\{a_{i_k,k}\}), \overline{g}^m(\{a_{j_h,h}\})) = d(g^m(a_{i_k,k}), g^m(a_{j_h,h})) < \varepsilon$$

for any $k,h \in \{1,2,\cdots,s\}$. Thus, $\overline{g}$ is strongly accessible in $w^e$-topology. □

**Remark 4.1.** (1) Let $g$ be a collectively Kato chaotic map on a compact metric space $(Y,d)$ in $w^e$-topology. Is collectively Kato chaotic $\overline{g}$ in $w^e$-topology? (2) Assume that $g$ is a continuous self-map on a compact metric space $(Y,d)$, and that $\overline{g}$ is collectively Kato chaotic in $w^e$-topology. Is collectively Kato chaotic $g$ in $w^e$-topology?(3) Let $g$ be a continuous self-map on a compact metric space $(Y,d)$ and $g$ be chaotic in the strong sense of Kato $w^e$-topology. Is $\overline{g}$ chaotic in the strong sense of Kato in $w^e$-topology? (4) Let $g$ be a continuous self-map on a compact metric space $(Y,d)$ and $\overline{g}$ be chaotic in the strong sense of Kato $w^e$-topology. Is $g$ chaotic in the strong sense of Kato in $w^e$-topology?

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